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Topological Realizations of Calkin Algebras on Frechet Domains of Unbounded Operator Algebras

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$\mathcal{D}$ sei ein dichter linearer Teilraum eines separablen Hilbertraumes und $\mathcal{L}^+(\mathcal{D})$ die maximale Op*-Algebra auf $\mathcal{D}$, versehen mit der gleichmäßigen Topologie $\tau_{\mathcal{D}}$. Wir setzen voraus, daß $\mathcal{D}$ bezüglich der Graphtopologie von $\mathcal{L}^+(\mathcal{D})$ ein Frechetraum ist. Weiter sei $\mathcal{C}(\mathcal{D})$ das zweiseitige *-Ideal aller Operatoren aus $\mathcal{L}^+(\mathcal{D})$, die beschränkte Teilmengen von $\mathcal{D}$ in relativ kompakte Teilmengen abbilden. Es wird untersucht, wann die Faktoralgebra $\mathcal{A}(\mathcal{D}) := \mathcal{L}^+(\mathcal{D})/\mathcal{C}(\mathcal{D})$, versehen mit der Faktorraumtopologie, eine topologische Realisierung als Op*-Algebra besitzt.

Пусть $\mathcal{D}$ плотное линейное подпространство сепарабельного гильбертова пространства и пусть $\mathcal{L}^+(\mathcal{D})$ максимальная Op*-алгебра над $\mathcal{D}$, снабжённая равномерной топологией τ_D. Предполагается, что $\mathcal{D}$ является пространством Фреше относительно топологии порождённой граф-нормами операторов из $\mathcal{L}^+(\mathcal{D})$. Через $\mathcal{C}(\mathcal{D})$ обозначается двусторонний *-идеал тех операторов из $\mathcal{L}^+(\mathcal{D})$, которые переводят ограниченные подмножества пространства $\mathcal{D}$ в относительно компактные подмножества. Исследуется вопрос о том, когда факторалгебра $\mathcal{A}(\mathcal{D}) := \mathcal{L}^+(\mathcal{D})/\mathcal{C}(\mathcal{D})$ снабжённая топологией факторпространства допускает топологическую реализацию как Op*-алгебра.

Let $\mathcal{D}$ be a dense linear subspace of a separable Hilbert space and let $\mathcal{L}^+(\mathcal{D})$ be the maximal Op*-algebra on $\mathcal{D}$ endowed with the uniform topology $\tau_{\mathcal{D}}$. Suppose $\mathcal{D}$ is a Frechet space with respect to the graph topology of $\mathcal{L}^+(\mathcal{D})$. Let $\mathcal{C}(\mathcal{D})$ denote the two-sided *-ideal of all operators in $\mathcal{L}^+(\mathcal{D})$ which map bounded subsets of $\mathcal{D}$ into relatively compact subsets. We study the question of when the quotient algebra $\mathcal{A}(\mathcal{D}) := \mathcal{L}^+(\mathcal{D})/\mathcal{C}(\mathcal{D})$, endowed with the quotient topology, has a topological realization as an Op*-algebra.

Introduction

Let $\mathcal{D}$ be a dense linear subspace of a separable complex Hilbert space $\mathcal{H}$ endowed with the graph topology t (see Section 1 for precise definitions). Suppose $\mathcal{D}[t]$ is a Frechet space. Let $\mathcal{C}(\mathcal{D})$ denote the set of all operators in $\mathcal{L}^+(\mathcal{D})$ which map each bounded subset of $\mathcal{D}[t]$ into a relatively compact subset of $\mathcal{D}[t]$. Then $\mathcal{C}(\mathcal{D})$ is a $\tau_{\mathcal{D}}$-closed two-sided *-ideal of $\mathcal{L}^+(\mathcal{D})$ which contains the finite rank operators in $\mathcal{L}^+(\mathcal{D})$ as a dense subset [15, 7]. (Note that in [15] the ideal $\mathcal{C}(\mathcal{D})$ is denoted by Vol (t, t).) The quotient algebra $\mathcal{A}(\mathcal{D}) := \mathcal{L}^+(\mathcal{D})/\mathcal{C}(\mathcal{D})$ is called the Calkin algebra on the domain $\mathcal{D}$. Let $\hat{\tau}$ denote the quotient topology on $\mathcal{A}(\mathcal{D})$ of $\mathcal{L}^+(\mathcal{D})$ $[\tau_{\mathcal{D}}]$. Obviously, $\mathcal{A}(\mathcal{D})$ $[\hat{\tau}]$ is a topological *-algebra. If $\mathcal{D} = \mathcal{H}$, then $\mathcal{A}(\mathcal{D}) = \mathcal{A}(\mathcal{H})$ is the usual Calkin algebra on the Hilbert space $\mathcal{H}$. It should be mentioned that if $\mathcal{D}[t]$ is a Montel space, then $\mathcal{C}(\mathcal{D}) = \mathcal{L}^+(\mathcal{D})$ and hence the Calkin algebra $\mathcal{A}(\mathcal{D})$ is trivial.

In his classical paper [3] Calkin constructed a class of faithful isometric *-representations of the C*-algebra $\mathcal{A}(\mathcal{H})$ (see [11] for a modern treatment). In this paper we investigate the corresponding problem for the Calkin algebra $\mathcal{A}(\mathcal{D})$ on the Frechet domain $\mathcal{D}[t]$: Does there exist a faithful *-representation π of $\mathcal{A}(\mathcal{D})$ which is a homeomorphism of $\mathcal{A}(\mathcal{D})$ $[\hat{\tau}]$ onto $\pi(\mathcal{A}(\mathcal{D}))$ $[\tau_{\mathcal{D}(\pi)}]$? For the domain $l_2 \otimes d$, d the space of all finite complex sequences, this problem has been considered in [9]. Note that $l_2 \otimes d[t]$ is not a Frechet space.

Let us briefly describe our main results concerning the above question.

Given a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we define in Section 2 a $*$-representation $\pi_{\mathcal{U}}$ of $\mathcal{A}(\mathcal{D})$ in a similar way as in the case $\mathcal{D} = \mathcal{H}$. We show that $\pi_{\mathcal{U}}$ is faithful and that $\pi_{\mathcal{U}}^{-1}$ is continuous (Theorem 2.1). Let τ_n denote the finest locally convex topology on $\mathcal{L}^+(\mathcal{D})$ for which the positive cone $\mathcal{L}^+(\mathcal{D})_+$ is normal [12]. If $\tau_n = \tau_{\mathcal{D}}$ on $\mathcal{L}^+(\mathcal{D})$, then each $*$-representation $\pi_{\mathcal{U}}$ is continuous and hence a homeomorphism (Theorem 2.2).

In Section 3 we obtain a converse of the latter in some sense. Suppose that the graph topology t on $\mathcal{D}$ is generated by a sequence of strongly commuting self-adjoint operators whose restrictions to $\mathcal{D}$ are in $\mathcal{L}^+(\mathcal{D})$. Under this additional assumption we prove that if $\tau_n \neq \tau_{\mathcal{D}}$ on $\mathcal{L}^+(\mathcal{D})$, then there is no continuous faithful $*$-representation of $\mathcal{A}(\mathcal{D})$ $[\hat{\tau}]$ (Theorem 3.1).

1. Preliminaries

In this section we collect some definitions and notations (see e.g. [8, 10]) needed later and we prove some preliminary lemmas.

1.1 Let $\mathcal{D}$ be a dense linear subspace of a complex Hilbert space $\mathcal{H}$ and let $\mathcal{L}^+(\mathcal{D}) :=\{a \in \text{End}\, \mathcal{D} : \mathcal{D} \subseteq \mathcal{D}(a^*) \text{ and } a^*\mathcal{D} \subseteq \mathcal{D}\}$. $\mathcal{L}^+(\mathcal{D})$ is a $*$-algebra endowed with the involution $a \to a^+ := a^* \mid \mathcal{D}$. An *Op$*$-algebra* $\mathcal{B}$ on $\mathcal{D}$ is a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$. In what follows we assume that $\mathcal{B}$ is an Op$*$-algebra on $\mathcal{D}$. Define $\underline{\mathcal{D}}(\mathcal{B}) = \bigcap \{\mathcal{D}(\bar{b}) : b \in \mathcal{B}\}$, where $\bar{b}$ is the closure of the operator b. The *graph topology* $t_{\mathcal{B}}$ is the locally convex topology on $\mathcal{D}$ defined by the seminorms $\|\varphi\|_b := \|b\varphi\| + \|\varphi\|$, $b \in \mathcal{B}$. In case $\mathcal{B} = \mathcal{L}^+(\mathcal{D})$ we simply write t for $t_{\mathcal{B}}$.

Let $(\varphi_n : n \in \mathbb{N})$ be a sequence of vectors $\varphi_n \in \mathcal{H}$ and let $\varphi \in \mathcal{H}$. Suppose $\mathcal{U}$ is a filter on $\mathbb{N}$. We write $\varphi = \text{w-lim}\, \varphi_n$ if $\lim_n \langle \varphi_n, \psi\rangle = \langle \varphi, \psi\rangle$ for all $\psi \in \mathcal{H}$ and $\varphi = \text{w-lim}_{\mathcal{U}}\, \varphi_n$ if $\lim_{\mathcal{U}} \langle \varphi_n, \psi\rangle = \langle \varphi, \psi\rangle$ for all $\psi \in \mathcal{H}$.

Lemma 1.1: *Suppose $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$. Let $(\varphi_n : n \in \mathbb{N})$ be a bounded sequence of vectors of $\mathcal{D}[t_{\mathcal{B}}]$. Let $\varphi := \text{w-lim}_{\mathcal{U}}\, \varphi_n$.*

(i) *Then, $\varphi \in \mathcal{D}(\mathcal{B})$ and $\bar{b}\varphi = \text{w-lim}_{\mathcal{U}}\, b\varphi_n$. In particular, if $0 = \text{w-lim}_{\mathcal{U}}\, \varphi_n$, then $0 = \text{w-lim}_{\mathcal{U}}\, b\varphi_n$ for each $b \in \mathcal{B}$.*

(ii) *If $\lim_{\mathcal{U}} \|\varphi_n\| = 0$, then $\lim_{\mathcal{U}} \|b\varphi_n\| = 0$ for each $b \in \mathcal{B}$.*

(iii) *If $\varphi = 0$ and if the set $\{\varphi_n\}$ is relatively compact in $\mathcal{D}[t_{\mathcal{B}}]$, then $\lim_{\mathcal{U}} \|b\varphi_n\| = 0$ for each $b \in \mathcal{B}$.*

Proof: (i) Suppose $b \in \mathcal{B}$. Since the set $\{b\varphi_n\}$ is bounded in the Hilbert space norm and $\mathcal{U}$ is an ultrafilter, $\lim_{\mathcal{U}} \langle b\varphi_n, \cdot\rangle$ is a continuous linear functional on $\mathcal{H}$. Hence there is a $\varphi_b \in \mathcal{H}$ such that $\varphi_b = \text{w-lim}_{\mathcal{U}}\, b\varphi_n$. For $\psi \in \mathcal{D}(b^*)$, this gives

$$\langle \varphi_b, \psi\rangle = \lim_{\mathcal{U}} \langle b\varphi_n, \psi\rangle = \lim_{\mathcal{U}} \langle \varphi_n, b^*\psi\rangle = \langle \varphi, b^*\psi\rangle.$$

Therefore, $\varphi \in \mathcal{D}(b^{**}) \equiv \mathcal{D}(\bar{b})$ and $\varphi_b = b^{**}\varphi \equiv \bar{b}\varphi$. Since $b \in \mathcal{B}$ is arbitrary, $\varphi \in \bigcap \{\mathcal{D}(\bar{b}) : b \in \mathcal{B}\} \equiv \mathcal{D}(\mathcal{B})$.

(ii) Since $\{\varphi_n\}$ is $t_{\mathcal{B}}$-bounded, $C_b := \sup \{\|b^+b\varphi_n\| : n \in \mathbb{N}\} < \infty$ for $b \in \mathcal{B}$. Now the assertion follows from

$$(\lim_{\mathcal{U}} \|b\varphi_n\|)^2 = \lim_{\mathcal{U}} \langle b^+b\varphi_n, \varphi_n\rangle \leqq C_b(\lim_{\mathcal{U}} \|\varphi_n\|) = 0.$$

(iii) Let $b \in \mathcal{B}$. Since $\{\varphi_n\}$ is relatively compact in $\mathcal{D}[t_{\mathcal{B}}]$, the set $\{b\varphi_n\}$ is relatively compact in $\mathcal{H}$. Given $\varepsilon > 0$, there is a finite rank projection F_ε on $\mathcal{H}$ such that $\|(I - F_\varepsilon)\, b\varphi_n\| \leqq \varepsilon$ for $n \in \mathbb{N}$. Since $0 = \text{w-lim}_{\mathcal{U}}\, b\varphi_n$ because of (i) and hence $\lim_{\mathcal{U}} \|F_\varepsilon b\varphi_n\| = 0$, we have $\lim_{\mathcal{U}} \|b\varphi_n\| \leqq \lim_{\mathcal{U}} \|(I - F_\varepsilon)\, b\varphi_n\| \leqq \varepsilon$, thus $\lim_{\mathcal{U}} \|b\varphi_n\| = 0$ ∎

The following corollary is of some interest in itself.

Corollary 1.2: *Suppose $\varphi \in \mathcal{H}$. If there is a bounded sequence $(\varphi_n : n \in \mathsf{N})$ in $\mathcal{D}[t_{\mathcal{B}}]$ such that $\varphi = \text{w-lim}\, \varphi_n$, then $\varphi \in \underline{\mathcal{D}}(\mathcal{B})$.*

Proof: Take an ultrafilter $\mathcal{U}$ on N which contains all sets $\{n \in \mathsf{N} : n \geqq k\}$, $k \in \mathsf{N}$. Then $\varphi = \text{w-lim}_{\mathcal{U}}\, \varphi_n$ and Lemma 1.1 (i) applies ∎

1.2 Next we briefly discuss the topologization of the Op*-algebra $\mathcal{B}$ on $\mathcal{D}$. Let $\mathcal{B}_h := \{b \in \mathcal{B} : b = b^+\}$. Suppose $b_1, b_2 \in \mathcal{B}_h$. We write $b_1 \geqq b_2$ if $\langle b_1\varphi, \varphi\rangle \geqq \langle b_2\varphi, \varphi\rangle$ for all $\varphi \in \mathcal{D}$. Define $\mathcal{B}_+ := \{b \in \mathcal{B} : b \geqq 0\}$ and $[b_1, b_2] := \{b \in \mathcal{B}_h : b_1 \leqq b \leqq b_2\}$. The *uniform topology* $\tau_{\mathcal{D}}$ is the locally convex topology on $\mathcal{B}$ defined by the seminorms

$$p_{\mathfrak{M}}(x) := \sup\, \{|\langle x\varphi, \psi\rangle| : \varphi, \psi \in \mathfrak{M}\},\ \mathfrak{M} \subset \mathcal{D}[t_{\mathcal{B}}] \text{ bounded.}$$

It has been introduced in [8]. We denote by τ_n the finest locally convex topology on $\mathcal{B}$ for which the positive cone $\mathcal{B}_+$ is normal. (All notions and facts concerning ordered vector spaces we need can be found in [12].) Since $\mathcal{B}_+$ is $\tau_{\mathcal{D}}$-normal [13], we have $\tau_{\mathcal{D}} \subseteqq \tau_n$. Let τ_0 denote the finest locally convex topology on $\mathcal{B}$ for which every order interval $[b_1, b_2]$, $b_1, b_2 \in \mathcal{B}_h$, is bounded. Since $\mathcal{B}_+$ is τ_n-normal, all order intervals are τ_n-bounded [12: p. 216] and hence $\tau_n \subseteqq \tau_0$. In [1] the topology τ_0 is called the *ϱ-topology.*

1.3 Let A be a $*$-algebra with unit element denoted by 1. By a *$*$-representation of* A *on* $\mathcal{D}$ we mean a $*$-homomorphism π of A into $\mathcal{L}^+(\mathcal{D})$ satisfying $\pi(1) = I$, where I is the identity map of $\mathcal{D}$. We then write $\mathcal{D}(\pi)$ for $\mathcal{D}$ and t_π for the graph topology of the Op*-algebra $\pi(\mathsf{A})$ on $\mathcal{D}(\pi)$. Suppose π is a $*$-representation of a topological $*$-algebra $\mathsf{A}[\tau]$. π is called *weakly continuous* if for each $\varphi \in \mathcal{D}(\pi)$ the linear functional $\langle \pi(\cdot)\, \varphi, \varphi\rangle$ is continuous on $\mathsf{A}[\tau]$. If π is a continuous mapping of $\mathsf{A}[\tau]$ onto $\pi(\mathsf{A})\, [\tau_{\mathcal{D}(\pi)}]$ we say π is *continuous*.

As above, let $\mathcal{B}$ be an Op*-algebra on $\mathcal{D}$. Let π be a $*$-representation of $\mathcal{B}$ on $\mathcal{D}(\pi)$. We say π is *positive* if $\pi(\mathcal{B}_+) \subseteqq \pi(\mathcal{B})_+$, i.e., if $b \in \mathcal{B}$ and $b \geqq 0$ on $\mathcal{D}$ always implies that $\pi(b) \geqq 0$ on $\mathcal{D}(\pi)$. A linear functional f on $\mathcal{B}$ is called positive if $f(b) \geqq 0$ for all $b \in \mathcal{B}_+$.

Lemma 1.3: *Each positive $*$-representation π of the Op*-algebra $\mathcal{B}$ is a continuous mapping of $\mathcal{B}[\tau_n]$ onto $\pi(\mathcal{B})\, [\tau_{\mathcal{D}(\pi)}]$.*

Proof: By the polarization formula it is easy to see [13] that the uniform topology $\tau_{\mathcal{D}(\pi)}$ on $\pi(\mathcal{B})$ is generated by the family of seminorms

$$p'_{\mathfrak{M}}(\pi(x)) := \sup\, \{|\langle \pi(x)\, \varphi, \varphi\rangle| : \varphi \in \mathfrak{M}\},\ \mathfrak{M} \subset \mathcal{D}(\pi)\, [t_\pi] \text{ bounded.}$$

Fix the bounded set $\mathfrak{M}$. Since the set $\{x \in \mathcal{B} : p'_{\mathfrak{M}}(\pi(x)) \leqq 1\}$ is absolutely convex and $\mathcal{B}_+$-saturated, it is a τ_n-neighborhood of zero in $\mathcal{B}$. This proves the continuity of π ∎

Lemma 1.4: *Suppose that $\mathcal{D}[t]$ is a Frechet space. Let π be a weakly continuous $*$-representation of $\mathcal{L}^+(\mathcal{D})\, [\tau_{\mathcal{D}}]$. Then:*

(i) *π is positive.*
(ii) *If $x \in \mathcal{L}^+(\mathcal{D})$ is bounded, then $\pi(x)$ is bounded on $\mathcal{D}(\pi)$ and $\|\pi(x)\| \leqq \|x\|$.*
(iii) *Suppose $x_n \in \mathcal{L}^+(\mathcal{D})$ for $n \in \mathsf{N}$. If $\{\|\cdot\|_{x_n} : n \in \mathsf{N}\}$ is a generating family for the graph topology t on $\mathcal{D}$, then $\{\|\cdot\|_{\pi(x_n)} : n \in \mathsf{N}\}$ is a generating family of seminorms for the graph topology t_π on $\mathcal{D}(\pi)$.*

Proof: (i) Suppose $x \in \mathcal{L}^+(\mathcal{D})_+$ and $\varphi \in \mathcal{D}(\pi)$. By [6: Theorem 6.1] there is a net $\{q_j\}$ of orthogonal projections $q_j \in \mathcal{L}^+(\mathcal{D})$ (that is, $q_j = q_j^+$ and $q_j = q_j^2$) such that $q_j\mathcal{H} \subseteqq \mathcal{D}$ for all j and $x = \tau_{\mathcal{D}}\text{-lim}\, q_jxq_j$. Let x_j denote the operator q_jxq_j on the Hilbert space $q_j\mathcal{H}$. Since $x \in \mathcal{L}^+(\mathcal{D})$, x_j is closed and hence bounded. Let y_j denote the positive square root of the bounded self-adjoint operator x_j on the Hilbert space $q_j\mathcal{H}$. Then $y_jq_j \in \mathcal{L}^+(\mathcal{D})$ and

$$\|\pi(y_jq_j)\,\varphi\|^2 = \langle\pi(q_jy_j{}^2q_j)\,\varphi, \varphi\rangle = \langle\pi(q_jxq_j)\,\varphi, \varphi\rangle \geqq 0.$$

Since π is weakly continuous, $\langle\pi(x)\,\varphi, \varphi\rangle = \lim \langle\pi(q_jxq_j)\,\varphi, \varphi\rangle \geqq 0$. That is, $\pi(x) \geqq 0$ on $\mathcal{D}(\pi)$.

(ii): First let $x \in \mathcal{L}^+(\mathcal{D})_h$. Since π is positive by (i) and $\pi(I) = I$, $\inf\{\lambda \in \mathsf{R}: -\lambda I \leqq \pi(x) \leqq \lambda I\} \leqq \inf\{\lambda \in \mathsf{R} - \lambda I \leqq x \leqq \lambda I\} = \|x\|$, which implies that $\pi(x)$ is bounded and $\|\pi(x)\| \leqq \|x\|$. For arbitrary $x \in \mathcal{L}^+(\mathcal{D})$ the assertion follows from $\|\pi(x)\|^2 = \|\pi(x^+x)\| \leqq \|x^+x\| = \|x\|^2$.

(iii): Suppose $x \in \mathcal{L}^+(\mathcal{D})$. By assumption, there are a positive constant C and a natural number s such that

$$\|x\varphi\|^2 \leqq C\left(\|\varphi\|^2 + \sum_{n=1}^{s} \|x_n\varphi\|^2\right) \text{ for all } \varphi \in \mathcal{D}.$$

Therefore,

$$y := C\left(I + \sum_{n=1}^{s} x_n{}^+x_n\right) - x^+x \in \mathcal{L}^+(\mathcal{D})_+ \text{ and } \pi(y) \geqq 0 \text{ on } \mathcal{D}(\pi).$$

The latter implies that

$$\|\pi(x)\,\varphi\|^2 \leqq C\left(\|\varphi\|^2 + \sum_{n=1}^{s} \|\pi(x_n)\,\varphi\|^2\right) \text{ for all } \varphi \in \mathcal{D}(\pi) \blacksquare$$

1.4 From now on we assume that $\mathcal{D}[t]$ is a Frechet space and that the underlying Hilbert space $\mathcal{H}$ is separable. To simplify the notation we adopt the following notational convention: We shall denote an operator whose domain contains $\mathcal{D}$ and its restriction to $\mathcal{D}$ by the same symbol. This will be mainly used in Section 3. Let $\mathcal{F}(\mathcal{D})$ denote the finite rank operators contained in $\mathcal{L}^+(\mathcal{D})$. For a linear subspace $\mathcal{D}_1$ of $\mathcal{H}$, let $\mathcal{F}(\mathcal{H}, \mathcal{D}_1)$ be the set of all bounded finite-ranked operators on $\mathcal{H}$ mapping $\mathcal{H}$ into $\mathcal{D}_1$. Moreover, we let $\mathfrak{B}_{\mathcal{D}_1} := \{\varphi \in \mathcal{D}_1 : \|\varphi\| \leqq 1\}$.

2. Generalized Calkin representations of $\mathcal{A}(\mathcal{D})$

2.1 Suppose that $\mathcal{U}$ is an ultrafilter on N. Let $\mathcal{D}_{\mathcal{U}}^\infty$ denote the set of all bounded sequences $(\varphi_n : n \in \mathsf{N}) = (\varphi_n)$ in the locally convex space $\mathcal{D}[t]$ satisfying $0 = \text{w-}\lim_{\mathcal{U}} \varphi_n$. Let $\mathcal{H}_{\mathcal{U}}^\infty$ be the set of all bounded sequences (φ_n) in $\mathcal{H}$ with $0 = \text{w-}\lim_{\mathcal{U}} \varphi_n$. $\mathcal{D}_{\mathcal{U}}^\infty$ and $\mathcal{H}_{\mathcal{U}}^\infty$ are vector spaces in the obvious way. Let $\mathcal{N}_{\mathcal{U}}$ be the set of all $(\varphi_n) \in \mathcal{H}_{\mathcal{U}}^\infty$ with $\lim_{\mathcal{U}} \|\varphi_n\| = 0$. We define a scalar product on the quotient space $\mathcal{D}_{\mathcal{U}} := \mathcal{D}_{\mathcal{U}}^\infty/\mathfrak{D}_{\mathcal{U}}^\infty \cap \mathcal{N}_{\mathcal{U}}$ by $\langle(\varphi_n), (\psi_n)\rangle := \lim_{\mathcal{U}} \langle\varphi_n, \psi_n\rangle$. In the same way, the quotient space $\mathcal{H}_{\mathcal{U}} := \mathcal{H}_{\mathcal{U}}^\infty/\mathcal{N}_{\mathcal{U}}$ becomes a Hilbert space (see e.g. [11: Section 2]). By an abuse of notation we denote the elements of the quotient spaces again by (φ_n). Since $\mathcal{D} \subseteq \mathcal{H}$, $\mathcal{D}_{\mathcal{U}}$ can be considered as a linear subspace of $\mathcal{H}_{\mathcal{U}}$.

Define $\varrho_{\mathcal{U}}(x)\,(\varphi_n) := (x\varphi_n)$ for $(\varphi_n) \in \mathcal{D}_{\mathcal{U}}$ and $x \in \mathcal{L}^+(\mathcal{D})$. Each operator $x \in \mathcal{L}^+(\mathcal{D})$ maps a bounded sequence in $\mathcal{D}[t]$ into a bounded sequence. By Lemma 1.1, (i) and (ii), $x\mathcal{N}_{\mathcal{U}} \subseteq \mathfrak{N}_{\mathcal{U}}$ and $x\mathcal{D}_{\mathcal{U}}^\infty \subseteq \mathcal{D}_{\mathcal{U}}^\infty$. Therefore, the above definition makes sense and defines a linear operator $\varrho_{\mathcal{U}}(x)$ which maps $\mathcal{D}_{\mathcal{U}}$ into $\mathcal{D}_{\mathcal{U}}$. It is straightforward to check that the mapping $x \to \varrho_{\mathcal{U}}(x)$ is a positive $*$-representation of $\mathcal{L}^+(\mathcal{D})$ on $\mathcal{D}_{\mathcal{U}}$.

Let j denote the quotient map of $\mathcal{L}^+(\mathcal{D})$ onto $\mathcal{A}(\mathcal{D}) = \mathcal{L}^+(\mathcal{D})/\mathcal{C}(\mathcal{D})$. Suppose $x \in \mathcal{C}(\mathcal{D})$ and $(\varphi_n) \in \mathcal{D}^\infty$. Then the set $\{x\varphi_n\}$ is relatively compact in $\mathcal{D}(t)$ and hence $\lim_{\mathcal{U}} \|x\varphi_n\| = 0$ by Lemma 1.1 (iii). This shows that $\mathcal{C}(\mathcal{D}) \subseteq \ker \varrho_{\mathcal{U}}$. Therefore, $\pi_{\mathcal{U}}(j(x)) := \varrho_{\mathcal{U}}(x)$ for $x \in \mathcal{L}^+(\mathcal{D})$ defines a $*$-representation of the $*$-algebra $\mathcal{A}(\mathcal{D})$ on $\mathcal{D}_{\mathcal{U}} \equiv \mathcal{D}(\pi_{\mathcal{U}})$.

2.2 Recall that an ultrafilter on N is said to be *free* if the intersection of all its members is empty.

Theorem 2.1: *Suppose that $\mathcal{U}$ is a free ultrafilter on* N. *Then $\pi_{\mathcal{U}}$ is a faithful $*$-representation of the Calkin algebra $\mathcal{A}(\mathcal{D})$. Its inverse $\pi_{\mathcal{U}}^{-1}$ is a continuous mapping of $\pi_{\mathcal{U}}(\mathcal{A}(\mathcal{D}))\left[\tau_{\mathcal{D}_{\mathcal{U}}}\right]$ onto $\mathcal{A}(\mathcal{D})$ $[\hat{t}]$.*

Proof: The quotient topology $\hat{t}$ on $\mathcal{A}(\mathcal{D})$ is generated by the seminorms

$$\hat{p}_{\mathfrak{M}}(\mathfrak{j}(x)) := \inf \{p_{\mathfrak{M}}(x+c) : c \in \mathcal{C}(\mathcal{D})\}, \mathfrak{M} \subset \mathcal{D}[t] \text{ bounded.}$$

Fix such a set $\mathfrak{M}$. Suppose for a moment we have shown that there exists a bounded subset $\mathfrak{N}$ (depending on $\mathfrak{M}$) of $\mathcal{D}_{\mathcal{U}}[t_{\varrho_{\mathcal{U}}}]$ such that

$$\hat{p}_{\mathfrak{M}}(\mathfrak{j}(x)) \leqq p_{\mathfrak{N}}(\varrho_{\mathcal{U}}(x)) \text{ for all } x \in \mathcal{L}^+(\mathcal{D}). \tag{1}$$

The latter means that

$$\hat{p}_{\mathfrak{M}}(a) \leqq p_{\mathfrak{N}}(\pi_{\mathcal{U}}(a)) \text{ for all } a \in \mathcal{A}(\mathcal{D}). \tag{2}$$

Since $\mathcal{C}(\mathcal{D})$ is $\tau_{\mathcal{D}}$-closed in $\mathcal{L}^+(\mathcal{D})$ and hence $\hat{t}$ is Hausdorff, it follows from (2) that $\ker \pi_{\mathcal{U}} = \{0\}$, that is, $\pi_{\mathcal{U}}$ is faithful. Moreover, (2) proves the continuity of $\pi_{\mathcal{U}}^{-1}$, and the proof would be complete.

It remains to show that there is a bounded set $\mathfrak{N}$ in $\mathcal{D}_{\mathcal{U}}[t_{\varrho_{\mathcal{U}}}]$ such that (1) is satisfied. According to [6: Theorem 4.1] there is a bounded self-adjoint operator z on $\mathcal{H}$ such that $\ker z = \{0\}$, $z\mathcal{H} \subseteq \mathcal{D}$ and $\mathfrak{M} \subseteq z\mathfrak{B}_{\mathcal{H}}$. If $x \in \mathcal{L}^+(\mathcal{D})$, then xz is a closed operator defined on $\mathcal{H}$ and hence bounded. Now fix an operator $x \in \mathcal{L}^+(\mathcal{D})$. Since $\mathcal{F}(\mathcal{D})$ is $\tau_{\mathcal{D}}$-dense in $\mathcal{C}(\mathcal{D})$, we obtain

$$\hat{p}_{\mathfrak{M}}(\mathfrak{j}(x)) \leqq \inf_{c \in \mathcal{F}(\mathcal{D})} p_{\mathfrak{M}}(x+c)$$

$$= \inf_{c \in \mathcal{F}(\mathcal{D})} \sup_{\varphi, \psi \in \mathfrak{B}_{\mathcal{H}}} |\langle (x+c)\, z\varphi, z\psi \rangle| = \inf_{c \in \mathcal{F}(\mathcal{D})} \|z(x+c)\, z\|.$$

Since $\ker z = \{0\}$, we have $\{cz : c \in \mathcal{F}(\mathcal{D})\} = \mathcal{F}(\mathcal{D})$. Moreover, $\{zc : c \in \mathcal{F}(\mathcal{D})\}$ is norm dense in $\mathcal{F}(\mathcal{H})$. Using these facts, we get

$$\hat{p}_{\mathfrak{M}}(\mathfrak{j}(x)) \leqq \inf_{c \in \mathcal{F}(\mathcal{D})} \|zxz + zc\|$$

$$= \inf_{c \in \mathcal{F}(\mathcal{H})} \|zxz + c\| = \inf_{c \in \mathcal{C}(\mathcal{H})} \|zxz + c\|. \tag{3}$$

On the other hand, let $\omega_{\mathcal{U}}$ denote the $*$-representation of $\mathsf{B}(\mathcal{H})$ on $\mathcal{H}_{\mathcal{U}}$ defined by $\omega_{\mathcal{U}}(y)\,(\varphi_n) := (y\varphi_n)$ for $(\varphi_n) \in \mathcal{H}_{\mathcal{U}}$ and $y \in \mathsf{B}(\mathcal{H})$. Since $\omega_{\mathcal{U}}$ obviously annihilates $\mathcal{C}(\mathcal{H})$, $\omega_{\mathcal{U}}$ defines a $*$-representation of the Calkin algebra $\mathcal{A}(\mathcal{H})$ on $\mathcal{H}_{\mathcal{U}}$ (see [11: Section 2]). Since $\mathcal{U}$ is assumed to be free and $\mathcal{A}(\mathcal{H})$ is simple, this $*$-representation of the C*-algebra $\mathcal{A}(\mathcal{H})$ is faithful and hence isometric. Since $zxz \in \mathsf{B}(\mathcal{H})$, this yields $\|\omega_{\mathcal{U}}(zxz)\| = \inf \{\|zxz + c\| : c \in \mathcal{C}(\mathcal{H})\}$. By (3), we obtain

$$\hat{p}_{\mathfrak{M}}(\mathfrak{j}(x)) \leqq \|\omega_{\mathcal{U}}(zxz)\| \text{ for all } x \in \mathcal{L}^+(\mathcal{D}). \tag{4}$$

Now define

$$\mathfrak{N} := \omega_{\mathcal{U}}(z)\, \mathfrak{B}_{\mathcal{H}_{\mathcal{U}}} \equiv \{(z\varphi_n) : (\varphi_n) \in \mathcal{H}_{\mathcal{U}} \text{ and } \|(\varphi_n)\|_{\mathcal{H}_{\mathcal{U}}} \leqq 1\}.$$

If $(\varphi_n) \in \mathcal{H}_{\mathcal{U}}$ and if $x \in \mathcal{L}^+(\mathcal{D})$, then xz is bounded on $\mathcal{H}$ and thus

$$\sup_{n \in \mathsf{N}} \|xz\varphi_n\| \leqq \|xz\| \sup_{n \in \mathsf{N}} \|\varphi_n\| < \infty.$$

This implies $\mathfrak{N} \subseteq \mathcal{D}_{\mathcal{U}}$. From

$$\|\varrho_{\mathcal{U}}(x)\,(z\varphi_n)\| = \|(xz\varphi_n)\|_{\mathcal{H}_{\mathcal{U}}} = \lim_{\mathcal{U}} \|xz\varphi_n\|$$

$$\leqq \|xz\| \lim_{\mathcal{U}} \|\varphi_n\| = \|xz\| \text{ for } (\varphi_n) \in \mathfrak{B}_{\mathcal{H}_{\mathcal{U}}} \text{ and } x \in \mathcal{L}^+(\mathcal{D}),$$

we see that $\mathfrak{N}$ is bounded in $\mathcal{D}_{\mathcal{U}}[t_{\varrho_{\mathcal{U}}}]$.

Finally, by (4), if $x \in \mathcal{L}^+(\mathcal{D})$, then

$$p_{\mathfrak{M}}(\mathfrak{j}(x)) \leqq \|\omega_{\mathcal{U}}(zxz)\| = \sup_{\varphi,\psi \in \mathfrak{B}_{\mathcal{H}_{\mathcal{U}}}} |\langle \omega_{\mathcal{U}}(xz)\,\varphi, \omega_{\mathcal{U}}(z)\,\psi\rangle|$$

$$= \sup_{(\varphi_n),(\psi_n) \in \mathfrak{B}_{\mathcal{H}_{\mathcal{U}}}} |\langle (xz\varphi_n), (z\psi_n)\rangle|$$

$$= \sup_{(\varphi_n),(\psi_n) \in \mathfrak{B}_{\mathcal{H}_{\mathcal{U}}}} |\langle \varrho_{\mathcal{U}}(x)\,(z\varphi_n), (z\psi_n)\rangle| = p_{\mathfrak{N}}(\varrho_{\mathcal{U}}(x)),$$

which proves (1). The proof of Theorem 2.1 is complete ∎

2.3 From Theorem 2.1 and Lemma 1.4 we obtain

Theorem 2.2: *Suppose that $\tau_n = \tau_{\mathcal{D}}$ on $\mathcal{L}^+(\mathcal{D})$. Let $\mathcal{U}$ be a free ultrafilter on* N. *Then, $\pi_{\mathcal{U}}$ is a faithful $*$-representation of $\mathcal{A}(\mathcal{D})$ and a homeomorphism of $\mathcal{A}(\mathcal{D})$ $[\hat{\tau}]$ onto $\pi_{\mathcal{U}}(\mathcal{A}(\mathcal{D}))$ $[\tau_{\mathcal{D}_{\mathcal{U}}}]$.*

1. In general the domain $\mathcal{D}_{\mathcal{U}}$ is not dense in $\mathcal{H}_{\mathcal{U}}$. 2. If the domain is of the form $\mathcal{D} = \cap \{\mathcal{D}(T^n) : n \in \mathsf{N}\}$ for some self-adjoint operator T on $\mathcal{H}$, then $\tau_n = \tau_{\mathcal{D}}$ on $\mathcal{L}^+(\mathcal{D})$ (see also Section 3).

3. Existence of continuous faithful $*$-representations of $\mathcal{A}(\mathcal{D})$ $[\hat{\tau}]$

3.1 We first recall the setup of [14: Section 4]. However, the notation is slightly changed.

Suppose a is a (bounded or unbounded) self-adjoint operator on the Hilbert space $\mathcal{H}$ with spectral decomposition $a = \int \lambda de(\lambda)$. Let $(f_k(t) : k \in \mathsf{N})$ be a sequence of real measurable functions on the spectrum $\sigma(a)$ of a. All measure-theoretic notions refer to the spectral measure of a. We assume that

$$f_1(t) = 1 \text{ and } f_k(t) \leqq f_k^2(t) \leqq f_{k+1}(t) \quad \text{a.e. on } \sigma(a) \text{ for } k \in \mathsf{N}. \tag{1}$$

Set $a_k = f_k(a)$ and $\mathcal{D} = \cap \{\mathcal{D}(a_k) : k \in \mathsf{N}\}$. Then, by (1), the operators a_k (more precisely, their restrictions to $\mathcal{D}$) are in $\mathcal{L}^+(\mathcal{D})$ and the graph topology t on $\mathcal{D}$ is generated by the seminorms $\|\cdot\|_{a_k}$, $k \in \mathsf{N}$.

In our next theorem the following condition $(*)$ plays an important role:

$(*)$ For each sequence $\gamma = (\gamma_k : k \in \mathsf{N})$ of positive numbers γ_k there is a $k = k_\gamma \in \mathsf{N}$ such that all functions f_n, $n \in \mathsf{N}$, are bounded on $\mathfrak{N}_k$, where $\mathfrak{N}_n := \{t \in \sigma(a) : f_1(t) \leqq \gamma_1, \ldots, f_n(t) \leqq \gamma_n\}$ for $n \in \mathsf{N}$.

The following assertions are equivalent:

(i) Condition $(*)$ is fulfilled.

(ii) $\tau_0 = \tau_{\mathcal{D}}$ on $\mathcal{L}^+(\mathcal{D})$.

(iii) $\tau_n = \tau_{\mathcal{D}}$ on $\mathcal{L}^+(\mathcal{D})$.

(iv) Each positive linear functional on $\mathcal{L}^+(\mathcal{D})$ is $\tau_{\mathcal{D}}$-continuous.

This is essentially [14: Theorem 4.1]. The equivalence of (i), (ii) and (iv) has been stated therein. Since $\tau_0 \supseteqq \tau_n \supseteqq \tau_{\mathcal{D}}$, (ii) $\Rightarrow$ (iii). Since each positive linear functional is τ_n-continuous, we have (iii) $\Rightarrow$ (iv).

3.2 The following theorem may be considered as a supplement to [14: Theorem 4.1]. Among other things it shows that if $\tau_n \neq \tau_{\mathcal{D}}$ on $\mathcal{L}^+(\mathcal{D})$, then there is no continuous faithful $*$-representation of $\mathcal{A}(\mathcal{D})[\hat{\tau}]$. In particular, the $*$-representations $\pi_{\mathcal{U}}$ occuring in Theorem 2.1 are not continuous.

Theorem 3.1: *Let $\mathcal{D}$ be as above. Then* (i) *is equivalent to each of the following conditions:*

(v) *There exists a faithful $*$-representation π of $\mathcal{A}(\mathcal{D})$ which is a homeomorphism of $\mathcal{A}(\mathcal{D})[\hat{\tau}]$ onto $\pi(\mathcal{A}(\mathcal{D}))[\tau_{\mathcal{D}(\pi)}]$.*

(v)′ *There exists a continuous faithful $*$-representation of $\mathcal{A}(\mathcal{D})[\hat{\tau}]$.*

(vi) *Each positive $*$-representation of $\mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]$ is continuous.*

(vi)′ *Each weakly continuous positive $*$-representation of $\mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]$ is continuous.*

Proof: Theorem 2.2 shows that (iii) $\Rightarrow$ (v). (iii) $\Rightarrow$ (vi) follows from Lemma 1.3. Since (v) $\Rightarrow$ (v)′ and (vi) $\Rightarrow$ (vi)′ are trivially fulfilled, it suffices to prove that (v)′ $\Rightarrow$ (i) and (vi)′ $\Rightarrow$ (i). Both proofs will be indirect (see e.g. the argument in [14: p. 366]).

(v)′ $\Rightarrow$ (i): Suppose that π is a continuous faithful $*$-representation of $\mathcal{A}(\mathcal{D})[\hat{\tau}]$. Then, $\varrho(x) := \pi(\mathfrak{j}(x))$, $x \in \mathcal{L}^+(\mathcal{D})$, defines a continuous $*$-representation of $\mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]$. To prove (i), we assume the contrary, that is, condition $(*)$ is not satisfied. Then there are a positive sequence $\gamma = (\gamma_k)$ and a sequence (i_k) of natural numbers such that f_{i_k} is not essentially bounded on the set $\mathfrak{R}_k$ for each $k \in \mathsf{N}$. There is no loss of generality if we assume that $\gamma_{k+1} > \gamma_k \geqq k$ and $i_k = k$ for all $k \in \mathsf{N}$. Then there are measurable subsets $\mathfrak{S}_{k,n}$, $n \in \mathsf{N}$, of $\mathfrak{R}_k$ of non-zero measure such that $f_{k+1}(t) \geqq \gamma_n$ a.e. on $\mathfrak{S}_{k,n}$ for all $k, n \in \mathsf{N}$. Let $\varphi_{k,n}$ be a unit vector from $e(\mathfrak{S}_{k,n})\mathcal{D}$.

Let Δ denote the family of all sequences $\delta = (\delta_k)$ of natural numbers δ_k satisfying $\delta_k \geqq k + 2$ for $k \in \mathsf{N}$. Fix a $\delta \in \Delta$. We first show that for $r \in \mathsf{N}$ and $\varphi \in \mathcal{D}(\varrho)$

$$\left\|\varrho(a_r)\,\varrho\left(e\left(\bigcup_{k\geqq r+1}\mathfrak{S}_{k,\delta_k}\right)\right)\varphi\right\| \leqq \gamma_r\,\|\varphi\| \tag{2}$$

and

$$\|\varrho(a_{r+1})\,\varrho(e(\mathfrak{S}_{r,\delta_r}))\,\varphi\| \geqq \gamma_{\delta_r}\,\|\varrho(e(\mathfrak{S}_{r,\delta_r}))\,\varphi\|. \tag{3}$$

For let χ denote the characteristic function of the set $\bigcup\{\mathfrak{S}_{k,\delta_k} : k \geqq r + 1\}$. By construction, $f_r(t)\,\chi(t) \leqq \gamma_r$ a.e. on $\sigma(a)$. Define a function g on $\sigma(a)$ by $g := (\gamma_r^2 - f_r^2\chi)^{1/2}$. Obviously, $g(a) \in \mathcal{L}^+(\mathcal{D})$. For $\varphi \in \mathcal{D}(\varrho)$, $\langle\varrho(g(a)^2)\,\varphi, \varphi\rangle = \|\varrho(g(a))\,\varphi\|^2 \geqq 0$ and hence

$$\begin{aligned}\|\varphi\|^2\,\gamma_r^2 &= \langle\varrho(\gamma_r^2 I)\,\varphi, \varphi\rangle\\ &\geqq \langle\varrho(f_r(a)^2\,\chi(a))\,\varphi, \varphi\rangle = \left\|\varrho(a_r)\,\varrho\left(e\left(\bigcup_{k\geqq r+1}\mathfrak{S}_{k,\delta_k}\right)\right)\varphi\right\|^2.\end{aligned}$$

(2) follows by the same argument.

Let q_δ be the orthogonal projection onto the closure of $\mathcal{D}_\delta := \text{l.h.}\,\{\varphi_{k,\delta_k} : k \in \mathsf{N}\}$. Next we prove that $q_\delta\mathcal{H} \subseteqq \mathcal{D}$. For let $r \in \mathsf{N}$. Each $\varphi \in \mathcal{D}_\delta$ can be written as a finite sum

$$\sum_{k=1}^{s} \lambda_k\varphi_{k,\delta_k}, \text{ where } \lambda_1, \ldots, \lambda_s \in \mathsf{C} \text{ and } s \in \mathsf{N},\ s > r.$$

Suppose $k, n \in \mathsf{N}$, $n > k$. Since $f_{k+1}(t) \geqq \gamma_{\delta_k} \geqq \gamma_{k+2} > \gamma_{k+1}$ a.e. on $\mathfrak{S}_{k,\delta_k}$ and $f_{k+1}(t) \leqq \gamma_{k+1}$ on $\mathfrak{S}_{n,\delta_n}$, it follows that $\mathfrak{S}_{k,\delta_k} \cap \mathfrak{S}_{n,\delta_n}$ has measure zero. Therefore, φ_{k,δ_k}

$\perp \varphi_{n,\delta_n}$ and $a_r\varphi_{k,\delta_k} \perp a_r\varphi_{n,\delta_n}$. Using the latter, we obtain

$$\|a_r\varphi\|^2 = \sum_{k=1}^{r} |\lambda_k|^2 \|a_r\varphi_{k,\delta_k}\|^2 + \sum_{k=r+1}^{s} |\lambda_k|^2 \|a_r\varphi_{k,\delta_k}\|^2$$

$$\leqq \max(\|a_r\varphi_{1,\delta_1}\|, \ldots, \|a_r\varphi_{r,\delta_r}\|, \gamma_r) \sum_{k=1}^{s} |\lambda_k|^2 = \max(\ldots) \|\varphi\|^2.$$

This implies $q_\varrho\mathcal{H} \subseteqq \mathcal{D}(a_r)$. Since $\mathcal{D} = \bigcap \{\mathcal{D}(a_r) : r \in \mathbb{N}\}$ by definition, this shows that $q_\varrho\mathcal{H} \subseteqq \mathcal{D}$.

We define $\mathfrak{N} := \bigcup \{\varrho(q_\delta)\,\mathfrak{B} : \delta \in \Delta\}$, where $\mathfrak{B} := \mathfrak{B}_{\mathcal{D}(\varrho)} \equiv \{\varphi \in \mathcal{D}(\varrho) : \|\varphi\| \leqq 1\}$. We prove that $\mathfrak{N}$ is bounded in $\mathcal{D}(\varrho)\,[t_\varrho]$. For take $r \in \mathfrak{N}$ and $\delta \in \Delta$. Let $c_{r,\delta}$ denote the orthogonal projection on $\mathcal{H}$ with range l.h. $\{\varphi_{1,\delta_1}, \ldots, \varphi_{r,\delta_r}\}$. Since obviously $a_rc_{r,\delta} \in \mathcal{C}(\mathcal{D})$, we have $a_rc_{r,\delta} \in \ker \varrho$. From

$$q_\delta - c_{r,\delta} = e\Big(\bigcup_{k \geqq r+1} \mathfrak{J}_{k,\delta_k}\Big)(q_\delta - e_{r,\delta})$$

and (2) we therefore obtain

$$\|\varrho(a_r)\,\varrho(q_\delta)\,\varphi\| = \|\varrho(a_r)\,\varrho(q_\delta - c_{r,\delta})\,\varphi\|$$

$$= \Big\|\varrho(a_r)\,\varrho\Big(e\Big(\bigcup_{k \geqq r+1} \mathfrak{J}_{k,\delta_k}\Big)\Big)\,\varrho(q_\delta - c_{r,\delta})\,\varphi\Big\|$$

$$\leqq \gamma_r \|\varrho(q_\delta - c_{r,\delta})\,\varphi\| \leqq \gamma_r \text{ for each } \varphi \in \mathfrak{B}.$$

By Lemma 1.4 (iii) the graph topology t_ϱ on $\mathcal{D}(\varrho)$ is generated by the seminorms $\|\cdot\|_{\varrho(a_r)}$, $r \in \mathbb{N}$. Therefore, the preceding proof shows that $\mathfrak{N}$ is bounded with respect to the graph topology t_ϱ.

Since the $*$-representation ϱ of $\mathcal{L}^+(\mathcal{D})\,[\tau_{\mathcal{D}}]$ is continuous, there exists a bounded subset $\mathfrak{M}$ of $\mathcal{D}(t)$ such that

$$p_{\mathfrak{N}}(\varrho(x)) \leqq p_{\mathfrak{M}}(x) \quad \text{for all} \quad x \in \mathcal{L}^+(\mathcal{D}). \tag{4}$$

Since $\mathfrak{M}$ is t-bounded, $C_r := \sup \{\|\varrho(a_r)\,\varphi\| : \varphi \in \mathfrak{N}\} < \infty$ for each $r \in \mathbb{N}$. We choose natural numbers δ_k such that $\delta_k \geqq k + 2$ and $\gamma_{\delta_k} \geqq C_{k+1}\,2^k$ for $k \in \mathbb{N}$. This is possible because $\gamma_n \geqq n$ for $n \in \mathbb{N}$. Define an operator x by $x := e(\cup \{\mathfrak{J}_{k,\delta_k} : k \in \mathbb{N}\})$. Clearly, $x \in \mathcal{L}^+(\mathcal{D})$. Our aim is to show that for this operator x (4) is not true. By (3), we have

$$\gamma_{\delta_r} \|\varrho(e(\mathfrak{J}_{r,\delta_r}))\,\varphi\| \leqq \|\varrho(a_{r+1})\,\varrho(e(\mathfrak{J}_{r,\delta_r}))\,\varphi\|$$

$$\leqq \|\varrho(a_{r+1})\,\varphi\| \leqq C_{r+1} \quad \text{for } r \in \mathbb{N} \text{ and } \varphi \in \mathfrak{N}.$$

That is,

$$\sup_{\varphi \in \mathfrak{M}} \|\varrho(e(\mathfrak{J}_{r,\delta_r}))\,\varphi\| \leqq C_{r+1}\gamma_{\delta_r}^{-1} \quad \text{for} \quad r \in \mathbb{N}.$$

Using this inequality, we obtain

$$p_{\mathfrak{M}}(x) = \sup_{\varphi,\psi \in \mathfrak{M}} \Big|\Big\langle e\Big(\bigcup_k \mathfrak{J}_{k,\delta_k}\Big)\varphi, \psi\Big\rangle\Big|$$

$$\leqq \sup_{\varrho \in \mathfrak{M}} \sum_{k=1}^{\infty} \|e(\mathfrak{J}_{k,\delta_k})\,\varphi\|^2 \leqq \sum_{k=1}^{\infty} C_{k+1}^2\gamma_{\delta_k}^{-2} \leqq \sum_{k=1}^{\infty} 2^{-2k} < 1. \tag{5}$$

Since a_rq_δ is a bounded operator on $\mathcal{H}$ for $r \in \mathbb{N}$ as shown above, the sequence $(\varphi_{k,\delta_k} : k \in \mathbb{N})$ is bounded in $\mathcal{D}(t)$. But the set $\{q_\delta\varphi_{k,\delta_k}\} = \{\varphi_{k,\delta_k}\}$ is certainly not relatively compact in $\mathcal{D}[t]$, since (φ_{k,δ_k}) is an orthonormal sequence in $\mathcal{H}$. This proves that

$q_\delta \notin \mathscr{C}(\mathscr{D}) = \ker \varrho$. Consequently,

$$1 = \sup_{\varphi \in \mathfrak{B}} \|\varrho(q_\delta)\, \varphi\|^2$$

$$= \sup_{\varphi \in \mathfrak{B}} \left| \left\langle \varrho \left(e \left(\bigcup_k \mathfrak{J}_{k,\delta_k} \right) \right) \varrho(q_\delta)\, \varphi, \varrho(q_\delta)\, \varphi \right\rangle \right| \leqq p_{\mathfrak{N}}(\varrho(x)).$$

Comparing (5) and (6) with (4), we obtain the desired contradiction.

(vi)′ → (i): This will be similar as the preceding proof. Again we assume that condition (*) is not fulfilled. We keep the notation introduced above. Let $\mathcal{U}$ be an arbitrary free ultrafilter on N. As already mentioned in Section 2, $\varrho_{\mathcal{U}}$ is a positive *-representation of $\mathscr{L}^+(\mathscr{D})$. It suffices to show that $\varrho_{\mathcal{U}}$ is weakly continuous, but not continuous. Let $\varphi = (\varphi_n) \in \mathscr{D}_{\mathcal{U}}$. By definition of $\mathscr{D}_{\mathcal{U}}$, the set $\mathfrak{M} := \{\varphi_n\}$ is bounded in $\mathscr{D}[t]$. If $x \in \mathscr{L}^+(\mathscr{D})$, then

$$|\langle \varrho_{\mathcal{U}}(x)\, \varphi, \varphi \rangle| = |\lim_{\mathcal{U}} \langle x\varphi_n, \varphi_n \rangle| \leqq \sup_{n \in \mathbf{N}} |\langle x\varphi_n, \varphi_n \rangle| \leqq p_{\mathfrak{M}}(x).$$

That is, $\varrho_{\mathcal{U}}$ is weakly continuous. From Theorem 2.1 we know that $\ker \varrho_{\mathcal{U}} = \mathscr{C}(\mathscr{D})$. Therefore, the preceding proof in the case $\varrho = \varrho_{\mathcal{U}}$ shows that $\varrho_{\mathcal{U}}$ is not $\tau_{\mathscr{D}}$-continuous ∎

Results similar to those proved in this paper are true for the topologies $\tau^{(\mathscr{D})}$ and τ^0 (see also [14]).

Addendum: After completing the manuscript the author has learned that in the case $\mathscr{D} = \cap \{\mathscr{D}(T^n) : n \in \mathbf{N}\}$, T a self-adjoint operator, the existence of a topological realization of $\mathscr{A}(\mathscr{D})\,[\hat{\tau}]$ has been independently obtained by F. Löffler and W. Timmermann in "The Calkin representation for a certain class of algebras of unbounded operators", Dubna-Preprint E 5-84-807, 1984.

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