Topological Realizations of Calkin Algebras on Frechet Domains of Unbounded **Operator Algebras**

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 $\mathcal D$ sei ein dichter linearer Teilraum eines separablen Hilbertraumes und $\mathcal{L}^+(\mathcal D)$ die maximale Op.-Algebra auf D, versehen mit der gleichmäßigen Topologie τ p. Wir setzen voraus, daß D bezüglich der Graphtopologie von $\mathcal{I}^+(D)$ ein Frechetraum ist. Weiter sei $\mathcal{E}(D)$ das zweiseitige •-Ideal aller Operatoren aus $\mathcal{L}^+(D)$, die beschränkte Teilmengen von D in relativ kompakte Teilmengen abbilden. Es wird untersucht, wann die Faktoralgebra $\mathcal{A}(\mathcal{D}) := \mathcal{L}^+(\mathcal{D})/\mathcal{E}(\mathcal{D})$, versehen mit der Faktorraumtopologie, eine topologische Realisierung als Op*-Algebra besitzt.

Пусть 2 плотное линейное подпространство сепарабельного гильбертова пространства и пусть $\mathcal{L}^+(\mathcal{D})$ максимальная Ор*-алгебра над \mathcal{D} , снабжённая равномерной топологией $\tau_{\rm D}$. Предполагается, что $\mathcal D$ является пространством Фреше относительно топологии порождённой граф-нормами операторов из $\mathcal{L}^+(D)$. Через $\mathcal{E}(D)$ обозначается двусторонний *-идеал тех операторов из $\mathcal{L}^+(\mathcal{D})$, которые переводят ограниченные подмножества пространства 2 в относительно компактные подмножества. Исследуется вопрос о том, когда факторалгебра $\mathcal{A}(\mathcal{D}):=\mathcal{L}^+(\mathcal{D})/\mathcal{E}(\mathcal{D})$ снабжённая топологией факторпространства допускает топологическую реализацию как Ор *- алгебра.

Let $\mathcal D$ be a dense linear subspace of a separable Hilbert space and let $\mathcal I^+(\mathcal D)$ be the maximal Ope-algebra on $\mathcal D$ endowed with the uniform topology $\tau \mathcal D$. Suppose $\mathcal D$ is a Frechet space with. respect to the graph topology of $\mathcal{L}^+(\mathcal{D})$. Let $\mathcal{E}(\mathcal{D})$ denote the two-sided \bullet -ideal of all operators in $\mathcal{L}^+(D)$ which map bounded subsets of D into relatively compact subsets. We study the question of when the quotient algebra $\mathcal{A}(\mathcal{D}) := \mathcal{L}^+(\mathcal{D})/\mathcal{E}(\mathcal{D})$, endowed with the quotient topology, has a topological realization as an Op+-algebra.

Introduction

Let D be a dense linear subspace of a separable complex Hilbert space $\mathcal X$ endowed with the graph topology t (see Section 1 for precise definitions). Suppose $\mathcal{D}[t]$ is a Frechet space. Let $\mathcal{E}(\mathcal{D})$ denote the set of all operators in $\mathcal{L}^+(\mathcal{D})$ which map each bounded subset of $\mathcal{D}[t]$ into a relatively compact subset of $\mathcal{D}[t]$. Then $\mathcal{E}(\mathcal{D})$ is a τ_p -closed two-sided *-ideal of $\mathcal{L}^+(\mathcal{D})$ which contains the finite rank operators in $\mathcal{L}^+(\mathcal{D})$ as a dense subset [15, 7]. (Note that in [15] the ideal $\mathcal{E}(\mathcal{D})$ is denoted by Vol (t, t) .) The quotient algebra $\mathcal{A}(\mathcal{D}) := \mathcal{L}^+(\mathcal{D})/\mathcal{E}(\mathcal{D})$ is called the Calkin algebra on the domain D. Let $\hat{\tau}$ denote the quotient topology on $\mathcal{A}(\mathcal{D})$ of $\mathcal{I}^+(\mathcal{D})$ [$\tau_{\mathcal{D}}$]. Obviously, $\mathcal{A}(\mathcal{D})$ [$\hat{\tau}$] is a topological *-algebra. If $\mathcal{D} = \mathcal{H}$, then $\mathcal{A}(\mathcal{D}) = \mathcal{A}(\mathcal{H})$ is the usual Calkin algebra on the Hilbert space \mathcal{H} . It should be mentioned that if $\mathcal{D}[t]$ is a Montel space, then $\mathcal{E}(\mathcal{D}) = \mathcal{L}^+(\mathcal{D})$ and hence the Calkin algebra $\mathcal{A}(\mathcal{D})$ is trivial.

In his classical paper [3] CALKIN constructed a class of faithful isometric *-representations of the C*-algebra $\mathcal{A}(\mathcal{H})$ (see [11] for a modern treatment). In this paper we investigate the corresponding problem for the Calkin algebra $\mathcal{A}(\mathcal{D})$ on the Frechet domain $\mathcal{D}[l]$: Does there exist a faithful *-representation π of $\mathcal{A}(\mathcal{D})$ which is a homeomorphism of $\mathcal{A}(\mathcal{D})$ [$\hat{\tau}$] onto $\pi(\mathcal{A}(\mathcal{D}))$ [$\tau_{\mathcal{D}(\pi)}$]? For the domain $l_2 \otimes d$, d the space of all finite complex sequences, this problem has been considered in [9]. Note that $l_2 \otimes d[l]$. is not a Frechet space.

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Let us briefly describe our main results concerning the above question:

Given'a free ultrafilter $\mathcal U$ on N, we define in Section 2a *-representation $\pi_{\mathcal U}$ of $\mathcal A(\mathcal D)$ in a similar way as in the case $\mathcal{D} = \mathcal{H}$. We show that $\pi_{\mathcal{U}}$ is faithful and that $\pi_{\mathcal{U}}^{-1}$ is continuous (Theorem 2.1). Let τ_n denote the finest locally convex topology on $L^*(\mathcal{D})$ for which the positive cone $L^*(\mathcal{D})_+$ is normal [12]. If $\tau_n = \tau_{\mathcal{D}}$ on $L^*(\mathcal{D})$, then each *-representation $\pi_{\mathcal{U}}$ is continuous and hence a homeomorphism (Theorem 2.2).

In Section 3 we obtain a converse of the latter in some sense. Suppose that the graph topology t on $\mathcal D$ is generated by a sequence of strongly commuting self-adjoint operators whose restrictions to $\mathcal D$ are in $\mathcal I^+(\mathcal D)$. Under this additional assumption we prove that if $\tau_n + \tau_{\mathcal{D}}$ on $\mathcal{L}^+(\mathcal{D})$, then there is no continuous faithful *-representation of $\mathcal{A}(\mathcal{D})$ [$\hat{\tau}$] (Theorem 3.1).

1. Preliminaries

In this section we collect some definitions and notations (see e.g. [8, 10]) needed later and we prove-some preliminary lemmas.

1.1 Let \mathcal{D} be a dense linear subspace of a complex Hilbert space \mathcal{H} and let $\mathcal{L}^+(\mathcal{D})$ $:=$ {a \in End $\mathcal{D}: \mathcal{D} \subseteq \mathcal{D}(a^*)$ and $a^*\mathcal{D} \subseteq \mathcal{D}$ }. $\mathcal{L}^+(\mathcal{D})$ is a *-algebra endowed with the involution $a \rightarrow a^+ := a^* | \mathcal{D}$. An *Op*algebra* \mathcal{B} on \mathcal{D} is a *-subalgebra of $\mathcal{L}^+(\mathcal{D})$. In what follows we assume that $\mathscr B$ is an Op*-algebra on $\mathscr D$. Define $\mathscr D(\mathscr B) = \bigcap {\mathscr D}(b)$ $b \in \mathscr{B}$, where \bar{b} is the closure of the operator b . The *graph topology* $t_{\mathscr{B}}$ is the locally 1.1 Let \mathcal{D} be a 'dense linear subspace of a complex Hilbert space \mathcal{H} and let $\mathcal{L}^+(\mathcal{D})$

:= $\{a \in \text{End } \mathcal{D} : \mathcal{D} \subseteq \mathcal{D}(a^*) \text{ and } a^*\mathcal{D} \subseteq \mathcal{D}\}$. $\mathcal{L}^+(\mathcal{D})$ is a *-algebra endowed with the

inv prove that if $\tau_n \neq \tau_{\mathcal{D}}$ on $\mathcal{I}^+(\mathcal{D})$, then there is no continuous faithful *-representation
of $\mathcal{A}(\mathcal{D})$ [$\hat{\tau}$] (Theorem 3.1).

1. Preliminaries

In this section we collect some definitions and notat

Let $(\varphi_n : n \in \mathbb{N})$ be a sequence of vectors $\varphi_n \in \mathcal{H}$ and let $\varphi \in \mathcal{H}$. Suppose \mathcal{U} is a filter on N. We write $\varphi = w$ -lim φ_n if $\lim_n \langle \varphi_n, \psi \rangle = \langle \varphi, \psi \rangle$ for all $\psi \in \mathcal{H}$ and φ $=\text{w-lim }_{\mathcal{U}} \varphi_n$ if $\lim_{\mathcal{U}} \langle \varphi_n, \psi \rangle = \langle \varphi, \psi \rangle$ for all $\psi \in \mathcal{H}$.

 $\mathbf{Lem}\,\mathbf{m}\,\mathbf{a}\,\mathbf{1}.1$: $Suppose\,\mathcal{U}\,is\,an\,ultrafilter\,on\,\mathbf{N}.\,Let\,\left(\mathbf{\varphi}_{n}\,;\,n\in\mathbf{N}\right)\,be\,a\,bounded\,sequence$ *of vectors of* $\mathcal{D}[t_{\mathcal{B}}]$ *. Let* $\varphi := \text{w-lim}_{\mathcal{U}} \varphi_n$. *i* different intervent we write $\varphi = w \cdot \lim_{n} \varphi_n$ if $\lim_{n} \langle \varphi_n, \psi \rangle = \langle \varphi, \psi \rangle$ for all $\psi \in \mathcal{H}$.
 i defining φ_n , $\psi \rangle = \langle \varphi, \psi \rangle$ for all $\psi \in \mathcal{H}$.
 Lemma 1.1: Suppose U is an ultrafilter on N. *Let*

- $0' = w\cdot \lim_{\mathcal{U}} b\varphi_n$ *for each* $b \in \mathcal{B}$.
- (ii) If $\lim_{u \to \infty} ||\varphi_n|| = 0$, then $\lim_{u \to \infty} ||b\varphi_n|| = 0$ for each $b \in \mathcal{B}$.
- *(iii)* \int *jor* each $b \in \mathcal{B}$. (iii) If $\varphi = 0$ and if the set $\{\varphi_n\}$ is relatively compact in $\mathcal{D}[t,\varphi]$, then $\lim u ||b\varphi_n|| = 0$

Proof: (i) Suppose $b \in \mathcal{B}$. Since the set ${b\varphi_n}$ is bounded in the Hilbert space norm **Proof:** (i) Suppose $b \in \mathcal{B}$. Since the set $\{b\varphi_n\}$ is bounded in the Hilbert space norm and \mathcal{U} is an ultrafilter, $\lim_{\mathcal{U}} \langle b\varphi_n, \cdot \rangle$ is a continuous linear functional on \mathcal{H} . Hence there is a φ there is a $\varphi_b \in \mathcal{H}$ such that $\varphi_b = w$ -lim_{ψ} b φ_n . For $\psi \in \mathcal{D}(b^*)$, this gives Therefore, $\varphi \in \mathcal{D}(b^{*})$ and $\varphi_{p} = w \cdot \lim_{\mathcal{U}} \varphi_{p}$, $\lim_{\mathcal{U}} \varphi_{p} = w \cdot \lim_{\mathcal{U}} \varphi_{p}$, then

(ii) $If \varphi = 0$ and if the set $\{\varphi_{n}\}$ is relatively compact in $\mathcal{D}[t_{\mathcal{B}}]$, then $\lim_{\mathcal{U}} ||b\varphi_{n}|| = 0$
 Froot. (i) suppose $b \in \mathcal{B}$. Since the set $\{o\psi_n\}$ is continuous linear functional on \mathcal{X} . Hence
there is a $\varphi_b \in \mathcal{X}$ such that $\varphi_b = w$ -lim ψ $b\varphi_n$. For $\psi \in \mathcal{D}(b^*)$, this gives
 $\langle \varphi_b, \psi \rangle = \lim_{\mathcal$

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\langle \varphi_b, \psi \rangle = \lim_{u \to b} \langle b\varphi_n, \psi \rangle = \lim_{u \to b} \langle \varphi_n, b^* \psi \rangle = \langle \varphi, b^* \psi \rangle.
$$

the assertion follows from $\langle \tilde{b} \rangle : b \in \mathcal{B} \rangle = \mathcal{D}(\mathcal{B})$.

ce $\{\varphi_n\}$ is $t_{\mathcal{B}}$ -bounded, $C_b := \sup \{||b^+b\varphi_n|| : n \in \mathbb{N}\}\right.$

(lim_U $||b\varphi_n|||^2 = \lim_{\mathcal{U}} \langle b^+b\varphi_n, \varphi_n \rangle \leq C_b(\lim_{\mathcal{U}} ||\varphi_n||) = 0$

t $b \in \mathcal{B}$. Since $\{m\}$ is relativ

$$
(\lim_{\mathcal{U}} ||b\varphi_n||)^2 = \lim_{\mathcal{U}} \langle b^+ b\varphi_n, \varphi_n \rangle \leq C_b(\lim_{\mathcal{U}} ||\varphi_n||) = 0.
$$

(iii) Let $b \in \mathcal{B}$. Since $\{\varphi_n\}$ is relatively compact in $\mathcal{D}[t_{\mathcal{B}}]$, the set $\{\varphi_n\}$ is relatively compact in \mathcal{H} . Given $\varepsilon > 0$, there is a finite rank projection F_{ε} on \mathcal{H} such that $|| (I - F_{\epsilon}) \, b\varphi_n || \leq \epsilon$ for $n \in \mathbb{N}$. Since $0 = w \cdot \lim_{\mathcal{U}} b\varphi_n$ because of (i) and hence $\lim_{\mathcal{U}} ||F_{\epsilon}b\varphi_n|| = 0$, we have $\lim_{\mathcal{U}} ||b\varphi_n|| \leq \lim_{\mathcal{U}} ||(I - F_{\epsilon}) b\varphi_n|| \leq \epsilon$, thus $\lim_{\mathcal{U}} ||b\varphi_n|| = 0$

The following corollary is of some interest in itself-

Corollary 1.2: *Suppose* $\varphi \in \mathcal{X}$. *If there is a bounded sequence* $(\varphi_n : n \in \mathbb{N})$ *in* $\mathcal{D}[t_{\mathcal{B}}]$ *such that* $\varphi = \text{w-lim } \varphi_n$ *, then* $\varphi \in \mathcal{D}(\mathcal{B})$ *.*

Fopological Realizations of Calkin
 Such that $\varphi = \mathbf{w}$ -lim φ_n , *then* $\varphi \in \mathcal{X}$. *If there is a bounded sequence* (φ ,
 outh that $\varphi = \mathbf{w}$ -lim φ_n , *then* $\varphi \in \mathcal{D}(\mathcal{X})$.
 Proof: Take an Proof: Take an ultrafilter U on N which contains all sets ${n \in N : n \geq k}, k \in N$. Then $\varphi = \text{w-lim}_{\mathcal{U}} \varphi_n$ and Lemma 1.1 (i) applies \blacksquare

1.2 Next we briefly discuss the topologization of the Op*-algebra \mathcal{B} on \mathcal{D} . Let \mathcal{B}_h Such that $\psi = w$ -lim φ_n , then $\varphi \in \mathcal{D}(\mathcal{B})$,

Proof: Take an ultrafilter \mathcal{U} on N which contains all sets $\{n \in \mathbb{N} : n \geq k\}$, $k \in \mathbb{N}$.

Then $\varphi = w$ -lim ψ φ_n and Lemma 1.1 (i) applies \blacksquare
 $\mathscr{B}: b = b^+$. Suppose $b_1, b_2 \in \mathscr{B}_h$. We write $b_1 \geq b_2$ if $\langle b_1 \varphi, b_2 \rangle$ effine $\mathscr{B}_+ := \{b \in \mathscr{B}: b \geq 0\}$ and $[b_1, b_2] := \{b \in \mathscr{B}_h : b_1 \leq \log y \tau_B \}$ is the locally convex topology on \mathscr{B} defined by t

If \mathcal{D} . Define $\mathcal{A}_+ := \{b \in \mathcal{B} : b \leq 0\}$ and $\{b_1, b_2\} := \{b \in \mathcal{B}_b : b_1 \leq b \leq b_2\}$. The m topology $\tau_{\mathcal{D}}$ is the locally convex topology on \mathcal{B} defined by the seminorms $\mathcal{P}_{\mathfrak{M}}(x) := \sup \{|\langle x\var$ It has been introduced in [8]. We denote by τ_n the finest locally convex-topology on $\mathscr B$ for which the positive cone $\mathscr B_+$ is normal. (All notions and facts concerning ordered vector spaces we need can be found in [12].) Since $\mathscr B_+$ is $\tau_{\mathscr D}$ -normal [13], we haveform topology τ_p is the locally convex topology on $\mathcal B$ defined by the seminorms
 $p_{\mathfrak{M}}(x) := \sup \{|\langle x\varphi, \psi \rangle| : \varphi, \psi \in \mathfrak{M}\}\$, $\mathfrak{M} \subset \mathcal D[t_{\mathcal B}]$ bounded.

It has been introduced in [8]. We denote by τ_n t $\tau_{\mathcal{D}} \subseteq \tau_n$. Let_i τ_0 denote the finest locally convex topology on \mathcal{B} for which every order τ_n -bounded [12: p. 216] and hence $\tau_n \subseteq \tau_0$. In [1] the topology τ_0 is called the ϱ *-topology*.

interval $[b_1, b_2], b_1, b_2 \in \mathcal{B}_h$, is bounded. Since \mathcal{B}_+ is τ_n -normal, all order intervals are τ_n -bounded [12: p. 216] and hence $\tau_n \subseteq \tau_0$. In [1] the topology τ_0 is called the *q-topology*.

1.3 Let A 1.3 Let A be a $*$ -algebra with unit element denoted by 1. By a $*$ -representation of A *on D* we mean a-*-homomorphism π of A into $\mathcal{L}^+(\mathcal{D})$ satisfying $\pi(1) = I$, where *I* is the identity map of \mathcal{D} . We then write $\mathcal{D}(\pi)$ for \mathcal{D} and t_{π} for the graph topology of the Op*-algebra $\pi(A)$ on $\mathcal{D}(\pi)$. Suppose π is a *-representation of a topological *-algebra $A[\tau]$. π is called *weakly continuous* if for each $\varphi \in \mathcal{D}(\pi)$ the linear functional $\langle \pi(\cdot) \varphi, \varphi \rangle$ π is *continuous*. **1.3** Let A be a \bullet
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 $\text{Op} \ast$ -algebra π ($\text{A}[\tau]$, π is called

is continuous or
 π is continuous.

As above, let

We say π is pos

that $\pi(b) \geq 0$ or
 $b \in \mathcal{B}_+$.

As above, let $\mathcal B$ be an Op*-algebra on $\mathcal D$. Let π be a *-representation of $\mathcal B$ on $\mathcal D(\pi)$.
We say π is *positive* if $\pi(\mathcal A_+) \subseteq \pi(\mathcal B)_+$, i.e., if $b \in \mathcal B$ and $b \ge 0$ on $\mathcal D$ always implies is continuous on A[τ]. If π is a continuous mapping of A[τ] onto $\pi(A)$ [$\tau_{\mathcal{D}(\pi)}$] we say π is *continuous*.
As above, let \mathcal{B} be an Op*-algebra on \mathcal{D} . Let π be a *-representation of $\mathcal{B$ that $\pi(b) \geq 0$ on $\mathcal{D}(\pi)$. A linear functional *f* on $\mathcal B$ is called positive if $f(b) \geq 0$ for all

Lemma 1.3: Each positive $*$ -representation π of the Op $*$ -algebra \mathcal{B} is a continuous *mapping of* $\mathscr{B}[\tau_n]$ *onto* $\pi(\mathscr{B})$ $[\tau_{\mathscr{D}(n)}]$.

 $\mathbf{Proof}\colon\mathbf{By}$ the polarization formula it is easy to see [13] that the uniform topology $\tau_{\mathcal{D}(\pi)}$ on $\pi(\mathcal{B})$ is generated by the family of seminorms $\mathcal{D}(\mathcal{$

$$
p'_{\mathfrak{M}}(\pi(x)) := \sup \left\{ |\langle \pi(x) \varphi, \varphi \rangle| : \varphi \in \mathfrak{M} \right\}, \mathfrak{M} \subset \mathcal{D}(\pi) \left[I_{\pi} \right] \text{ bounded.}
$$

Fix the bounded set \mathfrak{M} . Since the set $\{x \in \mathcal{B} : p'_m(\pi(x)) \leq 1\}$ is absolutely convex and \mathscr{B}_+ saturated, it is a τ_n -neighborhood of zero in \mathscr{B} . This proves the continuity of π

Lemma 1.4: Suppose that $\mathcal{D}[l]$ is a Frechet space.' Let π ' be a weakly continuous *-representation of $\mathcal{L}^+(\mathcal{D})$ [$\tau_{\mathcal{D}}$]. Then:

-
- (i) π is positive.
(ii) If $x \in \mathcal{L}^+(\mathcal{D})$ is bounded, then $\pi(x)$ is bounded on $\mathcal{D}(\pi)$ and $||\pi(x)|| \leq ||x||$.
- (iii) Suppose $x_n \in \mathcal{L}^+(\mathcal{D})$ for $n \in \mathbb{N}$. If $\{\|\cdot\|_{x_n} : n \in \mathbb{N}\}\$ is a generating family for the *graph topology t, on D, then* $\{||\cdot||_{n(x_n)} : n \in \mathbb{N}\}$ *is a generating family of seminorms for the graph topology* t_n *on* $\mathcal{D}(\pi)$ *.*

Proof: (i) Suppose $x \in \mathcal{L}^+(\mathcal{D})_+$ and $\varphi \in \mathcal{D}(\pi)$. By [6: Theorem 6.1] there is a net ${q_j}$ of orthogonal projections $q_j \in \mathcal{L}^+(\mathcal{D})$ (that is, $q_j = q_j^*$ and $q_j = q_j^*$) such that $q_j\mathscr{H}\subseteq \mathscr{D}$ for all $_j$ and $x = \tau_{\mathscr{D}}$ -lim q_jxq_j . Let x_j denote the operator q_jxq_j on the Hilbert space $q_i\mathscr{H}$. Since $x \in \mathscr{L}^+(D)$, x_j is closed and hence bounded. Let y_j denote the positive square root of the bounded self-adjoint operator x_j on the Hilbert space $q_j\mathcal{X}$. Then $y_i q_j \in \mathcal{L}^+(\mathcal{D})$ and Since $x \in \mathcal{L}^+(D)$, x_j is closed and hence bounded. I

of the bounded self-adjoint operator x_j on the 1

(y,q_j) φ ^{||2} = $\langle \pi(q_jy_j^2q_j) \varphi, \varphi \rangle = \langle \pi(q_jxq_j) \varphi, \varphi \rangle \ge 0$.

ceakly continuous. $\langle \pi(x) \varphi, \varphi \rangle =$

$$
\|\pi(y_i q_j) \varphi\|^2 = \langle \pi(q_j y_j^2 q_j) \varphi, \varphi \rangle = \langle \pi(q_j x q_j) \varphi, \varphi \rangle \geq 0.
$$

Since π is weakly continuous, $\langle \pi(x) \varphi, \varphi \rangle = \lim \langle \pi(q_j x q_j) \varphi, \varphi \rangle \ge 0$. That is, $\pi(x) \ge 0$ on $\mathcal{D}(\pi)$.

square root of the bounded self-adjoint operator x_j on the Hilbert space $q_j\mathcal{X}$. Then $||\pi(y_jq_j) \varphi||^2 = \langle \pi(q_jy_j^2q_j) \varphi, \varphi \rangle = \langle \pi(q_jxq_j) \varphi, \varphi \rangle \ge 0$.
Since π is weakly continuous, $\langle \pi(x) \varphi, \varphi \rangle = \lim \langle \pi(q_jxq_j) \var$ Since π is weakly continuous, $\langle \pi(x) \varphi, \varphi \rangle = \lim \langle \pi(q_j x q_j) \varphi, \varphi \rangle \ge 0$. That is, $\pi(x) \ge 0$

on $\mathcal{D}(\pi)$.

(ii): First let $x \in \mathcal{L}^+(\mathcal{D})_h$. Since π is positive by (i) and $\pi(I) = I$, inf $\{\lambda \in \mathbb{R} : -\lambda I \le$ $\begin{array}{l} \mathbb{C}^{c} \in q_{j}\mathscr{H}. \text{ Since } x\in \mathscr{L}^{+}(\mathscr{D}), x_{j} \text{ is clo} \ \mathbb{C}^{c} \in \mathscr{L}^{+}(\mathscr{D}) \text{ and } \mathbb{C}^{c} \in \mathscr{L}^{+}(\mathscr{D}) \text{ and } \mathbb{C}^{c} \in \mathscr{L}^{+}(\mathscr{D}) \text{ and } \mathbb{C}^{c} \in \mathscr{D}(\mathscr{H}) \text{ for all } x\in \mathscr{D}(\mathscr{H}) \text{ for all } x\in \mathscr{D}(\mathscr{H}) \$ ace $q_j \mathcal{X}$. Since $x \in \mathcal{X}^+(D)$, x_j is closed and hence bounded. Let y_j denote the positive
 $y_j \in \mathcal{X}^+(D)$ and
 $||\pi(y_j q_j) \varphi||^2 = \langle \pi(q_j y_j^2 q_j) \varphi, \varphi \rangle = \langle \pi(q_j x q_j) \varphi, \varphi \rangle \ge 0$.

nee π is weakly continuous, $\leq ||x^*x|| = ||x||^2$.
(iii): Suppose $x \in \mathcal{L}^+(D)$, By assumption; there are a positive constant C and a

natural, number *s* such that

$$
\|x\varphi\|^2 \leq C \left(\|\varphi\|^2 + \sum_{n=1}^s \|x_n \varphi\|^2 \right) \text{ for all } \varphi \in \mathcal{D}.
$$

Therefore,

$$
y := C\left(I + \sum_{n=1}^s x_n^+ x_n\right) - x^+ x \in \mathcal{L}^+(\mathcal{D})_+
$$
 and $\pi(y) \geq 0$ on $\mathcal{D}(\pi)$.

The latter implies that

$$
\|\pi(x) \varphi\|^2 \leq C \left(\|\varphi\|^2 + \sum_{n=1}^s \|\pi(x_n) \varphi\|^2 \right) \text{ for all } \varphi \in \mathcal{D}(\pi) \blacksquare
$$

1.4 From now on we assume that $\mathcal{D}[t]$ is a Frechet space and that the underlying Hilbert space $\mathscr H$ is separable. To simplify the notation we adopt the following notational convention: We shall denote an operator whose domain contains D and its restriction to $\mathcal D$ by the same symbol. This will be mainly used in Section 3. Let $\mathcal F(\mathcal D)$ denote the **1.4** From now on we assume that $\mathcal{D}[t]$ is a Frechet space and that the underlying
bert space \mathcal{H} is separable. To simplify the notation we adopt the following notatio
convention: We shall denote an operator whos be the set of all bounded finite-ranked operators on $\mathscr H$ mapping $\mathscr K$ into $\mathscr D_1$. More- $\begin{aligned} &\mathcal{F}^{\text{1}}(\mathcal{D})+\text{and }\pi(y)\geqq0\text{ on}\ &\mathcal{F}^{\text{1}}(\mathcal{D})+\text{and }\pi(y)\geqq0\text{ on}\ &\mathcal{F}^{\text{1}}(\mathcal{X}_n)\left.\varphi\right|^{2}\bigg)\text{ for all }\varphi\in\mathcal{D}(\pi)\text{ }\blacksquare\ &\mathcal{F}^{\text{1}}\text{ is a Frechet space and that }\mathcal{F}^{\text{2}}\text{ is not not possible to be a nontrivial if }\varphi\in\mathcal{D}^{\text{2}}\text{ is not possible to be a nontrivial if }\mathcal{D}$

2. Generalized Calkin representations of $\mathcal{A}(\mathcal{D})$

2.1 Suppose that $\mathcal U$ is an ultrafilter on N. Let $\mathcal D_{\mathcal U}^{\infty}$ denote the set of all bounded sequences $(\varphi_n : n \in \mathbb{N}) = (\varphi_n)$ in the locally convex space $\mathcal{D}[t]$ satisfying $0 = w\text{-}\lim_{\mathcal{U}} \varphi_n$. Let \mathcal{H}_{u}^{∞} be the set of all bounded sequences (φ_n) in \mathcal{H} with $0 = w\cdot \lim_{\mathcal{U}} \varphi_n$. \mathcal{D}_{u}^{∞} and \mathcal{H}_{u}^{∞} are vector spaces in the obvious way. Let \mathcal{N}_u be the set of all $(\varphi_n) \in \mathcal{H}_u^{\infty}$ with $\lim u ||\varphi_n||$ be the set of all bounded finite-ranked operators on \mathcal{H} mapping \mathcal{H} into \mathcal{D}_1 . More-

over, we let $\mathfrak{B}_{\mathfrak{D}_1} := {\varphi \in \mathcal{D}_1 : ||\varphi|| \leq 1}.$

2. Generalized Calkin representations of $\mathcal{A}(\mathcal{D})$

2.1 = 0. We define a scalar product on the quotient space $\mathcal{D}_{\mathcal{U}} := \mathcal{D}_{\mathcal{U}}^{\infty}/\mathcal{D}_{\mathcal{U}}^{\infty} \cap \mathcal{N}_{\mathcal{U}}$ by $\langle (\varphi_n), (\psi_n) \rangle := \lim_{\mathcal{U}} \langle \varphi_n, \psi_n \rangle$. In the same way, the quotient space $\mathcal{H}_{\mathcal{U}} := \mathcal{H}_{\mathcal{U$ a Hubert space (see e.g. [t 1: Section 2]). By an abuse of notation we denote the elements of the quotient spaces again by (φ_n) . Since $\mathcal{D} \subseteq \mathcal{H}$, \mathcal{D}_u can be considered as a linear subspace of $\mathcal{H}_{\boldsymbol{\mathcal{U}}}$. *xcs* ($\varphi_n : n \in \mathbb{N}$) = (φ_n) in the locally convex space $\mathcal{D}[l]$ satisfying $0 = w$ -lim ψ_n . Let \mathcal{H}_u^{ω} be the set of all bounded sequences (φ_n) in \mathcal{H} with $0 = w$ -lim ψ_n , \mathcal{D}_u^{ω} and $\mathcal{$

Define $\varrho_{\mathcal{U}}(x)(\varphi_n) := (\varphi_n)$ for $(\varphi_n) \in \mathcal{D}_{\mathcal{U}}$ and $x \in \mathcal{L}^+(\mathcal{D})$. Each operator $x \in \mathcal{L}^+(\mathcal{D})$ maps a bounded sequence in $\mathcal{D}[l]$ into a bounded sequence. By Lemma 1.1, (i) and (ii), $x\mathcal{N}u \subseteq \mathcal{W}u$ and $x\mathcal{D}u^{\infty} \subseteq \mathcal{D}u^{\infty}$. Therefore, the above definition makes sense and defines a linear operator $g_{\mathcal{U}}(x)$ which maps $\mathcal{D}_{\mathcal{U}}$ into $\mathcal{D}_{\mathcal{U}}$. It is straightforward to check that the mapping $x \to \varrho_u(x)$ is a positive *-representation of $\mathcal{L}^+(\mathcal{D})$ on \mathcal{D}_u .

Let *j* denote the quotient map of $\mathcal{L}^+(\mathcal{D})$ onto $\mathcal{A}(\mathcal{D}) = \mathcal{L}^+(\mathcal{D})/\mathcal{C}(\mathcal{D})$. Suppose $x \in$ (D) and $(\varphi_n) \in \mathcal{D}^{\infty}$. Then the set $\{x\varphi_n\}$ is relatively compact in $\mathcal{D}(t)$ and hence $\lim_{\mathcal{U}} ||x\varphi_n|| = 0$ by Lemma 1.1 (iii). This shows that $\mathcal{E}(\mathcal{D}) \subseteq \ker \varrho_{\mathcal{U}}$. Therefore, $\pi_{\mathcal{U}}(j(x)) := \varrho_{\mathcal{U}}(x)$ for $x \in \mathcal{L}^+(\mathcal{D})$ defines a *-representation of the *-algebra $\mathcal{A}(\mathcal{D})$ on strategy and $(\varphi_n) \in \mathcal{D}^{\infty}$.

i denote the quotient map of $\mathcal{L}^+(\mathcal{D})$ onto $\mathcal{A}(\mathcal{D}) = \mathcal{L}^+(\mathcal{D})/\mathcal{E}(\mathcal{D})$. Suppose $x \in$ and $(\varphi_n) \in \mathcal{D}^{\infty}$. Then the set $\{x\varphi_n\}$ is relatively compact in \math $\mathcal{D}_u = \mathcal{D}(\pi_u).$

2.2 Recall that an ultrafilter on N is said to be *free* if the intersection of all its members is empty.

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² *Recall that an ultrafilter on N* is said to be *free if the intersection of all its mem-*
 Theorem 2.1: Suppose that $\mathcal U$ *is a free ultrafilter on N. Then* $\$ **•** *Topological Realizations of Calkin Algebras* 485
 2.2 Recall that an ultrafilter on N is said to be *free* if the intersection of all its mem-

bers is empty.
 Theorem 2.1: *Suppose that U is a free ultrafilter o* $\pi_{\mathcal{U}}(\mathcal{A}(\mathcal{D}))$ $\sigma_{\mathcal{U}}$ onto $\mathcal{A}(\mathcal{D})$ [$\hat{\tau}$]. *i*nf *Calkin algebra* $A(D)$ *. Its inverse* πu^{-1} *is a conti* $A(D)$ [*t*].
 ent topology $\hat{\tau}$ on $A(D)$ is generated by the set in topology $\hat{\tau}$ on $A(D)$ is generated by the set in the parameter we have shown tha on N is said to be *free* if the in
that *U* is a *free ultrafilter on* N.
lgebra $A(D)$. Its inverse πu^{-1} is
logy $\hat{\tau}$ on $A(D)$ is generated by
 $x + c$: $c \in \mathcal{E}(D)$, $\mathfrak{M} \subset \mathcal{D}[t]$ be
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be *free* if the i

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pose for a m
n $\mathfrak M$ of $\mathcal D_u$
 $R(\varrho u(x))$ for al
 $u(u(x))$ for al
d in $\mathcal F^+(\mathcal D)$ em 2.1: Suppose that U is a free ultrafiller on N. Then $\pi_{\mathcal{U}}$ is a faithful \ast -
tion of the Calkin algebra $\mathcal{A}(\mathcal{D})$. Its inverse $\pi_{\mathcal{U}}^{-1}$ is a continuous mapping of
 $[\tau_{\mathcal{D}_{\mathcal{U}}}]$ onto $\mathcal{A}(\mathcal{D$

Proof: The quotient topology $\hat{\tau}$ on $\mathcal{A}(\mathcal{D})$ is generated by the seminorms

$$
\hat{p}_{\mathfrak{M}}(j(x)) := \inf \{p_{\mathfrak{M}}(x+c) : c \in \mathcal{E}(\mathcal{D})\}, \mathfrak{M} \subset \mathcal{D}[t] \text{ bounded.}
$$

Fix such a set \mathfrak{M} . Suppose for a moment we have shown that there exists a bounded subset \Re (depending on \Re) inf $\{p_{\mathfrak{M}}(x+c): c \in \mathcal{C}(\mathcal{X})\}$
ppose for a moment we
con \mathfrak{M}) of $\mathcal{D}u[\ell_{e\mathcal{U}}]$ such
 $p_{\mathfrak{M}}(e_{\mathcal{U}}(x))$ for all $x \in \mathcal{L}^+$ $\begin{align} \hat{p}_{\mathfrak{M}}(j(x)) := \text{if} \ \text{Fix such a set } \mathfrak{M}. \ \text{Sup} \ \text{subset} \ \mathfrak{N} \text{ (depending of } \hat{p}_{\mathfrak{M}}(j(x)) \leq p_{\mathfrak{N}} \ \text{The latter means that} \ \langle \quad \hat{p}_{\mathfrak{M}}(a) \leq p_{\mathfrak{M}}(a) \ \text{Since } \mathcal{E}(\mathfrak{N}) \text{ is realless} \end{align}$

$$
\hat{p}_{\mathfrak{M}}(\mathfrak{j}(x)) \leq p_{\mathfrak{R}}(\varrho_{\mathcal{U}}(x)) \text{ for all } x \in \mathcal{L}^+(\mathcal{D}). \tag{1}
$$

• The latter means that

$$
\hat{p}_{\mathfrak{M}}(a) \leq p_{\mathfrak{R}}(\pi_{\mathcal{U}}(a)) \text{ for all } a \in \mathcal{A}(\mathcal{D}).
$$

 $\hat{p}_{\mathfrak{M}}(j(x)) := \inf \{p_{\mathfrak{M}}(x + c) : c \in \mathcal{E}(\mathcal{D})\}, \mathfrak{M} \subset \mathcal{D}[t] \text{ bounded.}$

Fix such a set \mathfrak{M} . Suppose for a moment we have shown that there exists a bounded

subset \mathfrak{N} (depending on \mathfrak{M})) of $\mathcal{$ Since $\mathcal{E}(\mathcal{D})$ is $\tau_{\mathcal{D}}$ -closed in $\mathcal{L}^+(\mathcal{D})$ and hence $\hat{\tau}$ is Hausdorff, it follows from (2) that ker $\pi_{\mathcal{U}} = \{0\}$, that is, $\pi_{\mathcal{U}}$ is faithful. Moreover, (2) proves the continuity of π_{\math the proof would be complete.

It remains' to show that there is a bounded set \Re in $\mathcal{D}_{\mathcal{U}}[t_{e\mathcal{U}}]$ such that (1) is satisfied. According to [6: Theorem 4.1] there is a bounded self-adjoint operator *z* on \mathcal{H} $\hat{p}_{\mathfrak{M}}(a) \leq p_{\mathfrak{R}}(\pi_{\mathcal{U}}(a))$ for all $a \in \mathcal{A}(\mathcal{D})$. (2)

Since $\mathcal{E}(\mathcal{D})$ is $\tau_{\mathcal{D}}$ -closed in $\mathcal{L}^*(\mathcal{D})$ and hence $\hat{\tau}$ is Hausdorff, it follows from (2) that

ker $\pi_{\mathcal{U}} = \{0\}$, tha The latter means that
 $\ell_{\mathcal{D}}(a) \leq p_{\mathcal{B}}(\pi_{\mathcal{U}}(a))$ for all $a \in \mathcal{A}(\mathcal{D})$. (2)

Since $\mathcal{E}(\mathcal{D})$ is $\tau_{\mathcal{D}}$ -closed in $\mathcal{L}^*(\mathcal{D})$ and hence $\hat{\tau}$ is Hausdorff, it follows from (2) that

ker π tor defined on \mathcal{H} and hence bounded. Now fix an operator $x \in \mathcal{L}^+(\mathcal{D})$. Since $\mathcal{F}(\mathcal{D})$ is $\tau_{\mathcal{D}}$ -dense in $\mathcal{E}(\mathcal{D})$, we obtain $\partial_{\mathfrak{R}}(e_{\mathcal{U}}(x))$ for all $x \in \mathcal{L}^{+}(\mathcal{D})$.

it
 $(\pi_{\mathcal{U}}(a))$ for all $a \in \mathcal{A}(\mathcal{D})$.

ed in $\mathcal{L}^{+}(\mathcal{D})$ and hence $\hat{\tau}$ is Hausdorff, it follows from $(2, \pi_{\mathcal{U}})$ is faithful. Moreover, (2) p **c** $p_{\Re}(\pi_{\mathcal{U}}(a))$ for all $a \in \mathcal{A}(\mathcal{D})$.

losed in $\mathcal{L}^*(\mathcal{D})$ and hence $\hat{\tau}$ is Hausdorff, it follows

is, $\pi_{\mathcal{U}}$ is faithful. Moreover, (2) proves the continuit

e complete.

low that there is a b the proof would be complete.

It remains to show that there is

fied. According to [6: Theorem 4.1

such that ker $z = \{0\}$, $z\mathcal{H} \subseteq \mathcal{D}$ and

tor defined on \mathcal{H} and hence bound
 τ_2 -dense in $\mathcal{E}(\mathcal{D})$, **fied.** According to [6: Theorem 4.1] there is a bounded self-adjoint operator z on \mathcal{H}
such that ker $z = \{0\}$, $z\mathcal{H} \subseteq \mathcal{D}$ and $\mathfrak{M} \subseteq z\mathfrak{B}_{\mathcal{X}}$. If $x \in \mathcal{F}^+(Z)$, then zz is a closed opera-
to defi

$$
\hat{p}_{\mathfrak{M}}(j(x)) \leq \inf_{c \in \mathcal{F}(\mathcal{D})} p_{\mathfrak{M}}(x+c) \bullet
$$
\n
$$
= \inf_{c \in \mathcal{F}(\mathcal{D})} \sup_{\varphi, \psi \in \mathfrak{B}_{\mathscr{H}}} |\langle (x+c) \, z\varphi, z\psi \rangle| = \inf_{c \in \mathcal{F}(\mathcal{D})} |z(x+c) \, z|.
$$

Since ker $z = \{0\}$, we have $\{cz: c \in \mathcal{F}(\mathcal{D})\} = \mathcal{F}(\mathcal{D})$. Moreover, $\{zc: c \in \mathcal{F}(\mathcal{D})\}$ is norm dense in $\mathcal{F}(\mathcal{H})$. Using these facts, we get

$$
= \inf_{\epsilon \in \mathcal{F}(\mathcal{D})} \sup_{\varphi_* \varphi \in \mathfrak{B}_{\mathcal{H}}} |\langle (x + c) z \varphi, z \psi \rangle| = \inf_{\epsilon \in \mathcal{F}(\mathcal{D})} |z(x + c) z|.
$$

or $z = \{0\}$, we have $\{cz : c \in \mathcal{F}(\mathcal{D})\} = \mathcal{F}(\mathcal{D})$. Moreover, $\{zc : c \in \mathcal{F}(\mathcal{D})\}$ is
use in $\mathcal{F}(\mathcal{H})$. Using these facts, we get

$$
\hat{p}_{\mathfrak{M}}(j(x)) \leq \inf_{\epsilon \in \mathcal{F}(\mathcal{D})} ||zxz + zc||
$$

$$
= \inf_{\epsilon \in \mathcal{F}(\mathcal{H})} ||zxz + c|| = \inf_{\epsilon \in \mathcal{E}(\mathcal{H})} ||zxz + c||.
$$
(3)
to other hand, let $\omega_{\mathcal{U}}$ denote the * representation of $B(\mathcal{H})$ on $\mathcal{H}_{\mathcal{U}}$ defined by

On the other hand, let ω_{ℓ} denote the *-representation of $B(\mathcal{H})$ on \mathcal{H}_{ℓ} defined by $w(u(y)) (\varphi_n) := (y \varphi_n)$ for $(\varphi_n) \in \mathcal{H}_u$ and $y \in \mathsf{B}(\mathcal{H})$. Since ω_u obviously annihilates $\mathscr{E}(\mathscr{H})$; $\omega_{\mathscr{U}}$ defines a *-representation of the Calkin algebra $\mathscr{A}(\mathscr{H})$ on $\mathscr{H}_{\mathscr{U}}$ (see [11: Section 2]). Since \mathcal{U} is assumed to be free and $\mathcal{A}(\mathcal{H})$ is simple, this *-representation of the C*-algebra $\mathcal{A}(\mathcal{H})$ is faithful and hence isometric. Since $zxz \in \mathcal{B}(\mathcal{H})$, this yields $\|\omega_{\mathcal{U}}(zzz)\| = \inf \{\|zzz + c\| : c \in \mathcal{E}(\mathcal{H})\}$. By (3), we obtain $\hat{p}_{\mathfrak{M}}(i(x)) \leq \|\omega_{\mathcal{U}}(zzz)\|$ for all $x \in \mathcal{$ $\|\omega_{\mathcal{U}}(z\bar{z})\| = \inf \{\|\bar{z}\bar{z} + c\| : c \in \mathcal{E}(\mathcal{H})\}.$ By (3), we obtain we have $\{cz : c \in \mathcal{F}(\mathcal{D})\} = \mathcal{F}(\mathcal{D})\}$. Moreover, $\{zc : c \in \mathcal{S}(\mathcal{D})\}$.

Using these facts, we get

inf $\|zzz + zc\|$
 $\inf_{\epsilon \in \mathcal{F}(\mathcal{X})} \|zzz + c\| = \inf_{c \in \mathcal{F}(\mathcal{X})} \|zzz + c\|$.
 \downarrow , let $\omega\gamma$ denote the *-rep $\omega_u(y) (\varphi_n) := (y\varphi_n)$ for $(\varphi_n) \in \mathcal{H}_u$ and $y \in B(\mathcal{H})$. Since ω_u obverables $\theta(\mathcal{H})$; ω_u defines a *-representation of the Calkin algebra $\mathcal{A}(\mathcal{H})$ on tion 2]). Since \mathcal{U} is assumed to be free and $\$

$$
\hat{p}_{\mathfrak{M}}(i(x)) \leq ||\omega_{\mathcal{U}}(zxz)|| \text{ for all } x \in \mathcal{L}^{+}(\mathcal{D}).
$$

Now define

$$
\hat{p}_{\mathfrak{M}}(i(x)) \leq ||\omega_{\mathcal{U}}(zxz)|| \text{ for all } x \in \mathcal{L}^{+}(\mathcal{D}).
$$
\n
$$
\hat{p}_{\mathfrak{M}}(i(x)) \leq ||\omega_{\mathcal{U}}(zxz)|| \text{ for all } x \in \mathcal{L}^{+}(\mathcal{D}).
$$
\n
$$
\hat{\mathfrak{N}} := \omega_{\mathcal{U}}(z) \mathfrak{B}_{\mathcal{X}\mathcal{U}} \equiv \{(z\varphi_n) : (\varphi_n) \in \mathcal{H}_{\mathcal{U}} \text{ and } ||(\varphi_n)||_{\mathcal{X}\mathcal{U}} \leq 1\}.
$$

$$
||z\mathbf{x}z|| = \inf \{||z\mathbf{x}z + c|| : c \in b(\mathcal{X})\}.
$$

This implies $\mathfrak{R} \subseteq \mathfrak{D}_{\boldsymbol{u}}$. From

 $||| \psi(u(x) (z\varphi_n)|| = ||(xz\varphi_n)||_{\mathcal{X}_\mathcal{U}} = \lim_{u} ||xz\varphi_n||$

$$
\leq ||xz|| \lim_{\mathcal{U}} ||\varphi_n|| = ||xz|| \text{ for } (\varphi_n) \in \mathfrak{B}_{\mathscr{H}_\mathcal{U}} \text{ and } x \in \mathcal{L}^+(\mathcal{D}),
$$

we see that \Re is bounded in $\mathcal{D}_{u}[t_{e u}].$

Finally, by (4), if $x \in \mathcal{L}^+(\mathcal{D})$, then

$$
\hat{p}_{\mathfrak{M}}(i(x)) \leq ||\omega_{\mathcal{U}}(zxz)|| = \sup_{\varphi,\psi \in \mathfrak{B}} |\langle \omega_{\mathcal{U}}(xz) \varphi, \omega_{\mathcal{U}}(z) \psi \rangle|
$$

 sup $|\langle (xz\varphi_n), (z\psi_n) \rangle|$ $(\varphi_n), (\varphi_n) \in \mathfrak{B}_{\mathcal{H}_{2\ell}}$

$$
= \sup_{(\varphi_n),(\psi_n)\in \mathfrak{B}_{\mathscr{H}_{\mathscr{U}}}(\varphi_{\mathscr{U}}(x))} \langle z\varphi_n\rangle, (z\psi_n)\rangle = p_{\mathfrak{N}}(\varrho_{\mathscr{U}}(x)).
$$

which proves (1). The proof of Theorem 2.1 is complete \blacksquare

2.3 From Theorem 2.1 and Lemma 1.4 we obtain

Theorem 2.2: Suppose that $\tau_n = \tau_{\mathcal{D}}$ on $\mathcal{L}^+(\mathcal{D})$. Let U be a free ultrafilter on N. Then, π_u is a faithful *-representation of $\mathcal{A}(\mathcal{D})$ and a homeomorphism of $\mathcal{A}(\mathcal{D})$ [t] onto $\pi_{\mathcal{U}}(\mathcal{A}(\mathcal{D}))$ [$\tau_{\mathcal{D},\mathcal{U}}$].

1. In general the domain $\mathcal{D}u$ is not dense in $\mathcal{H}u$. 2. If the domain is of the form $\mathcal{D} = \cap \{ \mathcal{D}(T^n) : n \in \mathbb{N} \}$ for some self-adjoint operator T on H, then $\tau_n = \tau_D$ on $\mathcal{L}^+(\mathcal{D})$ (see also Section 3).

3. Existence of continuous faithful \bullet -representations of $\mathcal{A}(\mathcal{D})$ [$\hat{\tau}$]

3.1 We first recall the setup of [14: Section 4]. However, the notation is slightly changed.

Suppose a is a (bounded or unbounded) self-adjoint operator on the Hilbert space \mathcal{H} with spectral decomposition $a = \int \lambda de(\lambda)$. Let $(f_k(t): k \in \mathbb{N})$ be a sequence of real measurable functions on the spectrum $\sigma(a)$ of a. All measure-theoretic notions refer to the spectral measure of a. We assume that

$$
f_1(t) = 1 \text{ and } f_k(t) \leq f_k^2(t) \leq f_{k+1}(t) \qquad \text{a.e. on } \sigma(a) \text{ for } k \in \mathbb{N}. \tag{1}
$$

Set $a_k = f_k(a)$ and $\mathcal{D} = \cap \{\mathcal{D}(a_k) : k \in \mathbb{N}\}\)$. Then, by (1), the operators a_k (more precisely, their restrictions to D) are in $\mathcal{L}^+(\mathcal{D})$ and the graph topology t on D is generated by the seminorms $\|\cdot\|_{a_k}, k \in \mathbb{N}$.

In our next theorem the following condition $(*)$ plays an important role:

[For each sequence $\gamma = (\gamma_k : k \in \mathbb{N})$ of positive numbers γ_k there is a (*) $k = k_r \in \mathbb{N}$ such that all functions $f_n, n \in \mathbb{N}$, are bounded on \Re_k , where

$$
\{ \forall i_n := \{t \in \sigma(a) : f_1(t) \leq \gamma_1, \ldots, f_n(t) \leq \gamma_n \} \text{ for } n \in \mathbb{N}.
$$

The following assertions are equivalent:

 (i) Condition $(*)$ is fulfilled.

 (ii) $\tau_0 = \tau_{\mathfrak{D}}$ on $\mathscr{L}^+(\mathscr{D})$.

 $\tau_n = \tau_{\mathcal{D}}$ on $\mathcal{L}^+(\mathcal{D})$. (iii)

Each positive linear functional on $\mathcal{L}^+(\mathcal{D})$ is $\tau_{\mathcal{D}}$ -continuous. $(i\mathbf{v})$

This is essentially $[14:$ Theorem 4.1]. The equivalence of (i), (ii) and (iv) has been stated therein. Since $\tau_0 \supseteq \tau_n \supseteq \tau_2$, (ii) \Rightarrow (iii). Since each positive linear functional is τ_n -continuous, we have (iii) \Rightarrow (iv).

3.2 The following theorem may be considered as a supplement to [14: Theorem 4.1]. Among other things it shows that if $\tau_n \neq \tau_p$ on $\mathcal{L}^+(\mathcal{D})$, then there is no continuous faithful *-representation of $\mathcal{A}(\mathcal{D})$ [$\hat{\mathbf{t}}$]. In particular, the *-representations $\pi\mathcal{U}$ occuring in Theorem 2.1 are not continuous.

Theorem 3.1: Let $\mathcal D$ be as above. Then (i) is equivalent to each of the following con*ditions:*

- (v) **Theorem 2.1 are not continuous.**
 There 2.1: Let $\mathcal D$ be as above. Then (i) is equivalent to each of the following con-
 There exists a faithful *-representation π of $\mathcal A(\mathcal D)$ which is a homeomorphism

of \math *of* $\mathcal{A}(\mathcal{D})$ [$\hat{\tau}$] *onto* $\pi(\mathcal{A}(\mathcal{D}))$ [$\tau_{\mathcal{D}(n)}$].
There exists a continuous faithful *-representation of $\mathcal{A}(\mathcal{D})$ [$\hat{\tau}$]. (v) There exists a faithful *-representation π of $A(D)$ which is a homeomorphism
of $A(D)$ [i] onto $\pi(A(D))$ [$\pi_{D(n)}$].
(v) There exists a continuous faithful *-representation of $A(D)$ [i].
(vi) Each positive *-represen
-
-
- (vi) *Each positive *-representation of* $\mathcal{L}^+(\mathcal{D})$ [$\tau_{\mathcal{D}}$] *is continuous.* (vi)' *Each weakly continuous positive *-representation of* $\mathcal{L}^+(\mathcal{D})$ [*Each weakly continuous positive *-representation of* $\mathcal{L}^+(\mathcal{D})$ [τ ₂) *is continuous.*

Proof: Theorem 2.2 shows that (iii) \Rightarrow (v). (iii) \Rightarrow (vi) follows from Lemma 1.3.
Since $(v) \Rightarrow (v)'$ and $(vi) \Rightarrow (vi)'$ are trivially fulfilled, it suffices to prove that $(v)'$ (v) There exists a continuous passive v-representation of $\mathcal{L}^+(\mathcal{D})$ [$\tau_{\mathcal{D}}$] is continuous.

(vi)' Each positive *-representation of $\mathcal{L}^+(\mathcal{D})$ [$\tau_{\mathcal{D}}$] is continuous.

Proof: Theorem 2.2 shows that ($(v) \Rightarrow$ (i): Suppose that π is a continuous faithful *-representation of $\mathcal{A}(2)$ [$\hat{\tau}$]. Then, $\varrho(x) := \pi(j(x)), x \in \mathcal{L}^+(\mathcal{D}),$ defines a continuous *-representation of $\mathcal{L}^+(\mathcal{D})$ [τ ₂]. To prove (i) , we assume the contrary, that is, condition $(*)$ is not satisfied. Then there are a positive sequence $y = (y_k)$ and a sequence (i_k) of natural numbers such that f_i , is not essentially bounded on the set \Re_k for each $k \in \mathbb{N}$. There is no loss of generaprove (i), we assume the contrary, 1
prove (i), we assume the contrary, 1
are a positive sequence $\gamma = (\gamma_k)$ an
 f_{i_k} is not essentially bounded on the
lity if we assume that $\gamma_{k+1} > \gamma_k \ge$
surable subsets $\mathfrak{F}_{k,n}$, *k* and $i_k = k$ for all $k \in \mathbb{N}$. Then there are mea-Froof: Theorem 2.2 shows that $(iii) \Rightarrow (v)$. $(iii) \Rightarrow (vi)$ follows from Lemma 1.3.
Since $(v) \Rightarrow (v)'$ and $(vi) \Rightarrow (vi)'$ are trivially fulfilled, it suffices to prove that $(v)'$
 $\Rightarrow (i)$ and $(vi)' \Rightarrow (i)$. Both proofs will be indirect (see k, n for all $k, n \in \mathbb{N}$. Let $\varphi_{k,n}$ be a unit vector from $e(\mathfrak{F}_{k,n})$. \mathfrak{D} . $\epsilon \mathcal{L}^+(D)$, defines a continuous *-representation of \mathcal{L}^+

ime the contrary, that is, condition (*) is not satisfied

interesting that $\mathcal{L}^+(k)$ and a sequence (i_k) of natural number

by bounded on the set Since $(\mathbf{v}) \Rightarrow (\mathbf{v})$ and $(\mathbf{v}) \Rightarrow (\mathbf{v})$ are trivially fullmed, it so
 $\Rightarrow (\mathbf{v}) \Rightarrow (\mathbf{v}) \Rightarrow$ Suppose that π is a continuous faithful *-represent
 $x \in \mathcal{E}^+(\mathcal{D})$, defines a continuous *-representa

assume the contrary, that is, condition (*) is no

e sequence $\gamma = (\gamma_k)$ and a sequence (i_k) of naturality bou

Let Δ denote the family of all sequences $\delta = (\delta_k)$ of natural numbers δ_k satisfying $\delta_k \geq k + 2$ for $k \in \mathbb{N}$. Fix a $\delta \in \Delta$. We first show that for $r \in \mathbb{N}$ and $\varphi \in \mathcal{D}(\varrho)$ $\begin{array}{l} \text{if all } k \text{, } n \in \mathcal{A} \text{ denote } t \ k+2 \text{ for } k \end{array}$

and
$$
k, n \in \mathbb{N}
$$
. Let $\varphi_{k,n}$ be a unit vector from $e(\varphi_{k,n}) \in \mathbb{Z}$.
 denote the family of all sequences $\delta = (\delta_k)$ of natural numbers δ_k satisfying
 -2 for $k \in \mathbb{N}$. Fix a $\delta \in \Delta$. We first show that for $r \in \mathbb{N}$ and $\varphi \in \mathcal{D}(\varrho)$
\n
$$
\left\| \varrho(a_r) \varrho \left(e \left(\bigcup_{k \geq r+1} \Im_{k,\delta_k} \right) \right) \varphi \right\| \leq \gamma_r \|\varphi\|
$$
 (2)

 $\|\varrho(a_{r+1})\varrho(e(\mathfrak{F}_{r,\delta_r}))\varphi\| \geq \gamma_{\delta_r} \|\varrho(e(\mathfrak{F}_{r,\delta_r}))\varphi\|.$ **(3)**

For let- χ denote the characteristic function of the set $\bigcup {\mathfrak{F}_{k,\delta_k}} : k \geq r + 1$. By conand
 $\begin{aligned}\n\|\varrho(a_{r+1}) \varrho(e(\Im_{r,\delta_r})) \varphi\| &\geq \gamma_{\delta_r} \|\varrho(e(\Im_{r,\delta_r})) \varphi\|.\n\end{aligned}$

For let χ denote the characteristic function of the set $\bigcup {\Im_{k,\delta_k}} : k \geq r+1$. I

struction, $f_r(t) \chi(t) \leq \gamma_r$ a.e. on $\sigma(a)$. Define a funct (2) follows by the same argument.

Let q_a be the orthogonal projection onto the closure of $\mathcal{D}_s := \left\{ |e(x) - e^{i\theta} + e^{i\theta} \right\|^2 \leq \left\langle e^{i\theta} + e^{i\theta} \right\rangle \right\}$ (3) $\left\langle e^{i\theta} + e^{i\theta} \right\rangle \left\langle e^{i\theta} \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{$ and

For let- χ denoted

struction, $f_r(t)$ χ

obviously, $g(a)$
 $||\varphi||^2$ γ_r

(2) follows by the Let q_s be the

Next we prove sum
 $\sum_{k=1}^{s} \hat{\lambda}_k \varphi_i$

Suppose $k, n \in$

$$
\mathfrak{F}_{k,n} \text{ for all } k, n \in \mathbb{N}. \text{ Let } \phi_{k,n} \text{ be a unit vector from } e(\mathfrak{F}_{k,n}) \mathcal{D}.
$$

Let Δ denote the family of all sequences $\delta = (\delta_k)$ of natural numbers δ_k satisfying
 $\delta_k \geq k + 2$ for $k \in \mathbb{N}$. Fix a $\delta \in \Delta$. We first show that for $r \in \mathbb{N}$ and $\varphi \in \mathcal{D}(\varrho)$

$$
\left\| \varrho(a_r) \varrho \left(e \left(\bigcup_{k \geq r+1} \mathfrak{F}_{k,\delta_k} \right) \right) \varphi \right\| \leq \gamma_r \left\| \varphi \right\|
$$

$$
\left\| \varrho(a_{r+1}) \varrho(e(\mathfrak{F}_{r,\delta_r})) \varphi \right\| \geq \gamma_{\delta_r} \left\| \varrho(e(\mathfrak{F}_{r,\delta_r})) \varphi \right\|.
$$

For let χ denote the characteristic function of the set $\cup \{\mathfrak{F}_{k,\delta_k} : k \geq r+1\}$. By construction, $f_r(t) \chi(t) \leq \gamma_r$ a.e. on $\sigma(a)$. Define a function g on $\sigma(a)$ by $g := (\gamma_r^2 - f_r^2 \chi)^{1/2}$.
Obviously, $g(a) \in \mathcal{F}^+(\mathcal{D})$. For $\varphi \in \mathcal{D}(\varrho)$, $\langle \varrho(g(a)^2) \varphi, \varphi \rangle = ||\varrho(g(a)) \varphi||^2 \geq 0$ and hence
 $||\varphi||^2 \gamma_r^2 = \langle \varrho(\gamma_r^2 I) \varphi, \varphi \rangle$

$$
\geq \langle \varrho(f_r(a)^2 \chi(a)) \varphi, \varphi \rangle = ||\varrho(a_r) \varrho \left(e \left(\bigcup_{k \geq r+1} \mathfrak{F}_{k,\delta_k} \right) \right) \varphi||^2.
$$

(2) follows by the same argument.
Let g_δ be the orthogonal projection onto the closure of $\mathcal{D}_\delta := 1$.
 $\left\{ \varphi_{k,\$

Let q_{δ} be the orthogonal projection onto the closure of $\mathcal{D}_{\delta} := \text{l.h. } \{\varphi_{k,\delta_k}: k \in \mathbb{N}\}\$.
Next we prove that $q_{\delta}\mathscr{H} \subseteq \mathcal{D}$. For let $r \in \mathbb{N}$. Each $\varphi \in \mathcal{D}_{\delta}$ can be written as a finite $\geq \langle \varrho(f_r(a)^2 \chi(a)) \varphi, \varphi \rangle = \|\varrho(a_r) \varrho\|$
 \downarrow ws by the same argument.

be the orthogonal projection onto the c

prove that $q_s \mathcal{H} \subseteq \mathcal{D}$. For let $r \in \mathbb{N}$. Each
 $\sum_{k=1}^{s} \chi_{\varphi_k, \delta_k}$, where $\lambda_1, \ldots, \lambda_s \in$ (2) follows by the same argument.

Let q_{δ} be the orthogonal projection onto the Next we prove that $q_{\delta} \mathcal{H} \subseteq \mathcal{D}$. For let $r \in \mathbb{N}$. I

sum
 $\sum_{k=1}^{s} \hat{\lambda}_k \varphi_{k,\delta_k}$, where $\lambda_1, ..., \lambda_{\delta} \in \mathbb{C}$ and $s \$ $\leq (\ell_1, d) \cdot \chi(d) \mid \varphi, \varphi) = \|\ell(a_r) \cdot \ell\| \left(\sum_{k \geq r+1} \cdot \cdot \cdot \cdot k \cdot \epsilon_k\right)\| \cdot \varphi\|$

so the same argument.

to the orthogonal projection onto the closure of $\mathcal{D}_\delta := \text{I.h. } \{\varphi_{k,\delta_k} : k \in \mathbb{N}\}$.

prove that $q_\delta \mathcal{H} \subseteq \mathcal$

$$
\sum_{i=1}^n \lambda_k \varphi_{k,\delta_k}
$$
, where $\lambda_1, \ldots, \lambda_s \in \mathbb{C}$ and $s \in \mathbb{N}, s > r$.

√, *s*
^γ
has 1 Suppose $k, n \in \mathbb{N}$, $n > k$. Since $f_{k+1}(t) \geq \gamma_{\delta_k} \geq \gamma_{k+2} > \gamma_{k+1}$ a.e. on $\mathfrak{F}_{k,\delta_k}$ and $f_{k+1}(t)$ $\perp \varphi_{n,\delta_n}$ and $a_r \varphi_{k,\delta_k} \perp a_r \varphi_{n,\delta_n}$. Using the latter, we obtain

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\n
$$
\perp \varphi_{n,\delta_n} \text{ and } a_r \varphi_{k,\delta_k} \perp a_r \varphi_{n,\delta_n}.\text{ Using the latter, we obtain}
$$
\n
$$
||a_r \varphi||^2 = \sum_{k=1}^r |\lambda_k|^2 ||a_r \varphi_{k,\delta_k}||^2 + \sum_{k=r+1}^s |\lambda_k|^2 ||a_r \varphi_{k,\delta_k}||^2
$$
\n
$$
\leq \max (||a_r \varphi_{1,\delta_i}||, ..., ||a_r \varphi_{r,\delta_i}||, \gamma_r) \sum_{k=1}^s |\lambda_k|^2 = \max (...) ||\varphi||^2.
$$
\nThis implies $q_e \mathcal{H} \subseteq \mathcal{D}(a_r).$ Since $\mathcal{D} = \cap {\mathcal{D}(a_r) : r \in \mathbb{N}}$ by definition, this shows that $q_e \mathcal{H} \subseteq \mathcal{D}.$
\nWe define $\mathcal{R} := \bigcup \{ \varrho(q_\delta) \mathcal{B} : \delta \in \Delta \}$, where $\mathcal{B} := \mathcal{B}_{\mathcal{D}(\varrho)} = {\mathcal{D} \in \mathcal{D}(\varrho) : ||\varphi|| \leq 1}$.
\nWe present \mathcal{R} is bounded in $\mathcal{D}(c) [t_\varrho]$. For take $r \in \mathbb{N}$ and $\delta \in \Delta$. Let $c_r \beta$ denote the orthogonal projection on \mathcal{H} with range l.h. $\langle \varphi_{1,\delta_1}, ..., \varphi_{r,\delta_r} \rangle$. Since obviously $a_r c_r \beta \in \mathcal{E}(\mathcal{D})$, we have $a_r c_r \beta \in \text{ker } \varrho$. From
\n $q_\delta - c_{r,\delta} = e \bigcup_{k \geq r+1} \mathcal{S}_{k,\delta_k} \setminus (q_\delta - e_{r,\delta})$
\nand (2) we therefore obtain
\n $||\varrho(a_r) \varrho(q_\delta) \varphi|| = ||\varrho(a_r) \varrho(q_\delta - c_{r,\delta}) \varphi||$
\n $= ||\varrho(a_r) \varrho(e \mid \cup \mathcal{S}_{k,\delta_k})| q(q_\delta - c_{r,\delta}) \varphi||$

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 $\perp \varphi_{n,\delta_n}$ and $a_{\tau}\varphi_{k,\delta_k} \perp$
 $||a_{\tau}\varphi||^2 = \sum_{k=1}^{\tau}$
 \leq mai

This implies $q_e\mathcal{H} \subseteq \mathcal{L}$
 $q_e\mathcal{H} \subseteq \mathcal{D}$.

We define $\mathcal{R} := \cup$

We prove that \mathcal{R} is boothogonal projection We prove that 92 is $\mathbb{E}[\mathcal{U}_k] = \sum_{k=1}^n |\lambda_k|^2 ||a_r \varphi_{k,\delta_k}||^2 + \sum_{k=r+1}^s |\lambda_k|^2 ||a_r \varphi_{k,\delta_k}||^2$
 $\leq \max (||a_r \varphi_{1,\delta}||, ..., ||a_r \varphi_{r,\delta}||, \gamma_r) \sum_{k=1}^s |\lambda_k|^2 = \max (\dots) ||\varphi||^2.$

This implies $q_r \mathcal{H} \subseteq \mathcal{D}(a_r)$. Since $\mathcal{D} = \cap {\$ orthogonal projection on \mathcal{H} with range l.h. $\{\varphi_{1,\delta_1},\ldots,\varphi_{r,\delta_r}\}\)$. Since obviously $a_r c_{r,\delta}$ $\leq \max (\|a_r \varphi_{1,\delta_1}\|, \ldots,$

This implies $q_\varphi \mathcal{H} \subseteq \mathcal{D}(a_r)$. Since \mathcal{D} .
 $q_\varphi \mathcal{H} \subseteq \mathcal{D}$.

We define $\mathfrak{N} := \bigcup \{ \varrho(q_\delta) \mathfrak{B} : \delta \in$

We prove that \mathfrak{N} is bounded in $\mathcal{D}(\varrho)$

orthogonal proje in $\mathcal{D}($

i with ϱ . From $\mathfrak{F}_{k,\delta_{k}}$ $\left(\|\boldsymbol{a}_r \boldsymbol{\varphi}_{1,\delta_1}\|, \ldots, \|\boldsymbol{a}_r \boldsymbol{\varphi}_{r,\delta_r}\|, \gamma_r\right) \sum_{k=1}^s |\lambda_k|^2 = \max(\ldots) \|\boldsymbol{\varphi}\|^2$
 k_r). Since $\mathcal{D} = \bigcap \{\mathcal{D}(\boldsymbol{a}_r) : r \in \mathbb{N}\}$ by definition, the $\{\varrho(q_\delta) \mathcal{B} : \delta \in \Delta\}$, where $\mathcal{B} := \mathfrak{B}_{\mathcal{D}(\bold$

$$
q_{\delta}-c_{r,\delta}=e\left(\bigcup_{k\geq r+1}\mathfrak{F}_{k,\delta_k}\right)(q_{\delta}-e_{r,\delta})
$$

We prove that
$$
\Re
$$
 is bounded in $\mathcal{D}(\rho)$ [t_{e}]. For take $r \in \Re$ and $\delta \in \Delta$. Let
orthogonal projection on \mathcal{H} with range l.h, $\{\varphi_{1,\delta_1}, ..., \varphi_{r,\delta_r}\}$. Since $\epsilon \in \mathcal{E}(\mathcal{D})$, we have $a_r c_{r,\delta} \in \ker \varrho$. From
 $q_{\delta} - c_{r,\delta} = e \left(\bigcup_{k \geq r+1} \Im_{k,\delta_k} \right) \cdot (q_{\delta} - e_{r,\delta})$
and (2) we therefore obtain
 $||\varrho(a_r) \varrho(q_{\delta}) \varphi|| = ||\varrho(a_r) \varrho(q_{\delta} - c_{r,\delta}) \varphi||$
 $= ||\varrho(a_r) \varrho \varrho(e \left(\bigcup_{k \geq r+1} \Im_{k,\delta_k} \right)) \varrho(q_{\delta} - c_{r,\delta}) \varphi||$
 $\leq \gamma_r ||\varrho(q_{\delta} - c_{r,\delta}) \varphi|| \leq \gamma_r$ for each $\varphi \in \mathcal{B}$.
By Lemma 1.4 (iii) the graph topology t_{ϱ} on $\mathcal{D}(\varrho)$ is generated by the
 $||\cdot||_{e(a_r)}$, $r \in \mathbb{N}$. Therefore, the preceding proof shows that \Re is bounded
to the graph topology t_{ϱ} .
Since the * representation ϱ of $\mathcal{I}^+(\mathcal{D})$ [$\tau_{\mathcal{D}}$] is continuous, there exist
subset \Re of $\mathcal{D}(t)$ such that

By Lemma 1.4 (iii) the graph topology t_e on $\mathcal{D}(e)$ is generated by the seminorms $\|\cdot\|_{\rho(a_r)}$, $r \in \mathbb{N}$. Therefore, the preceding proof shows that \mathfrak{N} is bounded with respect to the graph topology l_{ρ} . *ps* f_e on $\mathcal{D}(e)$
 f_e **N**. Therefore, the preceding proof-shows
 f_e . Therefore, the preceding proof-shows
 f_e . Are $*$ -representation e of $\mathcal{F}^+(\mathcal{D})$ [$\tau_{\mathcal{D}}$] is cont
 f $\mathcal{D}(t)$ such that
 f

Since the *-representation ρ of $\mathcal{L}^+(\mathcal{D})$ [$\tau_{\mathcal{D}}$] is continuous, there exists a bounded subset \mathfrak{M} of $\mathfrak{D}(t)$ such that

 $\begin{pmatrix} 4 \end{pmatrix}$

$$
p_{\mathfrak{N}}(\varrho(x)) \leq p_{\mathfrak{M}}(x) \quad \text{for all} \quad x \in \mathcal{L}^+(\mathcal{D}).
$$

Since \mathfrak{M} is *l*-bounded, $C_r := \sup \{ ||\varrho(a_r) \varphi|| : \varphi \in \mathfrak{M} \} < \infty$ for each $r \in \mathbb{N}$. We choose $\|\cdot\|_{e(a_r)}, r \in \mathbb{N}$. Therefore, the preceding proof shows that \Re is bounded with respect
to the graph topology l_e .
Since the *-representation ϱ of $\mathcal{L}^+(\mathcal{D})$ [$\tau_{\mathcal{D}}$] is continuous, there exists a bou *By* Lemma 1.4 (iii) the graph topology t_0 on $\mathcal{D}(\varrho)$ is generated by the seminorms $\|\cdot\|_{e(a_i)}, r \in \mathbb{N}$. Therefore, the preceding proof shows that \mathcal{R} is bounded with respect to the graph topology t_0 .

S $\begin{array}{c} \text{Si} \ \text{subs} \ \text{sin} \text{c} \ \text{nat} \ \text{beca} \ x \in \mathbb{R} \end{array}$ • \mathfrak{M} is *l* bounded, $C_r := \sup \{ ||\varrho(a_r) \varphi|| : \varphi \in \mathfrak{M} \} < \infty$ for each $r \in \mathbb{N}$. We choosed in the sum that $\delta_k \geq k + 2$ and $\gamma_{\delta_k} \geq C_{k+1} 2^k$ for $k \in \mathbb{N}$. This is possible via $\gamma_n \geq n$ for $n \in \mathbb{N}$. because $\gamma_n \geq n$ for $n \in \mathbb{N}$. Define an operator x by $x := e(\cup \{\mathcal{F}_{k,\delta_k} : k \in \mathbb{N}\})$. Clearly, $x \in \mathcal{L}^+(\mathcal{D})$. Our aim is to show that for this operator x (4) is not true. By (3), we have $||\cdot||_{e(a_r)}, r \in N$. Therefore, the p
 $||\cdot||_{e(a_r)}, r \in N$. Therefore, the p

to the graph topology t_e .

Since the *-representation ϱ

subset \mathfrak{M} of $\mathcal{D}(t)$ such that
 $p_{\mathfrak{M}}(\varrho(x)) \leq p_{\mathfrak{M}}(x)$ for

Since \math *C* of $\mathcal{L}^+(\mathcal{D})$ $[\tau_{\mathcal{D}}]$ is continuous, there exists
 \mathbf{r} all $x \in \mathcal{L}^+(\mathcal{D})$.
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 \mathbf{C} and \mathbf{C} \mathbf{C}

$$
\begin{aligned} \left\|\rho(e(\mathfrak{F}_{r,\delta_r})\right)\varphi\right\| &\leq \left\|\varrho(a_{r+1})\,\varrho(e(\mathfrak{F}_{r,\delta_r})\right)\varphi\right\| \\ &\leq \left\|\varrho(a_{r+1})\,\varphi\right\| \leq C_{r+1} \quad \text{for } r \in \mathbb{N} \quad \text{and} \quad \varphi \in \mathfrak{M} \, . \end{aligned}
$$

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$$
\leq ||\varrho(a_{r+1}) \varphi|| \leq C_{r+1} \quad \text{for}
$$

$$
\sup_{\varphi \in \mathfrak{M}} ||\varrho(e(\mathfrak{F}_{r,\delta_r})) \varphi|| \leq C_{r+1} \gamma_{\delta_r}^{-1} \quad \text{for} \quad r \in \mathbb{N}.
$$

Using this inequality, we obtain

$$
\gamma_n \geq n \text{ for } n \in \mathbb{N}. \text{ Define an operator } x \text{ by } x := e(\cup \{\tilde{\chi}_{k,\delta_k} : k \in \mathbb{N}\}). \text{ Clearly,}
$$
\n
$$
\gamma_n \geq n \text{ for } n \in \mathbb{N}. \text{ Define an operator } x \text{ by } x := e(\cup \{\tilde{\chi}_{k,\delta_k} : k \in \mathbb{N}\}). \text{ Clearly,}
$$
\n
$$
\gamma_n \left\| \varrho(e(\tilde{\chi}_{r,\delta_r})) \varphi \right\| \leq \left\| \varrho(a_{r+1}) \varrho(e(\tilde{\chi}_{r,\delta_r})) \varphi \right\|
$$
\n
$$
\leq \left\| \varrho(a_{r+1}) \varrho \right\| \leq C_{r+1} \text{ for } r \in \mathbb{N} \text{ and } \varphi \in \mathfrak{M}.
$$
\n
$$
\sup_{\varphi \in \mathfrak{M}} \left\| \varrho(e(\tilde{\chi}_{r,\delta_r})) \varphi \right\| \leq C_{r+1} \gamma_{\delta_r}^{-1} \text{ for } r \in \mathbb{N}.
$$
\n
$$
\sup_{\varphi \in \mathfrak{M}} \left\| \varrho(e(\tilde{\chi}_{r,\delta_r})) \varphi \right\| \leq C_{r+1} \gamma_{\delta_r}^{-1} \text{ for } r \in \mathbb{N}.
$$
\n
$$
\lim_{\varphi, \varphi \in \mathfrak{M}} \sum_{k=1}^{\infty} \left\| e(\tilde{\chi}_{k,\delta_k}) \varphi, \varphi \right\|^{2} \leq \sum_{k=1}^{\infty} C_{k+1}^{2} \gamma_{\delta_k}^{-2} \leq \sum_{k=1}^{\infty} 2^{-2k} < 1. \tag{5}
$$
\n
$$
a_r q_\delta \text{ is a-bounded operator on } \mathcal{H} \text{ for } r \in \mathbb{N} \text{ as shown above, the sequence } \in \mathbb{N} \text{ is bounded in } \mathcal{D}(t). \text{ But the set } \{q_\delta \varphi_{k,\delta_k}\} = \{\varphi_{k,\delta_k}\} \text{ is certainly not relatively}
$$

Since a_1q_3 is a, bounded operator on $\mathcal H$ for $r \in \mathbb N$ as shown above, the sequence $(\varphi_{k,\delta_k}: k \in \mathbb{N})$ is bounded in $\mathcal{D}(l)$. But the set $\{q_{\delta}\varphi_{k,\delta_k}\} = \{\varphi_{k,\delta_k}\}\$ is certainly not relatively compact in $\mathcal{D}[l]$, since (φ_{k,δ_k}) is an orthonormal sequence in \mathcal{X} . This proves that

•
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 $\frac{1}{2}$

 $q_{\delta} \notin \mathcal{E}(\mathcal{D}) = \text{ker } \rho$. Consequently,

$$
1 = \sup_{\varphi \in \mathfrak{B}} ||\varrho(q_{\delta}) \varphi||^{2}
$$

=
$$
\sup_{\varphi \in \mathfrak{B}} \left| \left\langle \varrho \left(e \left(\bigcup_{k} \Im_{k,\delta_{k}} \right) \right) \varrho(\tilde{q}_{\delta}) \varphi, \varrho(q_{\delta}) \varphi \right\rangle \right| \leq p_{\mathfrak{B}}(\varrho(x)).
$$

Comparing (5) and (6) with (4), we obtain the desired contradiction.

 $(vi)' \rightarrow (i)$: This will be similar as the preceding proof. Again we assume that condition (*) is not fulfilled. We keep the notation introduced above. Let $\mathcal U$ be an arbitrary free ultrafilter on N. As already mentioned in Section 2, *ou* is a positive *-re-. presentation of $\mathcal{L}^+(\mathcal{D})$. It suffices to show that $\varrho_\mathcal{U}$ is weakly continuous, but not continuous. Let $\varphi = (\varphi_n) \in \mathcal{D}_{\mathcal{U}}$. By definition of $\mathcal{D}_{\mathcal{U}}$, the set $\mathfrak{M} := {\varphi_n}$ is bounded in $q_{\delta} \notin \mathscr{E}(\mathcal{D}) = \ker \varrho$. Consequently,
 $1 = \sup_{\varphi \in \mathfrak{B}} ||\varrho(q_{\delta}) \varphi||^2$
 $= \sup_{\varphi \in \mathfrak{B}} \left| \left\langle \varrho \left(e \left(\bigcup_{k} \Im_{k,\delta_k} \right) \right) \varrho(q_{\delta}) \varphi, \varrho(q_{\delta}) \varphi \right) \right\rangle$

Comparing (5) and (6) with (4), we obtain the d
 $\left(\$ (i): This will be
is not fulfilled.
is not fulfilled.
intrafilter on \mathbf{P}
is \mathbf{P} be the subset of $\mathcal{L}^*(\mathcal{D})$. It
Let $\varphi = (\varphi_n) \in \mathcal{L}^*(\mathcal{D})$, then
 $|\langle \varrho u(x) \varphi, \varphi \rangle|$ = urnal suppress to show that ϱ **u** is weakly continually suffices to show that ϱ **u** is weakly continually continually ∂ **u**. By definition of \mathcal{D} **u**, the set $\mathfrak{M} := \{ \varrho$
 $\lim_{n \in \mathbb{N}} \langle x\varphi_n, \varphi_n \rangle | \leq \sup_{$

$$
|\langle \varrho u(x) \varphi, \varphi \rangle| = |\lim_{\mathcal{U}} \langle x \varphi_n, \varphi_n \rangle| \leq \sup_{n \in \mathbb{N}} |\langle x \varphi_n, \varphi_n \rangle| \leq p_{\mathfrak{M}}(x).
$$

That is, $\varrho_{\mathcal{U}}$ is weakly continuous. From Theorem 2.1 we know that ker $\varrho_{\mathcal{U}} = \mathcal{E}(\mathcal{D})$. Therefore, the preceding proof in the case $\rho = \rho \gamma$ shows that $\rho \gamma$ is not $\tau \gamma$ -continuous **I**

Results similar to those proved in this paper are true for the topologies $\tau^{(2)}$ and τ^0 (see also $[14]$).

Addendum: After completing the manuscript the author has learned that in the case $\mathcal{D} = \cap \{ \mathcal{D}(T^n) : n \in \mathbb{N} \}$, T a self-adjoint operator, the existence of a topological realization of $\mathcal{A}(\mathcal{D})$ [$\hat{\tau}$] has been independently obtained by F. LöFFLER and W. TIMMERMANN in "The Calkin representation for a certain class of algebras of unbounded operators", Dubna-Preprint E tinuous. Let $\varphi = (\varphi_n) \in \mathcal{D}_\mathcal{U}$. By definition of $\mathcal{D}_\mathcal{U}$, the set $\mathfrak{M} := \mathcal{D}[t]$. If $x \in \mathcal{L}^+(\mathcal{D})$, then
 $|\langle \varrho u(x) \varphi, \varphi \rangle| = |\lim_{\mathcal{U}} \langle x\varphi_n, \varphi_n \rangle| \leq \sup_{\mathcal{R} \in \mathbb{N}} |\langle x\varphi_n, \varphi_n \rangle| \leq p_{\mathfrak{M}}$ 14]).

Addendum: After completing the manuscript the author has learned
 $D = \cap \{D(T^n) : n \in \mathbb{N}\}$, T a self-adjoint operator, the existence of a topolog
 $I(D)$ [i] has been independently obtained by F. LÖFFLER and W. TIMME

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