Zeitschrift für Analysis
und ihre Anwendungen
Bd. 5 (5) 1986, S. 437–444

On the Stability Property for a General Form of Variational Inequalities

LÊ VĂN CHỐNG

Für eine allgemeine Form von Variationsungleichungen in reflexiven Banach-Räumen werden unter Voraussetzungen über die Monotonie, Konkavität, Stetigkeit und Beschränktheit des parameter-abhängigen Problems Stabilitätskriterien angegeben. Einige Spezialfälle werden betrachtet.

Для общей формы вариационных неравенств в рефлексивных банаховых пространствах, при предположениях о монотонности, выпуклости, непрерывности и ограниченности зависящей от параметра проблемы доказываются критерии устойчивости. Обсуждаются некоторые специальные случаи.

Stability criteria for a general form of variational inequalities in reflexive Banach spaces are established under assumptions on the monotonicity, concavity, continuity and boundedness of the parameter-dependent problem. Some special cases are considered.

1. Introduction

For a parameter-dependent Problem (P_t) it is natural to raise the problem: Assume that Problem (P_{t^0}) admits a solution, when does a neighbourhood U of t^0 then exist such that for each $t \in U$ Problem (P_t) also admits a solution? What information about the solution set of Problem (P_t), $t \in U$, can be obtained?

In the case where (P_t) is an optimization problem, the above problem (stability problem) was investigated by many authors, e.g. KIRSCH [7], GOLLAN [5], BANK et al. [1]. For parametric optimization problems there was furthermore a lot of researches devoted to the extremal value function, e.g. GAUVIN and TOLLE [4], LEMPIO and MAURER [9], EKELAND and TEMAM [3], LEVITIN [10]. In the case of generalized equations, the stability problem and related problems were treated by ROBINSON [13], HOANG TUY [6], KUMMER [8]. A survey of parameter-dependent problems can be found in BANK et al. [1].

In this paper we are concerned with the qualitative stability problem in the case of a general form of variational inequalities. Specifically, the Problem (P_t) is here the following:

$$\begin{cases} \text{Find } x \in C \text{ such that} \\ f(x, y, t) \leqq 0 \text{ for all } y \in C;\ t \in T \end{cases} \qquad (P_t)$$

where C is a closed convex subset of a reflexive Banach space X; T is a metric space and $f : C \times C \times T \to \mathbf{R}$ is a function with certain properties of monotonicity, concavity and continuity ($\mathbf{R}$ is the set of all real numbers).

Throughout this paper, we denote by X a reflexive Banach space, by X^* the dual space of X, by $C \subset X$ a closed convex set, by T a metric space, by $t^0 \in T$ an accumulation point and by f a real-valued function on $C \times C \times T$. The compactness, closure, openness of a set in X and the continuity of a real-valued function on C are understood in the sense of the weak topology. The continuity of a real-valued function on $C \times T$ is understood in the sense of the weak topology on X and the metric topology on T.

2. Definitions and main results

In this section, some definitions used for the investigation below and stability criterions established for the parameter-dependent problem

$$\begin{cases} x \in C \\ f(x, y, t) \leqq 0 \quad \text{for all} \quad y \in C \end{cases} \qquad (\mathrm{P}_t)$$

will be given. The function $g: C \times C \to \mathbf{R}$ is said to be *monotone* if $g(x, x) \leqq 0$ and $g(x, y) + g(y, x) \geqq 0$ for all $x, y \in C$. g is said to be *hemicontinuous* if for arbitrary given $x, y \in C$ the function $g(x + \lambda(y - x), y)$ of the real variable $\lambda \in [0, 1]$ is lower semicontinuous (Mosco [12]). The set-valued mapping $\Gamma: T \to 2^x$ is said to be *upper semicontinuous at* $t^0 \in T$ if for each open set $\Omega \supset \Gamma(t^0)$ there is a neighbourhood V of t^0 such that $\Omega \supset \Gamma(t)$ for all $t \in V$ (Berge [2]). The *solution set mapping* $S: T \to 2^C$ is defined by

$$S(t) = \{x \in C: f(x, y, t) \leqq 0 \text{ for all } y \in C\}.$$

Problem (P_t) is said to be *stable at* t^0 if there is a neighbourhood U of t^0 such that $S(t)$ is non-empty, convex and compact for each $t \in U$ and the mapping $S: U \to 2^C$ is upper semicontinuous at t^0.

We introduce now the following assumptions (the topology considered on X is the weak topology, see Introduction).

Assumption 2.1: For each $t \in T$, $f(\cdot, \cdot, t)$ is a monotone and hemicontinuous function; for each $x \in C$, $f(x, \cdot, \cdot)$ is an upper semicontinuous function; for each $(x, t) \in C \times T$, $f(x, \cdot, t)$ is a concave function.

Assumption 2.2: There is a point $y_0 \in C$ such that the image set $N(t^0)$ of the mapping $N: T \to 2^C$ defined by

$$N(t) = \{x \in C: f(y_0, x, t) \geqq 0\}$$

is bounded.

In order to formulate and prove the main stability theorem, we need some preliminary considerations.

Theorem 2.1: *Under Assumptions* 2.1 *and* 2.2 *the following conditions are equivalent:*

(i) *There is a neighbourhood U of t^0 such that $N(U)$ is bounded in X.*
(ii) *N is upper semicontinuous at t^0.*
(iii) *There is an open set $\Omega \supset N(t^0)$ and a neighbourhood V of t^0 such that $\Omega \cap N(V)$ is bounded in X.*

The following lemmas are used for the proof of this theorem.

Lemma 2.1: *Let Assumption* 2.1 *and condition* (iii) *be satisfied. Then for any sequence $\{t_k\} \subset T$, $t_k \to t^0$, the sequence $\{x_k\} \subset X$, $x_k \in N(t_k) \setminus N(t^0)$, has an accumulation point contained in $N(t^0)$.*

Lemma 2.2: *Let $\Gamma : T \to 2^X$ be an upper semicontinuous mapping at t^0 with the closed and bounded image set $\Gamma(t^0)$. Then there is a neighbourhood V of t^0 such that $\Gamma(V)$ is bounded in X.*

Lemma 2.3 (Bank et al. [1: Lemma 2.2.2]): *Let $\Gamma : T \to 2^X$ be a mapping with the closed image set $\Gamma(t^0)$. Then Γ is upper semicontinuous at t^0 if and only if for any sequence $\{t_k\} \subset T$, $t_k \to t^0$, the sequence $\{x_k\} \subset X$, $x_k \in \Gamma(t_k) \setminus \Gamma(t^0)$, has an accumulation point contained in $\Gamma(t^0)$.*

Proof of Lemma 2.1: First we show that $\{x_k\}$ is bounded. Assume the contrary. Then there is a subsequence $\{x_{k'}\}$ with $\|x_{k'}\| \to \infty$. From $\{x_{k'}\}$ we now construct a bounded sequence $\{x_{k'}^{(d)}\}$,

$$x_{k'}^{(d)} = \frac{d}{\|x_{k'}\|} x_{k'} + \left(1 - \frac{d}{\|x_{k'}\|}\right) y_0 \tag{1.1}$$

where d is an arbitrary fixed number with $\|y_0\| < d \leqq \|x_k\|$ (y_0 is the point given in the assumption). Such a construction of $\{x_{k'}^{(d)}\}$ is always feasible since $\|x_{k'}\| \to \infty$. It is easy to check that

$$d - \|y_0\| \leqq \|x_{k'}^{(d)}\| \leqq d + \|y_0\|. \tag{1.2}$$

Since $t_{k'} \to t^0$ we have $t_{k'} \in V$ for all $k' \geqq k_0$ enough large (V is the neighbourhood given in (iii)). By $\{t_n\}$ we denote the sequence of $\{t_{k'}\}$ with $k' \geqq k_0$. So we get $\{t_n\} \subset V$ and $t_n \to t^0$. Since by the assumption $\Omega \cap N(V)$ is bounded, from (1.2) we can assume that $\{x_n^{(d_0)}\} \not\subset \Omega$ for d_0 enough large. By (1.2) $\{x_n^{(d_0)}\}$ has a convergent subsequence $\{x_{n'}^{(d_0)}\}$. Let $x_{n'}^{(d_0)} \to \bar{x}$. Since Ω is open and $x_{n'}^{(d_0)} \notin \Omega$ for all n', $\bar{x} \notin \Omega$. Hence, by $N(t^0) \subset \Omega$ follows $\bar{x} \notin N(t^0)$.

On the other hand $f(y_0, y_0, t^0) = 0$ (by the monotonicity of f). So we can write $f(y_0, y_0, t^0) > -\varepsilon$ for each $\varepsilon > 0$. Since the function $f(y_0, y_0, \cdot)$ is continuous at t^0, there is an index $k'(\varepsilon)$ of the index set of the sequence $\{t_{k'}\}$ such that $f(y_0, y_0, t_{k'}) > -\varepsilon$ for all $k' \geqq k'(\varepsilon)$. By $x_{k'} \in N(t_{k'})$ we have $f(y_0, x_k, t_{k'}) \geqq 0$. From (1.1), the last two inequalities and the concavity of f it follows $f(y_0, x_{k'}^{(d_0)}, t_{k'}) > -\varepsilon$. The upper semicontinuity of f then implies $f(y_0, \bar{x}, t^0) \geqq \varepsilon$ and since $\varepsilon > 0$ is arbitrary we get $f(y_0, \bar{x}, t^0) \geqq 0$. By the assumption that means $\bar{x} \in N(t^0)$. This contradicts $\bar{x} \notin N(t^0)$. Hence, the sequence $\{x_k\}$ is bounded.

Because of the boundedness $\{x_k\}$ has a convergent subsequence $\{x_{k'}\}$. Let $x_{k'} \to \hat{x}$. Since $x_{k'} \in N(t_{k'})$ we have $f(y_0, x_k, t_{k'}) \geqq 0$. By the upper semicontinuity of f then follows $f(y_0, \hat{x}, t^0) \geqq 0$, i.e. $\hat{x} \in N(t^0)$ ∎

Proof of Lemma 2.2: Assume the contrary: that for all neighbourhoods V of t^0, $\Gamma(V)$ is unbounded. Since $\Gamma(t^0)$ is bounded, we can then construct sequences $\{t_k\}$ and $\{x_k\}$ as follows:

Let $V_1 = B(t^0, r) \subset T$ be the ball with center t^0 and radius r. Since $\Gamma(V_1)$ is unbounded, we can take $x_1 \in \Gamma(V_1) \setminus \Gamma(t^0)$. There is then a point $t_1 \in V_1$ with $x_1 \in \Gamma(t_1)$. So, we have

$$t_1 \in V_1,$$
$$x_1 \in \Gamma(t_1) \setminus \Gamma(t^0).$$

Let $V_2 = \{t \in T: 2d(t, t^0) \leqq d(t_1, t^0)\}$ (here $d(\cdot, \cdot)$ denotes the distance function in the metric space T). Since $\Gamma(V_2)$ is unbounded, we can take $x_2 \in \Gamma(V_2) \setminus \big(\Gamma(t^0) \cup \{x \in C: \|x\| \leqq 2\|x_1\|\}\big)$. There is then a point $t_2 \in V_2$ with $x_2 \in \Gamma(t_2)$. So, we have

$$t_2 \in \{t \in T: 2d(t, t^0) \leqq d(t_1, t^0)\},$$
$$x_2 \in \Gamma(t_2) \setminus \big(\Gamma(t^0) \cup \{x \in C: \|x\| \leqq 2\|x_1\|\}\big).$$

Continuing this process, we then obtain for $k = 1, 2, \ldots$

$$t_{k+1} \in \{t : (k+1)\, d(t, t^0) \leqq d(t_k, t^0)\},$$
$$x_{k+1} \in \Gamma(t_{k+1}) \setminus \big(\Gamma(t^0) \cup \{x \in C : \|x\| \leqq (k+1)\, \|x_k\|\}\big).$$

By the above constructed sequences $\{t_k\} \subset T$ and $\{x_k\} \subset X$, $x_k \in \Gamma(t_k) \setminus \Gamma(t^0)$, it is easy to check that $t_k \to t^0$ and $\{x_k\}$ has no accumulation point. Hence, by Lemma 2.3 Γ is not upper semicontinuous at t^0. But this contradicts the assumption ∎

Proof of Theorem 2.1: (iii) ⇒ (ii): Let $\{t_k\} \subset T$ and $\{x_k\} \subset X$ be sequences with $t_k \to t^0$ and $x_k \in N(t_k) \setminus N(t^0)$. By Lemma 2.1 $\{x_k\}$ has an accumulation point contained in $N(t^0)$. By the upper semicontinuity of f it is easy to see that $N(t^0)$ is closed. Hence, by Lemma 2.3 N is upper semicontinuous at t^0. (ii) ⇒ (i): Since $N(t^0)$ is bounded (Assumption 2.2) and closed, the assertion follows from Lemma 2.2. (i) ⇒ (iii) is obvious ∎

Remark 2.1: As seen in the proof of Theorem 2.1, (iii) ⇒ (ii) and the proof of Lemma 2.1, the upper semicontinuity of the set-valued mapping N at t^0 follows from Assumption 2.1 and condition (iii). From this fact it is easy to derive the following criterion for the upper semicontinuity of a set-valued mapping:

Let φ be a real-valued and upper semicontinuous function on $C \times T$ such that $\varphi(\cdot, t)$ is concave for each $t \in T$. Let the set-valued mapping M be defined by $M(t) = \{x \in C : \varphi(x, t) \geqq 0\}$. Suppose that there is an open set $\Omega \subset X$ containing $M(t^0)$ and a neighbourhood V of t^0 such that $\Omega \cap M(V)$ is a bounded set in X. Then M is upper semicontinuous at t^0.

Using Theorem 2.1 we now establish stability criterions for Problem (P_t) above. We shall prove the following main stability theorem.

Theorem 2.2: *Let Assumptions 2.1 and 2.2 and one of the conditions* (i)—(iii) *of Theorem 2.1 be satisfied. Then Problem (P_t) is stable at t^0.*

The following results of Mosco [12] are used for the proof.

Lemma 2.4 [12: Theorem 3.1]: *Let $g : C \times C \to \mathbf{R}$ be a monotone and hemicontinuous function such that $g(x, \cdot)$ is concave and upper semicontinuous for each $x \in C$. Suppose that there exist a compact set $B \subset C$ and a point $y_0 \in B$ such that $f(x, y_0) > 0$ for each $x \in C \setminus B$ (the coerciveness condition). Then the solution set of the problem*

$$\begin{cases} x \in C \\ g(x, y) \leqq 0 \quad \text{for all } y \in C \end{cases}$$

is non-empty, convex and compact.

Lemma 2.5 [12: Lemma 3.1]: *Let $g : C \times C \to \mathbf{R}$ be a monotone and hemicontinuous function such that $g(x, \cdot)$ is concave and upper semicontinuous for each $x \in C$. Let*

$$G(y) = \{x \in C : g(x, y) \leqq 0\} \quad \text{and} \quad H(y) = \{x \in C : g(y, x) \geqq 0\}.$$

Then $\bigcap_{y \in C} G(y) = \bigcap_{y \in C} H(y)$.

Proof of Theorem 2.2: By Theorem 2.1 it is enough to show that Assumptions 2.1 and 2.2 and condition (i) imply the stability of Problem (P_t) at t^0. Since $N(U) \subset C$ is bounded (condition (i)), we can assume that $C(U)$ is contained in a compact set $B \subset C$. By the monotonicity of f we have $\{x \in C : f(x, y_0, t) \leqq 0\} \subset N(t)$. For each $t \in U$ it is then easy to see that $f(x, y_0, t) > 0$ for all $x \in C \setminus B$, i.e. the coerciveness condition in Lemma 2.4 is satisfied for Problem (P_t). By applying this Lemma it implies that the solution set $S(t)$ of (P_t) is non-empty, convex and compact.

We now show the upper semicontinuity of the mapping $S : U \to 2^B$ at t^0. Let $\{t_k\} \subset U$ and $\{x_k\} \subset B$ be sequences with $t_k \to t^0$ and $x_k \in S(t_k) \setminus S(t^0)$. Since B is compact, $\{x_k\}$ has a convergent subsequence $\{x_{k'}\}$. Let $x_{k'} \to \bar{x}$. Since $x_{k'} \in S(t_{k'})$ we have $f(x_{k'}, y, t_{k'}) \leqq 0$ for all $y \in C$ and hence, by the monotonicity of f, $f(y, x_{k'}, t_{k'}) \geqq 0$. From the upper semicontinuity of f follows $f(y, \bar{x}, t^0) \geqq 0$ and hence, by Lemma 2.5, $f(\bar{x}, y, t^0) \leqq 0$ for all $y \in C$, i.e. $\bar{x} \in S(t^0)$. $S(t^0)$ is here closed. Therefore, by Lemma 2.3, S is upper semicontinuous ∎

Remark 2.2: It is easy to see that Assumption 2.2 is contained in condition (iii). Hence, by Theorem 2.2, in the case where this condition is satisfied, Problem (P_t) is stable at t^0 if Assumption 2.1 is satisfied.

As we see in the proofs of Theorems 2.1 and 2.2, the set-valued mapping $N: T \to C$ plays an essential role. We are here interested in the question under which conditions the "level set" $N(U)$ is bounded for a neighbourhood U of t^0. Let us now consider a case where the set $N(U)$ is bounded.

Let $K \subset X$ be a cone with vertex a. K is said to be *pointed* if $a \notin \overline{\text{co}}\,(K \setminus B(a, 1))$, where $B(a, 1)$ denotes the ball with center a and radius 1. In the following lemma we give a property of the pointed cone, used for the stability consideration below.

Lemma 2.6: *If K is a pointed cone with vertex a, then there is a functional $l' \in X^*$ such that $l'(x) > l'(a)$ for all $x \in K \setminus \{a\}$ and the intersection of each hyperplane $(l', \beta) = \{x \in X : l'(x) = \beta, \beta \geqq l'(a)\}$ with K is a bounded set.*

Proof: Since K is a pointed cone we have $a \notin \overline{\text{co}}\,(K \setminus B(a, 1))$. Hence, there is a functional $l' \in X^*$ separating a and $\overline{\text{co}}\,(K \setminus B(a, 1))$ strictly such that with a suitable α we have

$$l'(a) < \alpha < l'(x) \quad \text{for all } x \in \overline{\text{co}}\,(K \setminus B(a, 1)). \tag{1.3}$$

Now we show that $l'(x) > l'(a)$, for all $x \in K \setminus \{a\}$. It is easy to see that $K \cap \{x \in X : l'(x) < l'(a)\} = \emptyset$. Assume the contrary: there is a point $\bar{x}$ of this intersection. Then, by a property of the cone we have $a + \lambda(\bar{x} - a) \in K \cap \{x \in X : l'(x) < l'(a)\}$, $\lambda > 0$, i.e. there is an $\hat{x} \in K \setminus B(a, 1)$ with $l'(\hat{x}) < l'(a)$. This contradicts (1.3). By an analogous argument we get $(K \setminus \{a\}) \cap (l', l'(a)) = \emptyset$. So, that means: $l'(x) > l'(a)$ for all $x \in K \setminus \{a\}$.

Since, by (1.3), the hyperplane (l', α) separates a and $K \setminus B(a, 1)$ strictly, the intersection of (l', α) with K cannot be contained in $K \setminus B(a, 1)$; it is contained in $K \cap B(a, 1)$. Hence, $(l', \alpha) \cap K$ is bounded. Now it is not difficult to show that $(l', \beta) \cap K$ for $\beta \geqq l'(a)$ is bounded, too. We assume here that $\beta > l'(a)$ (in the case $\beta = l'(a)$ it is easy to see that $(l', \beta) \cap K = \{a\}$ and hence bounded). Let $c = (\beta - l'(a))/(\alpha - l'(a))$, we have $c > 0$. Since $K - a$ is a cone with vertex 0, it follows then that $K - a = c(K - a)$. Since $(l', \alpha) - a = \{x \in X : l'(x) = \alpha - l'(a)\}$ it is easy to check that $(l', \beta) - a = c[(l', \alpha) - a]$. We then have

$$\begin{aligned} (l', \beta) \cap K - a &= [(l', \beta) - a] \cap (K - a) \\ &= c\{[(l', \alpha) - a] \cap (K - a)\} = c[(l', \alpha) \cap K - a]. \end{aligned}$$

Since $(l', \alpha) \cap K$ is bounded, $(l', \beta) \cap K$ is bounded ∎

Using Lemma 2.6 we can now prove the following theorem for the stability of Problem (P_t).

Theorem 2.3: *Let Assumptions 2.1 and 2.2 be satisfied. Suppose that $N(U)$ is contained in a pointed cone K. Then Problem* (P_t) *is stable at* t^0.

Proof: We show that the condition (iii) (given in Theorem 2.1) is here satisfied and hence the assertion follows from Theorem 2.2. Let $l' \in X^*$ be the functional which exists by Lemma 2.6 for the pointed cone K. Since $N(t^0)$ is bounded (Assumption 2.2) we have $\gamma = \sup \{l'(x) : x \in N(t^0)\} < +\infty$. Consider the hyperplane (l', α) with $\alpha > \gamma$. It is then easy to see that $l'(x) < \alpha$ for all $x \in N(t^0) \cup \{a\}$ where a is the vertex of K. By Lemma 2.6 $(l', \alpha) \cap K$ is bounded and hence $Q = \{x \in K : l'(x) < \alpha\}$ is also bounded. Since $N(U) \subset K$, it is then easy to see that $N(U) \cap \{x \in X : l'(x) < \alpha\} \subset Q$ is also bounded. Thus, condition (iii) is satisfied by taking the open set $\Omega = \{x \in X : l'(x) < \alpha\}$ and the neighbourhood $V = U$ ∎

Remark 2.3: According to Remark 2.1 it is here worth noticing that by using the pointed cone defined above we can derive the following criterion for the upper semicontinuity of a set-valued mapping:

Let φ be a real-valued and upper semicontinuous function on $C \times T$ such that $\varphi(\cdot, t)$ is concave for each $t \in T$. Let the set-valued mapping M be defined by $M(t) = \{x \in C: \varphi(x, t) \geqq 0\}$. Suppose that $M(t^0)$ is bounded and there is a neighbourhood V of t^0 such that $M(V)$ is contained in a pointed cone. Then M is upper semicontinuous.

By an argument analogous to that used in the proof of Theorem 2.3 it is easy to see that there is an open set Ω such that $\Omega \cap M(V)$ is bounded Hence, by Remark 2.1, follows the assertion.

The above-considered mapping M is a mapping with special structure. About criterions for the upper semicontinuity (and also lower semicontinuity) of general set-valued mappings we refer the reader, for example, to BERGE [2] and BANK et al. [1].

3. Special cases

In this section, stability criterions for some special cases will be given by using Theorems 2.2 and 2.3. Consider the following family of variational inequalities

$$\begin{cases} x \in C \\ \langle A(x, t) - v', x - y \rangle + \varphi(x, t) - \varphi(y, t) \leqq 0 \quad \text{for all } y \in C, \end{cases} \tag{3.1}$$

where A is an operator from $C \times T$ into X^*, φ is a real-valued function on $C \times T$ and $v' \in X^*$, is a given functional.

Proposition 3.1: *Let $A(\cdot, t)$ be monotone and hemicontinuous for each $t \in T$, $A(x, \cdot)$ continuous for each $x \in C$, φ* lower semi-continuous, *$\varphi(\cdot, t)$ convex for each $t \in T$. Suppose that there is a point $y_0 \in C$ such that the image set $N(t^0)$ of the set-valued mapping N defined by*

$$N(t) = \{x \in C: \langle A(y_0, t) - v', y_0 - x \rangle + \varphi(y_0, t) - \varphi(x, t) \geqq 0\}$$

is bounded. Moreover, suppose that there is a neighbourhood U of t^0 such that $N(U)$ is contained in a pointed cone.

Then Problem (3.1) *is stable at* t^0.

Proof: The assertion follows immediately from Theorem 2.3 ∎

Corollary 3.1: *Let $A: C \to X^*$ be a monotone and hemicontinuous operator, $\varphi : C \times T \to \mathbf{R}$ a lower semicontinuous function such that $\varphi(\cdot, t)$ is convex for each t, and $\alpha: T \to \mathbf{R}$ a continuous function. Suppose that $\varphi(x, t^0) \to +\infty$ as $\|x\| \to \infty$ and* *there is a point $y_0 \in C$ satisfying $Ay_0 = 0$, such that the set $\{x \in C : \varphi(x, t) \leqq \varphi(y_0, t)\}$ is contained in a pointed cone for all t in a neighbourhood of t^0.*

Then the problem

$$\begin{cases} x \in C \\ \alpha(t) \langle Ax, x - y\rangle + \varphi(x, t) - \varphi(y, t) \leqq 0 \quad \text{for all} \quad y \in C \end{cases}$$

is stable at t^0.

Proof: Apply Proposition 3.1 with $N(t) = \{x \in C : \varphi(x, t) \leqq \varphi(y_0, t)\}$ and $v' = 0$. Since $\varphi(x, t^0) \to +\infty$ as $\|x\| \to \infty$, $N(t^0)$ is bounded. The other assumptions are satisfied too ∎

Corollary 3.2: *Let C be contained in a pointed cone K with vertex 0 and let v' be a functional with $v'(x) \leqq 0$ for all $x \in K$. Let the operator A and function φ be given as in Proposition 3.1. Suppose that $\varphi(x, t^0) \to +\infty$ as $\|x\| \to \infty$ and there is a point $y_0 \in C$ satisfying $A(y_0, t^0) = 0$.*

Then Problem (3.1) is stable at t^0.

Proof: Apply Proposition 3.1 with $N(t) = \{x \in C : \langle v', x - y_0\rangle + \varphi(y_0, t) - \varphi(x, t) \geqq 0\}$. By the property of v' and φ it is easy to see that $N(t^0)$ is bounded. The other assumptions are satisfied too ∎

Let us now consider the following family of optimization problems

$$\text{Min}\,\{\varphi(x, t) : x \in C\}, \tag{3.2}$$

where as above φ is a real-valued function on $C \times T$. We write this problem in the form

$$\begin{cases} x \in C \\ \varphi(x, t) - \varphi(y, t) \leqq 0 \quad \text{for all} \quad y \in C. \end{cases} \tag{3.2'}$$

Problem (3.2) is said to be *stable at t^0* if Problem (3.2') is stable at t^0. From Proposition 3.1 it is easy to derive

Corollary 3.3: *Let φ be a lower semicontinuous function such that $\varphi(\cdot, t)$ is convex for each t. Suppose that $\varphi(x, t^0) \to +\infty$ as $\|x\| \to \infty$ and there is a point $y_0 \in C$ such that the set*

$$N(t) = \{x \in C : \varphi(x, t) \leqq \varphi(y_0, t)\}$$

is contained in a pointed cone for all t in a neighbourhood of t^0.

Then Problem (3.2) is stable at t^0.

The stability criterion in Corollary 3.3 is given only for the solution set mapping of convex optimization problems. A general stability theory for general optimization problems is given in Bank et al. [1], Gollan [2] and Kirsch [7].

Proposition 3.2: *Let X be a finite-dimensional space and let Assumptions 2.1 and 2.2 be satisfied. Then Problem (P_t) is stable at t^0.*

Proof: Since in a finite-dimensional space an open set is weakly open, condition (iii) given in Theorem 2.1 is satisfied by taking $\Omega = \{x \in X : \|x\| < r\} \supset N(t^0)$ for $r > 0$ enough large and $V \subset T$ to be an arbitrary neighbourhood of t^0. $\Omega \cap N(V)$ is then bounded. Thus, by Theorem 2.2 Problem (P_t) is stable at t^0 ∎

Example: Let (Q_t) be the following family of nonlinear complementarity problem

$$\begin{cases} x \in \mathbf{R}^n, \quad M(x, t) \in \mathbf{R}_+^n \\ x'M(x, t) = 0 \end{cases} \qquad (Q_t)$$

where $\mathbf{R}_+^n = \{x \in \mathbf{R}^n: x_1, x_2, \ldots, x_n \geqq 0\}$, $M: \mathbf{R}_+^n \times T \to \mathbf{R}^n$ is an operator such that for each $t \in T$, $M(\cdot, t)$ is monotone and hemicontinuous and for each $x \in \mathbf{R}_+^n$, $M(x, \cdot)$ is continuous (x' is the transposed vector of x). Assume that there is $y_0 \in \mathbf{R}_+^n$ such that the set $\{x \in \mathbf{R}_+^n: x'M(y_0, t_0)' \leqq \alpha\}$, $\alpha := y_0'M(y_0, t_0)$, is bounded. Then there exists a neighbourhood V of t_0 such that for each $t \in V$ the complementarity problem (Q_t) has a solution. If we denote by $\Gamma(t)$ the solution set of (Q_t), $t \in V$, then the set-valued mapping $\Gamma: t \to \Gamma(t)$ is upper semicontinuous at t_0.

It is here easy to see that $\mathbf{R}_+^n$ is a pointed cone. Hence, by applying Proposition 3.2 with $C = \mathbf{R}_+^n$, $f(x, y) = (x' - y')\, M(x, t)$ it implies that Problem (P_t) in this case is stable at t_0. The assertion follows then by the fact that a point $x \in \mathbf{R}_+^n$ is a solution of Problem (P_t) (in this case) if and only if it is a solution of the complementarity problem (Q_t), $t \in V$ (see LÜTHI [11]).

Some applications (e.g. to the obstacle problem, the free boundary problem) will be studied in another paper.

LITERATUR

[1] BANK, B., GUDDAT, J., KLATTE, D., KÜMMER, B., and K. TAMMER: Non-Linear Parametric Optimization. Berlin: Akademie-Verlag 1982.

[2] BERGE, C.: Topological spaces. Edinburgh—London: Oliver & Boyds 1963.

[3] EKELAND, I., and R. TEMAM: Convex analysis and variational problems. Amsterdam—Oxford: North-Holland 1978.

[4] GAUVIN, J., and J. W. TOLLE: Differential Stability in Non-Linear Programming. SIAM J. Control and Optim. **15** (1977), 294—311.

[5] GOLLAN, B.: Perturbation Theory for Abstract Optimization Problems. J. Optim. Theory and Appl. (JOTA) **35** (1981), 417—441.

[6] HOANG TUY: Stability property of a system of inequalities. Math. Operationsforsch. Statistik, Ser. Optim. **8** (1977), 27—39.

[7] KIRSCH, A.: Continuous Perturbations of Infinite Optimization Problems. J. Optim. Theory and Appl. (JOTA) **32** (1980), 171—182.

[8] KUMMER, B.: Generalized Equations: Solvability and Regularity. Preprint. Berlin: Sektion Math. der Humboldt-Universität, Preprint **30** (1982).

[9] LEMPIO, F., and H. MAURER: Differential Stability in Infinite-Dimensional Non-Linear Programming. Applied Math. and Optim. **6** (1980), 139—152.

[10] LEVITIN, E. S.: On the Local Perturbation Theory of Mathematical Programming in a Banach Space. Soviet Math. Doklady **16** (1975), 1354—1358.

[11] LÜTHI, H. J.: Komplementaritäts- und Fixpunktalgorithmen in der mathematischen Programmierung, Spieltheorie und Ökonomie (Lect. Notes in Econ. and Math. Systems 129). Berlin: Springer-Verlag 1976.

[12] MOSCO, U.: Implicit Variational Problems and Quasivariational Inequalities. Lect. Notes in Math. **543** (1975), 83—157.

[13] ROBINSON, S. M.: Stability theory for systems of inequalities, Part II: Differentiable Non-Linear Systems. SIAM J. Num. Anal. **13** (1976), 497—513.

Manuskripteingang: 13. 03. 1984; in revidierter Fassung 27. 08. 1984

VERFASSER:

Dr. LÊ VĂN CHÓNG
Institute of Mathematics
Nghĩa Đô, Từ Liêm
Hà Nội, R.S. Việt Nam