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A Variational Principle for Equations and Inequalities with Maximal Monotone Operators

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Gegeben sei eine Variationsungleichung mit einem monotonen Operator. Wir beschreiben die Lösungsmenge durch ein modifiziertes Variationsprinzip. Dieses bleibt auch dann gültig, wenn der Operator nicht Subdifferential einer konvexen Funktion ist. Es gestattet eine physikalische Interpretation und zeigt, in welchem Sinne die Monotonie des Operators zu „global stabilen" Lösungen der Variationsungleichung führt.

Пусть задано вариационное неравенство с монотонным оператором. Описываем множество решений модифицированным вариационным принципом, который остаётся верным даже если оператор не является субдифференциалом выпуклой функции. Этот вариационный принцип можно интерпретировать физически. Он показывает, в каком смысле монотонность оператора ведёт к „глобально стабильным" решениям вариационного неравенства.

Let be given a variational inequality with a monotone operator. We describe the set of solutions by a modified variational principle. This still remains valid if the operator is not a subdifferential of a convex function. It allows a physical interpretation and shows in which sense the monotonicity of the operator leads to "globally stable" solutions of the variational inequality.

1. Introduction

Let E be a real locally convex Hausdorff space with dual space E^*. The pairing between E and E^* is denoted by $\langle \cdot, \cdot \rangle$. Throughout this paper we shall assume that A is a possibly multivalued mapping from E into E^* which is *monotone*, i.e.

$$\langle \tilde{A}x - \tilde{A}y, x - y \rangle \geqq 0 \quad \text{for all} \quad \tilde{A}x \in Ax, \tilde{A}y \in Ay.$$

Moreover, M will always stand for a *convex* subset of the domain $D(A) := \{x \in E : Ax \neq \emptyset\}$.

The object of our investigation is the variational inequality

$$\left.\begin{array}{l} x \in M, \tilde{A}x \in Ax, \\ \langle \tilde{A}x, x - v \rangle \leqq 0 \quad \text{for all} \quad v \in M \end{array}\right\}. \tag{1}$$

By using the subdifferential ∂I_M of the indicator function of M, the inequality (1) can be written as

$$Ax + \partial I_M x \ni 0. \tag{2}$$

Another formulation of (1) is

$$\left.\begin{array}{l} x \in M, \tilde{A}x \in Ax, \\ \langle \tilde{A}x, w \rangle \geqq 0 \quad \text{for all} \quad w \in T_xM \end{array}\right\}. \tag{3}$$

Here T_xM denotes the *tangential cone* of the convex set M at the point $x \in M$, i.e. $T_xM = \overline{[0, \infty) \{M - \{x\}\}}$. Note that (3) furnishes a description of a local equilibrium in the presence of a constraint M (principle of d'Alembert-Fourier).

The aim of this paper is to describe the solutions to (1) by a variational principle. We do not suppose here A to be a subdifferential of a convex function. This variational principle admits a physical interpretation. Moreover, it shows in which sense the monotonicity of A leads to „globally stable" solutions of the variational inequality (1).

2. A saddle function associated to the operator A

In this section we attach a bivariate function $J_A : M \times M \to \mathbf{R}$ to the monotone operator A and study its properties.

Definition: We define the function $J_A : M \times M \to \mathbf{R}$ by

$$J_A(x, y) := \int_0^1 \left\langle \tilde{A}\big(x + t(y - x)\big), y - x \right\rangle dt,$$

where $\tilde{A}$ is an arbitrary single-valued section of A.

This definition is justified by the following

Lemma: *For each $x, y \in M$ and each section $\tilde{A}$ of A, the function*

$$[0, 1] \ni t \mapsto \left\langle \tilde{A}\big(x + t(y - x)\big), y - x \right\rangle$$

is Lebesgue-integrable. Moreover, the values of J_A do not depend on the choice of the section $\tilde{A}$.

Before proving this lemma we state some further results.

Theorem 1: *The function J_A has the following properties:*

1. *J_A is a skew-symmetric saddle function, i.e.*

$$J_A(x, y) = -J_A(y, x) \quad \textit{for all} \quad x, y \in M.$$

2. *The following estimate holds:*

$$\sup \{\langle f, y - x \rangle : f \in Ax\} \leqq J_A(x, y) \leqq \inf \{\langle g, y - x \rangle : g \in Ay\} \tag{4}$$

for all $x, y \in M$.

3. *We have*

$$J_A\big(x, \lambda x + (1 - \lambda) y\big) \leqq \lambda J_A(x, x) + (1 - \lambda) J_A(x, y)$$
$$= (1 - \lambda) J_A(x, y) \quad \textit{for} \quad \lambda \in [0, 1], \qquad x, y \in M. \tag{5}$$

In general, the function J_A is not concave-convex. This is shown by the following example, which is due to D. TIBA:

Let $E = \mathbf{R}^2$ be the Euclidean space and define

$$A(x_1, x_2) = [x_1, -x_1^2 + x_2] \quad \text{on} \quad D(A) = [-1, 1] \times \mathbf{R}.$$

Here the operator A is monotone but, for example, the function

$$\mathbf{R}^2 \ni y \mapsto J_A(0, y) = 1/2(y_1^2 + y_2^2) - 1/3(y_1^2 y_2)$$

is not convex.

Proof of the Lemma and of Theorem 1: Let $x, y \in M$ be arbitrarily fixed and define, for $t \in [0, 1]$,

$$\underline{h}(t) := \inf \left\{\langle f_t, y - y\rangle : f_t \in A\big(x + t(y - x)\big)\right\},$$

$$\tilde{h}(t) := \left\langle \tilde{A}\big(x + t(y - x)\big), y - x\right\rangle,$$

$$\bar{h}(t) := \sup \left\{\langle g_t, y - x\rangle : g_t \in A\big(x + t(y - x)\big)\right\}.$$

Obviously we have

$$\underline{h}(t) \leqq \tilde{h}(t) \leqq \bar{h}(t) \quad \text{for} \quad t \in [0, 1]. \tag{6}$$

On the other hand, we get

$$\underline{h}(t_3) \geqq \tilde{h}(t_2) \geqq \bar{h}(t_1) \quad \text{for} \quad 0 \leqq t_1 < t_2 < t_3 \leqq 1. \tag{7}$$

Indeed, in view of (6), the inequality (7) reduces to

$$\underline{h}(s) \geqq \bar{h}(t) \quad \text{for} \quad 0 \leqq t < s \leqq 1. \tag{7'}$$

But due to the monotonicity of A we have

$$\langle f_s, y - x\rangle - \langle g_t, y - x\rangle = \frac{1}{(s - t)} \left\langle f_s - g_t, \big(x + s(y - x)\big) - \big(x + t(y - x)\big)\right\rangle \geqq 0$$

for all $f_s \in A\big(x + s(y - x)\big)$ and $g_t \in A\big(x + t(y - x)\big)$.
This proves (7'). The inequalities (6) and (7) imply that $\underline{h}, \tilde{h}$ and $\bar{h}$ are monotone increasing functions which are finite on (0,1). Hence all these functions are Lebesgue-integrable. Moreover, (6) and (7) give rise to

$$\lim_{s \downarrow t} \underline{h}(s) = \lim_{s \downarrow t} \tilde{h}(s) = \lim_{s \downarrow t} \bar{h}(s) \quad \text{for all} \quad s \in (0, 1).$$

Hence, we obtain

$$\bar{h}(0) \leqq \int_0^1 \bar{h}(t)\, dt = \int_0^1 \tilde{h}(t)\, dt = \int_0^1 \underline{h}(t)\, dt \leqq \underline{h}(1)$$

(cf. R. T. Rockafeller [8]). This proves the Lemma and the estimate (4). The skew-symmetry of J_A is obvious. Thus it remains to check (5). For this purpose let $x, y \in M$ and $\lambda \in [0, 1]$ be fixed. By the monotonicity of $\tilde{h}$ we can conclude

$$\left\langle \tilde{A}\big(x + t(\lambda x + (1 - \lambda)\, y - x)\big), \big(\lambda x + (1 - \lambda)\, y\big) - x\right\rangle$$

$$= (1 - \lambda) \left\langle \tilde{A}\big(x + t(1 - \lambda)\,(y - x)\big), y - x\right\rangle$$

$$= (1 - \lambda)\, \tilde{h}\big(t(1 - \lambda)\big) \leqq (1 - \lambda)\, \tilde{h}(t).$$

Integrating here over [0, 1] yields

$$J_A\big(x, \lambda x + (1 - \lambda)\, y\big) \leqq (1 - \lambda)\, J_A(x, y).$$

The skew-symmetry of J_A implies $J_A(x, x) = 0$, so we get (5) as desired ∎

3. A variational principle

Now we can state the announced variational principle for the solutions to (1).

Theorem 2: *Let $A + \partial I_M$ be maximal monotone. Then the following conditions are equivalent:*

(i) *$x \in M$ solves the variational inequality* (1).

(ii) *$[x, x] \in M \times M$ is a saddle point of the skew-symmetric saddle function J_A, i.e., we have*

$$0 \leqq J_A(x, v) \quad \text{for all} \quad v \in M.$$

(iii) There *exists a neighbourhood $U(x)$ of x with*

$$0 \leqq J_A(x, v) \quad \text{for all} \quad v \in M \cap U(x).$$

(iv) *There exists an element $y \in M$ such that $[x, y]$ is a saddle point of J_A, i.e., we have*

$$J_A(v, y) \leqq J_A(x, y) \leqq J_A(x, w) \quad \text{for all} \quad v, w \in M.$$

Remarks: a) One can interpret $J_A(x, y)$, as the work (resp. the action) during a movement from the state x to the state y along the trajectory $\gamma(t) = x + t(y - x)$; $t \in [0, 1]$ (compare also Theorem 3).

b) The equivalence (ii) $\leftrightarrow$ (iii) shows that we can describe the solutions to (1) both by a local and a global criterion.

c) The maximality of $A + \partial I_M$ enters only into the proof of the implication (ii) $\to$ (i). The other implications of Theorem 2 remain true if one drops this assumption. Note that $A + \partial I_M$ is maximal monotone if E is a Banach space, M is closed, and the restriction of A to M is a radially continuous mapping having convex closed bounded values $\{Ax\}$ (cf. E. Krauss [2] and, for a single-valued mapping A, also R. T. Rockafellar [6, 7]).

Proof of Theorem 2: (i) $\to$ (ii): Let $x \in M$ be a solution to the variational inequality (1). According to the estimate (4), we obtain

$$0 \leqq \sup \{\langle f, v - x\rangle : f \in Ax\} \leqq J_A(x, v) \quad \text{for} \quad v \in M,$$

as desired.

(ii) $\to$ (iii): This implication is evident.

(iii) $\to$ (iv): Let us suppose

$$0 \leqq J_A(x, v) \quad \text{for all} \quad v \in M \cap U(x).$$

We show that one can set $y = x$ in (iv). Because of the skew-symmetry of J_A it remains to check $0 \leqq J_A(x, v)$ for all $v \in M$. For this purpose we assume $J_A(x, v_0) < 0$ for some $v_0 \in M$. If $\lambda \in (0, 1]$ is small enough we get $(1 - \lambda)\, x + \lambda v_0 \in U(x) \cap M$, and in connection with (5)

$$J_A\big(x, (1 - \lambda)\, x + \lambda v_0\big) \leqq \lambda J_A(x, v_0) < 0,$$

which is a contradiction to our assumption.

(iv) $\to$ (ii): Let $[x, y] \in M \times M$ be a saddle point of J_A. This means

$$J_A(v, y) \leqq J_A(x, y) \leqq J_A(x, w) \quad \text{for all} \quad v, w \in M.$$

Choosing here especially $v = y$ yields (compare Theorem 1)

$$0 = J_A(y, y) \leqq J_A(x, y) \leqq J_A(x, w) \quad \text{for all} \quad w \in M.$$

(ii) → (i): Let $x \in M$ satisfy $0 \leqq J_A(x, v)$ for all $v \in M$. As a consequence of (4) we get

$$0 \leqq \inf \{\langle g, v - x\rangle : g \in Av\} \quad \text{for all} \quad v \in M.$$

By the definition of ∂I_M this inequality implies

$$0 \leqq \langle f, v - x\rangle \quad \text{for all} \quad v \in M, f \in Av + \partial I_M v.$$

Since $A + \partial I_M$ was supposed to be maximal monotone, we can conclude $0 \in Ax + \partial I_M x$, i.e., x solves (1) ∎

Instead of the function J_A one can consider more generally a function

$$J_A{}^\gamma : M \times M \to \mathbf{R}; \quad J_A{}^\gamma(x, y) = \int_0^1 \langle \tilde{A}\gamma(t), \dot{\gamma}(t)\rangle \, dt,$$

where

$$\gamma = \gamma_{x,y} : [0, 1] \to M, \qquad \gamma(0) = x, \qquad \gamma(1) = y$$

is a sufficiently regular trajectory joining the points x and y. In order to avoid additional regularity assumptions on the operator A it is convenient to confine oneself to *polygonal trajectories* in M. These can be defined by

$$\gamma = \gamma_{x,y} : [0, 1] \to M; \qquad \gamma(t) = x_i + n(t - i/n)\, x_{i+1} \quad \text{for}$$
$$i/n \leqq t \leqq (i + 1)/n, \qquad i = 0, 1, \ldots, n,$$

where $x = x_0, x_1, x_2, \ldots, x_n = y$ is a finite subset of M. Hence

$$J_A{}^\gamma(x, y) = \int_0^1 \langle \tilde{A}\gamma(t), \dot{\gamma}(t)\rangle \, dt = \sum_{i=0}^{n-1} J_A(x_i, x_{i+1}),$$

i.e. J_A is well defined and does not depend on the choice of the section $\tilde{A}$ of A (compare the lemma in Section 2).

Now we show that in Theorem 2 it is not possible to replace the condition (8) by

$$\left.\begin{array}{l} 0 \leqq J_A{}^\gamma(x, v) \quad \text{for all} \quad v \in M \quad \text{and for all polygonal} \\ \text{trajectories } \gamma \text{ from } x \text{ to } y. \end{array}\right\} \qquad (8')$$

Theorem 3: *Let E be a Banach space and suppose that the restriction $A|_M$ of A is not contained in the subdifferential ∂p of a convex function $p : E \to \mathbf{R} \cup \{+\infty\}$, $p \not\equiv +\infty$. Then, for each $x, y \in M$ and for each natural number n, there exists a polygonal trajectory $\gamma : [0, 1] \to M$ between x and y with $J_A{}^\gamma(x, y) < -n$.*

Proof: By assumption $A|_M$ is not cyclically monotone, i.e. there exist a cyclic sequence $z_0, z_1, \ldots, z_n = z_0$ in $M \subseteqq D(A)$ and a sequence $\tilde{A}z_i \in Az_i$ $(i = 1, \ldots, n)$ with

$$\sum_{i=1}^{n} \langle \tilde{A}z_i, z_i - z_{i-1}\rangle < 0$$

(cf. R. T. Rockafellar [5, 9]). As a consequence of the estimate (4) we get

$$\sum_{i=1}^{n} J_A(z_{i-1}, z_i) := -\varepsilon < 0. \qquad (10)$$

Now we choose a natural number k with

$$k\varepsilon \geqq n + J_A(x, z_0) + J_A(z_0, y). \qquad (11)$$

Let the polygonal trajectory $\gamma : [0, 1] \to M$ be defined by $\gamma = \gamma_1 \cup k \cdot \gamma_2 \cup \gamma_3$, with $\gamma_1 = [x, z_0]$, $\gamma_3 = [z_0, y]$ and a polygonal trajectory γ_2 with the vertices $z_0, z_1, \ldots, z_n = z_0$. According to (10) and (11) we get

$$J_A{}^\gamma(x, y) = J_A(x, z_0) + k \left(\sum_{i=1}^{n} J_A(z_{i-1}, z_i) \right) + J_A(z_0, y) < -n \blacksquare$$

The skew-symmetric function $J_A : M \times M \to \mathbf{R}$ is not the only one allowing a characterization of the solutions to (1). In [3, 4] we showed that for each maximal monotone operator A from E into E^* there exists a skew-symmetric concave-convex closed saddle function $L : E \times E \to \mathbf{R} \cup \{\pm\infty\}$ such that $f \in Ax$ is equivalent to $[-f, f] \in \partial L(x, x)$. The concept of a closed saddle function which is used here is due to R. T. ROCKAFELLAR [10, 11] — compare also V. BARBU and TH. PRECUPANU [1].

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