Zeitschrift für Analysis
und ihre Anwendungen
Bd. 5 (6) 1986, S. 507–531

The Green Matrix for Strongly Elliptic Systems of Second Order with Continuous Coefficients

M. Fuchs

Wir untersuchen eine Verallgemeinerung des bekannten Konzepts der Green-Funktion für elliptische Gleichungen auf den Fall elliptischer Systeme zweiter Ordnung mit stetigen Koeffizienten. Wir beweisen Existenz und Eindeutigkeit einer solchen Green-Matrix G und diskutieren mit potentialtheoretischen Methoden das Wachstumsverhalten in der Nähe der singulären Diagonalen. Als eine mögliche Anwendung betrachten wir Systeme mit vektorwertigen Maßen als rechter Seite und leiten Darstellungsformeln für die Lösung ab, aus denen sich unter geeigneten Voraussetzungen neue Informationen gewinnen lassen.

Исследуется обобщение известной концепции функции Грина для эллиптических уравнений на случай эллиптических систем второго порядка с непрерывными коэффициентами. Доказываются существование и единственность такой матрицы Грина и теоретико-потенциальными методами обсуждается ее тип роста вблизи сингулярной диагонали. Как возможное применение рассматриваются системы с правой частью в виде вектор-значной меры и выводятся формулы представления для решения, из которых при подходящих предположениях можно получить дополнительную информацию.

We study a generalization of the Green function for elliptic equations to elliptic systems of second order with continuous coefficients. The existence and uniqueness of such a Green matrix as well as various estimates concerning the growth properties near the singular diagonal are proved. Moreover, one can derive representation formulas for solutions of elliptic systems and deduce from these further information about the solution even in the case when the right-hand side of the system is a vector-valued measure of bounded variation.

0. Introduction

In this paper we are concerned with the Green matrix for uniformly elliptic systems of the type

$$L_{ij}u^j :\doteq -D_\alpha(A^{ij}_{\alpha\beta}D_\beta u^j), \qquad i = 1, \ldots, N, \tag{0.1}$$

on a bounded domain $\Omega \subset \mathbf{R}^n$, $n \geqq 3$. We assume that the coefficients are continuous functions on $\bar{\Omega}$.

For a single elliptic operator ($N = 1$) the existence and the properties of a Green function have been completely analysed in a recent paper of Grüter and Widman [9], where also various applications are treated. Their main result is: There always exists a unique Green function g which satisfies

$$c\,|x - y|^{2-n} \leqq g(x, y) \leqq C\,|x - y|^{2-n} \tag{0.2}$$

with positive constants c, C. In the case of elliptic systems ($N > 1$) very little is known. By means of Fourier transforms it is easy to show that for operators (0.1) with constant coefficients there exists at least a fundamental matrix which is homogeneous of degree $2 - n$ (compare [13, 15, 17]). Apart from this, only the case of C^∞-coefficients is treated: John [13], for example, shows the existence of a local fundamental solution and his proof uses the smoothness of the coefficients in a very essential way so that his method cannot be applied to operators with continuous coefficients. Global constructions of abstract Green operators G associated with a boundary value problem can be found in the book of Hörmander [12], where one also finds the remark that G is related to a "kernel function", but the properties of the kernel are not examined in detail.

One could ask if it is worthwhile considering the case of continuous coefficients since we only deal with linear elliptic systems. This question has a very natural answer in the setting of nonlinear problems: Let $u:\mathring{\Omega} \to \mathbf{R}^N$ be a minimum of the functional

$$F(v) := \int_\Omega a^{ij}_{\alpha\beta}(.\,, v)\, D_\alpha v^i D_\beta v^j \, dx$$

with coefficients in $C^\infty(\Omega \times \mathbf{R}^N)$ satisfying the strong ellipticity condition. The Euler operator associated with F has the form (0.1) with $A(x) := a\big(x, u(x)\big)$. Since u is a minimum point of the functional F (defined on the Sobolev space $H^{1,2}$) the regularity theory for variational integrals implies $u \in C^{0,\alpha}$ at least on a great portion of Ω, compare [7], so that we arrive at an elliptic system with continuous coefficients. Despite of this fact it would be desirable to prove existence of Green's matrix for systems with bounded measurable coefficients. But the experience in elliptic regularity theory shows that the existence of a Green function (with estimates) is equivalent to regularity theorems for weak solutions to the homogeneous equation. Since such regularity results fail to hold in the vector-valued case (compare the counter examples in [7]) we have to restrict ourselves to continuous coefficients.

In Section 1 of our paper we collect known regularity results for weak solutions of the system

$$L_{ij}u^j = -D_\alpha F_\alpha^i \text{ on } \Omega, \qquad i = 1, \ldots, N, F \in L^p(\Omega)^{nN}. \tag{0.3}$$

The basic statement is: For $p > n$ the unique $\mathring{H}^{1,2}(\Omega)^N$-solution of (0.3) is continuous up to the boundary. Moreover, Section 1 contains various local estimates in L^p for weak solutions of (0.3) which will be useful later. Using the global regularity theorem, we solve in Section 2 the boundary value problem

$$L_{ij}u^j = \mu^i \text{ on } \Omega, \qquad i = 1, \ldots, N, u_{|\partial\Omega} = 0, \tag{0.4}$$

where μ is a vector-valued signed Radon measure of bounded total variation. Here we follow an idea of Littman, Stampacchia and Weinberger [14] which can roughly be described as "duality method". By choosing special measures μ we conclude existence (and uniqueness) of a Green matrix G to the operator $(L_{ij})_{1 \leq i,j \leq N}$ on the domain Ω. Simple properties of G such as certain symmetry relations and continuity on $\Omega \times \Omega \setminus \{(x, x) : x \in \Omega\}$ are investigated in Section 3. Here we make use of the precise local results summarized in Section 1.

Finally we show that the solution u to problem (0.4) can be written as the convolution $u = G * \mu$. In Section 4 we discuss the growth properties of G near the singular diagonal. Assuming a Hölder condition for the coefficients we show by a perturbation argument that G satisfies the standard estimate (compare (0.2))

$$|G(x, y)| \leq C\, |x - y|^{2-n} \tag{0.5}$$

at least locally in the interior of Ω, i.e. on small balls compactly contained in Ω. By Campanato type arguments we infer from (0.5) the local gradient bound $|\partial G(x, y)/\partial x| \leq C\, |x - y|^{1-n}$. In the final chapter we give two applications of Green's matrix: First we describe the behaviour of a weak solution u to the homogeneous system with zero boundary values having an isolated singularity at $y \in \Omega$, where u grows of order less than $|x - y|^{1-n}$. Then either u vanishes or grows exactly of order $|x - y|^{2-n}$. As a second application we show that the weak solution of (0.4) has certain regularity properties if the measure μ satisfies the condition

$$\sup_{y \in \Omega} \int_\Omega |x - y|^{2-n} \, d|\mu| < \infty.$$

Here we use the estimates from Section 4.

Notations: We make the following *general assumptions* (**GA**):

I. Ω is a bounded Lipschitz domain in $\mathbf{R}^n$, $n \geqq 3$.

II. Let $N \in \mathbf{N}$ be fixed and consider functions $A_{\alpha\beta}^{ij} \in L^\infty(\Omega)$; $i, j = 1, \ldots, N$; $\alpha, \beta = 1, \ldots, n$. We define

$$(L_{ij}) = \left(-D_\alpha(A_{\alpha\beta}^{ij} D_\beta)\right) \quad \text{and} \quad (L_{ij}^t) = \left(-D_\alpha(A_{\beta\alpha}^{ji} D_\beta)\right).$$

III. We assume that there are numbers $\lambda, \Lambda > 0$ such that

a) $\max \{\|A_{\alpha\beta}^{ij}\|_{L^\infty} : i, j = 1, \ldots, N; \alpha, \beta = 1, \ldots, n\} \leqq \Lambda$,

b) $A_{\alpha\beta}^{ij}(x) \, P_\alpha^i P_\beta^j \geqq \lambda \, |P|^2$ for all $x \in \Omega$, $P \in \mathbf{R}^{nN}$ (strong ellipticity).

Obviously III b) implies the weaker Legendre Hadamard condition

IV. $A_{\alpha\beta}^{ij}(x) \, w^i w^j \eta_\alpha \eta_\beta \geqq \lambda \, |w|^2 \, |\eta|^2$ for $x \in \Omega$, $\eta \in \mathbf{R}^n$, $w \in \mathbf{R}^n$.

Here and in the sequel we use *summation convention*: Greek (Latin) indices repeated twice are summed from 1 to $n(N)$. If D is an open subset of Ω, we denote by $L^p_{(\mathrm{loc})}(D)^M$, $H^{k,p}_{(\mathrm{loc})}(D)^M$, $\mathring{H}^{k,p}(D)^M$ the standard Lebesgue and Sobolev spaces of measurable functions $D \to \mathbf{R}^M$, which we norm in the usual way [1]. For balls $B = B_r(x_0)$ and functions $u \in H^{k,p}(B)^M$ we introduce the weighted norm

$$\|u\|_{H^{k,p}(B)} = \sum_{i=0}^{k} r^{-i} \, \|\nabla^{k-i} u\|_{L^p(B)}.$$

If $f \in L^1_{\mathrm{loc}}(\Omega)^N$, $F \in L^1_{\mathrm{loc}}(\Omega)^{nN}$, we call $u \in H^{1,1}_{\mathrm{loc}}(\Omega)^N$ *a weak solution of the system* $L_{ij} u^j = f^i - D_\alpha F^i$, $i = 1, \ldots, N$, on Ω if

$$\int_\Omega A_{\alpha\beta}^{ij} D_\alpha \Phi^i D_\beta u^j \, dx = \int_\Omega (\Phi^i f^i + F_\alpha^i D_\alpha \Phi^i) \, dx$$

for all $\Phi \in C_0^\infty(\Omega)^N$. For the adjoint operator we have obvious modifications. In the sequel we will denote all constants by the symbol C and it will be clear from the context on which parameters C depends.

1. Regularity results for linear elliptic systems

The purpose of this section is twofold: We firstly collect well-known global regularity results (compare [2, 15, 17]) for linear elliptic systems which play an essential role in proving the existence and uniqueness of a Green matrix G to the system under consideration. Secondly we establish local L^p-estimates for weak solutions from which we derive various properties of G.

We look at weak solutions $u : \Omega \to \mathbf{R}^N$ of the system

$$L_{ij} u^j = f^i - D_\alpha F_\alpha^i, \qquad i = 1, \ldots, N, \tag{1.1}$$

under the conditions

$$(\mathbf{GA}), \; A_{\alpha\beta}^{ij} \in C^0(\bar{\Omega}), \qquad i, j = 1, \ldots, N; \alpha, \beta = 1, \ldots, n. \tag{1.2}$$

With the exception of Corollary 2 to Theorem 1 all results of this section remain true if we drop the strong ellipticity condition (GA) IIIb).

Theorem 1 (Morrey [15: Thm. 6.4.8]): *For $1 < p, q < \infty$ let $u \in H^{1,p}(\Omega)^N$ be a weak solution of the system* (1.1) *with $f \in L^q(\Omega)^N$, $F \in L^q(\Omega)^{nN}$. Moreover assume that* (1.2) *holds and that $w \in H^{1,q}(\Omega)^N$ satisfies $u - w \in \mathring{H}^{1,p}(\Omega)^N$. Then $u \in H^{1,q}(\Omega)^N$ and there is a constant $C = C(n, N, \lambda, \Lambda, p, q, \omega, \Omega)$ such that*

$$\|u\|_{H^{1,q}(\Omega)} \leqq C(\|w\|_{H^{1,q}(\Omega)} + \|f\|_{L^q(\Omega)} + \|F\|_{L^q(\Omega)} + \|u\|_{L^1(\Omega)}). \tag{1.3}$$

The symbol ω denotes the modulus of continuity of the coefficients. If for example the $A^{ij}_{\alpha\beta}$ satisfy a uniform Hölder condition with exponent δ and Hölder constant L, then ω is determined by L and δ.

Corollary 1: *For any $1 < p \leqq \infty$ the homogeneous system associated with* (1.1) *has only the trivial solution in the space $\mathring{H}^{1,p}(\Omega)^N$, provided condition* (1.2) *is satisfied.*

Here and in Theorem 1 the continuity assumption on the coefficients cannot be dropped since there are counter-examples of Serrin [16] even in the case of a single equation.

Corollary 2: *Suppose that $n < p < \infty$ is given and that $u \in \mathring{H}^{1,2}(\Omega)^N$ is the unique Hilbert space solution to* (1.1) *with* (1.2), *where $f \in L^p(\Omega)^N$, $F \in L^p(\Omega)^{nN}$. Then $u \in C^0(\Omega)^N \cap \mathring{H}^{1,p}(\Omega)^N$ and we have the estimate*

$$\|u\|_{L^\infty(\Omega)}, \|u\|_{H^{1,p}(\Omega)} \leqq C(\|F\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}), \tag{1.4}$$

where the constant C depends on the same parameters as in Theorem 1. *Moreover, u has boundary values zero in the classical sense.*

Next we derive local versions of Theorem 1: Consider an arbitrary point $x_0 \in \Omega$, for simplicity we write 0 instead of x_0 in the sequel. We define for $i, j = 1, \ldots, N$ and $\zeta \in \mathbf{R}^n$

$$L_{0ij} = -A^{ij}_{\alpha\beta}(0)\, D_\alpha D_\beta,\ L_{0ij}(\zeta) = -A^{ij}_{\alpha\beta}(0)\, \zeta_\alpha \zeta_\beta,$$

$$L_0(\zeta) = \det\left(L_{0ij}(\zeta)\right)_{1\leq i,j\leq N},\ L_0^{ji}(\zeta) = \text{cofactor of } L_{0ij}(\zeta).$$

$L_{0ij}(\zeta)$, $L_0(\zeta)$, $L_0^{ji}(\zeta)$ are homogeneous polynomials of degree 2, $2N$, and $2N - 2$ respectively. Finally denote by $L_0(D)$, $L_0^{ji}(D)$ the differential operators associated with the polynomials $L_0(\zeta)$, $L_0^{ji}(\zeta)$. By means of Fourier transforms one easily produces a fundamental solution K to the operator $L_0(D)$ which has the following properties (compare [13: p. 69/70] and [15: p. 216/217]).

Lemma 1.1: *Under the assumptions* (1.2) *K is an analytic function on $\mathbf{R}^n \setminus \{0\}$, essentially homogeneous of degree $2N - n$. For all $\nu \in \mathbf{N}_0^n$, $|\nu| > 2N - n$, the estimate*

$$|D^\nu K(y)| \leqq C(n, N, \lambda, \Lambda, |\nu|)\, |y|^{2N-n-|\nu|}, \qquad y \in \mathbf{R}^n \setminus \{0\}, \tag{1.5}$$

holds. Moreover, K is an even function and satisfies $L_0(D)\, K = \delta_0$ (Dirac measure in 0) in the sense of distributions on $\mathbf{R}^n$.

The following lemma contains all the needed mapping properties of the potential operator related to the kernel K.

Lemma 1.2 [15: Thm. 6.2.1]: *Suppose that* (1.2) *is satisfied.*

a) *We define for $r > 0$, $f: B_r(0) \to \mathbf{R}$ the potential operator $Q_r(f)\,(x) = \int_{B_r(0)} K(x - y)\, f(y)\, dy$, $x \in B_r(0)$.*

(i) $Q_r: L^p(B_r(0)) \to H^{2N,p}(B_r(0))$ *is continuous for* $1 < p < \infty$.
(ii) *For all* $1 < p < \infty$; $f \in L^p(B_r(0))$ *and* $\nu \in \mathbb{N}_0^n$ *with* $|\nu| = 2N - 2$ *we have*

$$^*\|D^\nu (Q_r f)\|_{H^{2,p}(B_r(0))} \leqq C \|f\|_{L^p(B_r(0))}, \qquad C := C(n, N, \lambda, \Lambda, p).$$

(iii) $L_0(D)(Q_r f) = f$ *almost everywhere on* $B_r(0)$.

b) *Set* $E_k^j = L_0^{jk}(D)\, K$. *Then* E *is a fundamental matrix for the constant coefficient operator* $(L_{0ij})_{1 \leq i,j \leq N}$, *i.e.*

$$L_{0ij} E_k^j = \delta_{ik} \delta_0, \qquad i, k = 1, \ldots, N. \tag{1.6}$$

For $y \in \mathbb{R}^n \setminus \{0\}$ *and* $\nu \in \mathbb{N}_0^n$ *we have the estimate*

$$|D^\nu E(y)| \leqq C(n, N, \lambda, \Lambda, |\nu|)\, |y|^{2-n-|\nu|}. \tag{1.7}$$

Now we define potential solutions to the system

$$L_{0ij} u^j = f^i - D_\alpha F_\alpha^i, \qquad i = 1, \ldots, N. \tag{1.1$_0$}$$

Definition: Under the assumptions of the preceding lemma for $f \in L^p(B_r(0))^N$, $F \in L^p(B_r(0))^{nN}$, $1 \leqq p \leqq \infty$, $r > 0$, we define for $k = 1, \ldots, N$ and $x \in B_r(0)$

$$(P_r f)^k(x) := L_0^{kl}(D)\,(Q_r f^l)(x) = \int_{B_r(0)} E_l^k(x - y)\, f^l(y)\, dy,$$

$$(\tilde{P}_r F)^k(x) := L_0^{kl}(D)\,\{D_\alpha(Q_r F_\alpha^l)\}(x) = \int_{B_r(0)} (D_\alpha E_l^k)(x - y)\, F_\alpha^l(y)\, dy.$$

From Lemma 1.2 we conclude

Lemma 1.3: *If condition* (1.2) *is satisfied, then for all* $1 < p < \infty$ *and* $r > 0$ *the following statements hold:*

(i) *The linear operators* $P_r: L^p(B_r(0))^N \to H^{2,p}(B_r(0))^N$, $\tilde{P}_r: L^p(B_r(0))^{nN} \to H^{1,p}(B_r(0))^N$ *are continuous with bounds depending only on* n, N, p, λ *and* Λ, *provided the spaces* $H^{k,p}(B_r(0))^N$ *are normed by* $^*\|\cdot\|_{H^{k,p}}$, $k = 1, 2$.

(ii) *For any* $f \in L^p(B_r(0))^N$ *and* $F \in L^p(B_r(0))^{nN}$ *the function* $U := P_r(f) - \tilde{P}_r(F) \in H^{1,p}(B_r(0))^N$ *is a weak solution of* (1.1)$_0$ *on the ball* $B_r(0)$. *Moreover, if* f *and* F *have compact support in* $B_r(0)$ *and if* u *belongs to the class* $H^{1,1}$ *and is a weak solution of* (1.1)$_0$ *on* $B_r(0)$ *with compact support, then* $u = U$.

Before stating the local regularity theorem for weak solutions of (1.1) we introduce a class of operators which measure the deviation from the constant coefficient case.

Definition: Assume (1.2) and define for $1 < p < \infty$, $0 < r < \operatorname{dist}(0, \partial\Omega)$, $i = 1, \ldots, N$, $\alpha = 1, \ldots, n$ the *perturbation operators* $T_r^p: H^{1,p}(B_r(0))^N \to H^{1,p}(B_r(0))^N$,

$$T_r^p(u) = \tilde{P}_r(F), \quad F_\alpha^i = \left(A_{\alpha\beta}^{ij} - A_{\alpha\beta}^{ij}(0)\right) D_\beta u^j.$$

Introducing the *-norm on $H^{1,p}(B_r(0))^N$, we get from Lemma 1.3

$$^*\|T_r^p\| \leqq C(n, N, p, \lambda, \Lambda) \operatorname*{osc}_{B_r(0)} A, \quad \operatorname*{osc}_{B_r(0)} A := \|A - A(0)\|_{L^\infty(B_r(0))}. \tag{1.8}$$

and moreover

$$v := T_r^p u \text{ is a weak solution to the system}$$

$$L_{0ij} v^j = D_\alpha\left(\left(A_{\alpha\beta}^{ij} - A_{\alpha\beta}^{ij}(0)\right) D_\beta u^j\right), \qquad i = 1, \ldots, N, \text{ on } B_r(0). \tag{1.9}$$

In view of the uniform continuity of the coefficients on $\bar{\Omega}$ we conclude from (1.8) that the $^*\|\cdot\|$-norm of the perturbation operators T_r^p is smaller than some given ε uniformly for all base points $0 \in \Omega$, provided $0 < r < \operatorname{dist}(0, \partial\Omega)$ and r does not exceed a certain bound r_ε depending on n, N, p, λ, Λ, ε and the modulus of continuity of the coefficients, i.e. we have

$$^*\|T_r{}^p\| \leqq \varepsilon \text{ for all } 0 \in \Omega,\ 0 < r \leqq \min(r_\varepsilon, \operatorname{dist}(0, \partial\Omega)). \tag{1.10}$$

We now summarize our local regularity results.

Theorem 2 (compare [15: Thm. 6.4.3]): *Let $1 < p, q < \infty$, $f \in L^p_{\mathrm{loc}}(\Omega)^N$ and $F \in L^q_{\mathrm{loc}}(\Omega)^{nN}$ and suppose that $u \in H^{1,p}_{\mathrm{loc}}(\Omega)^N$ is a weak solution of (1.1) under the condition (1.2). Then u belongs to the space $H^{1,r}_{\mathrm{loc}}(\Omega)^N$ for*

$$r = \min\{q, s(p)\},\ s(p) = \begin{cases} np/(n-p) & \text{if } p < n \\ \infty & \text{if } p \geqq n. \end{cases}$$

From the proof of Theorem 2 we will deduce the following

Corollary: *Let $1 < p < n/(n-1)$, assume that (1.2) holds and define $q = np/\big(n-(n-1)p\big)$. Then there exist constants C depending on n, N, p, λ, Λ and R_0 determined by the same parameters and in addition by the modulus of continuity of the coefficients such that*

$$^*\|u\|_{H^{1,q}(B_r(x))} \leqq Cr^{1-n^*} \|u\|_{H^{1,p}(B_{2r}(x))} \tag{1.11}$$

$$^*\|u\|_{H^{1,2}(B_r(x))} \leqq Cr^{n/2r-n/p^*} \|u\|_{H^{1,p}(B_{2r}(x))} \tag{1.12}$$

for all balls $B_r(x)$, $x \in \Omega$, $0 < 2r < \min\big(2R_0, \operatorname{dist}(x, \partial\Omega)\big)$ and for all $u \in H^{1,p}\big(B_{2r}(x)\big)$ satisfying $Lu = 0$ on $B_{2r}(x)$. If u satisfies the homogeneous system only on the punctured ball $B_{2r}(x) \setminus \{x\}$, (1.11) becomes

$$\frac{1}{r}\|u\|_{L^q(T)} + \|\nabla u\|_{L^q(T)} \leqq Cr^{1-n^*}\|u\|^{H^{1,p}(B_{2r}(x))}, \tag{1.13}$$

where $T = B_r(x) \setminus B_{r/2}(x)$.

Proof of Theorem 2: (i) $p < n$ and $p \leqq s(p) \leqq q$: We have to show *that* $u \in H^{1,s}\big(B_r(x)\big)$, $s := s(p)$, for $x \in \Omega$ and sufficiently small values of r. To this purpose we may assume $x = 0 \in \Omega$ and take $2r < \operatorname{dist}(0, \partial\Omega)$. Furthermore, let $\tilde u := \eta u \in H^{1,p}\big(B_{2r}(0)\big)^N$, where $\eta \in C_0^\infty(B_{2r})$ is an arbitrary cut-off function. $\tilde u$ is a weak solution of the system

$$L_{0ij}\tilde u^j = \tilde f^i - D_\alpha(\tilde F_\alpha{}^i - G_\alpha{}^i), \qquad i = 1, \dots, N, \text{ on } B_{2r}(0),$$

where we defined for $i = 1, \dots, N$, $\alpha = 1, \dots, n$

$$\tilde f^i = \eta f^i + F_\alpha{}^i D_\alpha\eta - A^{ij}_{\alpha\beta} D_\alpha\eta D_\beta u^j, \qquad \tilde F_\alpha{}^i = A^{ij}_{\alpha\beta} D_\beta\eta + \eta F_\alpha{}^i,$$
$$G_\alpha{}^i = \big(A^{ij}_{\alpha\beta} - A^{ij}_{\alpha\beta}(0)\big) D_\beta \tilde u^j.$$

Since $\tilde f$, $\tilde F$, G have compact support in $B_{2r}(0)$, we infer from Lemma 1.3 (iii)

$$\tilde u - T^p_{2r}\tilde u = P_{2r}(\tilde f) - \tilde P_{2r}(\tilde F) =: \varphi. \tag{1.14}$$

Observing

$$P_{2r}(\tilde f) \in H^{2,p}\big(B_{2r}(0)\big)^N \subset H^{1,s}\big(B_{2r}(0)\big)^N, \qquad \tilde P_{2r}(\tilde F) \in H^{1,s}\big(B_{2r}(0)\big)^N$$

we get $\varphi \in H^{1,s}\big(B_{2r}(0)\big)^N$. According to (1.10) there exists R_0 depending on the stated parameters such that

$$^*\|T^p_{2r}\|,\ ^*\|T^s_{2r}\| \leqq 1/2 \quad \text{for} \quad 2r \leqq \min\big(2R_0, \operatorname{dist}(0, \partial\Omega)\big).$$

Fixing such a radius r, we see that the operators $Id - T^p_{2r}$, $Id - T^s_{2r}$ are one-to-one and onto. Consequently we find $w \in H^{1,s}\big(B_{2r}(0)\big)^N$ with the property $(Id - T^p_{2r})\,\tilde u$

$= (Id - T^s_{2r})\, w$. Using the fact that $Id - T^p_{2r}$ is an isomorphism of the space $H^{1,p}$, we see $\tilde{u} = w$, since $T^p_{2r} = T^s_{2r}$ on the space $H^{1,s}$. Choosing $\eta = 1$ on $B_r(0)$ we arrive at $u \in H^{1,s}(B_r(0))^N$.

(ii) $n \leqq p \leqq q$: Define $t = nq/(n + q) \in (1, n)$ and observe

$$u \in H^{1,t}_{\mathrm{loc}}(\Omega)^N, \quad F \in L^q_{\mathrm{loc}}(\Omega)^{nN}, \quad f \in L^t_{\mathrm{loc}}(\Omega)^N.$$

Since q is the Sobolev exponent corresponding to t the assertion of the theorem follows as in case (i), replacing p by t and s by q.

(iii) $p \geqq q$: This case is trivial ∎

Proof of the corollary: Again assume $0 \in \Omega$; according to (1.10) we find R_0 such that for $k = 0, \ldots, n - 1$

$$^*\|T^{s_k}_{2r}\| \leqq 1/2 \text{ for } 0 < 2r < \min\left(2R_0, \operatorname{dist}(0, \partial\Omega)\right), \qquad s_k = np/(n - kp).$$

Now let $u \in H^{1,p}(B_{2r}(0))^N$ satisfy $Lu = 0$ on the ball $B_{2r}(0)$. Using the notations from the proof of Theorem 2 we get according to (1.14) that

$$\tilde{u} - T^{s_k}_{2r}\tilde{u} = P_{2r}(\tilde{f}) - \bar{P}_{2r}(\tilde{F}) \qquad (k = 0, \ldots, n - 1), \tag{1.15}$$

where we suppose that r satisfies the above-stated smallness condition. Now choosing $\eta = 1$ on $B_{3r/2}(0)$, $|\nabla\eta| \leqq c/r$, (1.15) immediately implies the inequality

$$^*\|u\|_{1,s_1;3r/2} \leqq Cr^{-1}\, {}^*\|u\|_{1,p;2r},$$

where $^*\|.\|_{k,p;r}$ denotes the $*$-norm in $H^{k,p}(B_r(0))$. Inequality (1.11) now follows by a simple iteration argument, (1.12) is an easy consequence of (1.11) (use Hölder's inequality). To prove the last statement of the corollary we proceed as above, the only difference is that we use cut-off functions η with compact supports on rings centered at 0 ∎

2. Systems with vector-valued measures on the right-hand side

In this section we use the results of the preceding paragraph to prove existence and uniqueness of a weak solution to the boundary value problem

$$L_{ij}u^j = \mu^i,\ i = 1, \ldots, N, \text{ on } \Omega,\ u_{|\partial\Omega} = 0, \tag{2.1}$$

whenever μ^i, $i = 1, \ldots, N$, are prescribed signed Radon measures with finite variation. We will show that (2.1) admits a unique weak solution in the space

$$\mathring{H} = \{u : \Omega \to \mathbf{R}^N \mid u \in \mathring{H}^{1,r}(\Omega)^N \text{ for all } 1 \leqq r < n/(n - 1)\}.$$

The idea of the proof follows arguments of LITTMAN, STAMPACCHIA and WEINBERGER [14].

In the sequel we denote by $M(\Omega)$ the space of all signed Radon measures μ on Ω with finite total variation $|\mu|\,(\Omega)$; $M(\Omega)^N$ is the space of all $\mu = (\mu^1, \ldots, \mu^N)$ with components $\mu^i \in M(\Omega)$. Obviously $C_0{}^0(\Omega, \mathbf{R}^N)^*$ (dual space) is isomorphic to $M(\Omega)^N$ [4: Thm. 2.5.5]. From now on we will use the norms

$$|u|_{1,r} = \|\nabla u\|_{L^r(\Omega)}, \quad \|T\|_{-1,p} = \inf\{\|X\|_{L^p(\Omega)} : X \in L^p(\Omega)^{nN} \text{ represents } T\}$$

on the spaces $\mathring{H}^{1,r}(\Omega)^N$ and $H^{-1,p}(\Omega)^N$, respectively. Then there exists a natural isometric isomorphism

$$K_p : \mathring{H}^{1,p'}(\Omega, \mathbf{R}^N)^* \to H^{-1,p}(\Omega)^N, \qquad p' = p/(p - 1).$$

We therefore agree to identify the spaces $\mathring{H}^{1,p'}(\Omega, \mathbf{R}^N)^*$ and $H^{-1,p}(\Omega)^N$. In conclusion we can consider any function $u \in \mathring{H}^{1,p'}(\Omega)^N$ as an element of $H^{-1,p}(\Omega, \mathbf{R}^N)^*$ by defining $\langle u, U\rangle = \langle U, u\rangle$ for $U \in H^{-1,p}(\Omega)^N$.

For solving the boundary value problem (2.1) we assume

$$(\mathbf{GA}) \text{ and } A^{ij}_{\alpha\beta} \in C^0(\bar{\Omega}) \qquad (i, j = 1, \ldots, N; \alpha, \beta = 1, \ldots, n). \tag{2.2}$$

Then by the Lax-Milgram theorem the solution operator $S : H^{-1,2}(\Omega)^N \to \mathring{H}^{1,2}(\Omega)^N$, $v = S(T)$ being the unique $\mathring{H}^{1,2}$-solution to the adjoint system $L^t_{ij} v^j = T^i$, $i = 1, \ldots, N$, on Ω, is well-defined and continuous. The "duality method" introduced in [14] is based on the following principle: Corollary 2 to Theorem 1 implies that for values $n < p < \infty$ the solution operator S maps $H^{-1,p}(\Omega)^N$ into the space $C_0{}^0(\Omega)^N$ of functions $v : \Omega \to \mathbf{R}^N$ which are continuous on $\bar{\Omega}$ with boundary values zero. Therefore S^* (dual operator) maps linear functionals μ defined on a certain space of continuous functions on functions $u_\mu \in \mathring{H}^{1,p'}(\Omega)^N$, and we will show that u_μ is the natural solution to the system (2.1).

To be precise we consider for numbers $1 \leqq s \leqq r < \infty$, $p > n$ the embeddings

$$i_{rs}: \mathring{H}^{1,r}(\Omega)^N \to \mathring{H}^{1,s}(\Omega)^N, \qquad j_{rs}: H^{-1,r}(\Omega)^N \to H^{-1,s}(\Omega)^N,$$
$$h_p: \mathring{H}^{1,p}(\Omega)^N \to C_0{}^0(\Omega)^N$$

and define the linear operators

$$\Sigma_p: H^{-1,p}(\Omega)^N \to \mathring{H}^{1,p}(\Omega)^N, \qquad \Sigma_p(T) = S \circ j_{p2}(T),$$
$$S_p: H^{-1,p}(\Omega)^N \to C_0{}^0(\Omega)^N, \qquad S_p = h_p \circ \Sigma_p$$

with norms depending only on n, N, p, λ, Λ, Ω and the modulus of continuity of the coefficients. For exponents $p > n$ we look at the dual operator

$$S_p^*: C_0{}^0(\Omega, \mathbf{R}^N)^* \to H^{-1,p}(\Omega, \mathbf{R}^N)^*, \qquad \mu \to \mu \circ S_p.$$

Recalling the isomorphism mentioned above we have a continuous linear operator $S_p^*: M(\Omega)^N \to \mathring{H}^{1,p'}(\Omega)^N$ which satisfies

$$|S_p^*(\mu)|_{1,p'} \leqq C\, |\mu|\, (\Omega) \qquad \text{for all } \mu \in M(\Omega)^N$$

for some positive constant C which only depends on n, N, p, λ, Λ, Ω and the modulus of continuity of the coefficients.

It is now easy to show that $S_p^*(\mu)$, $\mu \in M(\Omega)^N$, induces the same element in $\mathring{H}^{1,1}(\Omega)^N$ for all values $p > n$. Let $q > p > n$ be arbitrary and observe the relations

$$\Sigma_p \circ j_{qp} = i_{qp} \circ \Sigma_q, \quad h_p \circ i_{qp} = h_q, \quad S_p = h_p \circ \Sigma_p, \quad S_q = h_q \circ \Sigma_q,$$
$$j^*_{qp} = i_{p'q'},$$

from which we get $S_q^* = i_{p'q'} \circ S_p^*$. This relation has the following interpretation.

Lemma 2.1: *If (2.2) is satisfied, then for arbitrary real p, $q > n$ and measures $\mu \in M(\Omega)^N$ we have $S_q^*(\mu) = S_p^*(\mu)$ in $\mathring{H}^{1,r}(\Omega)^N$ for $r = \min\{p/(p-1), q/(q-1)\}$. Therefore $S_p^*(\mu)$ induces a Sobolev function $u_\mu \in \mathring{H}^{1,1}(\Omega)^N$ which is contained in the space $\mathring{H}$.*

We can now state the main result of this section

Theorem 3: *Suppose that (2.2) holds and that $\mu \in M(\Omega)^N$ is given.*
(i) *The function u_μ defined in Lemma 2.1 is a weak solution of problem (2.1).*

(ii) *For* $r \in [1, n/(n-1))$ *there is a constant* C *depending on* n, N, r, Λ, λ, Ω *and the modulus of continuity of the coefficients such that*

$$\|u_\mu\|_{H^{1,r}(\Omega)} \leqq C\,|\mu|\,(\Omega).$$

(iii) *If* v *belongs to the space* $\mathring{H}^{1,1+\varepsilon}(\Omega)^N$ *for some* $\varepsilon > 0$ *and is a weak solution of problem* (2.1), *then* $v = u_\mu$.

Definition: Under the assumption (2.2) we call the function u_μ defined in the preceding theorem the *weak solution of the boundary value problem* (2.1).

As a simple consequence of Theorem 3 we have the following approximation lemma.

Lemma 2.2: *If* μ_m, μ, $m \in \mathbf{N}$, *belong to the space* $M(\Omega)^N$ *with the property*

$$\lim_{m\to\infty} \int_\Omega f^i\,d\mu_m{}^i = \int_\Omega f^i\,d\mu^i \quad \text{for all } f \in C_0{}^0(\Omega)^N$$

and if (2.2) *is satisfied, then we have weak convergence* $u_m \rightharpoonup u$ *for* $m \to \infty$ *in* $H^{1,p}(\Omega)^N$, $1 < p < n/(n-1)$, *for the corresponding solutions of the boundary value problem* (2.1).

3. Definition and first properties of the Green matrix

First we apply Theorem 3 to show existence and uniqueness of a Green matrix G of the system under consideration. Elementary properties of G such as continuity on domains that do not meet the singular diagonal follow directly from the local regularity theory stated in Section 1. Moreover, we prove a representation formula for the weak solution u_μ to problem (2.1): u_μ equals the convolution $G * \mu$ $\mathbf{L}^n$-a.e. on Ω ($\mathbf{L}^n$ means the n-dimensional Lebesgue measure on $\mathbf{R}^n$).

Definition: Assume that (2.2) holds. (i) For $y \in \Omega$, $k = 1, \ldots, N$, denote by $G_k(\cdot, y) \in \mathring{H}$ the *unique solution of* $L_{ij}v^j = \delta_{ik}\delta_y$, $i = 1, \ldots, N$, on Ω, $v_{|\partial\Omega} = 0$.

$$G\colon \Omega \times \Omega \to \mathbf{R}^{N^2},\ (x, y) \to \big(G_k{}^i(x, y)\big)_{1\le i,k\le N}$$

is called the *Green matrix for the operator* $(L_{ij})_{1\le i,j\le N}$ *on the domain* Ω.
(ii) For $y \in \Omega$, $\varrho > 0$ we define the *Radon measures*

$$\mu^i = \mathbf{L}^n\big(B_\varrho(y) \cap \Omega\big)^{-1} \cdot \delta_{ik} \cdot \mathbf{L}^n \llcorner B_\varrho(y) \cap \Omega, \quad i = 1, \ldots, N,$$

(cf. [4: p. 54]) and denote the *associated solution* of (2.1) by the symbol $G_k{}^\varrho(\cdot, y)$. We call G^ϱ the *mollified Green matrix* to $(L_{ij})_{1\le i,j\le N}$ on the domain Ω.
(iii) If the differential operator L is replaced by L^t, we use the notations ${}^tG$, ${}^tG^\varrho$.

The results from Section 2 are summarized in

Lemma 3.1: *If* (2.2) *is satisfied, then for all* $y \in \Omega$, $\varrho > 0$ *the following statements are true*:

(i) $G(\cdot, y) \in H^{1,r}_{\mathrm{loc}}(\Omega \setminus \{y\})^{N^2}$ *for all* $1 \leqq r < \infty$; *in particular* $G(\cdot, y) \in C^0 \times (\Omega \setminus \{y\})^{N^2}$.
(ii) $G^\varrho(\cdot, y) \in \mathring{H}^{1,r}(\Omega)^{N^2}$ *for all* $1 \leqq r < \infty$; *in particular* $G^\varrho(\cdot, y) \in C^{0,\alpha}(\Omega)^{N^2}$ *for all* $0 < \alpha < 1$.
(iii) *For each* $p \in [1, n/(n-1))$ *there exists a constant* C *depending on* n, N, p, λ, Λ, Ω *and the modulus of continuity of the coefficients such that*

$$\|G(\cdot, y)\|_{H^{1,p}(\Omega)},\ \|G^\varrho(\cdot, y)\|_{H^{1,p}(\Omega)} \leqq C.$$

(iv) $G^\varrho(\cdot, y) \rightharpoonup G(\cdot, y)$ *for* $\varrho \to 0$ *in* $H^{1,p}(\Omega)^{N^2}$ *for* $1 < p < n/(n-1)$.

The integrability properties of $G(\cdot, y)$ as stated in part (i) of the lemma can be improved.

Lemma 3.2: *Assume* (2.2) *and let $B_R(y)$ be a ball compactly contained in Ω. Then $G(\cdot, y) \in H^{1,r}(\Omega \setminus B_R(y))^{N^2}$ for any exponent $r < \infty$ and*

$$\|G(\cdot, y)\|_{H^{1,r}(\Omega \setminus B_R(y))} \leqq C \tag{3.1}$$

for some constant C depending on n, N, λ, Λ, Ω, r, $B_R(y)$ and the modulus of continuity of the coefficients. (3.1) *holds (with the same C) if $G(\cdot, y)$ is replaced by $G^\varrho(\cdot, y)$, provided $\varrho < R$.*

Corollary: *For $\varrho < R$ and $0 < \alpha < 1$ the $C^{0,\alpha}$-norm of $G^\varrho(\cdot, y)$ on $\Omega \setminus B_R(y)$ is estimated independent of ϱ.*

We omit the simple proof of this lemma. The corollary provides a useful tool in proving certain symmetry properties of Green's matrix.

Theorem 4: *Under the assumptions* (2.2) *we have for all points $x, y \in \Omega$ and integers $k, l = 1, \ldots, N$*

$$G(x, y) = \left({}^tG(y, x)\right)^{\mathrm{T}}, \qquad \text{i.e. } G_k{}^l(x, y) = {}^tG_l{}^k(y, x). \tag{3.2}$$

Corollaries: **1.** *If the coefficients are symmetric, $A_{\alpha\beta}^{ij} = A_{\beta\alpha}^{ji}$, then $G(x, y) = G(y, x)^T$ for all $x, y \in \Omega$.* 2. *For any $x, y \in \Omega$, $0 < \varrho < \operatorname{dist}(y, \partial\Omega)$,*

$$G^\varrho(x, y) = \fint_{B_\varrho(y)} \left({}^tG(z, x)\right)^{\mathrm{T}} dz = \fint_{B_\varrho(y)} G(x, z)\, dz. \tag{3.3}$$

Since ${}^tG(\cdot, x)$, for fixed x, is a continuous function on $\Omega \setminus \{x\}$, formula (3.2) shows continuity of $G(x, \cdot)$ as a function of the second argument, which is not a direct consequence of the definition. Corollary 2 justifies the name *mollified Green's matrix* for G^ϱ.

Proof of Theorem 4: Let $x \neq y \in \Omega$ and choose sequences (ϱ_ν), (σ_μ) tending to zero such that

$$G^{\varrho_\nu}(\cdot, y) \to G(\cdot, y), \quad {}^tG^{\sigma_\mu}(\cdot, x) \xrightarrow[\mu \to \infty]{} {}^tG(\cdot, x) \text{ a.e. on } \Omega. \tag{3.4}$$

Abbreviating $G^\nu = G^{\varrho_\nu}(\cdot, y)$, ${}^tG^\mu = {}^tG^{\sigma_\mu}(\cdot, x)$, $B_\nu = B_{\varrho_\nu}(y)$, $B_\mu = B_{\sigma_\mu}(x)$, we get from the definition of the mollified Green matrix

$$a_{\nu\mu}^{kl} := \fint_{B_\nu} {}^tG_l{}^{\mu k}\, dz = \fint_{B_\mu} G_k{}^{\nu l}\, dz, \qquad k, l = 1, \ldots, N. \tag{3.5}$$

We know ${}^tG^\mu \to {}^tG(\cdot, x) =: {}^tG$ for $\mu \to \infty$ in $H^{1,p}(\Omega)^{N^2}$, $1 < p < n/(n-1)$, and

$$\fint_{B_\mu} G^\nu\, dz \xrightarrow[\mu \to \infty]{} G^\nu(x) \text{ by the continuity of } G^\nu \text{ at } x,$$

so that (3.5) implies the relation

$$G_k{}^{\nu l}(x) = \fint_{B_\nu} {}^tG_l{}^k\, dz. \tag{3.6}$$

Since ${}^tG$ is continuous at the point y we may pass to the limit $\nu \to \infty$ in (3.6) to get

$${}^tG_l{}^k(y) = \lim_{\nu \to \infty} \left(\lim_{\mu \to \infty} a_{\nu\mu}^{kl}\right) \qquad \text{and} \qquad G_k{}^l(x) = \lim_{\mu \to \infty} \left(\lim_{\nu \to \infty} a_{\nu\mu}^{kl}\right). \tag{3.7}$$

Now (3.2) follows from (3.7) since $a_{\nu\mu}^{kl}$ converges uniformly in ν as μ tends to infinity. To prove this fix $0 < R < |x - y|$. Then, by the corollary of Lemma 3.2, there is a constant C independent of ν such that for any given $0 < \alpha < 1$ the $C^{0,\alpha}$-norm of

G^ν on the ball $B_R(x)$ is estimated by C. Arzela's theorem in combination with (3.4) implies

$$\|G^\nu - G\|_{L^\infty(B_{R/2}(x))} \xrightarrow[\nu\to\infty]{} 0, \text{ therefore } \left| \fint_{B_\mu} G^\nu\, dz - \fint_{B_\mu} G\, dz \right| \xrightarrow[\nu\to\infty]{} 0$$

uniformly with respect to μ ∎

Proof of the corollaries: $G(x, y) = G(y, x)^T$ is trivial since $G = {}^tG$ in the symmetric case. (3.3) follows from (3.6) by replacing ϱ_ν by $\varrho < \operatorname{dist}(y, \partial\Omega)$ ∎

As already remarked, the symmetry relation (3.2) implies certain continuity properties of Green's matrix. Now we are interested in the regularity of G as a function of two variables.

Theorem 5: *If* (2.2) *holds, then* G *is locally Hölder continuous on* $\Omega \times \Omega \setminus \{(x, x) : x \in \Omega\}$ *with any exponent* $0 < \alpha < 1$.

Proof: Let $0 < \alpha < 1$ and define $p = n/(n - \alpha)$, $q = n/(1 - \alpha)$. Consider two points $x \neq y$ in Ω and choose $r > 0$ so small that

(i) $r \leqq |x - y|/4, \quad B_{2r}(x) \cup B_{2r}(y) \subset \Omega$.

According to Theorem 4 and the corollaries, we have for points $(x', y') \in B_r(x) \times B_r(y)$, $\varrho = \max(|x - x'|, |y - y'|)$ the estimate

$$\begin{aligned} |G^\varrho(x', y') - G^\varrho(x, y)| &\leqq |G^\varrho(x', y') - G^\varrho(x, y')| + |G^\varrho(x, y') - G^\varrho(x, y)| \\ &\leqq \fint_{B_\varrho(y')} |{}^tG(u, x') - {}^tG(u, x)|\, du + \left| \fint_{B_\varrho(y')} {}^tG(u, x)\, du - \fint_{B_\varrho(y)} {}^tG(u, x)\, du \right| \\ &\leqq \fint_{B_\varrho(y')} |G(x', u) - G(x, u)|\, du + \fint_{B_\varrho(y')} |{}^tG(u, x) - {}^tG(y, x)|\, du \\ &\quad + \fint_{B_\varrho(y)} |{}^tG(u, x) - {}^tG(y, x)|\, du := a + b + c. \end{aligned}$$

In addition to (i) we require

(ii) $r \leqq R_0$, R_0 defined in the corollary to Theorem 2.

Applying estimate (1.11) to each function $G(\cdot, u)$, $u \in B_\varrho(y')$, we get by the Sobolev Embedding Theorem and Lemma 3.1 $\big(C_0 = C_0(n, N, \lambda, \Lambda, \alpha)\big)$

$$\begin{aligned} {}^*\|G(\cdot, u)\|_{H^{1,q}(B_r(x))} &\leqq C_0 r^{1-n}\, {}^*\|G(\cdot, u)\|_{H^{1,p}(B_{2r}(x))} \\ &\leqq C_0 r^{1-n} \|G(\cdot, u)\|_{H^{1,p}(\Omega)} \leqq C_1 r^{1-n}, \end{aligned}$$

where C_1 also depends on Ω and the modulus of continuity of the coefficients. This implies $a \leqq C_1 r^{1-n} \varrho^\alpha$. Replacing $G(\cdot, u)$ by ${}^tG(\cdot, x)$ in the above argument we get the same bound for b and c:

$$|G^\varrho(x', y') - G^\varrho(x, y)| \leqq C_1 r^{1-n} \varrho^\alpha. \tag{3.8}$$

To prove (3.8) for G we observe

$$\begin{aligned} |G(x', y') - G(x, y)| &\leqq |G(x', y') - G^\varrho(x', y')| + |G^\varrho(x', y') - G^\varrho(x, y)| \\ &\quad + |G^\varrho(x, y) - G(x, y)|. \end{aligned}$$

The first and the last term on the right-hand side are estimated as above, for the second term we use (3.8), so that

$$|G(x, y) - G(x', y')| \leqq C_1 r^{1-n} \varrho^\alpha \quad \text{for all } (x', y') \in B_r(x) \times B_r(y)$$ ∎

As a first application of Green's matrix we derive a representation formula for the weak solution of the boundary value problem

$$L_{ij}u^j = \mu^i, \quad i = 1, \ldots, N, \quad \text{on } \Omega, \; u_{|\partial\Omega} = 0 \quad \left(\mu \in M(\Omega)^N\right). \tag{3.9}$$

Theorem 6: *If (2.2) holds, $\mu \in M(\Omega)^N$ is a vector valued signed Radon measure of finite total variation and if $u \in \mathring{H}$ denotes the unique weak solution of (3.9), then for $\mathbf{L}^n$-almost all points $y \in \Omega$*

$$u^k(y) = \int_\Omega {}^tG_k{}^i(x, y)\, d\mu^i(x) = \int_\Omega G_i{}^k(y, x)\, d\mu^i(x), \qquad k = 1, \ldots, N. \tag{3.10}$$

For the proof we need

Lemma 3.3: *Assume that (2.2) is satisfied and that $\nu \geqq 0$ is a finite Radon measure on Ω. Then*

(i) *G is $(\nu \times \mathbf{L}^n)$-measurable on $\Omega \times \Omega$.*

(ii) *$\int_{\Omega\times\Omega} |G|\, d(\nu \times \mathbf{L}^n)$ is finite; especially the function $\Omega \ni y \to \int_\Omega G(x, y)\, d\nu(x)$ belongs to the space $L^1(\Omega)^{N^2}$.*

Proof of the lemma: (i) By Theorem 5 the statement follows if we show that $(\nu \times \mathbf{L}^n)\,(D) = 0$ for the diagonal $D = \{(x, x) : x \in \Omega\}$. To this purpose we choose a sequence of disjoint Borel sets $(A_i)_{i\in\mathbf{N}}$ such that $\Omega = A_1 \cup A_2 \cup \ldots$, $\mathbf{L}^n(A_i) \leqq \varepsilon$, where $\varepsilon > 0$ is given. We get

$$(\nu \times \mathbf{L}^n)\,(D) \leqq \sum_{i=1}^{\infty} (\nu \times \mathbf{L}^n)\,(A_i \times A_i) = \sum_{i=1}^{\infty} \nu(A_i)\, \mathbf{L}^n(A_i) \leqq \varepsilon\nu(\Omega),$$

and by the finiteness of ν we conclude $(\nu \times \mathbf{L}^n)\,(D) = 0$.

(ii) Since $|G|$ is a non-negative $(\nu \times \mathbf{L}^n)$-measurable function, Fubini's theorem implies

$$\int_{\Omega\times\Omega} |G(x, y)|\, d(\nu \times \mathbf{L}^n)\,(x, y) = \int_\Omega \left(\int_\Omega |G(x, y)|\, d\mathbf{L}^n(y)\right) d\nu(x)$$
$$= \int_\Omega \left(\int_\Omega |{}^tG(y, x)|\, d\mathbf{L}^n(y)\right) d\nu(x),$$

and by Lemma 3.1 the inner integral is bounded by a constant independent of x ∎

Proof of Theorem 6: We may assume that μ^i, $i = 1, \ldots, N$, are positive Radon measures of finite mass, otherwise we decompose $\mu^i = \mu_+{}^i - \mu_-{}^i$. Choose $y \in \Omega$, $1 \leqq k \leqq N$ and $0 < \varrho < \operatorname{dist}(y, \partial\Omega)$. Testing the weak form of (3.9) with ${}^tG_k{}^\varrho(\cdot, y) \in C^0 \cap \mathring{H}^{1,2}(\Omega)^N$ we get

$$\fint_{B_\varrho(y)} u^k\, dz = \int_\Omega {}^tG_k{}^{\varrho i}(x, y)\, d\mu^i(x) = \int_\Omega \left(\fint_{B_\varrho(y)} {}^tG_k{}^i(x, z)\, d\mathbf{L}^n(z)\right) d\mu^i(x)$$
$$= \fint_{B_\varrho(y)} \left(\int_\Omega {}^tG_k{}^i(x, z)\, d\mu^i(x)\right) d\mathbf{L}^n(z).$$

Here we used Lemma 3.3 with G replaced by ${}^tG$. Observing that $\mathbf{L}^n$-almost all points are Lebesgue points of u^k and the function

$$f^k : \Omega \ni z \to \int_\Omega {}^tG_k{}^i(x, z)\, d\mu^i(x),$$

we define Ω_0 to be the common set of Lebesgue points of u^k and f^k, $k = 1, \ldots, N$. Obviously, $\mathbf{L}^n(\Omega \setminus \Omega_0) = 0$ and (3.10) is valid for all $y \in \Omega_0$ ∎

4. Growth properties of Green's matrix

In the scalar case $N = 1$ most applications of Green's function g depend essentially on the growth properties of g near the singular diagonal, compare [9–11, 14]. The purpose of this section is to prove at least a local version of the standard estimate, i.e. we want to show

$$|G(x, y)| \leqq C\,|x - y|^{2-n}. \tag{4.1}$$

for points x, y in small balls compactly contained in Ω. This behaviour of G is suggested by the growth properties of the fundamental matrix E for systems with constant coefficients (see Lemma 1.2). For technical reasons (compare Lemma 4.3) the proof of (4.1) only works in the case of Hölder continuous coefficients so that we assume for the rest of this section

(GA) and there exist constants $0 < \alpha < 1$, $L \geqq 0$ such that

$$\sup\,\{|A(x) - A(y)|/|x - y|^\alpha : x \neq y \in \bar{\Omega}\} \leqq L. \tag{4.2}$$

Under this assumption the basic local estimate (4.1) is proven in Theorem 7. As a consequence of this theorem we derive inequalities concerning the behaviour of the first derivatives of Green's matrix near the diagonal.

The method of our proof is based on a pertubation argument: We freeze the coefficients at an arbitrary point $y \in \Omega$ and write $G(x, y) = E(x - y) + H_y(x)$, where E denotes the fundamental matrix for the operator with constant coefficients $A(y)$. Using the Hölder condition (4.2) it is possible to control the size of the perturbation H_y at least locally near y. We hope to be able to extend our technique to derive global estimates for G up to the boundary.

Theorem 7: *Suppose that* (4.2) *holds and let* $0 < \beta < 1$ *be given. Then there are constants* C_1 *depending on* $n, N, L, \lambda, \Lambda, \alpha, R_0$ *also depending on* β *and* C_2 *also depending on* β *and* Ω *such that*

$$|G(x, y)| \leqq C_1\,|x - y|^{2-n} + C_2 R^{1+\beta-n} \tag{4.3}$$

for all $x \in B_{2R}(y) \setminus \{y\}$, $y \in \Omega$, $0 < R \leqq \min\,(R_0, \operatorname{dist}\,(y, \partial\Omega)/4)$.

Corollary: *Let* $y \in \Omega$ *be given. Then there exists* R_y *depending on* $n, N, L, \lambda, \Lambda, \alpha$ *and* dist $(y, \partial\Omega)$ *such that*

$$|G(x, z)| \leqq C_1\,|x - z|^{2-n} + C_2 R_y^{1-n+\alpha} \tag{4.4}$$

for all $x, z \in B_{R_y}(y) \subset \Omega$. *Here* C_1, C_2 *are as in Theorem 7 with* $\beta = \alpha$.

(4.4) is the precise formulation of the local estimate (4.1): Since $R_y \to 0$ when y approaches the boundary $\partial\Omega$ we see that inequality (4.4) essentially depends on the location of the ball $B_{R_y}(y)$.

Since the proof of the theorem is lengthy we found it useful to proceed in several steps summarized as Lemma 4.1–4.3. From now on we assume that the assumptions of Theorem 7 are satisfied. Define $p = n/(n - \beta)$, $q = n/(1 - \beta)$, $p_0 = n/(n - \alpha)$ and assume that $y := 0$ is contained in Ω. According to (1.10) there exists R_0 such that

$$\|T^s_{4R}\| \leqq 1/2, \qquad s = p_0 \text{ and } s = np/(n - kp), \qquad k = 0, \ldots, n - 1, \tag{4.5}$$

for all $0 < R \leqq \min\bigl(R_0, \operatorname{dist}\,(0, \partial\Omega)/4\bigr)$. Fix R with (4.5) and for $1 \leqq k \leqq N$ let $G := G_k(\cdot, 0)$, $E := E_k$ and

$$w := G - T^p_{2R} G - E \in H^{1,p}\bigl(B_{2R}(0)\bigr)^N. \tag{4.6}$$

(E is defined in Lemma 1.2b)). Obviously, w belongs to the space $\bigcap \bigl\{H^{1,r}\bigl(B_{2R}(0)\bigr)^N : 1 \leqq r < n/(n - 1)\bigr\}$ and from (4.6) we infer $L_{0i}{}_j w^j = 0$ on $B_{2R}(0)$. In conclusion, w is analytic on $B_{2R}(0)$, especially bounded on any ball $B_r(0)$ with radius $r < 2R$. The following lemma shows boundedness of w on the whole ball $B_{2R}(0)$.

Lemma 4.1: *The function w is contained in the space $H^{1,q}(B_{2R}(0))^N$ and satisfies the estimate* $*\|w\|_{q,2R} \leqq C_2 R^{1-n}$.

Here and in the sequel we abbreviate $*\|\cdot\|_{s,r} = *\|\cdot\|_{H^{1,s}(B_r(0))}$.

Proof of the lemma: The corollary of Theor. 2 implies $*\|w\|_{q,R} \leqq C_2 R^{1-n} *\|w\|_{p,2R}$. To estimate the norm on the right-hand side we use the defining equation (4.6):

$$*\|G\|_{p,2R} \leqq \frac{1}{R} \left\{\mathbf{L}^n(B_{2R}(0))^\delta \|G\|_{L^{s(p)}(B_{2R}(0))}\right\} + \|\nabla G\|_{L^p(\Omega)}$$
$$\leqq C_2 \|\nabla G\|_{L^p(\Omega)} \leqq C_2, \quad \delta := \frac{1}{p} - \frac{n-p}{np}, \quad s(p) := \frac{np}{n-p},$$

by Lemma 3.1 (iii). From (4.5) and (1.7) we infer

$$*\|T_{2R}^p G\|_{p,2R} \leqq \frac{1}{2} *\|G\|_{p,2R} \leqq C_2, \qquad *\|E\|_{p,2R} \leqq C_2 R^{1-\beta},$$

in conclusion,

$$*\|w\|_{q,R} \leqq C_2 R^{1-n}. \tag{4.7}$$

To complete the proof of the lemma we have to show

$$\frac{1}{R} \|u\|_{L^q(T_{2R})} + \|\nabla u\|_{L^q(T_{2R})} \leqq C_2 R^{1-n} \tag{4.8}$$

for the functions $u := G,\ T_{2R}^p G,\ E$, where T_{2R} denotes the ring $B_{2R}(0) \setminus B_R(0)$. For $u := E$ (4.8) is already contained in (1.7), for $u := G$ we use estimate (1.13). Dropping all indices we let $\omega(y) \doteq A(y) - A(0)$ and write for $x \in B_{2R}(0)$

$$T_{2R}^p G(x) = \int_{B_{2R}(0)} \nabla^{2N-1} K(x-y)\, \omega(y)\, \nabla G(y)\, dy$$
$$= \int_{B_{R/2}(0)} \ldots dy + \int_{B_{2R} \setminus B_{R/2}} \ldots dy =: \varphi_1(x) + \varphi_2(x).$$

Obviously, $\varphi_2 = \tilde{P}_{2R}(F)$ for $F(y) := 0$ on $B_{R/2}(0)$; $F(y) := \omega(y) \nabla G(y)$ on $B_{2R}(0) \setminus B_{R/2}(0)$. Lemma 1.3 implies

$$*\|\varphi_2\|_{q,2R} \leqq C_2 \|F\|_{L^q(D)} \leqq C_2 \|\nabla G\|_{L^q(D)} \leqq C_2 R^{1-n}, \qquad D := B_{2R}(0) \setminus B_{R/2}(0),$$

by a version of inequality (1.13). Since φ_1 is of class C^1 on T_{2R} we get (4.8) for φ_1 by direct calculation. Combining (4.7) and (4.8), the assertion of the lemma follows ∎

Using $w \in H^{1,q}(B_{2R}(0))^N$, we can rewrite (4.6) in the form

$$G = E + \sum_{l=1}^{\infty} (T_{2R}^{p_0})^l E + \sum_{l=0}^{\infty} (T_{2R}^q)^l w \text{ on } B_{2R}(0), \tag{4.9}$$

where the last term on the right-hand side satisfies (use (4.5))

$$*\left\| \sum_{l=0}^{\infty} (T_{2R}^q)^l w \right\|_{q,2R} \leqq *\|w\|_{q,2R} \leqq C_2 R^{1-n},$$

and by the Sobolev Embedding Theorem we get

$$\sup_{B_{2R}(0)} \left| \sum_{l=0}^{\infty} (T_{2R}^q)^l w \right| \leqq C_2 R^{1-n+\beta}. \tag{4.10}$$

Estimate (1.7) implies $|E(x)| \leqq C_1 |x|^{2-n}$ for all $x \neq 0$, so that (4.3) follows from (4.9) and (4.10), provided the sum over l of all $(T_{2R}^{p_0})^l E$ has the correct growth. The calculation of the growth order of this remaining term is contained in

Lemma 4.2: *Suppose that σ, τ are real numbers satisfying $0 < \tau, \sigma < n$, $\sigma + \tau > n$. Then for all $x, y \in \mathbf{R}^n$*

$$\int_{\mathbf{R}^n} |x - z|^{-\sigma} |y - z|^{-\tau} dz \leqq C(n, \tau, \sigma) |x - y|^{n-\tau-\sigma}. \tag{4.11}$$

Lemma 4.3: *Let $u_l = (T_{2R}^{p_0})^l E$, $l \in \mathbf{N}_0$. Then there exists a constant $C_1 = C_1(n, N, L, \lambda, \Lambda, \alpha)$ such that for all balls $B_{4R}(0) \subset \Omega$ and all points $x, x_1, x_2 \in B_{2R}(0) \setminus \{0\}$, $R < 1$, the following estimates hold:*

$$\begin{aligned} &\text{(i)} \quad |\nabla u_l(x)| \leqq C_1^{l+1} R^{l\alpha} |x|^{1-n}, \\ &\text{(ii)} \quad |\nabla u_l(x_1) - \nabla u_l(x_2)| \leqq C_1^{l+1} R^{l\alpha} |x_1 - x_2|^\alpha \max(|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}). \end{aligned} \tag{4.12}$$

The proof of Lemma 4.2 is an easy calculation, whereas the proof of Lemma 4.3 is somewhat more involved. We therefore first finish the **proof of Theorem 7**: For $l \in \mathbf{N}$, $x \in B_{2R}(0) \setminus \{0\}$ we have

$$\begin{aligned} |(T_{2R}^{p_0})^l E(x)| &= \left| \int_{B_{2R}(0)} \nabla^{2N-1} K(x - y)\, \omega(y)\, \nabla u_{l-1}(y)\, dy \right| \\ &\leqq C_1 R^\alpha \int_{B_{2R}(0)} |x - y|^{1-n} C_1{}^l R^{(l-1)\alpha} |y|^{1-n} dy \leqq C_1{}^{l+1} R^{l\alpha} |x|^{2-n}, \end{aligned}$$

according to (4.11), (4.12). From this we infer

$$\left| \sum_{l=1}^{\infty} (T_{2R}^{p_0})^l E(x) \right| \leqq C_1 \sum_{l=1}^{\infty} (C_1 R^\alpha)^l |x|^{2-n}.$$

Requiring $C_1 R_0{}^\alpha \leqq 1/2$ (C_1 from Lemma 4.3) we arrive at (4.3) ∎

Before proceeding further let us give some comment on Lemma 4.3: In standard potential theory it is shown that Lebesgue, Sobolev and Hölder classes are reproduced by certain singular integral operators (compare for example [15]). Here we extend this reproduction property to a class of functions having an isolated singularity with prescribed growth order.

Proof of Lemma 4.3: We write for $x \in B_{2R}(0)$ and functions $u \in H^{1,p_0}(B_{2R}(0))^N$ dropping all indices

$$T_{2R}^{p_0} u(x) = \int_{B_{2R}(0)} \nabla^{2N-1} K(x - y)\, \omega(y)\, \nabla u(y)\, dy$$

and proceed by induction. For $l = 0$ (4.12) immediately follows from (1.7). Now assume that l is a positive integer and that for some constant D_{l-1} the estimates

$$\begin{aligned} &|\nabla u_{l-1}(x)| \leqq D_{l-1} |x|^{1-n}, \\ &|\nabla u_{l-1}(x_1) - \nabla u_{l-1}(x_2)| \leqq D_{l-1} |x_1 - x_2|^\alpha \max(|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}) \end{aligned} \tag{4.13}$$

hold for all points x, x_1, x_2 in $B_{2R}(0)$. Abbreviating $\Delta(y) = \nabla^{2N} K(y)$, $y \in \mathbf{R}^n \setminus \{0\}$, we have the following formula for the derivative of u_l (compare [15: Thm. 3.4.2b)], here and in the sequel C_1 denotes constants depending on the parameters stated in Lemma 4.3)

$$\begin{aligned} \nabla u_l(x) &= C_1 \omega(x) \nabla u_{l-1}(x) + \lim_{\varrho \to 0} \int_{B_{2R}(0) \setminus B_\varrho(x)} \Delta(x - y)\, \omega(y)\, \nabla u_{l-1}(y)\, dy \\ &=: f(x) + \lim_{\varrho \to 0} W_\varrho(x), \qquad x \in B_{2R}(0) \setminus \{0\}. \end{aligned} \tag{4.14}$$

a) We discuss $W(x) := \lim W_\varrho(x)$ $(\varrho \to 0)$ for fixed $x \in B_{2R}(0) \setminus \{0\}$: Choose $0 < \sigma < \varrho < |x|/2$ and apply [14: Thm. 2.6.5] to get.

$$W_\varrho(x) - W_\sigma(x) = \int_{B_\varrho(x)\setminus B_\sigma(x)} \Delta(x-y)\left(\varphi(y) - \varphi(x)\right) dy, \qquad \varphi := \omega \nabla u_{l-1}.$$

(4.13) gives the inequality

$$|\varphi(y) - \varphi(x)| \leqq C_1 D_{l-1}\{|x|^\alpha\, |x-y|^\alpha \max\left(|x|^{1-n-\alpha}, |y|^{1-n-\alpha}\right) + |y|^{1-n}\, |x-y|^\alpha\}. \tag{4.15}$$

By this we can estimate the above integral as follows:

$$\begin{aligned} |W_\varrho(x) - W_\sigma(x)| &\leqq C_1 D_{l-1}\Bigg(\int_{B_\varrho(x)\setminus B_\sigma(x)} |x-y|^{\alpha-n} \max\left(|x|^{1-n-\alpha}, |y|^{1-n-\alpha}\right) dy\, |x|^\alpha \\ &\quad + \int_{B_\varrho(x)\setminus B_\sigma(x)} |x-y|^{\alpha-n}\, |y|^{1-n}\, dy\Bigg) \\ &\leqq C_1 D_{l-1}\left(|x|^\alpha\, |x|^{1-n-\alpha} + |x|^{1-n}\right) \int_{B_\varrho(x)} |x-y|^{\alpha-n}\, dy \\ &\leqq C_1 D_{l-1}\, |x|^{1-n}\, \varrho^\alpha. \end{aligned}$$

Consequently, $\left(W_\tau(x)\right)_{\tau>0}$ is a Cauchy sequence, the limit $W(x)$ exists and satisfies for $x \in B_{2R}(0) \setminus \{0\}$, $\varrho < |x|/2$, $\varrho < 2R - |x|$ the inequality

$$|W_\varrho(x) - W(x)| \leqq C_1 D_{l-1} \varrho^\alpha\, |x|^{1-n}. \tag{4.16}$$

Furthermore we have for x and ϱ as above, using (4.11) and (4.15),

$$|W_\varrho(x)| \leqq \int_{B_{2R}(0)\setminus B_\varrho(x)} |\Delta(x-y)|\, |\varphi(y) - \varphi(x)|\, dy \leqq C_1 D_{l-1} R^\alpha\, |x|^{1-n}.$$

If we combine this result with (4.16) we arrive at

$$|W(x)| \leqq C_1 D_{l-1} R^\alpha\, |x|^{1-n}, \qquad x \in B_{2R}(0) \setminus \{0\}. \tag{4.17}$$

By (4.13), estimate (4.17) holds for the function f defined in (4.14). We thus have proven

$$|\nabla u_l(x)| \leqq C_1 D_{l-1}\, |x|^{1-n}\, R^\alpha \tag{4.18}$$

for all points x in the punctured ball $B_{2R}(0) \setminus \{0\}$.

b) We now derive the Hölder condition for ∇u_l: Let $x_1, x_2 \in B_{2R}(0) \setminus \{0\}$ be given and assume

$$\varrho := |x_1 - x_2| \leqq \min\left(|x_1|, |x_2|\right)/5. \tag{4.19}$$

One would like to argue as follows: By calculating ∇W_ϱ one gets a bound for $|W_\varrho(x_1) - W_\varrho(x_2)|$ and the Hölder condition for W is a consequence of (4.16). Unfortunately, (4.16) is restricted to the case $\varrho < 2R - |x_i|$, $i = 1, 2$, and this condition is obviously violated for x_1, x_2 near $\partial B_{2R}(0)$. To overcome this technical difficulty we extend the function ∇u_{l-1} to the punctured ball $B_{4R}(0) \setminus \{0\}$, assuming that (4.13) continues to hold for the extended function, which we also denote by ∇u_{l-1}. The constant D_{l-1} appearing in (4.13) has to be replaced by a constant of the form $C_1 D_{l-1}$, but this does not change the argument.

On $B_{4R}(0) \setminus \{0\}$ we define the functions

$$W_{1\varrho}(x) = \int_{B_{4R}(0)\setminus B_\varrho(x)} \Delta(x-y)\, \omega(y)\, \nabla u_{l-1}(y)\, dy,$$

$$W_1(x) = \lim_{\varrho \to 0} W_{1\varrho}(x),$$

$$W_2(x) = \int_{T_{2R}(0)} \Delta(x-y)\, \omega(y)\, \nabla u_{l-1}(y)\, dy.$$

Obviously, $W(x) = W_1(x) - W_2(x)$ on $B_{2R}(0)$, and the inequalities (4.16), (4.17) hold with W replaced by W_1 and $2R$ replaced by $4R$. Now a simple calculation shows ($\mathbf{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure)

$$\nabla W_{1\varrho}(x) = -\int\limits_{\partial B_\varrho(x)} \Delta(x-y)\left(\varphi(y)-\varphi(x)\right)\frac{y-x}{\varrho}\,d\mathbf{H}^n(y) + \int\limits_{B_{4R}(0)\setminus B_\varrho(x)} \nabla\Delta(x-y)\left(\varphi(y)-\varphi(x)\right)dy$$

for points $x \in B_{2R}(0) \setminus B_{2\varrho}(0)$. The resulting terms satisfy the estimates

$$\begin{aligned}&\left|\int\limits_{\partial B_\varrho(x)} \dots\, d\mathbf{H}^{n-1}(y)\right|\\ &\leqq C_1D_{l-1}\left\{|x|^\alpha \int\limits_{\partial B_\varrho(x)} |x-y|^{\alpha-n}\max\left(|x|^{1-n-\alpha}, |y|^{1-n-\alpha}\right)d\mathbf{H}^{n-1}(y)\right.\\ &\quad\left. + \int\limits_{\partial B_\varrho(x)} |x-y|^{\alpha-n}\,|y|^{1-n}\,d\mathbf{H}^{n-1}(y)\right\} \leqq C_1D_{l-1}R^\alpha\,|x|^{1-n-\alpha}\,\varrho^{\alpha-1},\end{aligned}$$

where we used (4.15) and the fact that $|y| \geqq |x|/2$ on $\partial B_\varrho(x)$. Let us write

$$B_{4R}(0)\setminus B_\varrho(x) = \left\{\left(B_{4R}(0)\setminus B_{|x|/2}(0)\right)\setminus B_\varrho(x)\right\} \cup B_{|x|/2}(0) =: \Omega_1 \cup \Omega_2 .$$

Observing $|y| \geqq |x|/2$ on Ω_1, $|x-y| \geqq |x|/2$ on Ω_2 we get

$$\begin{aligned}&\left|\int\limits_{\Omega_1} \nabla\Delta(x-y)\left(\varphi(y)-\varphi(x)\right)dy\right|\\ &\leqq C_1D_{l-1}\int\limits_{\Omega_1} |x-y|^{\alpha-1-n}\left(|x|^\alpha\max\left(|x|^{1-n-\alpha}, |y|^{1-n-\alpha}\right) + |y|^{1-n}\right)dy\\ &\leqq C_1D_{l-1}R^\alpha\,|x|^{1-n-\alpha}\int\limits_{\mathbf{R}^n\setminus B_\varrho(x)} |x-y|^{\alpha-1-n}\,dy = C_1D_{l-1}R^\alpha\,|x|^{1-n-\alpha}\,\varrho^{\alpha-1}\end{aligned}$$

and the same estimate holds for the integral over Ω_2. We have thus shown for points $x \in B_{2R}(0)\setminus B_{2\varrho}(0)$ that

$$|\nabla w_{1\varrho}(x)| \leqq C_1D_{l-1}R^\alpha\,|x|^{1-n-\alpha}\,\varrho^{\alpha-1}. \tag{4.20}$$

By integrating (4.20) over the path $\overrightarrow{x_1x_2}$ (which is contained in $B_{2R}(0)\setminus B_{2\varrho}(0)$) we conclude

$$|W_{1\varrho}(x_1) - W_{1\varrho}(x_2)| \leqq C_1D_{l-1}R^\alpha\,|x_1-x_2|^\alpha\max\left(|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}\right). \tag{4.21}$$

Applying (4.16) in the version for W_1 on the ball $B_{4R}(0)$, (4.21) holds for the function W_1 itself. It therefore remains to prove (4.21) for W_2. For $x \in B_{2R}(0)$ we have by definition

$$\nabla W_2(x) = \int\limits_{T_{2R}(0)} \nabla\Delta(x-y)\,\omega(y)\,\nabla u_{l-1}(y)\,dy .$$

From (4.13) we infer

$$|\nabla W_2(x)| \leqq C_1D_{l-1}R^{1-n+\alpha}\,\delta^{-1}(x), \qquad \delta(x) := 2R - |x| .$$

This estimate shows that ∇W_2 behaves well in the interior of the ball $B_{2R}(0)$. Introducing $\omega(x)\,\nabla u_{l-1}(x)$ in the expression for ∇W_2 and using (4.15), we see

$$|\nabla W_2(x)| \leqq C_1D_{l-1}R^\alpha\delta(x)^{\alpha-1}\,|x|^{1-n-\alpha} \tag{4.22}$$

for $x \in B_{2R}(0) \setminus \{0\}$. (4.22) has the advantage that ∇W_2 increases of lower order when $x \to \partial B_{2R}(0)$. As before let x_1, x_2 satisfy (4.19). We want to show

$$|W_2(x_1) - W_2(x_2)| \leqq C_1 D_{l-1} R^\alpha \, |x_1 - x_2|^\alpha \max(|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}) \tag{4.23}$$

and consider the following cases (compare [15: Thm. 2.6.6]):

1. $|x_i| \leqq 2R - \varrho$: Integration of (4.22) over the path $\overrightarrow{x_1x_2}$ implies (4.23).
2. $|x_1| > 2R - \varrho$, $|x_2| \leqq 2R - \varrho$: Consider the path $\overrightarrow{x_1x_3x_2}$, where x_3 is on the ray $\overrightarrow{0x_1}$ with $|x_3| = 2R - \varrho$. Observing

$$|x_1 - x_3| \leqq \varrho, \quad |x_2 - x_3| \leqq 2\varrho, \quad |x_2 + t(x_3 - x_2)| \geqq 3\,|x_2|/5, \quad 0 \leqq t \leqq 1,$$

we get the estimates

$$|W_2(x_2) - W_2(x_3)| \leqq C_1 D_{l-1} R^\alpha \, |x_2|^{1-n-\alpha} \, \varrho^\alpha,$$

$$|W_2(x_1) - W_2(x_3)| \leqq C_1 D_{l-1} R^\alpha |x_1 - x_3| \int_0^1 (\varrho - t\,|x_1 - x_3|)^{\alpha-1} \, dt \, |x_3|^{1-n-\alpha}$$

$$\leqq C_1 D_{l-1} R^\alpha \, |x_1 - x_3|^\alpha \, |x_2|^{1-n-\alpha}.$$

This proves (4.23) in the second case.

3. $|x_1| > 2R - \varrho$, $|x_2| > 2R - \varrho$: Choose x_3 on $\overrightarrow{0x_1}$, x_4 on $\overrightarrow{0x_2}$ with $|x_3| = |x_4| = 2R - \varrho$. We have

$$|x_1 - x_3| \leqq \varrho, \ |x_2 - x_4| \leqq \varrho, \ |x_3 - x_4| \leqq 3\varrho, \ |x_1 + t(x_3 - x_1)| \geqq 4\,|x_1|/5,$$
$$|x_2 + t(x_4 - x_2)| \geqq 4\,|x_2|/5, \ |x_3 + t(x_4 - x_3)| \geqq 2R - 4\varrho \geqq |x_1|/5$$
for $0 \leqq t \leqq 1$.

Similiar calculations as in the cases 1, 2 yield (4.23).

Collecting our results we have shown

$$|W(x_1) - W(x_2)| \leqq C_1 D_{l-1} R^\alpha \, |x_1 - x_2|^\alpha \max(|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}) \tag{4.24}$$

for points x_1, x_2 with (4.19). By (4.15), inequality (4.24) holds for the function f defined in (4.14). So it remains to consider the case $|x_1 - x_2| \geqq \min(|x_1|, |x_2|)/5$. But under this assumption the Hölder condition for ∇u_l is a trivial consequence of (4.18) ∎

In the scalar case Green's function g can be estimated from below in terms of $|x - y|^{2-n}$, compare [8]. We mention the following weaker result which is valid for $N > 1$.

Proposition: *Under the assumptions of Theorem 7 we have for all $y \in \Omega$ and $k = 1, \ldots, N$*

$$\limsup_{x \to y} |G_k(x, y)| \, |x - y|^{n-2} > 0. \tag{4.25}$$

Proof: We use the notations from the proof of Theorem 7. Since E is homogeneous of degree $2 - n$, $\limsup_{x \to 0} |E(x)| \, |x|^{n-2} = 0$ would imply $E = 0$. Therefore

$$|E(x_\nu)| \, |x_\nu|^{n-2} \geqq C, \qquad \nu \in \mathbf{N}, \tag{4.26}$$

for some constant $C > 0$ and a suitable sequence $x_\nu \to 0$. Recalling (4.9) we obtain the inequality

$$|G(x_\nu)| \geqq |E(x_\nu)| - \left| \sum_{l=1}^\infty (T_{2R}^{p_0})^l \, E(x_\nu) \right| - \left| \sum_{l=1}^\infty (T_{2R}^q)^l \, w(x_\nu) \right|$$

$$\geqq |E(x_\nu)| - C_1 \sum_{l=1}^\infty (C_1 R^\alpha)^l \, |x_\nu|^{2-n} - C_2 R^{1+\beta-n},$$

and by (4.26) we get

$$\limsup_{\nu\to\infty} |x_\nu|^{n-2} |G(x_\nu)| \geqq C - C_1 \sum_{l=1}^{\infty} (C_1 R^\alpha)^l,$$

which proves the proposition if R is chosen small enough ∎

Next we use the well-known Campanato technique (see [3, 7]) together with Theorem 7 to derive gradient bounds for Green's matrix.

Theorem 8: *Suppose that* (4.2) *holds and let* $0 < \beta < 1$ *be given. Then*

$$|\partial G(x, y)/\partial x| \leqq C_1 |x - y|^{1-n} + C_2 |x - y|^{-1} R^{1+\beta-n} \tag{4.27}$$

for all $x \in B_R(y) \setminus \{y\}$, $y \in \Omega$, $0 < R \leqq \min\big(R_0, \operatorname{dist}(y, \partial\Omega)/4\big)$. *Here* C_1, C_2, R_0 *are the constants appearing in Theorem* 7.

Proof: As before we may assume $y = 0 \in \Omega$ and use the notations from the proof of the preceding theorem. Let R satisfy the above hypothesis and choose $z \in B_R(0)$. Define $D = B_{|z|/8}(z)$; for $x_0 \in D$ and $0 < \varrho \leqq r \leqq \operatorname{diam}(D) = |z|/4$ let the function $v \in H^{1,2}\big(B_r(x_0)\big)^N$ be the solution of

$$-D_\gamma\big(A^{ij}_{\gamma\delta}(x_0)\, D_\delta v^j\big) = 0 \text{ on } B_r(x_0), \quad v_{\partial B_r(x_0)} = G_{|\partial B_r(x_0)},$$

which satisfies the Campanato estimate [(7: Thm. 2.1/p. 80])

$$\int_{B_\varrho(x_0)} |\nabla v|^2\, dx \leqq C_1(\varrho/r)^n \int_{B_r(x_0)} |\nabla v|^2\, dx \tag{4.28}$$

with $C_1 = C_1(n, N, \lambda, \Lambda)$. The function $w = G - v$ is a solution of

$$\int_{B_r(x_0)} A^{ij}_{\gamma\delta}(x_0)\, D_\gamma \Phi^i D_\delta w^j\, dx = -\int_{B_r(x_0)} \big(A^{ij}_{\gamma\delta} - A^{ij}_{\gamma\delta}(x_0)\big)\, D_\gamma \Phi^i D_\delta G^j\, dx$$

for all $\Phi \in \mathring{H}^{1,2}\big(B_r(x_0)\big)^N$. Inserting $\Phi = w$, a simple calculation shows

$$\int_{B_r(x_0)} |\nabla w|^2\, dx \leqq C_1 r^{2\alpha} \int_{B_r(x_0)} |\nabla G|^2\, dx. \tag{4.19}$$

Here C_1 has the former meaning. Combining (4.28) and (4.29) we get the following growth condition for

$$\varphi(t) = \int_{B_t(x_0)} |\nabla G|^2\, dx \colon \varphi(\varrho) \leqq C_1\big((\varrho/r)^n + r^{2\alpha}\big)\, \varphi(r),$$

and a well-known iteration lemma due to Giusti [7: Lemma 2.1/p. 87] implies for all $0 < \varrho \leqq r \leqq \operatorname{diam}(D)$

$$\varphi(\varrho) \leqq C_1(\varrho/r)^{n-\alpha}\, \varphi(r), \tag{4.30}$$

provided R_0 is sufficiently small. Since this smallness condition on R_0 involves the parameters $n, N, L, \lambda, \Lambda, \alpha$ we may assume that in the beginning R_0 has been chosen in the right way. Let

$$\psi(\varrho) := \int_{B_\varrho(x_0)} |\nabla G - (\nabla G)_{x_0,\varrho}|^2\, dx, \qquad (\nabla G)_{x_0,\varrho} := \fint_{B_\varrho(x_0)} \nabla G\, dx;$$

according to [7: Thm. 2.1/p. 80] the function v satisfies

$$\int_{B_\varrho(x_0)} |\nabla v - (\nabla v)_{x_0,\varrho}|^2\, dx \leqq C_1(\varrho/r)^{n+2} \int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0,r}|^2\, dx. \tag{4.31}$$

Comparing G and v as above, we get from (4.29)—(4.31)

$$\psi(\varrho) \leqq C_1\big((\varrho/r)^{n+2}\, \psi(r) + r^{2\alpha} \varphi(r)\big) \leqq C_1\big((\varrho/r)^{n+2}\, \psi(r) + r^{n+\alpha} d^{\alpha-n}\, \varphi(d)\big),$$

$d := \operatorname{diam}(D)$. From the iteration lemma cited above we infer

$$\psi(\varrho) \leqq C_1 \varrho^{n+\alpha}\{r^{-n-\alpha}\psi(r) + d^{\alpha-n}\varphi(d)\}, \qquad 0 < \varrho < r \leqq d.$$

Choosing $r := d$ we finally arrive at

$$\varrho^{-n-\alpha}\psi(\varrho) \leqq C_1 \, |z|^{-n-\alpha} \int_{B_{|z|/2}(z)} |\nabla G|^2 \, dx \tag{4.32}$$

for all $x_0 \in D$ and $0 < \varrho \leqq d$. Thus ∇G belongs to the Campanato space $L^{2,n+\alpha}(D)^{nN}$ (compare [7]), and from (4.32) we get the bound

$$\sup_D |\nabla G| \leqq C_1 \, |z|^{-n/2} \, \|\nabla G\|_{L^2(B_{|z|/2}(z))}.$$

The Dirichlet integral of G over $B_{|z|/2}(z)$ is controlled by

$$C_1 \, |z|^{-1} \, \|G\|_{L^2(B_{3|z|/4}(z))}.$$

Estimating the L^2-norm of G with the help of Theorem 7 we get (4.27) ∎

We just showed that ∇G belongs to the space $L^{2,n+\alpha}(D)$, i.e. $\nabla G \in C^{0,\alpha/2}(D)^{nN}$, but the proof of Theorem 8 contains more information: Take two points $z, \bar z \in B_{R/2}(0) \setminus \{0\}$, $|z| \leqq |\bar z|$. Defining $M = \sup\{|\nabla G(x)| : x \in B_{|z|/2}(z)\}$ we get $\psi(\varrho) \leqq C_1\big((\varrho/r)^{n+2}\,\psi(r) + r^{n+2\alpha}M^2\big)$ and by the iteration lemma

$$\psi(\varrho) \leqq C_1 \varrho^{n+2\alpha}\{d^{-n-2\alpha}\psi(d) + M^2\} \leqq C_1 \varrho^{n+2\alpha} d^{-2\alpha} M^2.$$

This implies

$$\sup\{|\nabla G(a) - \nabla G(b)|/|a-b|^\alpha : a \neq b \in \bar D\} \leqq C_1 M \, d^{-\alpha}. \tag{4.33}$$

Case 1: $\bar z \in \bar D$. Then (4.27) and (4.33) give

$$|\nabla G(z) - \nabla G(\bar z)| \leqq |z - \bar z|^\alpha \, \{C_1 \, |z|^{1-n-\alpha} + C_2 \, |z|^{-1-\alpha} \, R^{1+\beta-n}\}.$$

Case 2: $\bar z \notin \bar D$. Observing $|z - \bar z| \geqq |z|/8$, the above estimate is a trivial consequence of (4.27).

We state these facts in

Theorem 9: *If* (4.2) *is satisfied and if* $0 < \beta < 1$ *is given, then*

$$\begin{aligned}|\nabla_x G(x, y) - \nabla_x G(\bar x, y)| \leqq \{&C_1 \max\left(|x-y|^{1-n-\alpha}, |\bar x - y|^{1-n-\alpha}\right) \\ &+ C_2 \max\left(|x-y|^{-1-\alpha}, |\bar x - y|^{-1-\alpha}\right) R^{1+\beta-n}\} \, |x - \bar x|^\alpha\end{aligned}$$

for all $x, \bar x \in B_{R/2}(y) \setminus \{y\}$, $y \in \Omega$, $0 < R \leqq \min\big(R_0, \operatorname{dist}(y, \partial\Omega)/4\big)$. *Here* C_1, C_2 *and* R_0 *are as in Theorem* 7.

According to this estimate the α-Hölder constant for $\partial G(\cdot, y)/\partial x$ on rings $B_{2r}(y) \setminus B_r(y)$ with sufficiently small radius grows of order $r^{1-n-\alpha}$ when r becomes smaller.

5. Some applications

As Green's function for a single elliptic operator has become a useful tool in various fields (compare [9—11, 14]) we want to give two applications of Green's matrix. We start with the description of the behaviour of a week solution to a homogeneous elliptic system having an isolated singularity of prescribed growth. Then either the singularity is removable or of order $2 - n$. Our result corresponds to a well-known fact for harmonic functions.

Theorem 10: *Suppose that* (GA) *holds and that the coefficients* $A^{ij}_{\alpha\beta}$ *satisfy a Lipschitz condition on* $\bar{\Omega}$. *Let* $\partial\Omega$ *be of class* C^2 *and let* $u \in H^{1,1}_{\mathrm{loc}}(\Omega \setminus \{y\})^N$, $y \in \Omega$, *be a weak solution to the system* $L_{ij}u^j = 0$ *on* $\Omega \setminus \{y\}$ *having the following properties:*

$$\left.\begin{array}{l}\text{(i) There are constants } C,\ \varepsilon > 0 \text{ such that } |u(x)| \leqq C\,|x-y|^{\varepsilon+1-n} \text{ for points } x \text{ near } y.\\ \text{(ii) For some } \delta > 0 \text{ and all balls } B_\varrho(y) \text{ with sufficiently small radius } u\\ \text{belongs to the space } H^{1,1+\delta}\big(\Omega \setminus B_\varrho(y)\big)^N \text{ having boundary values zero on } \partial\Omega.\end{array}\right\} \tag{5.1}$$

Then $u = C_i G_i(\cdot, y)$ *on* Ω *with suitable constants* $C_i \in \mathbf{R}$.

Proof: We may assume $y = 0 \in \Omega$. It is easy to see (compare Lemma 3.2) that $u \in H^{1,r}\big(\Omega \setminus B_\varrho(0)\big)^N$ for all $1 \leqq r < \infty$ and small radii ϱ, and the difference quotient method gives $u \in H^{2,2}_{\mathrm{loc}}(\Omega \setminus \{0\})^N$. Since u vanishes at the boundary we have the stronger result $u \in H^{2,2}\big(\Omega \setminus B_\varrho(0)\big)^N$ (compare [8: proof of Thm. 8.12]; the technique described there also applies to elliptic systems). Moreover, (i) implies the gradient bound (see proof of Theorem 8)

$$|\nabla u(x)| \leqq C\,|x|^{\varepsilon-n} \tag{5.2}$$

for all points x near the origin, where C is independent of x. Now choose $0 < R_0 < \operatorname{dist}(0, \partial\Omega)$, $0 < \varrho < R < R_0$ and $z \in B_{R_0}(0) \setminus B_R(0)$. Let $w = {}^tG_k(\cdot, z)$. Then the relation $\int_\Omega A^{ij}_{\alpha\beta} D_\alpha w^i D_\beta \Phi^j\, dx = \Phi^k(z)$ holds for all $\Phi \in C_0^\infty(\Omega)^N$. Take a cut-off function $\eta \in C_0^\infty(\Omega)$ such that $\eta = 1$ on $T = B_{R_0}(0) \setminus B_R(0)$, $\eta = 0$ on $B_{R/2}(0)$ and insert $\Phi = \eta u$ in the above identity to get

$$u^k(z) = \int_T A^{ij}_{\alpha\beta} D_\alpha w^i D_\beta u^j\, dx + \int_{\Omega\setminus T} A^{ij}_{\alpha\beta} D_\alpha w^i D_\beta(\eta u^j)\, dx =: (1) + (2).$$

Since $L_{ij}u^j = 0$ on T we see that (ν is the outer normal to ∂T)

$$(1) = \int_{\partial T} A^{ij}_{\alpha\beta} w^i D_\beta u^j \nu_\alpha\, d\mathbf{H}^{n-1}.$$

Observing $w \in H^{2,2}(\Omega \setminus T)^N$ and $L^t_{ij} w^j = 0$ on $\Omega \setminus T$ we get

$$(2) = -\int_{\partial T} A^{ij}_{\alpha\beta} D_\alpha w^i u^j \nu_\beta\, d\mathbf{H}^{n-1}.$$

For functions $f, g : \partial B_r(0) \to \mathbf{R}^N$ we let

$$M(f, g) = A^{ij}_{\alpha\beta}(g^i D_\beta f^j \nu_\alpha - D_\alpha g^i f^j \nu_\beta), \quad \nu(x) = x/r.$$

Obviously, the above relations can be rewritten as

$$u^k(z) = \int_{\partial B_{R_0}(0)} M(u, w)\, d\mathbf{H}^{n-1} - \int_{\partial B_R(0)} M(u, w)\, d\mathbf{H}^{n-1}.$$

Since $u_{|\partial\Omega} = w_{|\partial\Omega} = 0$ the first integral vanishes, and integration by parts shows the identity

$$\int_{\partial B_R(0)} M(u, w)\, d\mathbf{H}^{n-1} = \int_{\partial B_\varrho(0)} M(u, w)\, d\mathbf{H}^{n-1},$$

and we arrive at the formula

$$u^k(z) = -\int_{\partial B_\varrho(0)} M(u, w)\, d\mathbf{H}^{n-1} \tag{5.3}$$

for all sufficiently small values of ϱ. By definition we have

$$\begin{aligned}\int_{\partial B_\varrho(0)} M(u, w)\, d\mathrm{H}^{n-1} &= w^i(0) \int_{\partial B_\varrho(0)} A^{ij}_{\alpha\beta} D_\beta u^j \nu_\alpha \, d\mathrm{H}^{n-1}\\ &+ \int_{\partial B_\varrho(0)} A^{ij}_{\alpha\beta}\big(w^i - w^i(0)\big) D_\beta u^j \nu_\alpha\, d\mathrm{H}^{n-1} - \int_{\partial B_\varrho(0)} A^{ij}_{\alpha\beta} D_\alpha w^i u^j \nu_\beta \, d\mathrm{H}^{n-1}\\ &=: w^i(0)\cdot a_i + b - c.\end{aligned}$$

Since $L^t_{ij} w^j = 0$ on $B_R(0)$, $m = \sup\{|\nabla w(x)| : x \in B_{R/2}(0)\}$ is finite and we infer from (5.1), (5.2) $|b| \leqq C\Lambda m \varrho \varrho^{n-1} \varrho^{\varepsilon-n} = \text{const}\, \varrho^\varepsilon$, $|c| \leqq \text{const}\, \varrho^\varepsilon$. Moreover,

$$a_i = \int_{\partial B_{R_0}(0)} A^{ij}_{\alpha\beta} D_\beta u^j \nu_\alpha \, d\mathrm{H}^{n-1} =: C_i.$$

Now passing to the limit $\varrho \to 0$, (5.3) becomes $u^k(z) = C_i G_i^k(z, 0)$ for all $z \in B_{R_0}(0)$. Corollary 1 of Theorem 1 implies $u = C_i G_i(\cdot, 0)$ all over Ω ∎

1. According to Theorem 10 the column vectors of Green's matrix $G(\cdot, y)$ form a basis of the space X of all non-trivial solutions with zero boundary values to the system $L_{ij} u^j = 0$ on $\Omega \setminus \{y\}$ satisfying the growth condition (5.1) which requires a growth order less than $|x - y|^{1-n}$, when $x \in \Omega$ approaches the point y. After Theorem 7 we remarked $\limsup\limits_{x\to y} |G(x, y)|\, |x - y|^{n-2} > 0$ and in Theorem 7 we proved an inequality of the form $|G(x, y)| \leqq C\, |x - y|^{2-n}$. On the other hand, $u \in X \setminus \{0\}$ cannot satisfy a local growth condition of the form $|u(x)| \leqq C\, |x - y|^{2+\varepsilon-n}$ for some positive ε (this would imply $\nabla u \in L^{n/(n-1)}$ near y and therefore $u = 0$, compare the following remark), so that the statement of Theorem 10 can be reformulated as follows: If u is a non trivial solution of the system $L_{ij} u^j = 0$ on $\Omega\{y\}$ with zero boundary values which increases of order less than $|x - y|^{1-n}$, when $x \to y$, then the growth order of u is exactly $|x - y|^{2-n}$.

2. Let us replace condition (5.1) by

$$u \in \mathring{H}^{1,1+\delta}(\Omega)^N \text{ for some } \delta > 0. \tag{5.1$'$}$$

For $\delta \geqq 1/(n - 1)$ an easy calculation shows that u is a weak solution on the whole domain and therefore vanishes identically. Consider the case $p := 1 + \delta \in \big(1, n/(n - 1)\big)$. For sufficiently small values of $|x - y|$ we get the inequalities (assume $y = 0$)

$$|\nabla u(x)| \leqq C\, |x|^{-n/2}\, \|\nabla u\|_{L^2(B_{|x|/2}(x))} \quad \text{(proof of Theorem 8)},$$

$$^*\|u\|_{H^{1,2}(B_{|x|/2}(x))} \leqq C\, |x|^{n/2}\, |x|^{-n/p}\, ^*\|u\|_{H^{1,p}(B_{|x|}(x))},$$

from which we get $|\nabla u(x)| \leqq C\, |x|^{-n/p}\, \|u\|_{H^{1,p}(\Omega)}$. Thus u satisfies a local growth condition of the form (5.2) and by integration we get (5.1) (i). Let us state this observation as

Corollary: *The statement of Theorem 10 continues to hold if the local growth condition in* (5.1) *is replaced by* (5.1)$'$.

We finally return to the representation formula proved in Theorem 6 and want to show that the solution to the boundary value problem (2.1) has certain regularity properties if the measure μ does not behave too bad.

Theorem 11: *Assume that* (4.2) *holds and that* $u \in \mathring{H}$ *is the unique weak solution of* (2.1) *with* $\mu \in M(\Omega)^N$, $\mu^i = \mu_+^{\ i} - \mu_-^{\ i}$, *satisfying*

$$\sum_{i=1}^N \int_\Omega |x - y|^{2-n}\, d(\mu_+^{\ i} + \mu_-^{\ i})\,(x) < \infty \text{ for all } y \in \Omega. \tag{5.5}$$

Then the following statements hold:

(i) *Each point* $y \in \Omega$ *is a Lebesgue point of* u, *i.e. the limit*

$$\overline{u}(y) := \lim_{\varrho\to 0} \fint_{B_\varrho(y)} u\, dx$$

exists for all y. Identifying u with the representative $\bar{u}$ we get

$$u^k(y) = \int_\Omega G_i^k(y, x)\, d\mu^i(x) \qquad (y \in \Omega,\, k = 1, \ldots, N).$$

(ii) *For all $\varepsilon > 0$ and all balls $B_R(y_0) \subset \Omega$ there exists a ball $B_r(y) \subset B_R(y_0)$ such that the oscillation of u on $B_r(y)$ is controlled by ε.*

(iii) *If* (5.5) *is replaced by the stronger condition*

$$\sup \left\{ \sum_{i=1}^{N} \int_\Omega |x - y|^{2-n}\, d(\mu_+{}^i + \mu_-{}^i)\,(x) : y \in \Omega \right\} < \infty, \tag{5.6}$$

then u is locally bounded.

Let $\Sigma \subset \Omega$ denote a compact $(n-1)$-dimensional manifold and define, for $A \subset \Omega$, $\mu^i(A) = \mathbf{H}^{n-1}(A \cap \Sigma)$, $i = 1, \ldots, N$. Obviously, (5.6) is satisfied, so that all statements of Theorem 11 hold. Such properties as local boundedness and generic continuity are not contained in Section 1: Theorem 2 for example only describes the behaviour of u on $\Omega \setminus \Sigma$, so that $u(x)$ could behave most irregularly in the limit $\Omega \setminus \Sigma \ni x \to x_0 \in \Sigma$.

The proof of Theorem 11 is based on ideas from [11: Thm. 2.3, 2.6], which we combine with our previous results. For further details we refer to [5: Section 5, Thm. 12]. Theorem 11 immediately applies to elliptic systems with quadratic growth, which are studied for example in [7, 10, 11].

Theorem 12: *Assume that* (4.2) *holds and let $u \in H^{1,2} \cap L^\infty(\Omega)^N$ be a weak solution of the system $L_{ij}u^j = f^i(\cdot, u, \nabla u)$, $i = 1, \ldots, N$, where f is a Caratheodory function satisfying $|f(x, y, p)| \leqq a\,|p|^2 + b$ with positive constants a, b. If*

$$\int_\Omega |\nabla u(x)|^2\, |x - y|^{2-n}\, dx \text{ is finite for all } y \in \Omega,$$

then each point is a Lebesgue point and u is of class C^1 on a dense open subset of Ω.

In [6] we had described another application of Green's matrix. Suppose that u is a weak minimum of the quadratic functional

$$Fv = \int_\Omega A_{\alpha\beta}^{ij} D_\alpha v^i D_\beta v^j\, dx$$

in a class of functions described by a side condition of the form $v \cdot e \geqq \psi$ for a fixed vector e and a smooth real-valued function ψ. If the coefficients satisfy (4.2), we had shown that u is regular in Ω. The arguments rest on a careful analysis of the sign properties of G combined with potential theoretic considerations.

6. Final remarks

Up to now we only considered the case $n \geqq 3$ and assumed the coefficients of the system to be continuous functions. In two dimensions the existence of Green's matrix can be proved for bounded measurable coefficients satisfying the strong ellipticity condition. This follows from the fact that for $n = 2$ the unique $\mathring{H}^{1,2}$-solution to the system $L_{ij}u^j = -D_\alpha F_\alpha{}^i$, $i = 1, \ldots, N$, on Ω with $F \in L^p(\Omega)^{nN}$ for some $p > 2$ is continuous with vanishing boundary values, so that the method of Section 2 applies (compare [5] for details). In general it is possible to construct Green's matrix by duality whenever regularity theorems are available. In higher dimensions such regularity results for systems with L^∞-coefficients are only true under additional smallness conditions. Thus for proving the existence of G the continuity hypothesis can be dropped in two dimensions and has to be replaced by a smallness condition in higher dimensions, respectively. But it seems to be impossible to derive the standard estimate for $G(x, y)$ in the more general situation: the technique used in this paper does not apply since Lemma 4.3 essentially rests on the Hölder continuity of the coefficient matrix. For continuous coefficients the corollary of Theorem 2 gives the information $|G(x, y)| \leqq C(\varepsilon)\, |x - y|^{\varepsilon+1-n}$ for all $0 < \varepsilon < 1$, but the right growth order can not be achieved by this simple argument.

We wish to remark that in the case $n = 2$ Theorem 7 has the analogue

$$|G(x, y)| \leqq C_1 \log (1/|x - y|) + C_2 R^{\beta-1} \log (R^{-1}) \tag{*}$$

for all $x \in B_{2R}(y)$, $y \in \Omega$, $0 < R \leqq \min (R_0, \operatorname{dist}(y, \partial\Omega)/4)$ (notations as in Theorem 7). The proof uses ideas from Section 4 combined with an appropriate modification of the local estimates in Section 1. From (*) one easily obtains gradient bounds and estimates for the Hölder norm of the first derivatives. For details we again refer to [5].

In this paper we studied Green's matrix for strongly elliptic operators of the form $L = (-D_\alpha(A^{ij}_{\alpha\beta} D_\beta))$. By a suitable extension of the method it is also possible to prove the existence of a Green matrix with the correct growth order for more general operators $\bar{L} = L + (B^{ij}_\alpha D_\alpha + a^{ij})$, provided the leading part L satisfies the Legendre-Hadamard condition and some mild regularity properties are imposed on the lower order terms. Since the details are somewhat technical, this generalization is discussed in a separate paper.

References

[1] Adams, R. A.: Sobolev Spaces. New York: Academic Press 1975.

[2] Agmon, S., Douglis, A., and L. Nirenberg: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, II. Comm. Pure Appl. Math. **12** (1959), 623–727 and **17** (1964), 35–92.

[3] Campanato, S.: Equazioni ellitiche del secondo ordine e spazi $L^{2,\lambda}$. Ann. Math. Pura Appl. **69** (1965), 321–380.

[4] Federer, H.: Geometric Measure Theory. Berlin—Heidelberg—New York: Springer Verlag 1969.

[5] Fuchs, M.: Die Green-Matrix für elliptische Systeme zweiter Ordnung in Divergenzform mit stetigen Koeffizienten. Inaugural-Dissertation. Düsseldorf: Universität Düsseldorf 1983.

[6] Fuchs, M.: Ein Regularitätssatz für ein lineares System von Variationsungleichungen mit einer Halbraumnebenbedingung. Z. Anal. Anw. **5** (1986), 47–57.

[7] Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems. Princeton (New Jersey): Princeton University Press. 1983.

[8] Gilbarg, D., and N. S. Trudinger: Elliptic Partial Differential Equations of Second Order. Berlin—Heidelberg—New York: Springer Verlag 1977.

[9] Grüter, M., and K. O. Widman: The Green function for uniformly elliptic equations. Manus. math. **37** (1982), 303–342.

[10] Hildebrandt, S., and K. O. Widman: Some regularity results for quasilinear elliptic systems of second order. Math. Z. **142** (1975), 67–86.

[11] Hildebrandt, S., and K. O. Widman: On the Hölder continuity of weak solutions of quasilinear elliptic systems of second order. Ann. Scuola Norm. Sup. Pisa (IV) **4** (1977), 145–178.

[12] Hörmander, L.: Linear Partial Differential Operators. Berlin—Heidelberg—New York: Springer Verlag 1976.

[13] John, F.: Plane Waves and Spherical Means. Berlin—Heidelberg—New York: Springer Verlag 1955.

[14] Littman, W., Stampacchia, G., and H. F. Weinberger: Regular points for elliptic equations with discontinuous coefficients. Ann. Scuola Norm. Sup. Pisa (III) **17** (1963), 43–77.

[15] MORREY, C. B. jr.: Multiple Integrals in the Calculus of Variations. Berlin—Heidelberg—New York: Springer Verlag 1966.
[16] SERRIN, J.: Pathological solutions of elliptic equations. Ann. Scuola Norm. Sup. Pisa 18 (1964), 385—387.
[17] SOLONNIKOV, V. A.: On general boundary problems for systems which are elliptic in the sense of A. Douglis and L. Nirenberg. I, II. Amer. Math. Soc. Trans. (Series 2) 56 (1966), 193—232, and Proceedings of the Steklov Institute of Mathematics 92 (1966), 269—339.

Manuskripteingang: 10. 12. 1984

VERFASSER:

Dr. MARTIN FUCHS

Mathematisches Institut der Universität
D-4000 Düsseldorf, Universitätsstraße 1

Buchbesprechung

G. HEINIG and K. ROST: **Algebraic Methods for Toeplitz-like Matrices and Operators** (Math. Forschung: Bd. 19). Berlin: Akademie-Verlag 1984, p. 212.

The book under review is divided into two parts: Part I — "Toeplitz and Hankel matrices" — contains 9 chapters (sections, according to the authors' terminology) and Part II — "Toeplitz-like operators" — contains 7 chapters. The book is supplied with a list of references (117 titles), subject and notation indices.

Many characteristic properties of Toeplitz matrices can be deduced from the fact that $AU_n - U_mA$ has at most rank two; here A is an $m \times n$ Toeplitz matrix and U_n is the matrix of forward shift in the space $\mathbf{C}^n$. This leads the authors to consider such operators A for which rank $AU - VA$ is small compared to rank A; U and V being some fixed operators. The authors call such operators A Toeplitz-like operators. Let us explain the contents of the book under review in more detail.

Part I is devoted to the algebraic theory of finite Toeplitz and Hankel matrices (T- and H-matrices). The main problems are the following:

1. Fast inversion algorithms.
2. Structure of T- and H-matrices, their rank and signature and the relation between T- and H-matrices.
3. Application to some problems of Wiener-Hopf equations theory.

Chapter 0 contains some facts which are utilized on the full length of the book; in particular the notions of T- and H-matrices are formulated and some special matrices of these classes are described.

In Chapter 1 the problem of inversion of T- and H-matrices is investigated. It is shown that one can reduce the last problem to the problem of the solution of two special linear algebraic systems, which the authors call "fundamental equations". If the fundamental equations are solvable then the given matrix is invertible and one receives a simple inversion formula. The right-hand side of the second fundamental equation depends on the given matrix. If this matrix satisfies some complementary conditions then it is possible to change the right-hand side of the equation by a certain fixed vector. Further, operators Δ and ∇ are introduced, each of which transforms any $m \times n$ matrix into an $(m + 1) \times (n + 1)$ matrix.