Zeitschrift für Analysis
und ihre Anwendungen
Bd. 5 (6) 1986, S. 534–553

Necessary Conditions for the Uniform Convergence and Abel-Summability of Eigenfunction Expansions with Irregular Ordinary Differential Bundles

G. Freiling

Wir betrachten nicht-selbstadjungierte Eigenwertprobleme

$$\left.\begin{aligned} l(y, \lambda) &= \sum_{j=1}^{n} \left(\sum_{k=0}^{j} p_{kj}(x)\, y^{(k)} \right) \lambda^{n-j} + \lambda^n y = 0 \\ U_\nu(y) &= 0 \ (1 \leqq \nu \leqq n) \end{aligned}\right\}$$

mit irregulären Zweipunkt-Randbedingungen — das bedeutet, daß die Greensche Funktion des Problems auf jedem Strahl in der komplexen λ-Ebene für $|\lambda| \to \infty$ exponentiell wachsen kann. Unter Verwendung asymptotischer Abschätzungen für die Eigenwerte $\lambda_{j\nu}$ und die Eigenfunktionen $\varphi_{j\nu}$ des Problems beweisen wir notwendige Kriterien für die gleichmäßige Konvergenz und die Abel-Summierbarkeit von Reihen der Form $\sum_{j=1}^{r} \sum_{\nu=1}^{\infty} a_{j\nu} \varphi_{j\nu}(x)$ auf $[x_0, x_1] \subset [0, 1]$.

Рассматриваются несамосопряженные задачи на собственные значения

$$\left.\begin{aligned} l(y, \lambda) &= \sum_{j=1}^{n} \left(\sum_{k=0}^{j} p_{kj}(x)\, y^{(k)} \right) \lambda^{n-j} + \lambda^n y = 0 \\ U_\nu(y) &= 0 \ (1 \leqq \nu \leqq n) \end{aligned}\right\}$$

с нерегулярными двуточечными краевыми условиями — это означает, что функция Грина проблемы может расти экспоненциально при $|\lambda| \to \infty$ на каждом луче комплексной λ — плоскости. Используя асимптотические оценки для собственных значений $\lambda_{j\nu}$ и функций $\varphi_{j\nu}$ проблемы, доказываются необходимые критерии для равномерной сходимости и суммируемости по Абелю рядов вида $\sum_{j=1}^{r} \sum_{\nu=1}^{\infty} a_{j\nu} \varphi_{j\nu}(x)$ на $[x_0, x_1] \subset [0, 1]$.

We consider non-selfadjoint eigenvalue problems

$$\left.\begin{aligned} l(y, \lambda) &= \sum_{j=1}^{n} \left(\sum_{k=0}^{j} p_{kj}(x)\, y^{(k)} \right) \lambda^{n-j} + \lambda^n y = 0 \\ U_\nu(y) &= 0 \qquad (1 \leqq \nu \leqq n) \end{aligned}\right\}$$

with irregular two-point boundary conditions — this means, that the Green function of the problem can grow exponentially for $|\lambda| \to \infty$ on every ray in the complex plane. Using asymptotic estimates for the eigenvalues $\lambda_{j\nu}$ and the eigenfunctions $\varphi_{j\nu}$ of the problem we deduce necessary conditions for the uniform convergence and the Abel-summability of the series $\sum_{j=1}^{r} \sum_{\nu=1}^{\infty} a_{j\nu} \varphi_{j\nu}(x)$ on $[x_0, x_1] \subset [0, 1]$.

1. Introduction

We consider non-selfadjoint eigenvalue problems

$$l(y, \lambda) = \sum_{j=1}^{n} \left(\sum_{k=0}^{j} p_{kj}(x)\, y^{(k)} \right) \lambda^{n-j} + \lambda^n y = 0, \tag{1.1a}$$

$$U_\nu(y) = 0 \qquad (\nu = 1, 2, \ldots, n) \tag{1.1b}$$

with irregular two-point boundary conditions. An eigenvalue problem of this type is called *normal* if there exist three rays, dividing the complex plane into sectors of opening angle $< \pi$, such that the Green function G of (1.1) satisfies on these rays for some $p \geqq 1 - n$ and $c > 0$ the estimate $|G(x, \xi, \lambda)| \leqq c\, |\lambda|^p$. If this estimate is satisfied in the entire complex plane, with the exception of circles of radius $\varepsilon > 0$ with centers at the eigenvalues of the problem (1.1), and if $p \geqq 1 - n$ is the smallest integer for which this estimate holds, then the problem (1.1) is said to be *almost-regular of order p*. An almost-regular problem of order $1 - n$ is said to be *regular*. Eigenvalue problems which are not almost-regular are said to be *irregular*.

Regular eigenvalue problems have already been discussed by TAMARKIN [13], who proved that the expansion of a given integrable function f into a series of eigenfunctions and associated functions of problem (1.1) behaves in an equiconvergent way to the trigonometric Fourier-expansion of f and that every function satisfying the boundary conditions (1.1b) can be expanded into a uniformly convergent series. Subsequent regular, almost-regular and normal eigenvalue problems have been investigated by many authors, recent results on this topic can be found in [6, 10, 12, 14]. The expansion problem for irregular eigenvalue problems has only been discussed in special cases of (1.1). WARD [15], EBERHARD [2, 3], KHROMOV [7, 8], BERGMANN [1] and WOLTER [16] proved that the eigenfunction expansions of irregular eigenvalue problems behave like power series and that only a very small class of functions can be expanded into a uniformly convergent series of eigen- and associated functions of irregular eigenvalue problems.

In a preceding paper the author [4] has proved a theorem on the completeness of the system of eigen- and associated functions of irregular differential bundles in $L_2[0, 1]$. In this paper we will give necessary conditions for the uniform convergence and the Abel-summability of series in eigenfunctions of irregular and non-normal differential bundles of type (1.1).

2. Assumptions and notations

Let the coefficients p_{kj} in (1.1a) satisfy the following *assumptions*:

(A₁) $p_{kj} \in C[0, 1]$ $(2 \leqq j \leqq n, 0 \leqq k \leqq j - 2)$, $p_{j-1,j}$ and p_{jj} are constant for $1 \leqq j \leqq n$ with $p_{nn} \neq 0$.

(A₂) The characteristic equation

$$\sum_{j=1}^{n} p_{jj}\omega^j + 1 = 0 \tag{2.1}$$

has n simple roots $\omega_1, \ldots, \omega_n$ with $\omega_j = R_j\, e^{i\alpha_j} \neq 0$ $(1 \leqq j \leqq n)$ and $\alpha_j \neq \alpha_k$ $(j \neq k)$.

(A₃) On each straight line in the complex plane there are at most two of the roots ω_j.

Remark 1: (i) In the case of variable coefficients $p_{jj}(x)$ and $p_{j-1,j}(x)$ we would have to impose very strong restrictions on the functions p_{jj} and therefore we omit the discussion of this case; the results of Section 4 are also valid if $p_{j-1,j} \in C^1[0, 1]$ (compare Remark 5).

(ii) The assumption $\alpha_j \neq \alpha_k$ for $j \neq k$ is not needed for the proofs of the results in this paper, but only to ensure that the system of eigen- and associated functions of (1.1) is complete in $L_2[0, 1]$.

The boundary conditions (if necessary after the substitution $x \to 1 - x$) are assumed to be given in the following form:

$$U_\nu(y) = U_{\nu 0}(y) \quad (1 \leqq \nu \leqq m > n - m \geqq 1),$$

$$U_\nu(y) = \mu_\nu U_{\nu 0}(y) + U_{\nu 1}(y) \quad (m + 1 \leqq \nu \leqq n)$$

with

$$\left.\begin{aligned} U_{\nu 0}(y) &= y^{(\varkappa_\nu)}(0) + \sum_{j=0}^{\varkappa_\nu - 1} \alpha_{\nu j} y^{(j)}(0), \\ U_{\nu 1}(y) &= y^{(\eta_\nu)}(1) + \sum_{j=0}^{\eta_\nu - 1} \beta_{\nu j} y^{(j)}(1), \end{aligned}\right\} \tag{2.2}$$

$0 \leqq \varkappa_1 < \varkappa_2 < \cdots < \varkappa_m \leqq n - 1$ and $0 \leqq \varkappa_\nu \leqq n - 1$ for $m + 1 \leqq \nu \leqq n$, $0 \leqq \eta_{m+1} < \eta_{m+2} < \cdots < \eta_n \leqq n - 1$ and with $\alpha_{\nu j}, \beta_{\nu j}, \mu_\nu \in \mathbf{C}$.

Notations: The λ-plane is divided by the lines $\operatorname{Re}(\lambda\omega_i) = \operatorname{Re}(\lambda\omega_j)$, $i \neq j$, into $2h$ sectors

$$S_k = \{\lambda \in \mathbf{C} \mid \gamma_k \leqq \arg \lambda \leqq \gamma_{k+1}\} \qquad (0 \leqq k \leqq 2h - 1)$$

with $0 \leqq \gamma_0 < \gamma_1 < \cdots < \gamma_{2h} = \gamma_0 + 2\pi$. For each $k \in \{0, 1, \ldots, 2h - 1\}$ there exists a permutation πk of $\{1, 2, \ldots, n\}$ with

$$\operatorname{Re}(\lambda\omega_{\pi k(1)}) \leqq \operatorname{Re}(\lambda\omega_{\pi k(2)}) \leqq \cdots \leqq \operatorname{Re}(\lambda\omega_{\pi k(n)}) \text{ for } \lambda \in S_k.$$

The set of all $k \in \{0, 1, \ldots, 2h - 1\}$ with $\operatorname{Re}(e^{i\gamma_k}\omega_{\pi k(m)}) = \operatorname{Re}(e^{i\gamma_k}\omega_{\pi k(m+1)})$ is denoted by $\{k_1, k_2, \ldots, k_r\}$, where $k_i \neq k_j$ for $i \neq j$. For $a_j \in \mathbf{C}$ and with the integers $\varkappa_\nu$ and η_ν introduced in (2.2) we define

$$D_0(a_1, \ldots, a_m) = \begin{vmatrix} a_1^{\varkappa_1} & \cdots & a_m^{\varkappa_1} \\ \vdots & & \vdots \\ a_1^{\varkappa_m} & \cdots & a_m^{\varkappa_m} \end{vmatrix},$$

$$D_1(a_1, \ldots, a_{n-m}) = \begin{vmatrix} a_1^{\eta_{m+1}} & \cdots & a_{n-m}^{\eta_{m+1}} \\ \vdots & & \vdots \\ a_1^{\eta_n} & \cdots & a_{n-m}^{\eta_n} \end{vmatrix},$$

$$D_2(a_1, \ldots, a_{n-m-1}) = \begin{vmatrix} a_1^{\eta_{m+1}} & \cdots & a_{n-m-1}^{\eta_{m+1}} \\ \vdots & & \vdots \\ a_1^{\eta_{n-1}} & \cdots & a_{n-m-1}^{\eta_{n-1}} \end{vmatrix}.$$

Using these notations we require for every $k \in \{k_1, \ldots, k_r\}$:

$$\text{(i)} \quad \operatorname{Re}(e^{i\gamma_k}\omega_{\pi k(m)}) > 0, \tag{2.3}$$

$$\left.\begin{aligned} \text{(ii)}\ D_{01k} &= D_0(\omega_{\pi k(1)}, \ldots, \omega_{\pi k(m)}) \neq 0, \\ D_{02k} &= D_0(\omega_{\pi k(1)}, \ldots, \omega_{\pi k(m-1)}, \omega_{\pi k(m+1)}) \neq 0, \end{aligned}\right\} \tag{2.4}$$

$$D_{11k} = D_1(\omega_{\pi k(m+1)}, \dots, \omega_{\pi k(n)}) \neq 0,$$
$$D_{12k} = D_1(\omega_{\pi k(m)}, \omega_{\pi k(m+2)}, \dots, \omega_{\pi k(n)}) \neq 0, \tag{2.4}$$
$$D_{21k} = D_2(\omega_{\pi k(m+2)}, \dots, \omega_{\pi k(n)}) \neq 0.$$

Remark 2: (i) Requirement (2.3) implies that the eigenvalue problems considered in this paper are irregular. It is possible to generalize our results stated in Section 4 and to include almost-regular eigenvalue problems (compare Remark 5).

(ii) Requirement (2.4) implies that the leading coefficients (with the highest powers of λ) in the main terms of the characteristic determinant and of the eigenfunctions do not vanish. If the coefficients p_{kj} are smooth enough and if we use more precise estimates of fundamental systems of solutions of (1.1a) (instead of Lemma 3.1), we can weaken requirement (2.4) in an obvious way.

(iii) Assumption (A_3) can be replaced by the following weaker assumption

$(A_3)^*$ $\mathrm{Re}\,(e^{i\gamma_k}\omega_{\pi k(m-1)}) < \mathrm{Re}\,(e^{i\gamma_k}\,\omega_{\pi k(m)}) < \mathrm{Re}\,(e^{i\gamma_k}\,\omega_{\pi k(m+2)})$
for every $k \in \{k_1, \dots, k_r\}$.

In this case [4: Satz 2.5] cannot be applied.

3. Auxiliary results

Here and in the sequel we use the notation $[A] = A + O(1/\lambda)$. Using the assumptions (A_1)—(A_3), we get the following estimates for a fundamental system of solutions of (1.1a).

Lemma 3.1 [13: p. 3]: *For every sector S_k, $0 \leqq k \leqq 2h - 1$, there exists a fundamental system $y_{k1}, \dots, y_{kn}$ of solutions of equation (1.1a) satisfying the estimates ($0 \leqq s$ $n - 1$)*

$$\left(\frac{\partial}{\partial x}\right)^s y_{k\nu}(x, \lambda) = (\lambda\omega_\nu)^s\, e^{\lambda\omega_\nu x}[\chi_\nu(x)] \tag{3.1}$$

for $(x, \lambda) \in [0, 1] \times S_k$. The functions $y_{k\nu}$ are analytic in λ for $\lambda \in S_k$ with $|\lambda|$ sufficiently large and the functions χ_ν can be represented in the form $\chi_\nu(x) = \tilde{c}_\nu\, e^{z_\nu x}$ with $\tilde{c}_\nu, z_\nu \in \mathbf{C} \setminus \{0\}$. Without loss of generality we assume $\chi_\nu(0) = 1$ and set $\sigma_\nu := \chi_\nu(1) \neq 0$ for $1 \leqq \nu \leqq n$.

Lemma 3.2: *Almost all eigenvalues of problem (1.1) are simple. The eigenvalues can be splitted into r sequences $(\lambda_{j\nu})_{\nu\in\mathbf{N}}$, $1 \leqq j \leqq r$, satisfying the estimates*

$$\lambda_{j\nu} = \frac{1}{\omega_{\pi k_j(m)} - \omega_{\pi k_j(m+1)}}\left\{2\pi(\nu + c_j)\, i + \log \theta_{k_j} + O\left(\frac{1}{\nu}\right)\right\},$$

where

$$\theta_\mu = \frac{D_{01\mu}D_{11\mu}}{D_{02\mu}D_{12\mu}}\,\frac{\sigma_{\pi\mu(m+1)}}{\sigma_{\pi\mu(m)}} \quad \textit{and} \quad c_j \in \mathbf{Z} \quad (1 \leqq j \leqq r).$$

Proof: Using (3.1) we get for $0 \leqq k \leqq 2h - 1$

$$U_\nu(y_{k\mu}) = [\lambda\omega_\mu]^{\varkappa_\nu} \quad (1 \leqq \nu \leqq m,\ 1 \leqq \mu \leqq n)$$

and

$$U_\nu(y_{kj}) = \begin{cases} [\lambda\omega_j]^{\eta_\nu}\,\sigma_j\, e^{\lambda\omega_j} & \text{if } \mathrm{Re}\,(\lambda\omega_j) > 0\\ \mu_\nu[\lambda\omega_j)]^{\varkappa_\nu} + [\lambda\omega_j]^{\eta_\nu}\,\sigma_j\, e^{\lambda\omega_j}\, j & \text{if } \mathrm{Re}\,(\lambda\omega_j) \leqq 0\end{cases} \quad (m + 1 \leqq \nu \leqq n).$$

Substituting this into the characteristic determinant, we obtain for $\lambda \in S_k$

$$\Delta_k(\lambda) = \begin{vmatrix} U_1(y_{\pi k(1)}) & \dots & U_1(y_{\pi k(n)}) \\ \vdots & & \vdots \\ U_n(y_{\pi k(1)}) & \dots & U_n(y_{\pi k(n)}) \end{vmatrix} = \Delta_{0k}(\lambda)\, \Delta_{1k}(\lambda) \tag{3.2}$$

with

$$\Delta_{0k}(\lambda) = \lambda^{\varkappa_1+\dots+\varkappa_m+\eta_{m+1}+\dots+\eta_n} \exp\{\lambda(\omega_{\pi k(m+2)} + \dots + \omega_{\pi k(n)}\} \prod_{j=m+2}^{n} \sigma_{\pi k(j)}$$

and

$$\Delta_{1k}(\lambda) = D_{01k} D_{11k}\, e^{\lambda\omega_{\pi k(m+1)}}[\sigma_{\pi k(m+1)}] - D_{02k} D_{12k}\, e^{\lambda\omega_{\pi k(m)}}[\sigma_{\pi k(m)}]. \tag{3.3}$$

For $0 < \varepsilon < \varepsilon_0 = 2^{-1} \min\{\gamma_{k+1} - \gamma_k \mid 0 \leqq k \leqq 2h - 1\}$ we set $T_{k,\varepsilon} = \{\lambda \in \mathbf{C} \mid \gamma_k - \varepsilon \leqq \arg\lambda \leqq \gamma_k + \varepsilon\}$. Since

$$\operatorname{Re}(\lambda\omega_{\pi k(m+1)}) > \operatorname{Re}(\lambda\omega_{\pi k(m)}) \quad \text{for} \quad \lambda \in S_k \setminus \bigcup_{j=1}^{r} T_{k_j,0},$$

we get from (3.2) and (3.3)

$$\Delta_k(\lambda) = \Delta_{0k}(\lambda)\, D_{01k} D_{11k}\, e^{\lambda\omega_{\pi k(m+1)}}\, [\sigma_{\pi k(m+1)}] \neq 0$$

for $\lambda \in S_k \setminus (T_{k_1,\varepsilon} \cup \dots \cup T_{k_r,\varepsilon})$ with $|\lambda| > C(k, \varepsilon)$. Hence, for $0 < \varepsilon \leqq \varepsilon_0$ almost all eigenvalues are lying in $T_{k_1,\varepsilon} \cup \dots \cup T_{k_r,\varepsilon}$. From

$$\omega_{\pi k_j-1(m)} = \omega_{\pi k_j(m+1)} \quad \text{and} \quad \omega_{\pi k_j-1(m+1)} = \omega_{\pi k_j(m)}$$

we obtain

$$D_{01k_j} = D_{02k_j-1},\ D_{02k_j} = D_{01k_j-1},\ D_{11k_j} = D_{12k_j-1},\ D_{12k_j} = D_{11k_j-1},$$

and this implies that both $\Delta_{1k_j}(\lambda)$ and $\Delta_{1k_j-1}(\lambda)$ have the form

$$\pm D_{01k_j} D_{11k_j}\, e^{\lambda\omega_{\pi k_j(m+1)}} \sigma_{\pi k_j(m+1)} \times \left([1] - \frac{[1]}{\theta_{k_j}} \exp\{\lambda(\omega_{\pi k_j(m)} - \omega_{\pi k_j(m+1)})\right).$$

Using these estimates and the method described in [11: §4.9], we get the assertion of Lemma 3.2. ∎

For $1 \leqq j \leqq r$ and $\nu \geqq \nu_0$ all eigenvalues $\lambda_{j\nu}$ are simple and we have $\lambda_{j\nu} \in S_k$, where $k \in \{k_j - 1, k_j\}$. Consequently,

$$\varphi_{j\nu}(x) = \begin{vmatrix} U_1(y_{\pi k(1)}) & \dots & U_1(y_{\pi k(n)}) \\ \vdots & & \vdots \\ U_{n-1}(y_{\pi k(1)}) & \dots & U_{n-1}(y_{\pi k(n)}) \\ y_{\pi k(1)}(x, \lambda) & \dots & y_{\pi k(n)}(x, \lambda) \end{vmatrix}_{\lambda=\lambda_{j\nu}}$$

defines an eigenfunction corresponding to the eigenvalue $\lambda_{j\nu}$.

Lemma 3.3: *There exist $K_{j\nu} \in \mathbf{C} \setminus \{0\}$ such that the eigenfunctions $\varphi_{j\nu}$ satisfy for each pair $a, b \in (0, 1)$ the estimates*

$$\varphi_{j\nu}^{(\mu)}(x) = K_{j\nu}\, e^{\lambda_{j\nu}\omega_{\pi k_j(m+1)}x} \chi_{\pi k_j(m+1)}(x)\, (\lambda_{j\nu}\omega_{\pi k_j(m+1)})^\mu \left(D_{01k_j} + O\left(\frac{1}{\nu}\right)\right)$$

$$- K_{j\nu}' e^{\lambda_{j\nu}\omega_{\pi k_j(m)}x} \chi_{\pi k_j(m)}(x)\, (\lambda_{j\nu}\omega_{\pi k_j(m)})^\mu \left(D_{02k_j} + O\left(\frac{1}{\nu}\right)\right)$$

for $1 \leqq j \leqq r$, $\nu \geqq \nu_0$, $0 \leqq \mu \leqq n-1$ *and uniformly for* $x \in [a, b]$. *Without loss of generality we assume henceforth* $K_{j\nu} = 1$ *for* $1 \leqq j \leqq r$, $\nu \in \mathbf{N}$.

Proof: On account of $\lambda_{j\nu} \in S_{k_j-1} \cup S_{k_j}$ for $\nu \geqq \nu_0$, we get similarly to the proof of Lemma 3.2 for $x \in [a, b]$

$$\varphi_{j\nu}(x) = \pm \Delta_{0k_j}(\lambda_{j\nu})\, D_{21k_j} \left\{ e^{\lambda_{j\nu}\omega_{\pi k_j(m+1)}x} \chi_{\pi k_j(m+1)}(x) \left(D_{01k_j} + O\left(\frac{1}{\lambda_{j\nu}}\right)\right) \right.$$
$$\left. - e^{\lambda_{j\nu}\omega_{\pi k_j(m)}x} \chi_{\pi k_j(m)}(x) \left(D_{02k_j} + O\left(\frac{1}{\lambda_{j\nu}}\right)\right)\right\}.$$

Since $O(1/\lambda_{j\nu}) = O(1/\nu)$, the desired estimate is valid for $\mu = 0$; the estimates for the derivatives are proved in an analoguous way, using Lemma 3.1 ▮

For the proof of the main results in Section 4 we have to estimate $|\varphi_{j\nu}(x)|$.

Lemma 3.4: *Let* $0 < \varepsilon < x_1 \leqq 1$. *For* $1 \leqq j \leqq r$ *and* $\nu \geqq \nu_0(\varepsilon)$ *there exist* $x_{j\nu} \in (x_1 - \varepsilon, x_1)$ *and* $K_0 > 0$ *such that*

$$|\varphi_{j\nu}^{(\mu)}(x_{j\nu})| \geqq K_0 \exp\{\mathrm{Re}\,(\lambda_{j\nu}\omega_{\pi k_j(m+1)})\, x_{j\nu}\}\, |\lambda_{j\nu}\omega_{\pi k_j(m+1)}|^\mu$$

for $1 \leqq j \leqq r$, $\nu \geqq \nu_0(\varepsilon)$ *and* $0 \leqq \mu \leqq n-1$.

Proof: We set

$$h_{j\nu\mu}(x) = 1 - \frac{D_{02k_j}}{D_{01k_j}} \frac{\chi_{\pi k_j(m)}(x)}{\chi_{\pi k_j(m+1)}(x)} \left(\frac{\omega_{\pi k_j(m)}}{\omega_{\pi k_j(m+1)}}\right)^\mu$$
$$\times \exp\{\lambda_{j\nu}(\omega_{\pi k_j(m)} - \omega_{\pi k_j(m+1)})\, x\}.$$

Since $\chi_\nu(x) = e^{z_\nu x}$ and

$$\lambda_{j\nu}(\omega_{\pi k_j(m)} - \omega_{\pi k_j(m+1)}) = 2\pi(\nu + c_j)\, i + \log \theta_{k_j} + O\left(\frac{1}{\nu}\right),$$

there exist $K_1 > 0$ and $x_{j\nu} \in (x_1 - \varepsilon, x_1)$ such that

$$|h_{j\nu\mu}(x_{j\nu})| \geqq K_1 \quad \text{for} \quad 1 \leqq j \leqq r,\ \nu \geqq \nu_0(\varepsilon) \quad \text{and} \quad 0 \leqq \mu \leqq n-1.$$

Combining this result with the estimate from Lemma 3.3, we get the desired result ▮

Remark 3: If requirements (2.3) and (2.4) were only fulfilled for $k \in I_0 \subset \{k_1, \ldots, k_r\}$, then the estimate of Lemma 3.4 would only be valid for $j \in I_0$.

From Lemma 3.4 we easily deduce

Lemma 3.5: *There exists a constant* $K_2 > 0$ *with*

$$|\varphi_{j\nu}^{(\mu)}(x)| \leqq K_2\, |\lambda_{j\nu}|^\mu \exp\{\mathrm{Re}\,(\lambda_{j\nu}\omega_{\pi k_j(m+1)})\, x$$

for $1 \leqq j \leqq r$, $\nu \geqq \nu_1$, $0 \leqq \mu \leqq n-1$ *and* $x \in [0, 1]$.

For the formulation of the results in Section 4 we need some additional notations. For $f = (f_1, \ldots, f_n)^T$ and $1 \leqq \nu \leqq n$ we set $(f)_\nu = f_\nu$ and define $U_\nu(f)$ by $U_\nu(f) = \big(U_\nu(f_1), \ldots, U_\nu(f_n)\big)^T$. Further we set

$$Y_\lambda = (y, \lambda y, \ldots, \lambda^{n-1} y)^T, \qquad D = \frac{d}{dx}, \qquad A_{n-j} = -\sum_{k=0}^{j} p_{kj}(x)\, D^k$$

and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ A_0 & A_1 & A_2 & \cdots & A_{n-1} \end{pmatrix}.$$

Then (1.1) is equivalent to the system $AY_\lambda = \lambda Y_\lambda$.

Definition: The vector-function $f = (f_1, \ldots, f_n)^T$ is called *A-analytic* on the interval $I \subset [0, 1]$ if $D^j A^q f(x)$ is defined for $x \in I$, $0 \leqq j \leqq n - 1$ and $q \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and if for every interval $[\alpha, \beta] \subset I$ there exists a constant $K = K(\alpha, \beta, f, A)$ such that

$$|D^j(A^q f)_\nu (x)| \leqq K^{q+j}(q + j)!$$

for $x \in [\alpha, \beta]$, $0 \leqq j \leqq n - 1$, $1 \leqq \nu \leqq n$ and $q \in \mathbf{N}_0$.

Example: Let $\varphi_{j\nu}$ be the eigenfunction corresponding to $\lambda_{j\nu}$ and $\Phi_{j\nu} := (\varphi_{j\nu}, \lambda_{j\nu}\varphi_{j\nu}, \ldots, \lambda_{j\nu}^{n-1}\varphi_{j\nu})^T$, then we have $D^\mu A^q \Phi_{j\nu} = \lambda_{j\nu}^q \Phi_{j\nu}^{(\mu)}$ for $0 \leqq \mu \leqq n - 1$ and $q \in \mathbf{N}_0$. Therefore, with Lemma 3.5, we can see that $\Phi_{j\nu}$ is A-analytic on the interval $[0, 1]$.

4. Main results

In this section we will prove necessary conditions for the uniform convergence and for the Abel-summability of series in eigenfunctions of (1.1). Since almost all eigenvalues are simple, we may assume without loss of generality that all eigenvalues are simple — otherwise we would have to include a finite set of associated functions.

For $a_{j\nu} \in \mathbf{C}$ and $J = \{1, \ldots, r\} \times \mathbf{N}$ we consider the series

$$\sum_{(j,\nu)\in J} a_{j\nu}\varphi_{j\nu}(x), \tag{4.1}$$

where the summation is performed in an arbitrary but fixed order.

Theorem 4.1: *If* (4.1) *converges uniformly on* $[x_0, x_1]$ *with* $0 \leqq x_0 < x_1 \leqq 1$, *then*

(i) *the series*

$$f_{\alpha\mu}(x) = \sum_{(j,\nu)\in J} a_{j\nu}\lambda_{j\nu}^\alpha D^\mu \varphi_{j\nu}(x) \qquad (0 \leqq \mu \leqq n - 1, \alpha \in \mathbf{N}_0)$$

are absolutely and uniformly convergent in any closed interval $[0, \beta] \subset [0, x_1)$;

(ii) $D^\mu f_{\alpha 0} = f_{\alpha\mu}$ *for* $0 \leqq \mu \leqq n - 1$ *and* $\alpha \in \mathbf{N}_0$;

(iii) $F_\tau := (f_{\tau 0}, f_{\tau+1,0}, \ldots, f_{\tau+n-1,0})^T$, $\tau \in \mathbf{N}_0$, *is A-analytic on the interval* $[0, x_1)$, *and*

$$U_s(A^\alpha F_\tau) = (0, \ldots, 0)^T \quad \textit{for} \quad 1 \leqq s \leqq m \textit{ and } \alpha, \tau \in \mathbf{N}_0.$$

Proof: On account of the uniform convergence of the series (4.1) on $[x_0, x_1]$ there exists a constant $K_3 > 0$ with

$$|a_{j\nu}\varphi_{j\nu}(x)| \leqq K_3 \quad \text{for} \quad x \in [x_0, x_1] \quad \text{and} \quad (j, \nu) \in J.$$

For fixed $\beta \in (0, x_1)$ we set $\varepsilon = 3^{-1}(x_1 - \beta)$. Applying Lemma 3.4, we obtain

$$|a_{j\nu}| \leq \frac{K_3}{K_0} \exp\{-\mathrm{Re}\,(\lambda_{j\nu}\omega_{\pi k_j(m+1)})x_{j\nu}\}$$
$$\leq \frac{K_3}{K_0} \exp\{-\mathrm{Re}\,(\lambda_{j\nu}\omega_{\pi k_j(m+1)})(\beta + 2\varepsilon)\}$$

for $1 \leq j \leq r$ and ν sufficiently large. Then, using Lemmata 3.2 and 3.5, we get from requirement (2.3) that (i) and (ii) are valid.

Since

$$AF_\tau = \sum_{(j,\nu)\in J} \begin{pmatrix} a_{j\nu}\lambda_{j\nu}^{\tau+1}\varphi_{j\nu} \\ a_{j\nu}\lambda_{j\nu}^{\tau+2}\varphi_{j\nu} \\ \vdots \\ a_{j\nu}\lambda_{j\nu}^{\tau+n-1}\varphi_{j\nu} \\ \sum_{m=0}^{n-1} a_{j\nu}\lambda_{j\nu}^{\tau+m} A_m\varphi_{j\nu} \end{pmatrix} = \lambda_{j\nu}F_\tau = F_{\tau+1},$$

we see that $D^\mu A^q F_\tau = D^\mu F_{\tau+q} = F_{\tau+q}^{(\mu)}$ is defined for $\tau \in \mathrm{N}_0$, $0 \leq \mu \leq n-1$ and $q \in \mathrm{N}_0$. Consequently,

$$U_s(A^\alpha F_\tau) = \sum_{(j,\nu)\in J} a_{j\nu}\lambda_{j\nu}^{\tau+\alpha}\big(U_s(\varphi_{j\nu}), \lambda_{j\nu}U_s(\varphi_{j\nu}), \ldots, \lambda_{j\nu}^{n-1}U_s(\varphi_{j\nu})\big)^{\mathrm{T}} = (0, \ldots, 0)^{\mathrm{T}}$$

for $1 \leq s \leq m$ and $\alpha, \tau \in \mathrm{N}_0$.

Finally, we have to estimate the components $f_{\tau+k,\mu}(x)$, $0 \leq k \leq n-1$, of $D^\mu F_\tau(x)$ for $x \in [0, \beta] \subset [0, x_1)$. With $\varepsilon = 3^{-1}(x_1 - \beta) > 0$ we get, as with $|a_{j\nu}|$,

$$|f_{\tau+k,\mu}(x)| \leq \sum_{(j,\nu)\in J} |\lambda_{j\nu}|^{\tau+k}\, |a_{j\nu}\varphi_{j\nu}^{(\mu)}|$$
$$\leq \sum_{(j,\nu)\in J} |\lambda_{j\nu}|^{\tau+k+\mu} \frac{KK_2}{K_0} \exp\{-\mathrm{Re}\,(\lambda_{j\nu}\omega_{\pi k_j(m+1)})\, 2\varepsilon\}$$
$$\leq K_4 \sum_{(j,\nu)\in J} |\lambda_{j\nu}|^{\tau+\mu} \exp\{-\mathrm{Re}\,(\lambda_{j\nu}\omega_{\pi k_j(m+1)})\, \varepsilon\}$$
$$\leq K_5(\tau+\mu)!\left(\frac{1}{\delta(\varepsilon)}\right)^{\tau+\mu},$$

where $K_4, K_5 > 0$ and $\delta(\varepsilon) = 2^{-1}\varepsilon \min\{\mathrm{Re}\, e^{i\gamma k_j}\omega_{\pi k_j(m+1)} \mid 1 \leq j \leq r\}$. The last estimate follows from Lemma 3.2 in the same way as the proof of [8: Theorem 2] ∎

Definition: $\sum_{(j,\nu)\in J} a_{j\nu}\varphi_{j\nu}$ is *uniformly Abel-summable to order* 1 *on the interval* $I \subset [0, 1]$ *and has the limit* f if for every $t > 0$ the series

$$u(x, t) = \sum_{(j,\nu)\in J} a_{j\nu} \exp\{i\lambda_{j\nu}(\omega_{\pi k_j(m)} - \omega_{\pi k_j(m+1)})\, t\}\, \varphi_{j\nu}(x)$$

is uniformly convergent for $x \in I$ and if $\lim_{t\to 0} u(x, t) = f(x)$ uniformly for $x \in I$.

Remark 4: In the case of regular eigenvalue problems it can be shown that the expansions of sufficiently smooth functions into series of eigenfunctions of such problems is Abel-summable to order 1 [5]. The following theorem shows that this result cannot be valid in the case of non-normal eigenvalue problems of type (1.1). In the case of irregular eigenvalue problems, results have up till now only been published on the Abel-summability to order $\alpha > 1$ (compare [9]).

Theorem 4.2: *If the series* (4.1) *is uniformly Abel-summable on the interval* $[x_0, x_1]$, $0 \leqq x_0 < x_1 \leqq 1$, *with the limit* $f_{00}(x)$, *then the assertions* (i)—(iii) *of Theorem* 4.1 *are valid.*

Proof: Because of Theorem 4.1 it is sufficient to show that for every $\beta \in (0, x_1)$ the series (4.1) is uniformly convergent on $[0, \beta]$ and has the limit $f_{00}(x)$. Since (4.1) is uniformly Abel-summable on $[x_0, x_1]$, there exists for ever $t > 0$ a constant $C_t > 0$ such that

$$|a_{j\nu} \exp \{i\lambda_{j\nu}(\omega_{\pi k_j(m)} - \omega_{\pi k_j(m+1)})\, t\}\, \varphi_{j\nu}(x)| \leqq C_t$$

for $(j, \nu) \in J$ and $x \in [x_0, x_1]$. Therefore we obtain with Lemma 3.4 for $t > 0$ and $(j, \nu) \in J$

$$\begin{aligned} |a_{j\nu}| &\leqq \frac{C_t}{K_2} \exp \left\{-\operatorname{Re}\,(\lambda_{j\nu}\omega_{\pi k_j(m+1)})\, x_{j\nu} - \operatorname{Re}\,\{i\lambda_{j\nu}(\omega_{\pi k_j(m)} - \omega_{\pi k_j(m+1)})\}\, t\right\} \\ &= \frac{C_t}{K_2} \exp \left\{-\operatorname{Re}\,(\lambda_{j\nu}\omega_{\pi k_j(m+1)})\, x_{j\nu} + 2\pi(\nu + c_j)\left(1 + O\left(\frac{1}{\nu}\right)\right) t,\right. \end{aligned}$$

where $x_{j\nu} \in (x_1 - \varepsilon, x_1)$ and $\varepsilon = 3^{-1}(x_1 - \beta)$. If we choose $d > 0$ and $\delta > 0$ so that

$$d \operatorname{Re}\,(\lambda_{j\nu}\omega_{\pi k_j(m+1)}) > 2\pi(\nu + c_j)\,(1 + \delta) \quad \text{for} \quad 1 \leqq j \leqq r,\ \nu \geqq \nu_0,$$

then we get with $t_0 = \varepsilon/d$

$$\begin{aligned} |a_{j\nu}| &\leqq \frac{C_{t_0}}{K_2} \exp \left\{-\operatorname{Re}\,(\lambda_{j\nu}\omega_{\pi k_j(m+1)})\,(\beta + 2\varepsilon) + 2\pi(\nu + c_j)\,\frac{1}{d}\,\varepsilon\left(1 + O\left(\frac{1}{\nu}\right)\right)\right. \\ &\leqq \frac{C_{t_0}}{K_2} \exp \{-\operatorname{Re}\,(\lambda_{j\nu}\omega_{\pi k_j(m+1)})\,(\beta + \varepsilon)\} \qquad \text{for } 1 \leqq j \leqq r,\ \nu \geqq \nu_0. \end{aligned}$$

Hence, applying Lemma 3.5, we conclude that the series (4.1) is absolutely and uniformly convergent on the interval $[0, \beta]$.

Further we have

$$\begin{aligned} R_n(\beta) &= \sup_{x \in [0,\beta]} |f_{00}(x) - \sum_{j=1}^{r} \sum_{\nu=1}^{n} a_{j\nu}\varphi_{j\nu}(x)| \\ &\leqq \sup_{x \in [0,\beta]} |f_{00}(x) - u(x, t)| + \sup_{x \in [0,\beta]} |u(x, t) - \sigma_n(x, t)| \\ &\quad + \sup_{x \in [0,\beta]} |\sigma_n(x, t) - \sum_{j=1}^{r} \sum_{\nu=1}^{n} a_{j\nu}\varphi_{j\nu}(x)| \\ &=: A(\beta, t) + B_n(\beta, t) + C_n(\beta, t), \end{aligned}$$

where

$$\sigma_n(x, t) = \sum_{\substack{(j,\nu) \in J \\ 1 \leqq \nu \leqq n}} a_{j\nu} \exp \{i\lambda_{j\nu}(\omega_{\pi k_j(m)} - \omega_{\pi k_j(m+1)})\, t\}\, \varphi_{j\nu}(x).$$

With $\varepsilon_1 > 0$, $n \geqq N(\varepsilon_1)$, $x \in [0, \beta]$ and $t > 0$ we have

$$\begin{aligned} |u(x, t) - \sigma_n(x, t)| &\leqq \sum_{j=1}^{r} \sum_{\nu=n+1}^{\infty} |a_{j\nu}\, e^{i\lambda_{j\nu}(\omega_{\pi k_j(m+1)})t}\, \varphi_{j\nu}(x)| \\ &\leqq \sum_{j=1}^{r} \sum_{\nu=n+1}^{\infty} |a_{j\nu}\varphi_{j\nu}(x)| < \frac{\varepsilon_1}{3}. \end{aligned}$$

Thus, for $t < t_1(\varepsilon_1)$, $n \geqq N(\varepsilon_1)$ and $\beta \in (0, x_1)$

$$A(\beta, t) + B_n(\beta, t) + C_n(\beta, t) \leqq \frac{\varepsilon_1}{3} + \frac{\varepsilon_1}{3} + \frac{\varepsilon_1}{3} = \varepsilon_1$$

and

$$\lim_{n\to\infty} \sum_{(j,\nu)\in J} a_{j\nu}\varphi_{j\nu}(x) = f_{00}(x) \quad \text{for} \quad x \in [0, \beta] \blacksquare$$

Remark 5: The results of this paper can be generalized in the following way:

(i) Let requirements (2.3) and (2.4) be only fulfilled for $k \in I_0 \subset \{k_1, \ldots, k_r\}$; in this case we consider, instead of (4.1), series of the form

$$\sum_{(j,\nu)\in I_0\times\mathbf{N}} a_{j\nu}\varphi_{j\nu}(x),$$

and we can prove like in Section 4 that the assertions of Theorems 4.1 and 4.2 remain valid, if J is replaced by $I_0 \times \mathbf{N}$.

(ii) If the boundary conditions depend polynomially on λ and if these boundary conditions can be normalized so that their main part is equivalent to the boundary conditions (1.1b), we can prove for this type of problem results analogous to those obtained in Sections 3 and 4.

(iii) We have assumed for simplicity that $p_{j-1,j}$ $(1 \leqq j \leqq n)$ is constant. If we assume $p_{j-1,j} \in C^1[0, 1]$, then the functions χ_ν in formula (3.1) are exponential functions, depending continuously on the coefficients p_{jj} and $p_{j-1,j}$ (compare [13: p. 4]); in this case the statements of Theorems 4.1 and 4.2 are also valid, since the formula $\chi_\nu(x) = e^{z_\nu x}$ has only been used in the proof of Lemma 3.4 and since this proof can be carried over to the more general case.

REFERENCES

[1] Bergmann, J.: Nichtreguläre Randwertaufgaben. Beitr. z. Anal. 7 (1975), 71–85.

[2] Eberhard, W.: Die Entwicklungen nach Eigenfunktionen irregulärer Eigenwertprobleme mit zerfallenden Randbedingungen. Math. Z. 86 (1964), 205–214.

[3] Eberhard, W.: Die Entwicklungen nach Eigenfunktionen irregulärer Eigenwertprobleme mit zerfallenden Randbedingungen II. Math. Z. 90 (1965), 126–137.

[4] Freiling, G.: Zur Vollständigkeit des Systems der Eigenfunktionen und Hauptfunktionen irregulärer Operatorbüschel. Math. Z. 188 (1984), 55–68.

[5] Gasymov, M. G., and M. G. Dzavadov: Multiple completeness of a part of the set of eigenfunctions and adjoint functions of differential operator bundles. Soviet Math. Dokl. 13 (1972), 521–524. Transl. from Dokl. Akad. Nauk SSSR 203 (1972), 1235–1237.

[6] Heisecke, G.: Rand-Eigenwertprobleme $N(y) = \lambda P(y)$ bei λ-abhängigen Randbedingungen. Mitt. Math. Sem. Gießen 145 (1980), 1–74.

[7] Khromov, A. P.: Differentialoperator with irregular splitting boundary conditions. Math. Notes 19 (1976), 451–456. Transl. from Matem. Zametki 19 (1976), 763–772.

[8] Хромов, А. П.: Разложение по собственным функциям обыкновенных линейных дифференциальных операторов с нерегулярными распадающимися краевыми условиями. Мат. сб. 170 (1966), 310–329.

[9] Kostyuchenko, A. G., and A. A. Shkalikov: Summability of eigenfunction expansions of differential operators and convolution operators. Funct. Anal. 12 (1978), 262–276. Transl. from Funkc. Analiz i Ego Prilozh. 12 (1978) 4, 24–40.

[10] Mennicken, R.: Spectral theory for certain operator polynomials. In: Functional Analysis, Holomorphy an Approximation Theory II (Ed.: G. I. Zapata). Elsevier Science Publishers 1984, 203–243.

[11] Naimark, M. A.: Linear differential operators, Part I. New York: Ungar 1967.

[12] Shkalikov, A. A.: Boundary-value problems for ordinary differential equations with a parameter in the boundary conditions. Funct. Anal. 16 (1982), 324–326. Transl. from Funkc. Analiz i Ego Prilozh. 16 (1982) 4, 92–93.

[13] TAMARKIN, J.: Some general problems of the theory of ordinary linear differential equations and expansion of an arbitrary function in series of fundamental functions. Math. Z. 27 (1927), 1—54.

[14] VAGABOV, A. I.: Multiple summability in eigenfunctions and associated functions for irregular differential pencils. Soviet Mat. Dokl. 21 (1980), 917—921. Transl. from Dokl. Akad. Nauk SSSR 252 (1980), 1300—1303.

[15] WARD, L. E.: A third order irregular boundary value problem and the associated series. Trans. Amer. Math. Soc. 34 (1932), 417—434.

[16] WOLTER, M.: Über die Entwicklungen nach Eigenfunktionen N-irregulärer Eigenwertprobleme mit zerfallenden Randbedingungen. Math. Z. 178 (1981), 99—113.

o

Manuskripteingang: 30. 10. 1984

VERFASSER:

Prof. Dr. GERHARD FREILING
Fachbereich Mathematik
der Universität-Gesamthochschule Duisburg
D-4100 Duisburg, PF 101629

Buchbesprechung

H. KURKE, J. MECKE, H. TRIEBEL and R. THIELE (eds.): **Recent Trends in Mathematics, Reinhardsbrunn 1982.** Proc. Conf. Reinhardsbrunn (GDR), October 11—13, 1982 (Teubner-Texte zur Mathematik: Vol. 50). Leipzig: BSB B. G. Teubner Verlagsgesellschaft 1983.

Die seit Jahrzehnten stattfindende rapide Entwicklung der Mathematik — vor allem ihrer verschiedensten Techniken — ruft oft Ratlosigkeit und ein Gefühl von Verlorenheit hervor. Deswegen sind Wegweiser und Orientierungsmittel jeder Art von großem Nutzen. Inzwischen sind regelmäßige Serien entstanden, die diesem Ziele dienen sollen ...

Die zu besprechende Publikation enthält "Invited Lectures" verschiedenster Art.

I.

1. Es gibt übliche kurze Berichte und Abhandlungen von L. BĂDESCU, S. V. BOCHKARIEV, Z. CIESIELSKI, G. CASPER und W. TREBELS, K.-P. HADELER und K. DIETZ, C. PROCESI (*Computing with Matrices*), Š. SCHWABIK, K. UENO, T. F. BANCHOFF (*Computer Graphics in Geometric Research*) und S. PRÖSSDORF.

2. Berichte über den Stand der Forschung auf speziellen (engen) Gebieten:

a) W. DICKMEIS und R. J. NESSEL: *Quantitative Banach-Steinhaus Theorems — a Survey*, in dem auch sehr interessant auf die Vorgeschichte der Theorie eingegangen wird;

b) M. COSTABEL, E. STEPHAN und W. L. WENDLAND: *Zur Randintegralmethode für das erste Fundamentalproblem der ebenen Elastizitätstheorie auf Polygongebieten;*

c) V. HUBER-DYSON: *Decision Problems in Group Theory;*

d) D. G. SIMADER: *Selbstadjungiertheitsprobleme und spektraltheoretische Fragen für Schrödinger-Operatoren;*

e) H. WALLIN: *Interpolation by Rational Functions with Free Poles in the Complex Plane;*

f) H. G. FEICHTINGER: *Banach Spaces of Distributions Defined by Decomposition Methods and Some of Their Applications.*