(2)

 (3)

A Variational Principle for Equations and Inequalities with Maximal **Monotone Operators**

E. KRAUSS

Gegeben sei eine Variationsungleichung mit einem monotonen Operator. Wir beschreiben die Lösungsmenge durch ein modifiziertes Variationsprinzip. Dieses bleibt auch dann gültig, wenn der Operator nicht Subdifferential einer konvexen Funktion ist. Es gestattet eine physikalische Interpretation und zeigt, in welchem Sinne die Monotonie des Operators zu "global stabilen" Lösungen der Variationsungleichung führt.

Пусть задано вариационное неравенство с монотонным оператором. Описываем множество решений модифицированным вариационным принципом, который остаётся верным даже если оператор не является субдифференциалом выпуклой функции. Этот вариационный принцип можно интерпретировать физически. Он показывает, в каком смысле монотонность оператора ведёт к ,,глобально стабильным" решениям вариационного неравенства.

Let be given a variational inequality with a monotone operator. We describe the set of solutions by a modified variational principle. This still remains valid if the operator is not a subdifferential of a convex function. It allows a physical interpretation and shows in which sense the monotonicity of the operator leads to "globally stable" solutions of the variational inequality.

1. Introduction

Let E be a real locally convex Hausdorff space with dual space E^* . The pairing between E and E^* is denoted by $\langle \cdot, \cdot \rangle$. Throughout this paper we shall assume that A is a possibly multivalued mapping from E into E^* which is monotone, i.e.

$$
\langle \tilde{A}x - \tilde{A}y, x - y \rangle \geq 0' \text{ for all } \tilde{A}x \in Ax, \tilde{A}y \in Ay.
$$

Moreover, M will always stand for a convex subset of the domain $D(A) := \{x \in E : Ax$ \pm 0.

The object of our investigation is the variational inequality

$$
x \in M, \tilde{A}x \in Ax, \qquad \qquad \}
$$

\n
$$
\langle \tilde{A}x, x - v \rangle \leq 0 \quad \text{for all} \quad v \in M
$$
 (1)

By using the subdifferential ∂I_M of the indicator function of M, the inequality (1) can be written as

$$
Ax+\partial I_{M}x\ni 0.
$$

Another formulation of (1) is

$$
\begin{aligned}\nx \in M, & \tilde{A}x \in Ax, \\
\langle \tilde{A}x, w \rangle &\geq 0 \quad \text{for all} \quad w \in T_xM\n\end{aligned}\bigg\}.
$$

Here T_xM denotes the *tangential cone* of the convex set M at the point $x \in M$, i.e. $T_xM = [0, \infty)$ $\{M - \{x\}\}\$. Note that (3) furnishes a description of a local equilibrium in the presence of a constraint M (principle of d'Alembert-Fourier). **in the presence of a constraint Alene of the convex set M at the point x**
 $T_xM = [0, \infty) \{M - \{x\}\}\.$ Note that (3) furnishes a description of a local equin the presence of a constraint M (principle of d'Alembert-Fourier).

The aim of this paper is to describe the solutions to (1) by a variational principle. We do not suppose here *A* to be a subdifferential of a convex function. This variational principle admits a physical interpretation. Moreover, it shows in which sense the nionotonicity of *A* legds to ,,globally stable" solutions of the variational in-432 E. KRAUSS

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 $T_xM = [0, \infty)$ $\{M - \}$

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pointraint** M **(principle of d'Alembert-Fourier).

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solutions of the variational in-
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2. A saddle function associated to the operator A

In this section we attach a bivariate function $J_A : M \times M \to \mathbb{R}$ to the monotone operator *A* and study its properties.

Definition: We define the function $J_A: M \times M \rightarrow \mathbb{R}$ by

$$
J_A(x, y) := \int\limits_0^1 \langle \tilde{A}(x + t(y - x)), y - x \rangle dt,
$$

where \tilde{A} is an arbitrary single-valued section of A .

This definition is justified by the following

Lemma: For each $x, y \in M$ *and each section* \widetilde{A} *of A, the function*

$$
[0, 1] \ni t \mapsto \langle \tilde{A}(x + t(y - x)), y = x \rangle
$$

is Lebesque-integrable. Moreover, the values Of "A do not depend on the choice Of the section A. is Lebesque-integrable. Moreover, the values of J_A do not de
section \tilde{A} .
Before proving this lemma we state some further results.
Theorem 1: The function J_A has the following properties:
1. J_A is a skew-symm *••• A* is an arbitrary single-valued section of A.
 Find definition is justified by the following
 Lemma: *For each* $x, y \in M$ and each section \tilde{A} of A , the function
 $[0, 1] \ni t \mapsto (\tilde{A}(x + t(y - x)), y - x)$
 is Lebes 10, 11 \rightarrow $(A(x + i(y - x)), y - x)$
 is Lebesgue-integrable. Moreover, the values of J_A

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Before proving this lemma we state some further

Theorem 1: The function J_A has the following γ

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1. *JA is a skew-symmetric saddle function, i.e.*

$$
J_A(x, y) = -J_A(y, x) \quad \text{for all} \quad x, y \in M.
$$

. The /ollowimj estimate holds:

$$
\begin{aligned}\n\text{Proving this lemma we state some further results.} \\
\text{rem 1: The function } J_A \text{ has the following properties:} \\
\text{is a skew-symmetric saddle function, i.e.} \\
J_A(x, y) &= -J_A(y, x) \quad \text{for all} \quad x, y \in M. \\
\text{Equation: } \begin{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \begin{aligned}\n&\text{Equation: } \end{aligned}\n&\text{Equation: } \
$$

for all $x, y \in M$.
3. We have \int_{0}^{1}

more proving this lemma we state some further results.

\n(eorem 1: The function
$$
J_A
$$
 has the following properties:

\n
$$
J_A
$$
 is a skew-symmetric saddle function, i.e.\n
$$
J_A(x, y) = -J_A(y, x) \quad \text{for all} \quad x, y \in M.
$$
\nThe following estimate holds:

\n
$$
\sup \{ \langle f, y - x \rangle : f \in Ax \} \leq J_A(x, y) \leq \inf \{ \langle g, y - x \rangle : g \in Ay \} \qquad (4)
$$
\nif $x, y \in M$.

\nWe have

\n
$$
J_A(x, \lambda x + (1 - \lambda) y) \leq \lambda J_A(x, x) + (1 - \lambda) J_A(x, y)
$$
\n
$$
= (1 - \lambda) J_A(x, y) \quad \text{for} \quad \lambda \in [0, 1], \qquad x, y \in M.
$$
\n(5)

\ngeneral, the function J_A is not concave-convex. This is shown by the following

\nple, which is due to D. TIBA:

\n
$$
J_A(x, \lambda x + (1 - \lambda) y) = \lambda J_A(x, y) \quad \text{for} \quad \lambda \in [0, 1], \qquad \lambda \
$$

In general, the function J_A is not concave-convex. This is shown by the following example, which is due to D. TIBA:

Let $E = \mathbb{R}^2$ be the Euclidean space and define

 $A(x_1, x_2) = [x_1, -x_1^2 + x_2]$ on $D(A) = \{-1, 1\} \times \mathbb{R}$.

A Variational Principle 433

one but, for example, the function Here the operator *A* is monotone but, for example, the function

operator A is monotone but, for example, the f
\n
$$
\mathbf{R}^2 \ni y \mapsto J_A(0, y) = 1/2(y_1^2 + y_2^2) - 1/3(y_1^2 y_2)
$$

is not convex.

Proof of the Lemma and of Theorem 1: Let $x, y \in M$ be arbitrarily fixed A Variational Principle

Here the operator A is monotone but, for example, the function
 $\mathbb{R}^2 \ni y \mapsto J_A(0, y) = 1/2(y_1^2 + y_2^2) - 1/3(y_1^2y_2)$

is not convex.

Proof of the Lemma and of Theorem 1: Let $x, y \in M$ be arbitra *-* -S

A Variational Principle 433
\nHere the operator A is monotone but, for example, the function
\n
$$
\mathbf{R}^2 \ni y \mapsto J_A(0, y) = 1/2(y_1^2 + y_2^2) - 1/3(y_1^2y_2)
$$
\nis not convex.
\nProof of the Lemma and of Theorem 1: Let $x, y \in M$ be arbitrarily fixed
\nand define, for $t \in [0, 1]$,
\n
$$
\underline{h}(t) := \inf \{ \langle f_0, y - y \rangle : f_1 \in A(x + t(y - x)) \},
$$
\n
$$
\overline{h}(t) := \{ \overline{A}(x + t(y - x)), y - x \},
$$
\n
$$
\overline{h}(t) := \sup \{ \langle g_1, y - x \rangle : g_1 \in A(x + t(y - x)) \}.
$$
\nObviously we have
\n
$$
\underline{h}(t) \leq \overline{h}(t) \leq \overline{h}(t) \text{ for } t \in [0, 1].
$$
\nOn the other, hand, we get
\n
$$
\underline{h}(t_0) \geq \overline{h}(t_1) \text{ for } 0 \leq t_1 < t_2 < t_3 \leq 1.
$$
\n(7)
\nIndeed, in view of (6), the inequality (7) reduces to
\n
$$
\underline{h}(s) \geq \overline{h}(t) \text{ for } 0 \leq t < s \leq 1.
$$
\n(7)
\nBut due to the monotonicity of A we have

$$
\underline{h}(t) \leq \overline{h}(t) \leq \overline{h}(t) \quad \text{for} \quad t \in [0, 1].
$$

Obvious
On the compared in the set of the set

$$
\underline{h}(t) \leq \overline{h}(t) \leq \overline{h}(t) \quad \text{for} \quad t \in [0, 1].
$$
\nthere are the following matrices:

\n
$$
\underline{h}(t_3) \geq \overline{h}(t_2) \geq \overline{h}(t_1) \quad \text{for} \quad 0 \leq t_1 < t_2 < t_3 \leq 1.
$$

Indeed, in view of (6), the inequality *(7)* reduces to

$$
\underline{h}(s) \geq \overline{h}(t) \quad \text{for} \quad 0 \leq t < s \leq 1. \tag{7}
$$

But due to the monotonicity of A we have

$$
\underline{h}(s) \geq \overline{h}(t) \quad \text{for} \quad 0 \leq t < s \leq 1. \tag{7}
$$
\nBut due to the monotonicity of A we have

\n
$$
\langle f_s, y - x \rangle - \langle g_t, y - x \rangle = \frac{1}{(s - t)} \langle f_s - g_t, (x + s(y - x)) - (x + t(y - x)) \rangle \geq 0
$$

for all $f_s \in A(x + s(y - x))$ *and* $g_t \in A(x + t(y - x))$ *.*

This proves (7'). The inequalities (6) and (7) imply that \underline{h} , \underline{h} and \overline{h} are monotone increasing functions which are finite on (0,1). Hence all these functions are Lebesgueintegrable. Moreover, (6) and *(7)* give rise to Let₃ $\geq \bar{h}(t_2) \geq \bar{h}(t_1)$ for $0 \leq t_1 < t_2 < t_3 \leq 1$. (7)

n view of (6), the inequality (7) reduces to
 $\underline{h}(s) \geq \bar{h}(t)$ for $0 \leq t < s \leq 1$. (7)

to the monotonicity of A we have
 \Rightarrow \Rightarrow \Rightarrow $\frac{1}{(s-t)} \langle$ $(s - t)$
 $(s - t)$
 $(s - x)$ and g_t
 s inequalities
 $s(t)$ and (7)
 $\lim_{s \to t} h(s) = \lim_{s \to t} h(s)$ There a, in view of (b), the inequality (*f*) educts
 $\underline{h}(s) \geq \overline{h}(t)$ for $0 \leq t < s \leq 1$.

But due to the monotonicity of *A* we have
 $\langle f_s, y - x \rangle - \langle g_t, y - x \rangle = \frac{1}{(s-t)} \langle f_s - g_t, (x \rangle)$

for all $f_s \in A(x + s(y - x))$ and $g_t \in$ *f***h** $f(x) = \frac{1}{(s-t)} \left\langle f_s - g_t, (x + s(y-x)) - (x + s(y-x)) \right\rangle$
 *f*_n $f_s(y-x)$ and $g_t \in A(x+t(y-x))$.
 *f*he inequalities (6) and (7) imply that *h*, *h* and *h* is which are finite on (0,1). Hence all these function

by the inequalities

$$
\lim_{s \downarrow t} \underline{h}(s) = \lim_{s \downarrow t} \overline{h}(s) = \lim_{s \downarrow t} \overline{h}(\varepsilon) \quad \text{for all} \quad s \in (0, 1)
$$

Hence, we obtain.

$$
\overline{h}(0) \leq \int_{0}^{1} \overline{h}(t) dt = \int_{0}^{1} \overline{h}(t) dt = \int_{0}^{1} \underline{h}(t) dt \leq \underline{h}(1)
$$

(cf. R. T. ROCKAFELLER [8]). -This proves the Lemma and the estimate (4). The skewsymmetry of J_A is obvious. Thus it remains to check (5). For this purpose let $x, y \in M$

$$
\lim_{s \to i} \underline{h}(s) = \lim_{s \to i} \overline{h}(s) = \lim_{s \to i} \overline{h}(s)
$$
 for all $s \in (0, 1)$.
\nHence, we obtain
\n
$$
\overline{h}(0) \leq \int_{0}^{1} \overline{h}(t) dt = \int_{0}^{1} \overline{h}(t) dt = \int_{0}^{1} \underline{h}(t) dt \leq \underline{h}(1)
$$
\n
$$
(cf. R. T. RockAFELLER [8]). This proves the Lemma and the estimate (4). The skew-symmetry of J_A is obvious. Thus it remains to check (5). For this purpose let $x, y \in A$ and $\lambda \in [0, 1]$ be fixed. By the monotonicity of \overline{h} we can conclude
\n
$$
\langle \overline{A}(x + t(\lambda x + (1 - \lambda) y - x)), (\lambda x + (1 - \lambda) y) - x \rangle
$$
\n
$$
= (1 - \lambda) \langle \overline{A}(x + t(1 - \lambda) (y - x)), y - x \rangle
$$
\n
$$
= (1 - \lambda) \overline{h}(t(1 - \lambda)) \leq (1 - \lambda) \overline{h}(t).
$$
\nIntegrating here over [0, 1] yields
\n
$$
J_A(x, \lambda x + (1 - \lambda) y) \leq (1 - \lambda) J_A(x, y).
$$
\nThe skew-symmetry of J_A implies $J_A(x, x) = 0$, so we get (5) as desired 1
\n28 Analysis, Bd. 5, Hejt 5 (1980)
$$

$$
J_A(x, \lambda x + (1 - \lambda) y) \leq (1 - \lambda) J_A(x, y).
$$

The skew-symmetry of J_A implies $J_A(x, x) = 0$, so we get (5) as desired **I**

3. A variational principle

Now we can state the announced variational principle for the solutions to (1).

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 Theorem 2: Let $A + \partial I_M$ be maximal monotone. Then the following conditions are
 equivalent: **2.434 E.** KRAUSS
 2.6. A variational principle

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Theorem 2: Let $A + \partial I_M$ be maximal monotone. Then

equivalent:

(i) $x \in M$ solves the variational inequality **234 E.** KRAUSS
 We we can state the announced variational principle for the Theorem 2: Let $A + \partial I_M$ be maximal monotone. Then the equivalent:

(i) $x \in M$ solves the variational inequality (1).

(ii) $[x, x] \in M \times M$ is a

(i) $x \in M$ solves the variational inequality (1).

(ii) $[x, x] \in M \times M$ *is a saddle point of the skew-symmetric saddle function* J_A , *i.e.*, *we have*

(iii) There *exists a neighbourhood* $U(x)$ *of x with*

$$
0 \leq J_A(x, v) \text{ for all } v \in M.
$$

where exists a neighborhood $U(x)$ of x and $0 \leq J_A(x, v)$ for all $v \in M \cap U(x)$.

(iv) There exists an element $y \in M$ *such that* $[x, y]$ *is a saddle point of* J_A *, i.e., we have*

 $J_A(v, y) \leq J_A(x, y) \leq J_A(x, w)$ for all $v, w \in M$.

Remarks: a) One can interpret $J_A(x, y)$, as the work (resp. the action) during a movement from the state x to the state y along the trajectory $\gamma(t) = x + t(y - x)$; $t \in [0, 1]$ (compare also Theorem 3).

b) The equivalence (ii) \leftrightarrow (iii) shows that we can describe the solutions to (1) both

(ii) $[x, x] \in M \times M$ is a saddle point of the skew-symmetric saddle function J_A , i.e.
we have
 $0 \leq J_A(x, v)$ for all $v \in M$.
(iii) There exists a neighbourhood $U(x)$ of x with
 $0 \leq J_A(x, v)$ for all $v \in M \cap U(x)$.
(iv) There exi $(ii) \rightarrow (i)$. The other implications of Theorem 2 remain true if one drops this assumption. Note that $A + \partial I_M$ is maximal monotone if E is a Banach space, M is closed, and the restriction of A to M is a radially continuous mapping having convex closed bounded values $\{Ax\}$ (cf. F. KRAUSS [2] and, for a single-valued mapping A, also **R. T. ROCKAFELLAR** [6, 7]). and the restriction of A to M is a radially continuous in
bounded values $\{Ax\}$ (cf. F. KRAUSS [2] and, for a s
R. T. ROCKAFELLAR [6, 7]).
Proof of Theorem 2: (i) \rightarrow (ii): Let $x \in M$ be a s
equality (1). According to t b) The equivalence (ii) \leftrightarrow (iii) shows that we can describe th
 r a local and a global criterion.

c) The maximality of $A + \partial I_M$ enters only into the pro

(i) \rightarrow (i). The other implications of Theorem 2 remain true (ii) \rightarrow (i). The other implications of Theorem 2
tion. Note that $A + \partial I_M$ is maximal monoton
and the restriction of A to M is a radially cont
bounded values $\{Ax\}$ (cf. E. KRAUSS [2] and,
R. T. ROCKAFELLAR [6, 7]).

Proof of Theorem 2: (i) \rightarrow (ii): Let $x \in M$ be a solution to the variational in-CKAFELLAR [6, 7]).

of Theorem 2: (i) \rightarrow (ii): Let $x \in M$ be a solution to

(1). According to the estimate (4), we obtain
 $0 \leq \sup \{(f, v - x) : f \in Ax\} \leq J_A(x, v)$ for $v \in M$,

$$
0 \leq \sup \left\langle f, v - x \right\rangle : f \in Ax \right\} \leq J_A(x, v) \quad \text{for} \quad v \in M,
$$

as desired.

-

(iii) \rightarrow (iv): Let us suppose

We show that one can set $y = x$ in (iv). Because of the skew-symmetry of J_A it' remains to check $0 \leq J_A(x, v)$ for all $v \in M$. For this purpose we assume $J_A(x, v_0) < 0$ **for** some $v_0 \in M$. If $\lambda \in (0, 1)$ is small enough we get $(1 - \lambda)x + \lambda v_0 \in U(x)$ n and in connection with (5) email
or so:
.nd in
.

$$
J_A(x, (1 - \lambda) x + \lambda v_0) \leq \lambda J_A(x, v_0) < 0,
$$

which is a contradiction to our assumption.

(i) definition of $J_A(x, (1 - \lambda)x + \lambda v_0) \leq \lambda J_A(x, v_0) < 0$,

iich is a contradiction to our assumption.

(iv) \rightarrow (ii): Let $[x, y] \in M \times M$ be a saddle point of J_A . This means

$$
J_A(x, (1 - \lambda) x + \lambda v_0) \le \lambda J_A(x, v_0) < 0,
$$

a contradiction to our assumption.
(ii): Let $[x, y] \in M \times M$ be a saddle point of J_A .

$$
J_A(v, y) \le J_A(x, y) \le J_A(x, w) \text{ for all } v, w \in M.
$$

Choosing here especially $v = y$ yields (compare Theorem 1)

 $0 = J_A(y, y) \leq J_A(x, y) \leq J_A(x, w)$ for all $w \in M$.

 (10)

iuw

(ii) \rightarrow (i): Let $x \in M$ satisfy $0 \leq J_A(x, v)$ for all $v \in M$. As a consequence of (4) we get (ii) \rightarrow (i): Let $x \in M$ satisfy $0 \le$
we get
 $0 \le \inf \{ \langle g, v - x \rangle : g \in Av \}$
By the definition of ∂I_M this inequa
 $0 \le \langle f, v - x \rangle$ for all v
Since $A + \partial I_M$ was supposed to be
 $+ \partial I_M x$, i.e., x solves (1) \blacksquare
Instead of

$$
0 \leq \inf \left\{ \left\langle g, v - x \right\rangle : g \in Av \right\} \text{ for all } v \in M.
$$

By the definition of ∂I_M this inequality implies

$$
0 \leq \langle f, v - x \rangle \quad \text{for all} \quad v \in M, f \in Av + \partial I_M v.
$$

 $0 \le \inf \{ \langle g, v - x \rangle : g \in Av \}$ for all $v \in M$.
By the definition of ∂I_M this inequality implies
 $0 \le \langle f, v - x \rangle$ for all $v \in M, f \in Av + \partial I_M v$.
Since $A + \partial I_M$ was supposed to be maximal monotone, we can conclude $0 \in A_T$
 $+ \partial I_M x$, $+ \partial I_M x$, i.e., *x* solves (1)

Instead of the function *JA* **one** can consider more generally a function *[0, 1] M,* y(0)=x, y(l)y *•JA : M x* R; *JA(X, y)* = *f KAy(t), (t) dl,*

 $\begin{aligned} \mathbf{y} & = \mathbf{y} \mathbf{y} \\ \mathbf{y} & = \mathbf{y} \mathbf{y} \end{aligned}$

$$
\dot{\gamma} = \gamma_{x,y} : [0,1] \to M, \qquad \gamma(0) = x, \qquad \gamma(1) = y
$$

is a sufficiently regular trajectory joining the points x and y . In order to avoid additional regularity assumptions on the operator *.4* it is convenient to confine oneself to *polygonal trajectories* **in** *M.* These can he defined by $\dot{\gamma} = \gamma_{x,y}: [0, 1] \rightarrow M,$ $\gamma(0) = x,$ $\gamma(1) = y$

ciently regular trajectory joining the points x and y. In
 i regularity assumptions on the operator A it is convenien
 ygonal trajectories in M. These can be defined by
 1ar trajectory joining
assumptions on the op
ctories in M. These ca
 $11 \rightarrow M$; $\gamma(t) = 1$
 \Rightarrow $\gamma(t) = 1$
 \Rightarrow $\gamma(t) = 0$
 \therefore $x_n = y$ is a finite:
 $\langle \tilde{A} \gamma(t), \dot{\gamma}(t) \rangle dt = \sum_{i=0}^{n-1} \langle \tilde{A} \gamma(t), \dot{\gamma}(t) \rangle dt$ $\gamma = \gamma_{x,y} : [0, 1] \rightarrow M,$ $\gamma(0) = x,$ $\gamma(1) = y$

is a sufficiently regular trajectory joining the points x and y. In order to avo

additional regularity assumptions on the operator A it is convenient to confine on

self to *p*

$$
\gamma = \gamma_{x,y} : [0, 1] \to M; \qquad \gamma(t) = x_i + n(t - i/n) x_{i+1} \quad \text{for}
$$

\n
$$
i/n \le t \le (i + 1)/n, \qquad i = 0, 1, ..., n,
$$

\ne $x = x_0, x_1, x_2, ..., x_n = y$ is a finite subset of M. Hence
\n
$$
J_A^2(x, y) = \int_0^1 \langle \tilde{A} \gamma(t), \gamma(t) \rangle dt = \sum_{i=0}^{n-1} J_A(x_i, x_{i+1}),
$$

where $x = x_0, x_1, x_2, ..., x_n = y$ is a finite subset of *M*. Hence

$$
J_A^{\prime\prime}(x, y) = \int\limits_0^1 \langle \tilde{A}\gamma(t), \dot{\gamma}(t) \rangle dt = \sum_{i=0}^{n-1} J_A(x_i, x_{i+1})
$$

IA is well defined and does not depend on the choice of the section \tilde{A} of A (com-

2 it is not possible to replace the condition (8) by
 $0 \leq J_A^{\gamma}(x, v)$ for all $v \in M$ and for all polygonal

1 trajectories γ

 $0 \leq J_A^{\gamma}(x, v)$ for all $v \in M$ and for all polygonal)

 $i/n \le t \le (i + 1)/n$,

where $x = x_0, x_1, x_2, ..., x_n = y$ is a
 $J_A \lambda(x, y) = \int_0^1 \langle \tilde{A} \gamma(t), \dot{\gamma}(t) \rangle dx$

i.e. J_A is well defined and does not a

pare the lemma in Section 2).

Now we show that in Theorem 2
 $0 \le J_A \lambda(x, v)$ for all Theorem 3: Let E be.a Banach space and suppose that the restriction $A|_M$ of A is *rot contained in the subdifferential* ∂p *of a convex function* $p : E \to \mathbf{R} \cup \{+\infty\}$ *,* not contained in the subdifferential ∂p of a convex function $p : E \to \mathbf{R} \cup \{+\infty\}$,
 $p \equiv +\infty$. Then, for each $x, y \in M$ and for each natural number n, there exists a poly-

gonal trajectory $\gamma : [0, 1] \to M$ between x an *gonal trajectory* γ : $[0, 1] \rightarrow M$ *between x and y with* $J_A^{\gamma}(x, y) < -n$. *J* and in the subdifferential ∂p of a
J and in the subdifferential ∂p of a
Phen, for each x, y \in *M* and for each
ory γ : [0, 1] \rightarrow *M* between x and y
Ny assumption $A|_M$ is not cyclical
 $z_1, ..., z_n =$

Proof: By assumption $A|_M$ is not cyclically monotone, i.e. there exist a cyclic
sequence $z_0, z_1, ..., z_n = z_0$ in $M \subseteq D(A)$ and a sequence $\tilde{Az}_i \in Az_i$ $(i = 1, ..., n)$ with *a*, *joi* each $x, y \in M$ and for each natural $\gamma : [0, 1] \rightarrow M$ between x and y with J_A ⁿ
assumption $A|_M$ is not cyclically mono
 $\ldots, z_n = z_0$ in $M \subseteq D(A)$ and a sequence
 $z_i, z_i - z_{i-1}$ < 0
saFELLAR [5, 9]). As a conseq rith

(10)

(11)

(11) gonal trajectory

Proof: By

sequence z_0, z_1 ,
 $\sum_{i=1}^{n} \langle \tilde{A} \rangle$

(cf. R. T. Rock
 $\sum_{i=1}^{n} J_A$

Now we choose
 $k \varepsilon \ge 28*$

$$
\sum_{i=1}^n \langle \tilde{A}z_i, z_i - z_{i-1} \rangle < 0
$$

(cf. R. T. **ROCKAFELLAR** [5, 9]). As a consequence of the estimate (4) we get

$$
\sum_{i=1}^n J_{\lambda}(z_{i-1},z_i):=-\varepsilon<0.
$$

Now we choose a natural number *k* with

$$
k\varepsilon \geq n + J_A(x,z_0) + J_A(z_0,y).
$$

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Let the polygonal trajectory $\gamma : [0, 1] \to M$ be defined by $\gamma = \gamma_1$
 $\gamma_1 = [x, z_0], \gamma_3 = [z_0, y]$ and a polygonal trajectory γ_2 with the vert \cup k \cdot γ_2 \cup γ_3 , with $y_1 = [x, z_0], y_3 = [z_0, y]$ and a polygonal trajectory y_2 with the vertices $z_0, z_1, \ldots, z_n = z_0$. According to (10) and (11) we get

\n- \n
$$
y_0
$$
, $y_3 = [z_0, y]$ and a polygonal trajectory y_2 with the vertices $z_n = z_0$. According to (10) and (11) we get\n
\n- \n $J_A'(x, y) = J_A(x, z_0) + k \left(\sum_{i=1}^n J_A(z_{i-1}, z_i) \right) + J_A(z_0, y) < -n$ \n
\n- \n**EVALUATE:** The equation $J_A: M \times M \to \mathbb{R}$ is not the only one.\n
\n

The skew-symmetric function $J_A: M \times M \to \mathbb{R}$ is not the only one allowing a characterization of the solutions to (1). In [3, 4] we showed that for each maximal monotone operator A from E into E^* there exists a skew-symmetric concave-convex closed saddle function $L: E \times E \to \mathbf{R} \cup \{\pm \infty\}$ such that $f \in Ax$ is equivalent to $[-f, f] \in \partial L(x, x)$. The concept, of a closed saddle function which is used here is due to R. T. ROCKAFELLAR $[10, 11]$ — compare also V. BARBU and TH. PRECUPANU $[1]$.

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