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Comparison Theorems for Conjugate Points of Sturm-Liouville Differential Equations

E. MÜLLER-PFEIFFER

Es werden Vergleichssätze für Lösungen Sturm-Liouvillescher Differentialgleichungen bezüglich ihrer Nullstellen auf einem endlichen Intervall bewiesen.

Доказываются теоремы сравнения решений дифференциальных уравнений Штурма-Лиувилля относительно их нулей на конечном интервале.

Comparison theorems for solutions of Sturm-Liouville equations are proved concerning their zeros on a finite interval.

Consider the differential equations

 $\quad \text{and} \quad$

$$
-(P(x) u')' + Q(x) u = 0, \qquad -a \le x \le a,
$$

$$
-(p(x) u')' + q(x) u = 0, \qquad -a \le x \le a,
$$

$$
(1)
$$

$$
(2)
$$

 $a \in \mathbb{R}$, where the coefficients are real-valued continuous functions and, additionally, p, P are positive piecewise and continuously differentiable. The points $x_1, x_2 \in [-a, a]$ are said to be *conjugate points* for equation (1) or (2) if there exists a nontrivial solution u of the corresponding equation with $u(x_1) = 0 = u(x_2)$. We prove the following

Theorem 1: Assume the following:

(i) P and Q are even functions on $[-a, a]$,

- (ii) $p \leq P$ on $[-a, a]$,
- (iii) $\int_{[-\sigma,\sigma]} \int_{-\sigma,\sigma} \int_{(-\sigma,\sigma]} Q \, dx$ for all σ, τ with $0 < \tau < \sigma < a$.

If there exists a solution u of (1) with $u(-a) = 0 = u(a)$ and $u(x) > 0$, $-a < x < a$, then there exists a pair of conjugate points for equation (2) on $[-a, a]$.

Proof: The solution u of (1) belongs to $C^T[-a, a]$ (cp. [7: p. 25], for instance). By using the Sturm comparison theorem (cp. [7]) it is easily seen that such u is an even function. By means of u we construct a test function for the quadratic form of equation (2). Let $\alpha \in [0, a)$ be a point with $u'(\alpha) = 0$ and $u' \leq 0$ on $[\alpha, a]$. (Note that there can exist several points α with this property.) It is easily seen that the function

 $w(x) = \begin{cases} u(\alpha), & x \in (-\alpha, \alpha) \\ u(x), & x \in [-a, a] \setminus (-\alpha, \alpha) \end{cases}$

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belongs to the Sobolev spaces $\mathring{W}_2^{-1}(-a, a)$ and $W_2^{-2}(-a, a)^{-1}$. The sequilinear forms of the equations (1) and (2) defined on $\mathring{W}_2^{-1}(-a, a)$ are closed. In the following, by means of w_i , the quadratic form of (2) will be estimated. Thus, by using Fubini's theorem, the function 126 E. MÜLLER-PEEIFFER

belongs to the Sobolev spaces $\ddot{W}_2^{-1}(-a, a)$ and $W_2^{-2}(-a, a)^1$). I

of the equations (1) and (2) defined on $\ddot{W}_2^{-1}(-a, a)$ are close

by means of w_i , the quadratic form of (2) will be esti *h*(*b*) **h**) **h h** sobolev spaces $W_2^{-1}(-a, a)$ and $W_2^{-2}(-a, a)^1$. The sequilinear forms puations (1) and (2) defined on $W_2^{-1}(-a, a)$ are closed. In the following, sof w_i , the quadratic form of (2) will be estimate

$$
h(y) := \sup \{x \in [0, a] \mid w^2(x) \geq y\}, \qquad 0 \leq y \leq u^2(\alpha),
$$

and the hypotheses (ii) and (iii), we obtain

$$
\int_{-a}^{a} [p(w')^{2} + qw^{2}] dx = \int_{-a}^{a} [(p - P) (w')^{2} + (q - Q) w^{2}] dx + \int_{-a}^{a} [P(w')^{2} + Qw^{2}] dx
$$

\n
$$
\leq \int_{-a}^{a} (q - Q) w^{2} dx + P u' u |_{-\frac{a}{a}}^{a} + \int_{-a}^{a} [-(P u')' + Q u] u dx
$$

\n
$$
+ u^{2}(\alpha) \int_{-\alpha}^{a} Q dx + P u' u |_{\alpha}^{a} + \int_{-a}^{a} [-(P u')' + Q u] u dx
$$

\n
$$
= \int_{-a}^{a} (q - Q) w^{2} dx + u^{2}(\alpha) \int_{-a}^{a} Q dx
$$

\n
$$
= \int_{-a}^{a} [q - Q) w^{2} dx + u^{2}(\alpha) \int_{-a}^{a} Q(x) dx
$$

\n
$$
= \int_{-a}^{a} \int_{0}^{u(x)} [q(x) - Q(x)] dx + \int_{-a}^{a} Q(x) dx dx
$$

\n
$$
= \int_{0}^{u^{2}(x)} (\int_{-h(y)}^{h(y)} q(x) dx - \int_{-a}^{a} Q(x) dx) dy
$$

\n
$$
= \int_{0}^{u^{2}(x)} (\int_{[-h(y),h(y)]} q(x) dx - \int_{[-h(y),h(y)] \setminus (-a, a)} Q(x) dx) dy \leq 0.
$$

\nTherefore,
\n
$$
\inf \left\{ \int_{-a}^{a} (p |q'|^{2} + q |q|^{2}) dx : q \in C_{0}^{\infty}, ||q|| = 1 \right\}
$$

Therefore,

$$
\inf \left\{ \int_{-a}^{a} (p \, |\varphi'|^2 + q \, |\varphi|^2) \, dx \colon \varphi \in C_0^{\infty}, ||\varphi|| = 1 \right\}.
$$

is either less or equal zero.

In the first case it follows that there exists a nontrivial solution v of (2) having
least two zeros on $(-a, a)$ (cp. [5]). In the second case the (normalized) func-
m w is realizing the infimum, and, consequently, it is at least two zeros on $(-a, a)$ (cp. [5]). In the second case the (normalized) function *w* is realizing the infimum, and, consequently, it is a solution of (2). This can be proved as follows Let *A* be the Friedrichs extension of the operator first case it follows that there exists a nontrivial solution v of (2) have zeros on $(-a, a)$ (cp. [5]). In the second case the (normalized) realizing the infimum, and, consequently, it is a solution of (2). Thit is a solu

$$
A_0\varphi = -(p\varphi')' + q\varphi, \qquad \varphi \in C_0^{\infty}(-a,a).
$$

be proved as follows. Let *A* be the Friedrichs extension of the operator $A_0\varphi = -(p\varphi')' + q\varphi$, $\varphi \in C_0^{\infty}(-a, a)$.
Because of $(A_0\varphi, \varphi) \ge 0^2$, $\varphi \in C_0^{\infty}(-a, a)$, the operator $A^{1/2}$, $D(A^{1/2}) = \mathring{W}_2^{-1}(-a, a)$ can be defined (cp. [3]). Then, it follows from

$$
0 = \int_{0}^{1} [p(w')^{2} + qw^{2}] dx = ||A^{1/2}w||^{2}
$$

1) $\hat{W}_2^{-1}(-a, a)$ is the completion of $C_0^{\infty}(-a, a)$ in the W_2^{-1} -norm.

²) (...) and $\|\cdot\|$ denote inner product and norm in the Hilbert space $L_2(-a, a)$, respectively.

that $A^{1/2}w = 0$ and $0 = A^{1/2}(A^{1/2}w) = Aw$. By the first representation theorem (see [3: p. 322]) we have $\int_{-a}^{a} (\rho w \overline{v} + q w \overline{v}) dx = (Aw, v) = 0$ for every $v \in \mathring{W}_2^{-1}(-a, a)$, (see [3: p. 322]) we have Comparis

hat $A^{1/2}w = 0$ and $0 = A^{1/2}(A^{1/2}w) = Aw$. By the first rep

see [3: p. 322]) we have
 $\int_{-a}^{a} (\rho w' \overline{v}' + q w \overline{v}) dx = (Aw, v) = 0$ for every $v \in \mathring{W}_2$

and by integration by parts it follows that

$$
\int (p w' \overline{v}' + q w \overline{v}) dx = (A w, v) = 0 \text{ for every } v \in \mathring{W}_2^1(-a, a),
$$

and by integration by parts it follows that

$$
\int [-(pw')'+qw]\,\overline{v}\,dx=0 \quad \text{for all}\ \ v\in \mathring{W}_2^{-1}(-a,a).
$$

Hence, we obtain $-(pw')' +qw = 0$. The solution *w* of (2) has zeros at the end points of the interval $[-a, a]$. (ii) $\int_{-a}^{a} (-p w')' + q w \bar{v} \, dx = 0$ for all $v \in W_2^{-1}(-a, a)$.

Hence, we obtain $-(p w')' + q w = 0$. The solution w of (2) has zeros at the end

points of the interval $[-a, a]$.

We state that in both cases there exists a pair o

We state that in both cases there exists a pair of conjugate points for equation (2) on $[-a, a]$. This completes the proof of Theorem 1 *f***)** $f(x) = f(x)$ $f(x) = f(x)$, $f(x) = f$

If a point $\alpha \in [0, a)$ with the named properties is known, Theorem 1 can be modified as follows.

Corollary 1: *Assume the following*:

- (i) *P* and *Q* are even functions on $[-a, a]$, (ii) $p \le P$ on $[-a, a]$.
-

Corollary 1: Assume the following:

(i) P and Q are even functions on $[-a, a]$,

(ii) $p \le P$ on $[-a, a]$.
 Assume that there exists a solution u, of (1) with $u(-a)$
 $-a < x < a$; $u'(\alpha) = 0$ and $u' \le 0$ on $[\alpha, a]$. If, additi If a point $\alpha \in [0, a)$ with the named

d as follows.

Corollary 1: Assume the following

(i) P and Q are even functions on [

(ii) $p \le P$ on $[-a, a]$.

sume that there exists a solution
 $a < x < a$; $u'(\alpha) = 0$ and $u' \le 0$ o

$$
\int_{-\sigma,\sigma]} q \, dx \leq \int_{[-\sigma,\sigma] \setminus (-a,a)} Q \, dx \quad \text{for all} \quad \sigma \in (\alpha, a),
$$

then there exists a pair of conjugate points for equation (2) on $[-a, a]$ *.*

Proof: Compare the proof of Theorem I I

$$
\int_{0}^{1} Q \, dx \leq 0 \quad \text{when} \quad \sigma \in (0, a)
$$
 (3)

f ary 1: Assume the following:
 f Q *dx* even functions on $[-a, a]$,
 $\leq P$ on $[-a, a]$.
 fhat there exists a solution u. of (1) with $u(-a) = 0 = u(a)$; $u(x) > 0$,
 $a, u'(x) = 0$ and $u' \leq 0$ on $[x, a]$. *If, additionally* the solution u is monotone decreasing on $[0, a]$. This can be seen as follows. By setting $v = -Pu'y^{-1}$ it follows from $uv' + u'v = (uv)' = -(Pu')' = -Qu$ that *v* satisfies the Riccati differential equation $v' = -Q + P^{-1}v^2$. Because of $u'(0) = 0$ we have $v(0) = 0$ and $v' \equiv -Q + P^{-1} v^2$ implies *x* is monotone decreasing on [0, *a*]. This can be seen as follows. By
 u is monotone decreasing on [0, *a*]. This can be seen as follows. By
 $-Pu'u^{-1}$ it follows from $uv' + u'v = (uv)' = -(Pu')' = -Qu$ that *v*

Riccati different $\int Q \, dx \leq 0$ when $\sigma \in (0, a)$

solution u is monotone decreasing on [0, *a*]. This can be

solution u is monotone decreasing on [0, *a*]. This can be

sing $v = -P u' u^{-1}$ it follows from $u v' + u' v = (u v)' = -($

sing $v(0) = 0$

$$
v(x) = -\int_{0}^{x} Q \, dt + \int_{0}^{x} P^{-1} v^2 \, dt, \qquad 0 \le x \le a. \tag{4}
$$

In view of (3) and (4) we obtain $v \ge 0$ and, consequently, $u' \le \infty$ [0, *a*]. Therefore, by assuming (3), the point α can be chosen equal to zero. Thus; we obtain the following result from corollary 1.

Corollary 2: Assume the following:

$$
v(x) = -\int_{0}^{x} \sqrt{a}u + \int_{0}^{x} \sqrt{b}u, \quad 0
$$

or view of (3) and (4) we obtain $v \ge 0$ and
one, by assuming (3), the point α can be c
the following result from Corollary 1.
Corollary 2: Assume the following:
(i) P and Q are even functions on $[-a, a]$,
(ii) $p \le P$ on $[-a, a]$,
(iii) $\int_{-a}^{a} q \, dx \le \int_{-5}^{a} Q \, dx \le 0$ for all $a \in (0, a)$.

 $\frac{1}{\sqrt{2}}$

1/ there exists a solution u of (1) with $u(-a) = 0 = u(a)$ and $u(x) > 0$, $-a < x < a$, *then there exists a pair of conjugate points for equation (2) on* $[-a, a]$ *.*

there exists a solution u of (1) with $u(-a) = 0 = u(a)$ and $u(x) > 0$, $-a < x < a$,
n there exists a pair of conjugate points for equation (2) on $[-a, a]$.
A similar result of Corollary 2 was obtained by Fix [2: Th. 2] under the hy *If there exists a solution u of* (1) with $u(-a) = 0 = u(a)$ and $u(x) > 0$, $-a < x < a$, then there exists a pair of conjugate points for equation (2) on $[-a, a]$.
A similar result of Corollary 2 was obtained by FINK [2: Th. 2] und [4: Th.1.3]. *for exists a pair of conjugate points for equation (2) on* $[-a]$
 f (2) on $[-a]$
 f (1) In the special case $p = P = 1$ and $Q \le 0$ Corollary 2 is

1.3].
 f (1) *for 3:* Consider the equations (1) *and (2) on* [0, *a]*

Corollary 3: *Consider the equations* (1) *and* (2) *an* [0, a] *and assume the following:*

7.

- (i) $p \leq P \text{ on } [0, a],$
- (ii) $\int q\,dx \leq \int Q\,dx$ for all σ, τ with $0 < \tau < \sigma < a$
- *or*

If there exists a solution u of (1) *on* $[0, a]$ *with* $u'(0) = u(a) = 0$ *and* $u(x) > 0, 0 \le x < a$, *then every solution v of* (2) *on* $[0, a]$ *with* $v(0) > 0$ *and* $v'(0) \le 0$ *has a zero on* $(0, a]$. (i) $p \le P$ on $[0, a]$,

(ii) $\int_{0}^{a} q \, dx \le \int_{0}^{a} Q \, dx$ for all σ, τ with $0 < \tau < \sigma < a$

or

(ii) $\int_{0}^{a} q \, dx \le \int_{0}^{a} Q \, dx \le 0$ for all $\sigma \in (0, a)$.

If there exists a solution u of (1) on $[0, a]$ with $u'(0) = u(a) = 0$

If there exists a solution u of (1) *on* $[0, a]$ with $u'(0) = u(a) = 0$ and $u(x) > 0$, $0 \le x < a$, then every solution v of (2) *on* $[0, a]$ with $v(0) > 0$ and $v'(0) \le 0$ has a zero on $(0, a]$.

Proof: By using the Sturm compa Proof: By using the Sturm comparison theorem it is easily seen that it is sufficient
to prove Corollary 3 under the hypothesis $v'(0) = 0$ in place of $v'(0) \leq 0$. Then, extend the functions p, P, q, Q , and the solutions u and v as even functions on $[-a, a]$ and use Theorem 1 (or Corollary 2) with the aid of Sturm's comparison. theorem In the functions p, P, q, Q , and the solutions u and v as even functions on $[a, a]$ and use Theorem 1 (or Corollary 2) with the aid of Sturm's comparison eorem 1
In the following the hypothesis $p \le P$ is to be weakened.

continuously differentiable on $(0, a]$.
Theorem 2: Assume the following:
(i) P and Q are even functions on [
(ii) Q, P' \leq 0 on $(0, a]$,

Theorem 2: *Assume the following*:

- *(i)* P and Q are even functions on $[-a, a]$,
-
- *(ii)* $Q, P' \leq 0$ *on* $(0, a]$,
 (iii) $\int_{1-a,a] \setminus (-a,a)} [P p] dx \geq 0$ *and* $\int_{-a}^{a} [Q q] dx \geq 0$ *for all* $a, 0 < a < a$. Fractional state of the hypothesis

invously differentiable on $(0, a)$

peorem 2: Assume the followin

i) P and Q are even functions on

i) Q, P' ≤ 0 on $(0, a)$,

i) $\int_{[-a,a] \setminus (-a,a)} [P - p] dx \geq 0$ and \int_{-a}

ere exist

If there exists a solution u of (1) with $u(-a) = 0 = u(a)$ and $u(x) > 0, -a < x < a$, *then there exists a pair of conjugate points for equation (2) on* $[-a, a]$ *.*

Proof: $Q \leq 0$ implies inequality (3). Therefore, as shown in the proof of Corollary 1, we have $u' \leq 0$ on [0, *a*] and by means of $P' \leq 0$ it follows from $Pu'' = -P'u'$. $+$ Qu on (0, a), that $u'' \leq 0$ on (0, a). Consequently, the derivative u' is monotone decreasing on [0, *a*]. The function $(u')^2$ is even and monotone increasing on [0, *a*]. Hence, by means of the hypothesis (iii) we obtain

$$
\int_{-a}^{a} (p - P) (u')^2 dx = \int_{-a}^{a} \int_{0}^{u(x)} [p(x) - P(x)] dx dy \le 0
$$
\n
$$
= \int_{-a}^{a} (u'(x))^2 dx - \int_{-a}^{a} \int_{0}^{u'(x)} [p(x) - P(x)] dy dx
$$
\n
$$
= \int_{0}^{u'(a)} (-a a)(-b(y), b(y)) dx dy \le 0
$$

Comparison Theorems 429
where $h(y) = \sup\{x \in [0, a] \mid [u'(x)]^2 \le y\}$, $0 \le y \le [u'(a)]^2$. Analogously (compare the proof of Theorem 1), we get $\int (q - Q) u^2 dx \leq 0$. Thus, it follows that Comparison Theorems

where $h(y) = \sup_x \{x \in [0, a] \mid [u'(x)]^2 \le y\}$, $0 \le y \le [u'(a)]^2$. Analogously (comparison of Theorem 1), we get $\int_a^a (q - Q) u^2 dx \le 0$. Thus, it follows that
 $\int_a^a [p(u')^2 + qu^2] dx = \int_a^a [(p - P)(u')^2 + (q - Q) u^2] dx \le 0$.

$$
\int_{-a}^{a} [p(u')^{2} + qu^{2}] dx = \int_{-a}^{a} [(p - P) (u')^{2} + (q - Q) u^{2}] dx \leq 0.
$$

Corollary 4: *Consider the equations* (1) *and* (2) *on* [0, a] *and assume the following:* Finally, finish the proof as the proof of Theorem 1 **1**
Corollary 4: *Consider the equations* (1) and (2) or
ing:
(i) Q, $P' \le 0$ °on [0, a],

(ii) $\int P dx \ge \int p dx$ and $\int Q dx \ge \int q dx$ for all $\sigma \in (0, a)$.

(ii) $\int_{\sigma}^{a} P dx \ge \int_{\sigma}^{a} p dx$ and $\int_{0}^{a} Q dx \ge \int_{0}^{a} q dx$ for all $\sigma \in (0, a)$.
 if there exists a solution u of (1) with $u'(0) = 0 = u(a)$ and $u(x) > 0$, $0 \le x < a$, then every solution of (2) with $v(0) > 0$ and $v'(0) \le 0$ (ii) $\int_{\sigma}^{\tau} P dx \ge \int_{\sigma}^{\tau} p dx$ and $\int_{0}^{t} Q dx \ge \int_{0}^{t} q dx$ for all $\sigma \in (0, a)$.
 if there exists a solution u of (1) with $u'(0) = 0 = u(a)$ and $u(x) > 0$,

then every solution of (2) with $v(0) > 0$ and $v'(0) \le 0$ has a z If there exists a solution u of (1) with $u'(0) = 0 = u(a)$ and $u(x) > 0$, $0 \le x < a$, then every solution of (2) with $v(0) > 0$ and $v'(0) \le 0$ has a zero on (0, a).
Proof: Compare the proof of Corollary 3 \blacksquare

Corollaries 3 and 4 are generalizations of theorems of NEHARI [6] and LEIGHTON [4: Th. 1.1]. Corollary 4: Consider the equations (1) and (2) on [

7:

(i) Q, P' $\leq 0^{\circ}$ on [0, a],

(ii) $\int_{\sigma}^{a} P dx \geq \int_{\sigma}^{a} p dx$ and $\int_{0}^{a} Q dx \geq \int_{0}^{a} q dx$ for all $\sigma \in (0, 1)$

there exists a solution u of (1) with $u'(0) =$ **f**: Compare the proof of Corollary 3 **f**

aries 3 and 4 are generalizations of theorems of NEH

llary 5: If the inequality
 $\therefore \sup_{\sigma \in (0,a)} \frac{1}{\sigma} \int_{0}^{\sigma} q \, dx \leq -\frac{\pi^2}{4a^2} \sup_{\sigma \in (0,a)} \frac{1}{a-\sigma} \int_{\sigma}^{a} p \, dx$ exists a

y solution

: Compa

ies 3 and

lary 5 :

sup
 $\frac{1}{\sigma}$

n every s then every solut
Proof: Com
Corollaries 3 a
Corollary 5
*s*up
 $\frac{1}{\sigma \in (0, a)}$
holds, then ever
(0, a].
Proof: Defi there exists a solution u of (1) with $u'(0) =$

en every solution of (2) with $v(0) > 0$ and $v'(0)$

Proof: Compare the proof of Corollary 3

Corollaries 3 and 4 are generalizations of theorems corollary 5: If the inequalit Example the proof of Corollary 3 **a**

ios 3 and 4 are generalizations of theorems of NEHARI [6] and LEIGHTON [4: Th. 1.1]

lary 5: If the inequality

sup $\frac{1}{\sigma} \int_{0}^{a} q \, dx \leq -\frac{\pi^2}{4a^2} \sup_{\sigma \in (0,a)} \frac{1}{a-\sigma} \int_{0}^{a} p \$

(5) 600.0 4i a((O,a) *a -* ^a 0 *holds, then every v o/* (2) *on* [0, *a] with* (0)> 0 *and v'(0)* ⁰*has a. zero on y s*

Corollaries 3 and 4 are generalizations of theorems of NEHARI [6] and LEGHTON [4: Th. 1.1].
\nCorollary 5: If the inequality
\n
$$
\sup_{\sigma \in (0,a)} \frac{1}{\sigma} \int_{0}^{a} q \, dx \leq -\frac{\pi^2}{4a^2} \sup_{\sigma \in (0,a)} \frac{1}{a-\sigma} \int_{0}^{a} p \, dx
$$
\n(s) holds, then every solution v of (2) on [0, a].
\n*holds* (6)
\n1. Proof: Define
\n
$$
\varrho = \sup_{\sigma \in (0,a)} \frac{1}{a-\sigma} \int_{0}^{a} p \, dx
$$
\nand set $P \equiv \varrho$ and $Q \equiv -\pi^2 \varrho/4a^2$. The function $u = \cos(\pi x/2a)$ is a solution of the differential equation
\n
$$
-\varrho u'' - \frac{\pi^2 \varrho}{4a^2} u = 0, \qquad 0 \leq x \leq a,
$$
\nwith the properties supposed in Corollary 4. Hypothesis (i) of Corollary 4 is fulfilled. It follows from (5) and (6) that (ii) also holds. This proves Corollary 5

and set $P \equiv \varrho$ and $Q \equiv -\pi^2 \varrho / 4a^2$. The function $u = \cos (\pi x / 2a)$ is a solution of the and set $P \equiv \rho$ and $Q \equiv -\pi^2 \rho / 4a^2$. The fund
differential equation
 $-\rho u'' - \frac{\pi^2 \rho}{4a^2} u = 0$, $0 \le x \le$
with the properties supposed in Corollary
filled. It follows from (5) and (6) that (ii) a
Example: Consider the di $\begin{align*}\n a_1^2(x) - y^2 \end{align*}$
 $\begin{align*}\n a_2^2(x) &= -\pi^2 \frac{q}{4a^2}. \text{ The function } u = \cos(\pi x/2a) \text{ is a solution of the } \end{align*}$
 $\begin{align*}\n -\frac{\pi^2 \frac{q}{4a^2}}{4a^2} u &= 0, \quad 0 \le x \le a, \\
 \text{It is supposed in Corollary 4. Hypothesis (i) of Corollary 4 is full-
from (5) and (6) that (ii) also holds. This proves Corollary 5
sider the differential equation\n\end{align*}$
 \begin

$$
-eu'' - \frac{\pi^2 \varrho}{4a^2} u = 0, \qquad 0 \le x \le a,
$$

with the properties supposed in Corollary 4. Hypothesis (i) of Corollary 4 is fulfilled. It follows from (5) and (6) that (ii) also holds. This proves Corollary 5

$$
-((1-x2) u')' - 6u = 0, \qquad 0 \le x \le a < 1.
$$

Since.

 $\frac{1}{2}$

$$
-eu'' - \frac{\pi^2 g}{4a^2} u = 0, \quad 0 \le x \le a,
$$

e properties supposed in Corollary 4. Hypothesis (i) of Coroll
to follows from (5) and (6) that (ii) also holds. This proves Coroll
inple: Consider the differential equation

$$
-((1 - x^2) u')' - 6u = 0, \quad 0 \le x \le a < 1.
$$

$$
\frac{1}{a-a} \int_a^a (1 - x^2) dx \le \frac{1}{a} \int_0^a (1 - x^2) dx = 1 - \frac{a^2}{3}, \quad a \in (0, a),
$$

/

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we have $q = 1 - a^2/3$. Inequality (5) holds when $a \ge a_1 = (3\pi^2/(72 + \pi^2))^{1/2} \approx 0,6014$.

Hence, every solution v of (7) with $v(0) > 0$ and $v'(0) \le 0$ has a zero on $(0, a_1]$. Because Hence, every solution *v* of (7) with $v(0) > 0$ and $v'(0) \le 0$ has a zero on $(0, a_1]$. Because
 $v = 1 - 3x^2$ is a solution of (7) with $v'(0) = 0$, the smallest *a* is $a_0 = 3^{-1/2} \approx 0.5774$. We
 $v = 1 - 3x^2$ is a solution *vie* have $q = 1 - a^2/3$. Inequality (5) holds when $a \ge a_1 = (3\pi^2/(72 + \pi^2))^1$.

Hence, every solution v of (7) with $v(0) > 0$ and $v'(0) \le 0$ has a zero on $(0, a, v = 1 - 3x^2)$ is a solution of (7) with $v'(0) = 0$, the smalles Hence, every solution v of (7) with $v(0) > 0$ and $v'(0) \le 0$ has a zero on $(0, a_1]$. Because $v = 1 - 3x^2$ is a solution of (7) with $v'(0) = 0$, the smallest a is $a_0 = 3^{-1/2} \approx 0.5774$. We state that a_1 is a good approximate value for a_0 . we have $g = 1 - a^2/3$. Inequ

Hence, every solution v of (7)
 $v = 1 - 3x^2$ is a solution of (7)

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