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Boundedness of Anisotropic Pseudo-Differential Operators in Function Spaces of Besov-Hardy-Sobolev Type

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Es werden Pseudodifferentialoperatoren in anisotropen Funktionenräumen vom Besov-Hardy-Sobolev-Typ $B^{\bar{s}}_{p,q}$ und $F^{\bar{s}}_{p,q}$ betrachtet. Dazu wird eine Verallgemeinerung der klassischen Hörmanderschen Pseudodifferentialoperatoren für den anisotropen Fall eingeführt und ein Satz über die Beschränktheit dieser anisotropen Pseudodifferentialoperatoren in den entsprechenden anisotropen Funktionenräumen bewiesen.

Рассматриваются псевдодифференциальные операторы в анизотропных функциональных пространствах типа Бесова-Харди-Соболева $B^{\bar{s}}_{p,q}$ и $F^{\bar{s}}_{p,q}$. Для этого обобщаются классические псевдодифференциальные операторы Хёрмандера на анизотропный случай и доказывается предложение об ограниченности этих анизотропных псевдодифференциальных операторов в соответствующих анизотропных функциональных пространствах.

This paper is concerned with pseudo-differential operators in the anisotropic function spaces of Besov-Hardy-Sobolev type $B^{\bar{s}}_{p,q}$ and $F^{\bar{s}}_{p,q}$. An anisotropic generalization of the classical Hörmander class of pseudo-differential operators is introduced and a theorem about the boundedness of these anisotropic pseudo-differential operators in associated anisotropic function spaces is proved.

There are several results concerning the boundedness of pseudo-differential operators in function spaces. From the results of HÖRMANDER [12], CALDERÓN and VAILLANCOURT [6] and CHING [7] it follows that pseudo-differential operators of Hörmander class $S^0_{\varrho,\delta}$ are bounded in $L_2(\mathbf{R}^n)$ if and only if $0 \leqq \delta \leqq \varrho \leqq 1$ and $(\varrho, \delta) \neq (1, 1)$. The problem of boundedness of pseudo-differential operators of class $S^m_{\varrho,\delta}$ in $L_p(\mathbf{R}^n)$ has been studied by KAGAN [15], KUMANO-GO and NAGASE [18], ILLNER [14], FEFFERMAN [8] and others. Then there was a development in two directions. On the one hand there were defined more general classes of pseudo-differential operators — see for example BEALS [1], HÖRMANDER [13] — and results were proved on the boundedness of these pseudo-differential operators in $L_2(\mathbf{R}^n)$ and $L_p(\mathbf{R}^n)$, respectively [2—4]. On the other hand, the classical pseudo-differential operators $S^m_{\varrho,\delta}$ were considered in more complicated function spaces, as by GOLDBERG [9], BUI HUY QUI [5], NILSSON [20] in local Hardy spaces and Hardy-Triebel spaces $F^s_{p,q}$. A relatively final result was obtained by PÄIVÄRINTA [21] in the case of isotropic Hardy-Triebel spaces and Sobolev-Besov spaces: The pseudo-differential operators of class $S^0_{1,\delta}$ are bounded in $F^s_{p,q}$ and $B^s_{p,q}$ if $0 \leqq \delta < 1$ and $0 < p, q < \infty$.

This paper is concerned with pseudo-differential operators in anisotropic function spaces of Besov-Hardy-Sobolev type. These spaces are a general scale of anisotropic function spaces (like their isotropic "counterpart"), containing for example the anisotropic Bessel-potential spaces and the anisotropic Sobolev spaces — see [26].

Of course an anisotropic structure in the function space requires an adequate anisotropic structure in the definition of the pseudo-differential operators. The symbol classes $S^m_{a;\varrho,\delta}$, defined in the following, are a natural generalization of the symbol class $S^m_{\varrho,\delta}$ introduced by Hörmander [10, 11] and Kohn and Nirenberg [16].

The results obtained here are related to those in [21].

1. Definition and basic properties of anisotropic pseudo-differential operators

Let $a = (a_1, \ldots, a_n)$ be a fixed n-tuple of positive numbers and $a_1 + \cdots + a_n = n$. The *anisotropic distance* of $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ from the origin is defined by

$$|x|_a = (|x_1|^{2/a_1} + \cdots + |x_n|^{2/a_n})^{1/2}.$$

Now, fixing an anisotropy a, let us define an associated class of pseudo-differential operators as follows.

Definition: We say that a C^∞-function $p(x, \xi)$ defined on $\mathbf{R}^{2n} = \mathbf{R}_x^n \times \mathbf{R}_\xi^n$ is a *symbol of class* $S^m_{a;\varrho,\delta}$ $(-\infty < m < \infty;\ 0 \leqq \delta \leqq \varrho \leqq 1,\ \delta < 1)$ if for any multi-indices α, β there exists a constant $c_{\alpha\beta}$ such that

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq c_{\alpha\beta}(1 + |\xi|_a)^{m - \|\alpha\|\varrho + \|\beta\|\delta}$$

in $\mathbf{R}^{2n}$, where

$$p^{(\alpha)}_{(\beta)}(x, \xi) = \partial_\xi{}^\alpha D_x{}^\beta p(x, \xi),$$

$$\|\alpha\| = \sum_{j=1}^n a_j\alpha_j, \qquad \|\beta\| = \sum_{j=1}^n a_j\beta_j.$$

From the definition of the anisotropic distance we obtain

$$c_1(1 + |\xi|)^{\frac{1}{\max a_j}} \leqq 1 + |\xi|_a \leqq c_2(1 + |\xi|)^{\frac{1}{\min a_j}} \tag{1}$$

and as a consequence of $\min a_j > 0$ there exists the imbedding

$$S^m_{a;\varrho,\delta} \subset S^{m'}_{\varrho',\delta'} \quad \text{for} \quad \delta < \frac{\min a_j}{\max a_j},$$

where $m' = \dfrac{m}{\min a_j}$, $\varrho' = \varrho\,\dfrac{\min a_j}{\max a_j}$, $\delta' = \delta\,\dfrac{\max a_j}{\min a_j}$ and $S^{m'}_{\varrho',\delta'}$ denotes the Hörmander class of pseudo-differential operators.

As usual, the *pseudo-differential operator* $P(x, D_x)$ *with the symbol* $p(x, \xi)$ is defined by

$$P(x, D_x)\, u(x) = \int e^{ix\xi} p(x, \xi)\, \hat{u}(\xi)\, d\xi \quad \text{for} \quad u \in S(\mathbf{R}^n),$$

where

$$\hat{u}(\xi) = \int e^{-iy\xi} u(y)\, dy \quad \text{and} \quad d\xi = (2\pi)^{-n}\, d\xi.$$

Because of (1), this operator maps $S(\mathbf{R}^n)$ continuously into itself and we may extend it to a continuous operator from $S'(\mathbf{R}^n)$ into $S'(\mathbf{R}^n)$. The mapping $p(x, \xi) \to P(x, D_x)$ is a bijection. We shall represent the symbol $p(x, \xi)$ of $P(x, D_x)$ also by $\sigma(P)\,(x, \xi)$.

For $p \in S^m_{a;\varrho,\delta}$ and $l = a_1\gamma_1 + \cdots + a_n\gamma_n$ where γ is an arbitrary multi-index, we define the *semi-norms* $|p|_l^{(m)}$ by

$$|p|_l^{(m)} = \max_{\|\alpha+\beta\|\leq l} \sup_{\mathbf{R}^{2n}} \{|p_{(\beta)}^{(\alpha)}(x, \xi)| (1 + |\xi|_a)^{-(m-\|\alpha\|\varrho+\|\beta\|\delta)}\}.$$

Then $S^m_{a;\varrho,\delta}$ is a Frechet space with these semi-norms, and we have for any $p(x, \xi) \in S^m_{a;\varrho,\delta}$

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leq |p|^{(m)}_{\|\alpha+\beta\|}(1 + |\xi|_a)^{m-\|\alpha\|\varrho+\|\beta\|\delta} \quad \text{and} \quad p_{(\beta)}^{(\alpha)} \in S^{m-\|\alpha\|\varrho+\|\beta\|\delta}_{a;\varrho,\delta}.$$

The symbols

$$\left(\sum_{j=1}^{n} (1 + \xi_j^2)^{k/2a_j}\right)^{s/k}$$

are an interesting example, since these symbols define lifting operators in the anisotropic function spaces, which will be defined in the second section. It is not hard to show that for any $s \in \mathbf{R}$ the symbols belong to the class $S^s_{a;1,0}$, provided that $k \in \mathbf{R}$ is such that $k/2a_j \in \mathbf{N}$ $(j = 1, 2, \ldots, n)$. To get this result only for n-tuples $a = (a_1, a_2, \ldots, a_n)$ with rational components is unnatural. But it follows by the above kind of definition of the symbols — there are only integer derivatives, and this is not compatible with an irrational anisotropy in some direction. We get some simple but non-trivial examples if we consider $\mathbf{R}^2$ with the anisotropy $a = (a_1, a_2)$ and $a_1 = 1/2$, $a_2 = 3/2$. Then $(\xi_1^6 + \xi_2^2)/((1 + \xi_1^2)^3 + (1 + \xi_2^2)) \in S^0_{a;1,0}$ and

$$D_\xi^\alpha D_x^\beta(\xi_1^6 + \xi_2^2 + \varrho^{2m}(x)\, \xi_2^6) \cdot (1 + \xi_1^6 + \xi_2^2 + \varrho^{2m}(x)\, \xi_2^6)^{-1} \in S^{-\|\alpha\|+\|\beta\|\delta}_{a;1,\delta}$$

for all multi-indices α and β, if $m > 2$ is an integer, $\delta = 2/m$ and ϱ is a function belonging to $B(\mathbf{R}^2) = \{\varphi\colon \sup_x |D_x^\gamma \varphi(x)| \leq c_\gamma$ for every multi-index $\gamma\}$.

Theorem 1: *Assume that* $0 \leq \delta < \varrho \leq 1$. *Let* $P_1(x, D_x) \in S^{m_1}_{a;\varrho,\delta}$ *and* $P_2(x, D_x) \in S^{m_2}_{a;\varrho,\delta}$. *Then* $P(x, D_x) = P_1(x, D_x)\, P_2(x, D_x)$ *belongs to* $S^{m_1+m_2}_{a;\varrho,\delta}$ *and the symbol* $\sigma(P)\,(x, \xi)$ *is expressed by*

$$\sigma(P)\,(x, \xi) = \text{Os-}\iint e^{-iy\eta} p_1(x, \xi + \eta)\, p_2(x + y, \xi)\, dy\, d\eta.$$

(Os-$\iint$ *denotes an oscillatory integral in the sense of* Kumano-go [17].) *Moreover, if we set*

$$p_\alpha(x, \xi) = p_1^{(\alpha)}(x, \xi)\, p_{2(\alpha)}(x, \xi) \in S^{m_1+m_2-(\varrho-\delta)\|\alpha\|}_{a;\varrho,\delta}$$

and

$$r_{\gamma,\theta}(x, \xi) = \text{Os-}\iint e^{-iy\eta} p_1^{(\gamma)}(x, \xi + \theta\eta)\, p_{2(\gamma)}(x + y, \xi)\, dy\, d\eta,$$

then for any N we have the expansion formula

$$\sigma(P)\,(x, \xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} p_\alpha(x, \xi) + N \sum_{|\gamma|=N} \int_0^1 \frac{(1 - \theta)^{N-1}}{\gamma!} r_{\gamma,\theta}(x, \xi)\, d\theta,$$

where $\{r_{\gamma,\theta}\}_{|\theta|\leq 1}$ *is for any γ and any real number $\varkappa > 0$ a bounded set in* $S^{m_1+m_2+\varkappa-\|\gamma\|(\varrho-\delta)}_{a;\varrho,\delta}$. *Furthermore, for any l and any $\varkappa > 0$ there exist constants c, c' and a positive number $L(l, a, \varkappa, n)$ independent of θ such that*

$$|p_\alpha|_l^{(m_1+m_2-\|\alpha\|(\varrho-\delta))} \leq c\, |p_1|^{(m_1)}_{l+\|\alpha\|}\, |p_2|^{(m_2)}_{l+\|\alpha\|} \tag{2}$$

and

$$|r_{\gamma,\theta}|_l^{(m_1+m_2+\varkappa-\|\gamma\|(\varrho-\delta))} \leq c'\, |p_1|^{(m_1)}_{l'}\, |p_2|^{(m_2)}_{l'} \tag{3}$$

hold for all $l' \geq L$.

This theorem contains the fundamental properties of anisotropic pseudo-differential operators which will be needed in this paper. It should be remarked that the number $\varkappa$ in (3) will be zero if we have an anisotropy $a = (a_1, \ldots, a_n)$ with rational components. We need the number $\varkappa > 0$ in the general case only for technical reasons — see also the first example. Hence (3) is not the best possible result, but it is sufficient for the proof of our main result in Section 3. In the isotropic case, which means $a_1 = \cdots = a_n = 1$, all results (with $\varkappa = 0$) may be found in KUMANO-GO [17; Chapter 2]. The proof of the theorem in the anisotropic case turns out to be analogous to the proof in [17] for the isotropic case and therefore it is omitted. There are only some technical complications — especially in proving (3).

The following lemma, which gives estimates of the kernels of pseudo-differential operators with bounded support in ξ, will be useful later on in the proof of the theorem in Section 3.

Lemma: *Let $q(x, \xi) \in S^{-\infty} \cap S^0_{a;1,\delta}$ and $\delta < 1$. Then the associated kernel $K(x, y) = \int e^{iy\xi} q(x, \xi)\, d\xi$ belongs to $C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and we can write*

$$Q(x, D_x)\, u(x) = \int K(x, x-y)\, u(y)\, dy \quad \text{for} \quad u \in S(\mathbf{R}^n).$$

If in addition $\operatorname{supp}_\xi q \subset \{\xi : 2^{k-j} \leqq |\xi|_a \leqq 2^{k+j}\}$ for arbitrary fixed j and k, then for every $M > 0$ and l with $2l > M + n/\min a_j + 2 \max a_j + 1$ there exists a constant c which is independent of $q(x, \xi)$ and k such that

$$\int |K(x, x-y)|\, (1 + 2^k\, |x-y|_a)^M\, dy \leqq c\, |q|_l^{(0)}. \tag{4}$$

Proof: If $2l' > M + n/\min a_j$, then it is easy to show that

$$\int |K(x, x-y)|\, (1 + 2^k\, |x-y|_a)^M\, dy$$
$$\leqq c(l', n)\, 2^{-kn} \sup_y |K(x_i, 2^{-ka_i} y_i)|\, (1 + |y|_a^{2l'})$$

holds. Now we have

$$|y|_a^{2l'} \leqq c_{l'} (|y_1|^{(2/a_1)l'} + \cdots + |y_n|^{(2/a_n)l'})$$

and

$$|y_p|^{\gamma_p} K(x_i, 2^{-ka_i} y_i) = \int e^{i(2^{ka_i} y_i, \xi_i)} 2^{ka_p \gamma_p} D_{\xi_p}^{\gamma_p} q(x, \xi)\, d\xi$$

for every $\gamma_p \in \mathbf{N}$ and $p = 1, 2, \ldots, n$. Setting $\gamma_p = [(2/a_p)\, l'] + 2$ we get

$$\int |K(x, x-y)|\, (1 + 2^k\, |x-y|_a)^M\, dy$$
$$\leqq c 2^{-kn} \sum_{p=1}^n \Big(\sup_{\substack{y \\ y_p \leqq 1}} 2\, |K(x_i, 2^{-ka_i} y_i)| + \sup_{\substack{y \\ y_p \geqq 1}} |K(x_i, 2^{-ka_i} y_i)\, y_p^{\gamma_p}| \Big)$$
$$\leqq c \sum_{p=1}^n \Big(2^{-kn} \sup_y 2 \int |q(x, \xi)|\, d\xi + 2^{-kn} \sup_y \int 2^{ka_p \gamma_p} D_{\xi_p}^{\gamma_p} q(x, \xi)|\, d\xi \Big)$$
$$\leqq c 2^n 2^{-kn} \int_{|\xi|_a < 2^{k+j}} |q(x, \xi)|\, d\xi + c \sum_{p=1}^n 2^{-kn + ka_p \gamma_p} \int_{2^{k-j} < |\xi|_a < 2^{k+j}} |D_{\xi_p}^{\gamma_p} q(x, \xi)|\, d\xi.$$

Moreover we notice that

$$|D_{\xi_p}^{\gamma_p} q(x, \xi)| \leqq |q|^{(0)}_{a_p \gamma_p} (1 + |\xi|_a)^{-\gamma_p a_p} \quad \text{and} \quad \int_{|\xi|_a < 2^{k+j}} d\xi \leqq c 2^{(k+j)n}.$$

This proves that

$$\int |K(x, x-y)|\, (1 + 2^k\, |x-y|_a)^M\, dy \leqq c(l, j, n)\, |q|_l^{(0)}$$

holds if $l > 2l' + 2 \max a_j + 1$, $2l' > M + n/\min a_j$ where $c(l, j, n)$ is a constant which is independent of $q(x, \xi)$ and k ∎

Corollary: *Let* $q(x, \xi) = q_1(x, \xi)\, q_2(x, \xi)$, $q_1 \in S^{-\infty} \cap S^{m_1}_{a;1,\delta}$, $q_2 \in S^{m_2}_{a;1,\delta}$ *and* $\operatorname{supp}_\xi q_1 \subset \{\xi : 2^{k-j} \leqq |\xi|_a \leqq 2^{k+j}\}$. *$K(x, y)$ denotes again the associated kernel-function. Then for every* $M > 0$ *and* $l > M + n/\min a_j + 2 \max a_j + 1$ *there exists a constant* c *which is independent of* $q(x, \xi)$ *and* k *such that for all* $m \in \mathbf{R}$ *with* $-m \geqq m_2$

$$\int |K(x, x - y)|\, (1 + 2^k |x - y|_a)^M\, dy \leqq c\, |q_1|_l^{(m)}\, |q_2|_l^{(-m)}. \tag{5}$$

This is a consequence of (4) and (2) ∎

2. Definition and basic properties of anisotropic function spaces

Let $\Phi^a(\mathbf{R}^n)$ be the collection of all systems $\varphi = \{\varphi_k\}_{k=0}^\infty \subset S(\mathbf{R}^n)$ with

(i) $\operatorname{supp} \varphi_0 \subset \{\xi : |\xi|_a \leqq 2\}$,

$\operatorname{supp} \varphi_k \subset \{\xi : 2^{k-1} \leqq |\xi|_a \leqq 2^{k+1}\}$ if $k = 1, 2, \ldots,$ (6)

(ii) for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ there exists a positive number c_α such that

$$2^{k\|\alpha\|}\, |D^\alpha \varphi_k(\xi)| \leqq c_\alpha \quad \text{if} \quad k = 0, 1, \ldots \quad \text{and} \quad \xi \in \mathbf{R}^n, \tag{7}$$

(iii) $\sum_{k=0}^{\infty} \varphi_k(\xi) = 1$ if $\xi \in \mathbf{R}^n$.

This is a smooth anisotropic resolution of unity adapted to the given anisotropy $a = (a_1, \ldots, a_n)$. For simplicity one may assume that

$$\varphi_k(\xi_1, \ldots, \xi_n) = \varphi(2^{(-k+1)a_1}\xi_1, \ldots, 2^{(-k+1)a_n}\xi_n) \quad \text{if} \quad k = 1, 2, \ldots,$$

where φ is an appropriate C^∞-function with $\operatorname{supp} \varphi \subset \{\xi : 2^{-2} < |\xi|_a < 1\}$.

Now the *anisotropic function spaces* will be defined as follows:

Let $-\infty < s < \infty$ and $\bar{s} = (s_1, \ldots, s_n)$ with $s_1 = s/a_1, \ldots, s_n = s/a_n$. If $0 < p, q \leqq \infty$, then

$$B^{\bar{s}}_{p,q}(\mathbf{R}^n) = \{f \in S' : \|f \mid B^{\bar{s}}_{p,q}\| = \|\{2^{ks}\varphi_k(D)\, f\} \mid l_q(L_p)\|\},$$

If $0 < p < \infty$ and $0 < q \leqq \infty$, then

$$F^{\bar{s}}_{p,q}(\mathbf{R}^n) = \{f \in S' : \|f \mid F^{\bar{s}}_{p,q}\| = \|\{2^{ks}\varphi_k(D)\, f\} \mid L_p(l_q)\|,$$

where $\{\varphi_k(\xi)\}_{k=0}^\infty \in \Phi^a$ and $\varphi_k(D)\, f = F^{-1}\varphi_k(\xi)\, Ff$, F denotes the Fourier transform in $S'(\mathbf{R}^n)$. The spaces $B^{\bar{s}}_{p,q}$ and $F^{\bar{s}}_{p,q}$ are quasi-Banach spaces (Banach spaces if $1 \leqq p, q \leqq \infty$). They are independent of the chosen system $\varphi \in \Phi^a(\mathbf{R}^n)$; different systems φ and $\bar{\varphi}$ lead to equivalent quasi-norms. If $0 < p, q < \infty$, then $S(\mathbf{R}^n)$ is dense in them — see [26: Chapter 10.1]. In particular we have

$$F^{\bar{s}}_{p,2} = H_p^{\bar{s}} \quad \text{if} \quad 1 < p < \infty$$

and

$$F^{\bar{s}}_{p,2} = H_p^{\bar{s}} = W_p^{\bar{s}} \quad \text{if} \quad 1 < p < \infty \quad \text{and} \quad s = (s_1, \ldots, s_n)$$

is an n-tuple of natural numbers. $H_p^{\bar{s}}$ and $W_p^{\bar{s}}$ denote the classical anisotropic Bessel-potential spaces and the anisotropic Sobolev spaces, respectively — see [19, 24].

We have the following maximal inequality. Suppose that

$$f_k^*(x) = \sup_{y\in\mathbf{R}^n} \frac{|(F^{-1}\varphi_k(\xi)\, Ff)\,(y)|}{(1+2^k\,|x-y|_a)^M}. \tag{8}$$

Then

$$\|\{2^{ks}f_k^*(x)\} \mid L_p(l_q)\| \leqq c\,\|f \mid F^{\bar{s}}_{p,q}\| \tag{9}$$

holds for $M > n/\min(p, q)$ and all $f \in F^{\bar{s}}_{p,q}$. This is a consequence of [25: Sections 2.3.2 and 2.5.2].

We recall the interpolation theorem:

$$B^{\bar{s}}_{p,q} = (F^{\bar{s}_0}_{p,q_0}, F^{\bar{s}_1}_{p,q_1})_{\theta,q}$$

if $0 < p < \infty$, $0 < q, q_0, q_1 \leqq \infty$ and $s = (1-\theta)\,s_0 + \theta s_1$ with $s_0 \neq s_1$ and $0 < \theta < 1$ — see [24].

The functions $\varphi_k(D)$ are pseudo-differential operators of order $-\infty$, and from (6) and (7) we obtain for any $m \in \mathbf{R}$

$$|D^\alpha\varphi_k(\xi)| \leqq c_{\alpha,m}2^{-km}(1+|\xi|_a)^{m-\|\alpha\|}, \qquad k = 0, 1, \ldots,$$

where the constant $c_{\alpha,m}$ is independent of k. Therefore we have

$$|\varphi_k|_l^{(m)} \leqq c_{l,m}2^{-km} \quad \text{for} \quad k = 0, 1, \ldots, \tag{10}$$

where $c_{l,m}$ does not depend on k.

Let us consider function systems

$$\psi = \{\psi_k\}_{k=0}^\infty \quad \text{with (7), i.e.} \quad 2^{k\|\alpha\|}\,|D^\alpha\psi_k(\xi)| \leqq c_\alpha,$$

and

$$\operatorname{supp}\psi_0 \subset \{\xi : |\xi|_a < 2^j\}, \qquad \operatorname{supp}\psi_k \subset \{\xi : 2^{k-j} < |\xi|_a < 2^{k+j}\}$$

for a fixed real number $j \geqq 1$ instead of (6). Then we have the same estimates of the semi-norms of $\psi_k(D)$ as in (10). The constant $c_{l,m}$ depends now on j, too, but is indenpendent of k. Moreover, the quasi-norm $\|\{2^{ks}\psi_k(D)\,f\} \mid L_p(l_q)\|$ defined by use of the system ψ is an equivalent quasi-norm in $F^{\bar{s}}_{p,q}(\mathbf{R}^n)$.

3. The main result

We are now in the position to prove a theorem about boundedness of pseudo-differential operators in anisotropic function spaces.

Theorem 2: *Let $P(x, D_x) \in S^0_{a;1,\delta}$ and $\delta < 1$. Then for all p, q and s with $0 < p < \infty$, $0 < q \leqq \infty$ and $-\infty < s < \infty$*

$$P(x, D_x) : F^{\bar{s}}_{p,q}(\mathbf{R}^n) \to F^{\bar{s}}_{p,q}(\mathbf{R}^n)$$

is a linear and continuous operator. Moreover, there exist real numbers l and $c > 0$, both independent of $P(x, D_x)$, such that for $f \in F^{\bar{s}}_{p,q}(\mathbf{R}^n)$

$$\|P(x, D_x)\,f \mid F^{\bar{s}}_{p,q}\| \leqq c\,|p|_l^{(0)}\,\|f \mid F^{\bar{s}}_{p,q}\|.$$

Proof: Let $\{\varphi_k\}_{k=0}^\infty \in \Phi^a(\mathbf{R}^n)$. Then we have

$$\|P(x, D_x)\,f \mid F^{\bar{s}}_{p,q}\|^\varphi = \|\{2^{ks}\varphi_k(D)\,P(x, D_x)\,f\}_{k=0}^\infty \mid L_p(l_q)\|.$$

As a consequence of the composition rule of pseudo-differential operators we see that

$$\varphi_k(D)\, P(x, D_x) = P(x, D_x)\, \varphi_k(D) + R_k(x, D_x)$$
$$= P(x, D_x)\, \varphi_k(D)\, \psi_k(D) + \sum_{j=0}^{\infty} R_k(x, D_x)\, \varphi_j(D)\, \psi_j(D). \qquad (11)$$

$R_k(x, D_x)$ denotes the pseudo-differential operator which equals the "commutator" of $\varphi_k(D)$ and $P(x, D_x)$, $\{\psi_j\}_{j=0}^{\infty}$ is the function system with $\psi_j(\xi) = \varphi_{j-1}(\xi) + \varphi_j(\xi) + \varphi_{j+1}(\xi)$ if $j = 0, 1, \ldots$ $(\varphi_{-1} = 0)$. In particular, $\psi_j(\xi) = 1$ if $\xi \in \operatorname{supp} \varphi_j$. We have also used

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1 \quad \text{and} \quad \sigma\big(P(x, D_x)\, \varphi_j(D)\big) = \sigma\big(P(x, D_x)\big) \cdot \varphi_j(\xi).$$

Now we consider the first term on the right-hand side of (11). Let $f \in F^{\bar{s}}_{p,q}$ and $f_k = \psi_k(D)\, f$. Then we obtain

$$|P(x, D_x)\, \varphi_k(D)\, f_k| = \left|\int K_k(x, x - y)\, f_k(y)\, dy\right|,$$

where

$$K_k(x, x - y) = \int e^{i(x-y)\xi} p(x, \xi)\, \varphi_k(\xi)\, d\xi.$$

Hence, by the lemma and the corollary proved in Section 1 and by (10) we have the estimate

$$\begin{aligned} &\big|\big(P(x, D_x)\, \varphi_k(D)\, f_k\big)(x)\big| \\ &\leqq \int |K_k(x, x - y)|\, (1 + 2^k |x - y|_a)^M \frac{|f_k(y)|}{(1 + 2^k |x - y|_a)^M}\, dy \\ &\leqq c\, |p|_l^{(0)}\, f_k^*(x) \end{aligned} \qquad (12)$$

for $M > n/\min(p, q)$ and l large enough, where $f_k^*(x)$ denotes the maximal function defined by (8). From (12) and (9) we deduce that

$$\begin{aligned} &\|\{2^{ks} P(x, D_x)\, \varphi_k(D)\, f\}_{k=0}^{\infty} \mid L_p(l_q)\| \\ &\leqq c |p|_l^{(0)}\, \|\{2^{ks} f_k^*\}_{k=0}^{\infty} \mid L_p(l_q)\| \leqq c' |p|_l^{(0)}\, \|f \mid F^{\bar{s}}_{p,q}\|^{\psi}, \end{aligned}$$

where the constant $c' > 0$ does not depend on $P(x, D_x)$.

Now we estimate the second pseudo-differential operator in (11). Again we set $f_j = \psi_j(D)\, f$ and

$$K_{k,j}(x, x - y) = \int e^{i(x-y)\xi} r_k(x, \xi)\, \varphi_j(\xi)\, d\xi.$$

Then by (5) we have the inequality

$$\begin{aligned} &\big|\big(R_k(x, D_x)\, \varphi_j(D)\, f_j\big)(x)\big| \\ &\leqq \int |K_{k,j}(x, x - y)|\, (1 + 2^j |x - y|_a)^M \frac{|f_j(y)|}{(1 + 2^j |x - y|_a)^M}\, dy \\ &\leqq c\, |r_k|_l^{(s-\varepsilon)}\, |\varphi_j|_l^{\circ\,(-s+\varepsilon)}\, f_j^*(x) \end{aligned} \qquad (13)$$

for $M > n/\min(p, q)$, l large enough and arbitrary $\varepsilon \in \mathbf{R}$. To estimate the seminorm $|r_k|_l^{(s-\varepsilon)}$ we apply Theorem 1 to $R_k(x, D_x) = \varphi_k(D)\, P(x, D_x) - P(x, D_x)\, \varphi_k(D)$ and get by (2) and (3) the estimate $|r_k|_l^{(s-\varepsilon)} \leqq c\, |p|_{l'}^{(0)}\, |\varphi_k|_{l'}^{(s+\varepsilon)}$ with $\varkappa < (1 - \delta) \min a_j$ and $2\varepsilon = (1 - \delta) \min a_j - \varkappa > 0$. In view of this estimate we obtain from (13) and (10) that

$$\big|\big(2^{ks} R_k(x, D_x)\, \varphi_j(D)\, f_j\big)(x)\big| \leqq c 2^{-k\varepsilon} 2^{j(s-\varepsilon)}\, |p|_{l'}^{(0)}\, f_j^*(x).$$

It is easy to verify that

$$\sum_{j=0}^{\infty} 2^{-j\varepsilon} 2^{js} f_j^*(x) \leqq c_\varepsilon \left(\sum_{j=0}^{\infty} |2^{js} f_j^*(x)|^q \right)^{1/q} \quad \text{if} \quad q < \infty$$

and

$$\sum_{j=0}^{\infty} 2^{-j\varepsilon} 2^{js} f_j^*(x) \leqq c_\varepsilon \sup_j 2^{js} |f_j^*(x)| \quad \text{if} \quad q = \infty.$$

This is obvious if $0 < q \leqq 1$; if $q > 1$ it follows from Hölder's inequality. Thus we obtain by use of the maximal inequality (9) that

$$\left\| \left\{ \sum_{j=0}^{\infty} 2^{ks} R_k(x, D_x)\, \varphi_j(D)\, f_j \right\}_{k=0}^{\infty} \mid L_p(l_q) \right\|$$

$$\leqq c\, |p|_l^{(0)}\, \|\{2^{js} f_j^*\}_{j=0}^{\infty} \mid L_p(l_q)\| \leqq c'\, |p|_l^{(0)}\, \|f \mid F_{p,q}^{\bar{s}}\|^{\varphi}.$$

The constant $c' > 0$ again does not depend on $P(x, D_x)$.

The proof will be finished by the remark that the quasi-norm $\|\cdot\|^{\psi}$ defined by the function system $\{\psi_j\}_{j=0}^{\infty}$ is equivalent to the quasi-norm $\|\cdot\|^{\varphi}$ defined by the system $\{\varphi_j\}_{j=0}^{\infty}$. ∎

Remarks: 1. By the interpolation theorem for anisotropic spaces [24], we have immediately

$$\|P(x, D_x)\, f \mid B_{p,q}^{\bar{s}}\| \leqq c\, |p|_l^{(0)}\, \|f \mid B_{p,q}^{\bar{s}}\| \tag{14}$$

if $0 < p < \infty$, $0 < q \leqq \infty$ and $-\infty < s < \infty$. In the case $p = \infty$ and $0 < q \leqq \infty$ (14) is also true. This follows immediately by the proof of the theorem, using the maximal inequality for the space $B_{\infty,q}^{s}(\mathbf{R}^n)$ [25: Sections 2.3.2 and 2.5.2].

2. The theorem contains the results about Fourier multipliers for non-homogeneous anisotropic spaces in [23, 25]. Indeed, the constant l in the theorem is larger than necessary in this special case.

3. We have $F_{p,2}^{\bar{0}}(\mathbf{R}^n) = L_p(\mathbf{R}^n)$ if $1 < p < \infty$. Therefore we obtain from the theorem some extensions of the results in [8, 14, 15, 18]. For example the pseudo-differential operator

$$\frac{\xi_1^6 + \varrho^{2m}(x)\, \xi_2^6}{1 + \xi_1^6 + \xi_2^2 + \varrho^{2m}(x)\, \xi_2^6} \in S_{a;1,\delta}^{0}$$

is bounded in $L_p(\mathbf{R}^2)$ if $2 < m \in \mathbf{N}$, $\delta = 2/m$ and $\varrho \in B(\mathbf{R}^2)$. In the sense of the isotropic Hörmander class of pseudo-differential operators it belongs at most to $S_{\varrho,\delta}^{0}$ with $\varrho \leqq 1/3$.

4. The use of maximal functions in the proof goes back to PÄIVÄRINTA [21]. He proved the boundedness of pseudo-differential operators of $S_{1,\delta}^{0}$ ($\delta < 1$) in the isotropic function spaces $F_{p,q}^{s}(\mathbf{R}^n)$ and $B_{p,q}^{s}(\mathbf{R}^n)$. But in contrast to the previous results we have now an explicit estimate of the operator-norm $\|P(x, D_x) : L(F_{p,q}^{\bar{s}}, F_{p,q}^{\bar{s}})\|$ by semi-norms of the symbol $p(x, \xi)$ and a constant c which is independent of $P(x, D_x)$. This is very useful in order to deal with other questions, for example results about convergence. It was possible to get this result by an estimate of KUMANO-GO [17: Chapter 2, Lemma 2.4] for the remainder term which is obtained by the composition of pseudo-differential operators and a generalization of it to the anisotropic case — inequality (3).

5. Similar results were meanwhile obtained also by M. YAMAZAKI (see J. Fac. Sci. Univ. Tokyo, Sect. IA Math., **33** (1986), 131–174).

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