The Green Matrix for Strongly Elliptic Systems of Second Order Estischrift

with Green Matrix for Strongly Elliptic Systems of Second Order

with Continuous Coefficients

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with Continuous Coe
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Wir untersuchen eine Verailgemeinerung des bekannten Konzepts der Green-Funktion für elliptische Gleichungen auf den Fall elliptischer Systemezweiter Ordnung mit stetigen Koeffizienten. Wir beweisen Existenz und Eindeutigkeit einer solchen Green-Matrix G und disku. tieren mit potentialtheoretischen Methoden das Wachstumsverhalten in der Nähe der singulären Diagonalen. Als eine mögliche Anwendung betrachten wir Système mit vektorwertigen Mal3en als rechter Seite und leiten Darstellungsformeln für die Losung ab, ausdenen sich unter geeigneten Voraussetzungen neue Inlormationen gewinnen lassen.

Исследуется обобщение известной концепции функции Грина для эллиптических уравнений на случай эллиптических систем второго порядка с непрерывными коэффициен. тами. Доказываются существование и единственность такой матрицы Грина и теоретикопотенциалъ̀ными методами обсуждается ее тип роста вблизи сингулярной диагонали.
Как возможное применение рассматриваются системы с правой частью в виде векторвееденее Voraussetzungen neue Informationen gewinnen lassen.
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та значной меры и выводятся формулы представления для решения, из которых при подходящих предположениях можно получить дополнительную информацию.

We study a generalization of the Green function for elliptic equations to elliptiè systems of second order with continuous coefficients. The existence and uniqueness of such a Green matrix as well as various estimates concerning the growth properties near the singular diagonal are proved. Moreover, one can derive representation formulas for solutions of elliptic systems and deduce from these further information about the solution even in the case when the right-hand side of the system is a vector-valued measure of bounded variation. Нак возможное применение рассматриваются системы
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0. Introduction

In this paper we are concerned with the Green matrix for uniformly elliptic systems of the type

$$
L_{i j} u^{j} := -D_{\mathfrak{a}} (A^{ij}_{\mathfrak{a} \beta} D_{\beta} u^{j}), \qquad i = 1, ..., N,
$$

on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$. We assume that the coefficients are continuous functions on \overline{Q} .

For a single elliptic operator $(N = 1)$ the existence and the properties of a Green function have been completely analysed in a recent paper of GRUTER and WIDMAN [9], where also various applications are-treated. Their main result is: There always exists a unique Green function g which satisfies pe
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 $g(x, y) \leq C |x - y|^{2-n}$
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$$
|x - y|^{2-n} \le g(x, y) \le C |x - y|^{2-n}
$$

with positive constants c, C . In the case of elliptic systems $(N > 1)$ very little is known. By means of Fourier transforms it is easy to show that for operators (0.1) with constant coefficients there exists at least a fundamental matrix which is homogeneous of degree $2 - \dot{n}$ (compare [13, 15, 17]). Apart from this, only the case of C^{∞} -coefficients is treated: Jonn [13], for

(0.2)

example, shows the existence of a local fundamental solution and his proof uses the smoothness of the coefficients in a very essential way so that his method cannot be applied to operators with continuous coefficients. Global constructions of abstract Green operators G associated with a boundary value problem can be found in the book of HöRMANDER [12], where one also finds the remark that G is related to a "kernel function", but the properties of the kernel are not examined in detail. *M.* Fucus
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 erandary value problem can be found in the boo

One could ask if it is worthwhile considering the case of continuous coefficients since we only deal with linear elliptic systems. This question has a very natural answer in the setting of nonlinear problems: Let $u: \Omega \to \mathbb{R}^N$ be a minimum of the functional

$$
F(v) := \int\limits_{\Omega} a^{ij}_{\alpha\beta}(\cdot, v) D_{\alpha}v^{i}D_{\beta}v^{j} dx
$$

with coefficients in $C^{\infty}(\Omega \times \mathbb{R}^N)$ satisfying the strong ellipticity condition. The Euler operator associated with *F* has the form (0.1) with $A(x) := a(x, u(x))$. Since *u* is a minimum point of the functional F (defined on the Sobolev space $H^{1,2}$) the regularity theory for variational integrals implies $u \in C^{0,a}$ at least on a great portion of Ω , compare [7], so that we arrive at an elliptic system with continuous coefficients. Despite of this fact it would be desirable to prove existence of Green's matrix for systems with bounded measurable coefficients. But the experience in elliptic regularity theory shows that the existence of a Green function (with estimates) is equivalent to regularity theorems for weak solutions to the homogeneous equation. Since such regularity results fail to hold in the vector-valued case (compare the counter examples in [7]) we have to restrict ourselves to continuous coefficients. *'L11u* = *—Dj.i* on Q, *ⁱ*= 1, ..., *N, F E L' (Q)'.* tion. The Euler
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in My and the existence of a Green function (with estimates) is equivalent to regularity theorems for weak solutions to the homogen

In Section 1 of our paper we collect known regularity results for weak solutions of the system

$$
L_i u^j = -D_a F_a^i \text{ on } \Omega, \qquad i = 1, ..., N, F \in L^p(\Omega)^{nN}.
$$
 (0.3)

The basic statement is: For $p>n$ the unique $\hat{H}^{1,2}(\Omega)^N$ -solution of (0.3) is continuous up to the boundary. Moreover, Section 1 contains various local estimates in *LP* for weak solutions of (0.3) which will be useful later. Using the global regularity theorem, we solve in Section 2 the boundary value problem

$$
L_{ij}u^{j} = \mu^{i} \text{ on } \Omega, \qquad i = 1, ..., N, u_{|\partial\Omega} = 0,
$$
\n(0.4)

where μ is a vector-valued signed Radon measure of bounded total variation. Here we follow an idea of LITTMAN, STAMPACCHIA and WEINBERGER [14] which can roughly be described as "duality method". By choosing special measures μ we conclude existence (and uniqueness) of a Green matrix G to the operator $(L_{ij})_{1 \leq i,j \leq N}$ on the domain Q. Simple properties of *0* such as certain symmetry relations and continuity on $\Omega \times \Omega \setminus \{x, x\}$: $x \in \Omega$ are investigated in Section 3. Here we make use of the precise local results summarized in Section 1. *Compared Section 1*
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Finally we show that the solution *u* to problem (0.4) can be written as-the convolution $u = G * \mu$. In Section 4 we discuss the growth properties of G near the singular diagonal. Assuming a Hölder condition for the coefficients we show by a perturbation argument that G satisfies the standard estimate (compare (0.2)) $G * \mu$. In S
Assuming
ment that $|G(x, y)| \le$
ocally in the $\frac{1}{2}$

$$
|G(x, y)| \leq C |x - y|^{2-n}
$$
 (0.5)

at least locally in the interior of Q , i.e. on small balls compactly contained in Q . By Campanato type arguments we infer from (0.5) the local gradient bound $|\partial G(x, y)|\partial x|$ $\leq C |x - y|^{1-n}$. In the final chapter we give two applications of Green's matrix: First we describe the behaviour of a weak solution u to the homogeneous system with zero boundary values having an isolated singularity at $y \in \Omega$, where u grows of order

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less than $|x - y|^{1-n}$.
second application w
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 $|x - y|^{1-n}$. Then either u vanishes opplication we show that the weak sol

the measure μ satisfies the condition

sup $\int_{\theta} |x - y|^{2-n} d|\mu| < \infty$.

use the estimates from Section 4. The Green Matrix for Strongly E
 i y_1^{1-n} . Then either *u* vanishes or grows exactly of

tion we show that the weak solution of (0.4) has

measure μ satisfies the condition
 $\int |x - y|^{2-n} d|\mu| < \infty$.

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\sup_{y\in\Omega}\int\limits_{\Omega}|x-y|^{2-n}\,d|\mu|<\infty
$$

Here we use the estimates from Section 4.

Notations: We make the following *general assumptions (GA):*

EXECUTE: $\int_{y \in \Omega} |x - y|^{2-n} d|\mu| < \infty$.

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I. Ω is a bounded Lipschitz domain in \mathbb{R}^n , $n \geq 3$.

II. Let $N \in \mathbb{N}$ be fixed and consider functions $A_{\$

$$
\alpha, \beta = 1, ..., n. \text{ We define}
$$
\n
$$
(L_{ij}) = \left(-D_a(A_{ab}^{ij}D_{\beta})\right) \text{ and } (L_{ij}^t) = \left(-D_a(A_{ba}^{ji}D_{\beta})\right).
$$
\nWe assume that there are numbers $\lambda, \Lambda > 0$ such that\na) max $\{\|A_{ab}^{ij}\|_{L^{\infty}}: i, j = 1, ..., N; \alpha, \beta = 1, ..., n\} \le$ \nb) $A_{ab}^{ij}(x) P_a^i P_{\beta}^j \ge \lambda |P|^2 \text{ for all } x \in \Omega, P \in \mathbb{R}^{nN} \text{ (stroObviously III b) implies the weaker Legendre Hadan$

III. We assume that there are numbers $\lambda, \Lambda > 0$ such that

 $|P|^2$ for all $x \in \Omega$, $P \in \mathbb{R}^{nN}$ (strong ellipticity).

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\n- III. We assume that there are numbers
$$
\lambda, \Lambda > 0
$$
 such that
\n- a) max $\{\|A_{ij}^{ij}\|_{L^{\infty}}: i, j = 1, ..., N; \alpha, \beta = 1, ..., n\} \leq \Lambda$,
\n- b) $A_{ab}^{ij}(x) P_a^i P_j^j \geq \lambda |P|^2$ for all $x \in \Omega$, $P \in \mathbb{R}^{n}$ (strong elements) Obviously IIIb) implies the weaker Legendre Hadamard IV. $A_{ab}^{ij}(x) w^i w^j \eta_a \eta_\beta \geq \lambda |w|^2 |\eta|^2$ for $x \in \Omega$, $\eta \in \mathbb{R}^n$, $w \in \mathbb{R}^n$. Here and in the sequel we use summation convention: Greek (Latin
\n

Here and in the sequel we use *summation convention*: Greek (Latin) indices repeated twice are summed from 1 to $n(N)$. If D is an open subset of Ω , we denote by $L_{loc}^n(D)^M$, b) $A_{i\beta}^{ij}(x) P_a^i P_j^j \geq \lambda |P|^2$ for all $x \in \Omega$, $P \in \mathbb{R}^{nN}$ (strong ellipticity).

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IV. $A_{i\beta}^{ij}(x) w^i w^j \eta_a \eta_\beta \geq \lambda |w|^2 |\eta|^2$ for $x \in \Omega$, $\eta \in \mathbb{$ tions $u \in H^{k,p}(B)^M$ we introduce the weighted norm IV. $A_{i\beta}^{ij}(x) w^i w^j \eta_a \eta_i$

Here and in the sequel

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 $H_{(10c)}^{k,p}(D)^M$, $\dot{H}^{k,p}(D)^M$ th

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 $||u||_{H^{k,p}(B)} = \sum_{i=0}^k$

If $f \in L_{10c}^1(\Omega)^N$, F *b)* $A_{\alpha\beta}^{ij}(x) P_{\alpha} \,^i P_{\beta}^j \geq \lambda |P|^2$ for all $x \in \Omega$, $P \in \mathbb{R}^{nN}$ (strong ellip

Obviously III b) implies the weaker Legendre Hadamard co

IV. $A_{\alpha\beta}^{ij}(x) w^{ij} \eta_{\alpha} \eta_{\beta} \geq \lambda |w|^2 |\eta|^2$ for $x \in \Omega$, $\eta \in \mathbb{$

$$
||u||_{H^{k,p}(B)} = \sum_{i=0}^k r^{-i} ||\nabla^{k-i}u||_{L^p(B)}.
$$

If $f \in L_{loc}^1(\Omega)^N$, $F \in L_{loc}^1(\Omega)^{nN}$, we call $u \in H_{loc}^{1,1}(\Omega)^N$ a weak solution of the system $L_{ij}u^j$

$$
\int\limits_{\Omega} A^{ij}_{\alpha\beta}D_{\alpha}\Phi^i D_{\beta}u^{\dagger}\,dx=\int\limits_{\Omega} (\Phi^i)^i + F_{\alpha}^i D_{\alpha}\Phi^i)\,dx
$$

for all $\Phi \in C_0^{\infty}(\Omega)^N$. For the adjoint operator we have obvious modifications. In the sequel we will denote all constants by the symbol *C* and it will be clear from the context on which- parameters *C* depends. 1. Regularity results for linear elliptic systems
 $u \in H^{k,p}(B)^M$ we introduce the weighted norm
 $||u||_{H^{k,p}(B)} = \sum_{i=0}^{k} r^{-i} ||\nabla^{k-i}u||_{L^{p}(B)}$.

If $f \in L_{\text{loc}}^1(\Omega)^N$, $F \in L_{\text{loc}}^1(\Omega)^{nN}$, we call $u \in H_{\text{loc}}^{1,1}(\Omega)^N$

The purpose of this section is twofold: We firstly collect well-known global regularity results (compare [2, 15,17]) for linear elliptic systems which play an essential role in proving the existence and uniqueness of a Green matrix G to the system under consideration. Secondly we establish local L^p -estimates for weak solutions from which we derive various properties of G . for all $\Phi \in C_0^{\infty}(\Omega)^N$. For t
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1. Regularity results for li
The purpose of this section is
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1. Regularity results for linear elliptic syste (hich parameters C depends.

Trity results for linear elliptic systems

se of this section is twofold: We firstly collect well-known global regularity results

2, 15, 17]) for linear elliptic systems which play an essenti

We look at weak solutions $u:Q \to \mathbb{R}^N$ of the system

$$
L_{ij}u^{j} = f^{i} - D_{a}F_{a}^{i}, \qquad i = 1, ..., N,
$$
\n(1.1)

$$
(GA), A_{a\beta}^{ij} \in C^{0}(\bar{\Omega}), \qquad i, j = 1, ..., N; \alpha, \beta = 1, ..., n.
$$
 (1.2)

With the exception of Corollary 2 to Theorem 1 all results of this section remain true if we drop the strong ellipticity condition **(GA) III b).**

Theorem 1 (MORREY [15: Thm. 6.4.8]): For $1 < p, q < \infty$ let $u \in H^{1,p}(\Omega)^N$ be a *weak solution of the system* (1.1) with $f \in L^q(\Omega)^N$, $F \in L^q(\Omega)^{nN}$. Moreover assume that (1.2) *holds and that* $w \in H^{1,q}(\Omega)^N$ *satisfies* $u - w \in \mathring{H}^{1,p}(\Omega)^N$. Then $u \in H^{1,q}(\Omega)^N$ and is exception of Corollary 2 to Theorem 1 all results of this section remain true if we
trong ellipticity condition (GA) IIIb).

Em 1 (MORREY [15: Thm. 6.4.8]): For $1 < p, q < \infty$ let $u \in H^{1,p}(\Omega)^N$ be a

tion of the system

$$
||u||_{H^{1,q}(\Omega)} \leq C(||w||_{H^{1,q}(\Omega)} + ||f||_{L^q(\Omega)} + ||F||_{L^q(\Omega)} + ||u||_{L^1(\Omega)}).
$$
\n(1.3)

there is a constant $C = C(n, N, \lambda, A, p, q, \omega, \Omega)$ *such that*
 $\|u\|_{H^{1,q}(\Omega)} \leq C(\|w\|_{H^{1,q}(\Omega)} + \|f\|_{L^q(\Omega)} + \|F\|_{L^q(\Omega)} +$

The symbol ω denotes the modulus of continuity of the coe

satisfy a uniform Hölder condition with The symbol ω denotes the modulus of continuity of the coefficients. If for example the A^{ij}_{α} satisfy a uniform Hölder condition with exponent δ and Hölder constant *L*, then ω is determined
by *L* and δ . **.:**

Corollary 1: For any $1 < p \leq \infty$ the homogeneous system associated with (1.1) has *only the trivial solution in the space* $\hat{H}^{1,p}(\Omega)^N$ *, provided condition (1.2) is satisfied.*

Here and in Theorem 1 the continuity assumption on the coefficients cannot be dropped since there are counter-examples of **SERRIN** [16] even in the case of a single equation.

Corollary 2: *Suppose that* $n < p < \infty$ *is given and that* $u \in \mathring{H}^{1,2}(\Omega)^N$ *is the unique Hilbert space solution to* (1.1) with (1.2), where $\check{f} \in L^p(\Omega)^N$, $F \in L^p(\Omega)^{nN}$. Then $u \in C^0(\Omega)^N$.
 $\cap \check{H}^{1,p}(\Omega)^N$ and we have the estimate
 $||u||_{L^{\infty}(\Omega)}, ||u||_{H^{1,p}(\Omega)} \leq C(||F||_{L^p(\Omega)} + ||f||_{L^p(\Omega)}),$ (1.4) $\int f^{(1,p)}(Q)^N$ *and we have the estimate* Examples of SERRIN [16] even in the case of $\text{Var}_Y 2$: *Suppose that* $n < p < \infty$ is given and that us accusolation to (1.1) with (1.2), where $f \in L^p(\Omega)^N$, $F \in N$ and we have the estimate $||u||_{L^{\infty}(\Omega)}, ||u||_{H^{1,p}(\Omega)} \leq C(||F||$ $L^{1}(\Omega)$).

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 $L^p(\Omega)^{nN}$. The

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||u||_{L^{\infty}(\Omega)}, ||u||_{H^{1,p}(\Omega)} \leq C(||F||_{L^{p}(\Omega)} + ||f||_{L^{p}(\Omega)}), \qquad (1.4)
$$

where the constant C depends on the same parameters as in Theorem 1. *Moreover, u boun'dary values zero in the classical sense.*

-Next we derive local versions of Theorem 1: Consider an arbitrary point $x_0 \in \Omega$, for simplicity we write 0 instead of x_0 in the sequel. We define for $i, j = 1, ..., N$ and € *Rn Louis 2.1.5 in the existent sense.*

We derive local versions of Theorem 1: C
 $L_{0ij} = -A_{\alpha\beta}^{ij}(0) D_{\alpha}D_{\beta}$, $L_{0ij}(\zeta) = -A_{\alpha\beta}^{ij}(0)$

$$
L_{0ij} = -A_{\alpha\beta}^{ij}(0) D_{\alpha}D_{\beta}, L_{0ij}(\zeta) = -A_{\alpha\beta}^{ij}(0) \zeta_{\alpha}\zeta_{\beta},
$$

$$
L_{0ij} = \{x_a_{\beta}(0) \cup \beta_a \cup \beta, L_{0ij}(s) = -x_a_{\alpha\beta}(0) \cup \beta_a \cup \beta\},
$$

$$
L_0(\zeta) = \det (L_{0ij}(\zeta))_{1 \le i,j \le N}, L_0^{ji}(\zeta) = \text{cofactor of } L_{0ij}(\zeta).
$$

 $L_{0,i}(\zeta), L_0(\zeta), L_0(\zeta)$ are homogeneous polynomials of degree 2, 2N, and $2N - 2$ respectively. Finally denote by $\widetilde{L}_0(D)$, $L_0^{\widetilde{J}t}(D)$ the differential operators associated with the polynomials $L_0(\zeta)$, $L_0^{ji}(\zeta)$. By means of Fourier transforms one easily produces a fundamental solution K to the operator $L_0(D)$ which has the following properties (compare [13: p. 69/70] and [15: p. 216/217]). If $L_0(f, L_0^H(\zeta))$ are homogeneous polynomials of degree 2, 2N, and $2N - 2$ re-

I. Finally denote by $L_0(D)$, $L_0^H(D)$ the differential operators associated with

iomials $L_0(\zeta)$, $L_0^{\mathcal{H}}(\zeta)$. By means of Fourie

Lemma 1.1: Under the assumptions (1.2) K is an analytic function on $\mathbb{R}^n \setminus \{0\}$, *essentially homogeneous of degree* $2N - n$. For all $\nu \in N_0$ ⁿ, $|\nu| > 2N - n$, the estimate

$$
|D^r K(y)| \leq C(n, N, \lambda, \Lambda, |\nu|) |y|^{2N-n-|\nu|}, \qquad y \in \mathbf{R}^n \setminus \{0\}, \tag{1.5}
$$

holds. Moreover, K is an even function and satisfies $L_0(D)$ $K = \delta_0$ *(Dirac measure in*) *in the sense of distributions on Rⁿ.* $|D^*K(y)| \leq C(n, N, \lambda, A, |\nu|)$ bolds. Moreover, *K* is an even function at
in the sense of distributions on \mathbb{R}^n .
The following lemma contains all the
operator related to the kernel *K*.
Lemma 1.2 [15; Thm. 6.2.1]: Supp

The following lemma contains all the needed mapping properties of the potential

the polynomials $L_0(\zeta)$, $L_0^{ji}(\zeta)$. By means of Fourier transforms one easily produces
a fundamental solution *K* to the operator $L_0(D)$ which has the following properties
(compare [13: p. 69/70] and [15: p. 216/217 $\begin{aligned} \text{1: a random variable } \{13: \text{p. } 69/7\} \ \text{Lem}\text{m}\text{m}\text{a}\text{ }\text{1.1}: \text{ } Und\text{essentially homogeneous}\} \ |\text{D}^*K(y)| &\leq \text{holds}. \ \text{Moreover, } K \text{ is in the sense of distribution} \ \text{The following lemma}\text{operator related to the Lemma 1.2 [15: Ta) We define for } r > 0 \ x \in B_r(0). \end{aligned}$ *b* operator $L_0(D)$ which has the foll
p. 216/217]).
ptions (1.2) *K* is an analytic funct
 $N - n$. For all $\nu \in N_0$ ⁿ, $|\nu| > 2N -$
 $|\nu|$] $|y|^{2N-n-|\nu|}$, $y \in \mathbb{R}^n \setminus \{0\}$,
tion and satisfies $L_0(D)$ $K = \delta_0$ (*Di*
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(i) $Q_r: L^p(B_r(0)) \to H^{2N,p}(B_r(0))$ *is continuous for* $1 < p < \infty$. *(ii) For all* $1 < p < \infty$; $f \in L^p(B_r(0))$ *and* $v \in N_0$ ^{*n*} *with* $|v| = 2N - 2$ *we have* The Green Matrix for Strongly Elliptic $\mathcal{L}(B_r(0)) \to H^{2N,p}(B_r(0))$ is continuous for $1 < p < \infty$.
 $\mathcal{L}(B_r(0)) \to H^{2N,p}(B_r(0))$ and $v \in \mathbb{N}_0^n$ with $|v| = 2N - 2$.
 $\mathcal{L}^* ||D(Q_r f)||_{H^{2,p}(B_r(0))} \leq C ||f||_{L^p(B_r(0))}, \qquad C := C(n, N, \lambda, A, p$ (i) $Q_r: L^p(B_r(0)) \to H^{2N,1}$

(ii) For all $1 < p < \infty$;

* $||D(Q_r f)||_{H^{2,p}(\beta)}$

(iii) $L_0(D) (Q_r f) = f$ almo

b) Set $E_k^j = L_0^{jk}(D)$ F

operator $(L_{0ij})_{1 \leq i,j \leq N}$, i.e.
 $L_{0ij}E_k^j = \delta_{ik}\delta_{0j}$ $Q_r: L^p(B_r(0)) \to H^{2N,p}(B_r(0))$ is continuous for $1 < p < \infty$.

For all $1 < p < \infty$; $f \in L^p(B_r(0))$ and $v \in N_0$ ⁿ with $|v| = 2N - 2$ we ha
 $* ||D(Q_r)||_{H^{2,p}(B_r(0))} \leq C ||f||_{L^p(B_r(0))}$, $C := C(n, N, \lambda, A, p)$.

i) $L_0(D) (Q_r f) = f$ almost everywh

*
$$
||D(Q_t)||_{H^{2,p}(\beta_r(0))} \leq C ||f||_{L^p(B_r(0))},
$$
 $C := C(n, N, \lambda, \Lambda, p).$

(iii) $L_0(D)$ ($Q_t f$) = *f* almost everywhere on $B_r(0)$.

b) Set $E_k^i = L_0^{jk}(D)$ K. Then E is a fundamental matrix for the constant coefficient operator $(L_{0i})_{1 \le i,j \le N}$, i.e. The Green Matrix for Strongly Elliptic Systems 511
 $P(B_r(0)) \rightarrow H^{2N,p}(B_r(0))$ is continuous for $1 < p < \infty$.
 $lll 1 < p < \infty; f \in L^p(B_r(0))$ and $v \in N_0^n$ with $|v| = 2N - 2$ we have
 $* ||D(Q_r f)||_{H^{2,p}(B_r(0))} \leq C ||f||_{L^p(B_r(0))}, \qquad C := C(n, N, \lambda, \$ The Green Matrix for Strongly Elliptic Systems 511
 $(B_r(0)) \rightarrow H^{2N,p}(B_r(0))$ is continuous for $1 < p < \infty$.
 $\mathcal{U} \mathbf{1} < p < \infty; f \in L^p(B_r(0))$ and $v \in \mathbb{N}_0^n$ with $|v| = 2N - 2$ we have
 $* ||D(Q_r f)||_{H^{2,p}(B_r(0))} \leq C ||f||_{L^p(B_r(0))}, \qquad$

$$
L_{0i j} E_k^j = \delta_{ik} \delta_{0}, \qquad i, k = 1, \ldots, N. \qquad (1.6)
$$

For $y \in \mathbb{R}^n \setminus \{0\}$ *and* $v \in \mathbb{N}_0^n$ *we have the estimate* $|D^r E(y)| \leq C(n, N, \lambda, A, |\nu|) |y|^{2-n-|\nu|}$.

Now we define potential solutions to the syste
 $L_{0ij}u^j = f^i - D_a F_a^i$, $i = 1, ..., N$.

Definition: Under the assum

$$
|D'E(y)| \le C(n, N, \lambda, \Lambda, |\nu|) |y|^{2-n-|\nu|}.
$$
 (1.7)

$$
L_{0ij}u^j = f^i - D_a F_a^i, \qquad i = 1, ..., N. \tag{1.1}_0
$$

Definition: Under the assumptions of the preceding lemma for $f \in L^p(B_r(0))^N$, $F \in L^p(B_r(0))^{nN}, 1 \leq p \leq \infty, r > 0$, we define for $k = 1, ..., N$ and $x \in B_r(0)$

\n- (iii)
$$
L_0(D)(Q_t f) = f
$$
 almost everywhere on $B_r(0)$.
\n- b) Set $E_k^j = L_0^{jk}(D) K$. Then E is a fundamental matrix for the constant coefficient operator $(L_{0ij})_{1 \leq i,j \leq N}$, $i.e.$
\n- $L_{0ij}E_k^j = \delta_{ik}\delta_0$, $i, k = 1, ..., N$.
\n- (1.6) For $y \in \mathbb{R}^n \setminus \{0\}$ and $v \in \mathbb{N}_0^n$ we have the estimate $|D'E(y)| \leq C(n, N, \lambda, A, |\nu|) |y|^{2-n-|\nu|}$.
\n- (1.7) Now we define potential solutions to the system $L_{0ij}u^j = f^i - [D_aF_a^i, \quad i = 1, ..., N$.
\n- Definition: Under the assumptions of the preceding lemma for $f \in L^p(B_r(0))^N$, $F \in L^p(B_r(0))^n$, $1 \leq p \leq \infty$, $r > 0$, we define for $k = 1, ..., N$ and $x \in B_r(0)$, $(P_r)^k(x) := L_0^{kl}(D)(Q_r^l)(x) = \int_{B_r^l}(x-y)f'(y) dy$, $E_r(0)$
\n- $(\tilde{P}_r F)^k(x) := L_0^{kl}(D) \{D_a(Q_r F_a^l)\}(x) = \int_{B_r(0)} (D_a E_l^k)(x-y) F_a^l(y) dy$.
\n- From Lemma 1.2 we conclude
\n- in an in a 1.3: If condition (1.2) is satisfied, then for all $1 < p < \infty$ and $r > 0$ the following statements hold:
\n- (i) The linear operators $P_r: L^p(B_r(0))^N \rightarrow H^{2,p}(B_r(0))^N$

From Lemma 1.2 we conclude

(i) The linear operators $P_r: L^p(B_r(0))^N \to H^{2,p}(B_r(0))^N$, $\tilde{P}_r: L^p(B_r(0))^{nN} \to H^{1,p}(B_r(0))^N$ *are continuous with bounds depending only on n, N,* p, *2 and A, provided the spaces'* $H^{k,p}(B_r(0))^N$ are normed by $*$ *f*₁. *J*_H_K.₂, *k* = 1, 2.

(ii) For any $f \in L^p(B_r(0))^N$ and $F \in L^p(B_r(0))^n$ ike function $U := P_r(f) - \tilde{P}_r(F) \in L^p(B_r(0))^n$

 $H^{1,p}(B_r(0))^N$ is a weak solution of $(1.1)_0$ on the ball $B_r(0)$. Moreover, if f and F have compact *support in* $B_r(0)$ and if u belongs to the class $H^{1,1}$ and is a weak solution of (1.1) on $B_r(0)$ with compact support, then $u = U$. *following statements hold:*

(i) The linear operators $P_r: L^p(B_r(0))^N$

are continuous with bounds depending on
 $H^{k,p}(B_r(0))^N$ are normed by $*||.||_{H^{k,p}}, k =$

(ii) For any $f \in L^p(B_r(0))^N$ and $F \in L$
 $H^{1,p}(B_r(0))^N$ is a weak s b)^{*x*} is a weak solution of (1.1) ₀ on the ball $B_r(0)$. Moreover, if f and F have con
n $B_r(0)$ and if u belongs to the class $H^{1.1}$ and is a weak solution of (1.1)
h compact support, then $u = U$.
stating the loc *n* $B_r(0)$ and if u belongs to the class $H^{1,1}$ and is a weak solution of (1.1)
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stating the local regularity theorem for weak solutions of (1.1) we intro

operators which measure the

Before stating the local regularity theorem for weak solutions of (1.1) we introduce a class of operators which measure the deviation from the constant coefficient case.

Definition: Assume (1.2) and define for $1 < p < \infty$, $0 < r <$ dist (0, $\partial\Omega$), *i* Before stating the local regularity theorem for weak solutions of (1.1) we introduce a class of operators which measure the deviation from the constant coefficient case.

Definition: Assume (1.2) and define for $1 < p < \$ Let one stating the local regularity theorem for weak solutions of (1.1) we introduce

a class of operators which measure the deviation from the constant coefficient case.

Definition: Assume (1.2) and define for $1 < p < \infty$

$$
T_r^{\ p}(u) = \tilde{P}_r(F), \quad F_a^i = \left(A_{a\beta}^{ij} - A_{a\beta}^{ij}(0)\right)D_\beta u^j.
$$

Introducing the *-norm on $H^{1,p}(B_r(0))^N$, we get from Lemma 1.3

$$
T_r^p(u) = \tilde{P}_r(F), \quad F_a^i = \left(A_{a\beta}^{ij} - A_{a\beta}^{ij}(0)\right) D_\beta u^i.
$$

lucing the *-norm on $H^{1,p}(B_r(0))^N$, we get from Lemma 1.3
* $||T_r^p|| \leq C(n, N, p, \lambda, \Lambda) \operatorname{osc}_{B_r(0)} A$, $\operatorname{osc}_{B_r(0)} A := ||A - A(0)||_{L^\infty(B_r(0))}$. (1.8)
every

and moreover

$$
L_{0i}v^{j} = D_{\alpha}\big((A_{\alpha\beta}^{ij} - A_{\alpha\beta}^{ij}(0)) D_{\beta}u^{j}\big), \qquad i = 1, ..., N, \text{ on } B_{r}(0).
$$
 (1.9)

*L,i L,i L*_i *L,i L*ⁱ *L,i L*ⁱ *L,i D*ⁱ *L,i D*ⁱ *L,i D*ⁱ *L,i L*_i *D*_i *D*ⁱ *L*_i *D*ⁱ *L*_i *D*ⁱ *L*_i *D*ⁱ *L*_i $v := T_r^p u$ is a weak solution to the system
 $L_{0ij}v^j = D_a((A_{\alpha\beta}^{ij} - A_{\alpha\beta}^{ij}(0)) D_\beta u^j), \qquad i = 1, ..., N,$ on $B_r(0)$. (1.9)

In view of the uniform continuity of the coefficients on $\overline{\Omega}$ we conclude from (1.8) that the
 $||\$ $v := T_r^p u$ is a weak solution to the system
 $L_{0ij}v^j = D_a((A_{\alpha\beta}^{ij} - A_{\alpha\beta}^{ij}(0)) D_\beta u^j),$ $i = 1, ..., N$, on $B_r(0)$. (1.9)

In view of the uniform continuity of the coefficients on \overline{Q} we conclude from (1.8) that the

*|

ing on $n, N, p, \lambda, \Lambda$, ε and the modulus of continuity of the coefficients, i.e. we have

$$
||T_r^p|| \leq \varepsilon
$$
 for all $0 \in \Omega$, $0 < r \leq \min(r_n, \text{dist}(0, \partial \Omega))$.

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We now summarize our local regularity results.

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N, p, λ , Λ , ϵ and the modulus of continuity of the coefficients, i.e. we have
 $||T_r|| \leq \epsilon$ for all $0 \in \Omega$, $0 < r \leq \min(r_e, \text{dist}(0, \partial \Omega))$. (1.10)

w summarize our local regularity results.

com 2 (compare [$-$ Theorem 2 (compare [15: Thm. 6.4.3]): Let $1 < p, q < \infty, f \in L_{\mathrm{loc}}^p(\Omega)^N$ and $F \in L^p(\Omega)$ $L^q_{\text{loc}}(\Omega)$ ^{nN} and suppose that $u \in H^{1,p}_{\text{loc}}(\Omega)$ ^N is a weak solution of (1.1) under the condition (1.2). *Then u belongs to the space* $H_{loc}^{1,r}(\Omega)^N$ for *h*, *A*, *e* and the modulus of continuity of the coeffice $\leq \epsilon$ for all $0 \in \Omega$, $0 < r \leq \min(r_e, \text{dist}(0, \partial \Omega))$.

Imarize our local regularity results.

(compare [15: Thm. 6.4.3]): Let $1 < p, q <$

uppose that $u \in H_{\text{loc}}^{1,p}$ tulus of continuity of the coefficity $0 < r \leq \min (r_s, \text{dist } (0, \partial \Omega)).$
regularity results.
hm. 6.4.3]): Let $1 < p, q < \infty$
 $\mathcal{E}^p_{\alpha}(\Omega)^N$ is a weak solution of (1.1)
 $(\Omega)^N$ for
 $= \begin{cases} np/(n-p) & \text{if } p < n \\ \infty & \text{if } p \geq n. \end{cases}$
we

$$
r = \min \{q, s(p)\}, s(p) = \begin{cases} \frac{np}{n-p} & \text{if } p < n \\ \infty & \text{if } p \geq n \end{cases}
$$

From-the-proof of Theorem 2 we will deduce the following

Corollary: Let $1 < p < n/(n - 1)$, *assume that* (1.2) holds and define $q = np/$ $(n - (n - 1) p)$. Then there exist constants C depending on n, N, p, λ , Λ and R_0 deter*mined by the same parameters and in addition by the modulus* of *continuity of the coeffic*_{*c*} $||T_r^p|| \leq \epsilon$ for all $0 \in \Omega$, $0 < r \leq \min(r_s, \text{dist}(0, \partial \Omega))$

We now summarize our local regularity results.

Theorem 2 (compare [15: Thm. 6.4.3]): Let $1 < p$, q
 $L_{\text{loc}}^q(\Omega)^{nN}$ and suppose that $u \in H_{\text{loc}}^{1,q}(\$ $r = \min \{q, s(p)\}, s(p) = \begin{cases} np/(n-p) & \text{if } p < n \\ \infty & \text{if } p \geq n. \end{cases}$
 hhe-proof of Theorem 2 we will deduce the following
 $|\arg y: Let 1 < p < n/(n-1),$ assume that (1.2) holds and d
 -1) p). Then there exist constants *C* depending on n, cm 2 (compare [15: Thm. 6.4.3]): Let $1 < p, q < \infty, f \in L_{loc}^{p}(\Omega)$ ^N and $F \in$

and suppose that $u \in H_{loc}^{12}(\Omega)^N$ is a weak solution of (1.1) under the condition (1.2).

Longs to the space $H_{loc}^{12}(\Omega)^N$ for
 $r = \min \{q, s(p)\}, s$ From the proof of Theorem 2 we will ∞

Corollary: Let $1 < p < n/(n - 1)$, ∞
 $(n - (n - 1) p)$. Then there exist constant

mined by the same parameters and in addit

cients such that
 $*||u||_{H^{1,q}(B_r(x))} \leq Cr^{1-n^*} ||u||_{H^{1,p}(B_r)}$ Corollar
 $\begin{array}{r} (n-1) \ (n-2) \ (n-1) \ (n-1)$ Corollary: Let $1 < p < n |(n - 1)$, assume that $(1.2 \n n - (n - 1) p)$. Then there exist constants C depending on
mined by the same parameters and in addition by the modul
cients such that
 $*||u||_{H^{1,q}(B_r(x))} \leq C r^{1-n}$. $||u||_{H^{1,p}(B_r(x$

$$
^*||u||_{H^{1,q}(B_r(x))} \leq C r^{1-n^*} ||u||_{H^{1,p}(B_{1r}(x))}
$$
\n(1.11)

$$
^*||u||_{H^{1,1}(B_r(x))} \leq Cr^{n/2}r^{-n/p^*}||u||_{H^{1,p}(B_{1r}(x))}
$$
\n(1.12)

for all balls $B_r(x)$, $x \in \Omega$, $0 < 2r < \min(2R_0, \text{dist}(x, \partial \Omega))$ and for all $u \in H^{1,p}(B_{2r}(x))$ satisfying $Lu = 0$ on $B_{2r}(x)$. If u satisfies the homogeneous system only on the punctured *IIII.* $\lim_{t \to (B_r(x))} \leq C r^{1-n^*} ||u||_{H^{1,p}(B_{1r}(x))}$
 $\lim_{t \to (B_r(x))} \leq C r^{n/2} r^{-n/p^*} ||u||_{H^{1,p}(B_{1r}(x))}$
 $\lim_{t \to (B_r(x), x \in \Omega, 0 < 2r < \min (2R_0, \text{dist } (x, \partial \Omega))$ and
 $\lim_{t \to 0} \lim_{t \to (0, T)} E_r(x)$. If u satisfies the homogeneous system

$$
-\frac{1}{r} \|u\|_{L^{q}(T)} + \|\nabla u\|_{L^{q}(T)} \leqq C r^{1-n^{\bullet}} \|u\|^{H^{1,p}(B_{1r}(x))}, \qquad (1.13)
$$

-

•

where $T = B_r(x) \setminus B_{r/2}(x)$.
Proof of Theorem 2: (i) $p < n$ and $p \leq s(p) \leq q$: We have to show *that* $u \in$ $H^{1,s}(B_r(x)),$ $s := s(p)$, for $x \in \Omega$ and sufficiently small values of *r*. To this purpose we may assume $x = 0 \in \Omega$ and take $2r <$ dist $(0, \partial\Omega)$. Furthermore, let $\tilde{u} := \eta u \in$ where $\eta \in C_0^{\infty}(B_{2r})$ is an arbitrary cut-off function. \tilde{u} is a weak solu-
 i = $\tilde{f}^i - D_a(\tilde{F}_a^i - G_a^i)$, $i = 1, ..., N$, on $B_{2r}(0)$,

ned for $i = 1, ..., N$, $\alpha = 1, ..., n$
 $\eta f^i + F_a^i D_a \eta - A_{\alpha\beta}^{ij} D_a \eta D_\beta u^j$, \til ^{*} $||u||_{H^{1,q}(B_r(x))} \leq C r^{1-n}$. $||u||_{H^{1,p}(B_{tr}(x))}$

* $||u||_{H^{1,p}(B_r(x))} \leq C r^{n/2} r^{-n/p^*} ||u||_{H^{1,p}(B_{tr}(x))}$

for all balls $B_r(x)$, $x \in \Omega$, $0 < 2r < \min \{2R_0$, dist $(x, \partial \Omega) \}$ and

satisfying $Lu = 0$ on $B_{2r}(x)$. If u satisfies t tion of the system $\|\|u\|_{H^{1,1}(B,t^2)} \leq C r^{n/2} r^{-n/p} \cdot \|\|u\|_{H^{1,p}(B,t^2)}$
 Lu = 0 on $B_{2r}(x)$. If *u* satisfies the
 $|\sum_{i} |x_i|, (1,11)$ becomes
 $1 \leq r$, $|x_i|, (1,11)$ becomes
 $\frac{1}{r} \|\|u\|_{L^q(T)} + \|\nabla u\|_{L^q(T)} \leq C r^{1-n^*} \|u\|$
 $= B_r(x) \set$ for all balls $B_r(x)$, $x \in \Omega$, $0 < 2r < \min\{2R_0, \text{dist}(x, \partial \Omega)\}\$ and for all $u \in H^{1,p}[L]$
satisfying $Lu = 0$ on $B_{2r}(x)$. If u satisfies the homogeneous system only on the pund
ball $B_{2r}(x) \setminus \{x\}$, (1.11) becomes
 $-\frac{1$ $E = B_r(x) \setminus B_{r/2}(x)$.

of Theorem 2: (i) $p < n$ and $p \le s(p) \le q$. We have to show that $u \in (n, s) = s(p)$, for $x \in \Omega$ and sufficiently small values of r. To this purpose we $x = 0 \in \Omega$ and take $2r <$ dist $(0, \partial \Omega)$. Furthermore $H^{1,s}(B_r(x)), s := s(p),$
 $H^{1,s}(B_r(x)), s := s(p),$
 $H^{1,p}(B_{2r}(0))^N$, where \imath
 $L_{0i}, \tilde{u}^j = \tilde{f}^i$ –
 $L_{0i}, \tilde{u}^j = \tilde{f}^i$ –

where we defined for
 $\tilde{f}^i = \eta f^i + F$
 $G_a{}^i = \left\langle A^i g_s - \tilde{g}^i \right\rangle$

Since \tilde{f} , \tilde{F} ,

$$
L_{0i}i\tilde{u}^j=\tilde{f}^i-D_{\mathfrak{a}}(\tilde{F}_{\mathfrak{a}}^i-G_{\mathfrak{a}}^i),\qquad i=1,\ldots,N, \text{ on }B_{2r}(0),
$$

*L*_{0i},
$$
\tilde{u}^j = \tilde{f}^i - D_c(\tilde{F}_s^i - G_s^i), \qquad i = 1, ..., N
$$
, on $B_{2r}(0)$,
\nwhere we defined for $i = 1, ..., N$, $\alpha = 1, ..., n$
\n $\tilde{f}^i = \eta f^i + F_s^i D_s \eta - A_{2g}^{ij} D_s \eta D_g u^j, \qquad \tilde{F}_s^i = A_{2g}^{ij} D_g \eta + \eta F_s^i,$
\n $G_s^i = (A_{2g}^{ij} - A_{2g}^{ij}(0)) D_g \tilde{u}^j.$
\nSince \tilde{f} , \tilde{F} , G have compact support in $B_{2r}(0)$, we infer from Lemma 1.3 (iii)
\n $\tilde{u} - T_{2r}^p \tilde{u} = P_{2r}(\tilde{f}) - \tilde{P}_{2r}(\tilde{F}) =: \varphi$:
\nObserving
\n $P_{2r}(\tilde{f}) \in H^{2,p}(B_{2r}(0))^N \subset H^{1,s}(B_{2r}(0))^N, \qquad \tilde{P}_{2r}(\tilde{F}) \in H^{1,s}(B_{2r}(0))^N$
\nwe get $\varphi \in H^{1,s}(B_{2r}(0))^N$. According to (1.10) there exists R_0 depending on the parameters such that
\n $*||T_{2r}^{p}||, *||T_{2r}^{s}|| \leq 1/2$ for $2r \leq \min(2R_0, \text{dist}(0, \partial \Omega))$.
\nFixing such a radius \tilde{r} , we see that the operators $Id - T_{2r}^p$, $Id - T_{2r}^s$ are on
\nand onto. Consequently we find $w \in H^{1,s}(B_{2r}(0))^N$ with the property (*Id*

Since \tilde{f} , \tilde{F} , G have compact support in $B_{2r}(0)$, we infer from Lemma 1.3 (iii)

$$
\tilde{u} - T_{2r}^p \tilde{u} = P_{2r}(\tilde{f}) - \tilde{P}_{2r}(\tilde{F}) =: \varphi.
$$
\n(1.14)

Observing

$$
\begin{aligned}\n\tilde{F}, & G \text{ have compact support in } B_{2r}(0), \text{ we infer from Lemma 1.3,} \\
\tilde{u} - T_{2r}^p \tilde{u} &= P_{2r}(\tilde{f}) - \tilde{P}_{2r}(\tilde{F}) =: \varphi: \\
\text{g} \\
P_{2r}(\tilde{f}) & \in H^{2,p}(B_{2r}(0))^N \subset H^{1,s}(B_{2r}(0))^N, \qquad \tilde{P}_{2r}(\tilde{F}) \in H^{1,s}(B_{2r}(0))^N \\
&\in H^{1,s}(B_{2r}(0))^N \text{ According to (1.10), there exists } B_{2r} \text{ depending on}\n\end{aligned}
$$

we get $\varphi \in H^{1,\bullet}(B_{2r}(0))^N$. According to (1.10) there exists R_0 depending on the stated parameters such that $\ddot{}$

$$
^*\|T_{2r}^p\|, ^*\|T_{2r}^{\bullet}\|\leqq 1/2 \quad \text{for}\quad 2r\leqq \min\left(2R_0,\, \text{dist }(0,\,\partial \Omega)\right).
$$

Fixing such a radius \dot{r} , we see that the operators $Id - T_{2r}^p$, $Id - T_{2r}^s$ are one-to-one and onto. Consequently we find $w \in H^{1,s}(B_{2r}(0))^N$ with the property $(Id - T_{2r}^p)$ \tilde{u}

= $(Id - T_{2r}^s) w$. Using the fact that $Id - T_{2r}^p$ is an isomorphism of the space $H^{1,p}$, we
see $\tilde{u} = w$, since $T_{2r}^p = T_{2r}^s$ on the space $H^{1,s}$. Choosing $\eta = 1$ on $B_r(0)$ we arrive
at $u \in H^{1,s}(B_r(0))^N$.
(ii) n see $\tilde{u} = w$, since $T_{2r}^p = T_{2r}^s$ on the space $H^{1,s}$. Choosing $\eta = 1$ on $B_r(0)$ we arrive at $u \in H^{1,s}(B_r(0))^N$. The Green Matrix for Strongly Elliptic Syste
 $= (Id - T_{2r}^{s}) w$. Using the fact that $Id - T_{2r}^{p}$ is an isomorphism of the space $u = w$, since $T_{2r}^{p} = T_{2r}^{s}$ on the space $H^{1,s}$. Choosing $\eta = 1$ on $B_{r}(0)$

at $u \in H^{$ The Green Matri
 $(Id - T_{2r}^{s}) w$. Using the fact that $Id - T_{2r}^{p}$ is
 $\tilde{u} = w$, since $T_{2r}^{p} = T_{2r}^{s}$ on the space $H^{1,s}$.
 $u \in H^{1,s}(B_r(0))^{N}$.

(ii) $n \leq p \leq q$: Define $t = nq/(n + q) \in (1, n$
 $u \in H^{1,1}_{loc}(\Omega)^{N}$, $F \in L$ see $\tilde{u} = w$, since $T_{gr}^p = T_{2r}^t$ on the space $H^{1,s}$. Choosing $\eta = 1$ on $B_r(0)$ we arrition $u \in H^{1,s}(B_r(0))^N$.

(ii) $n \leq p \leq q$: Define $t = nq/(n + q) \in (1, n)$ and observe
 $u \in H^{1,s}_{loc}(\Omega)^N$, $F \in L^{s}_{loc}(\Omega)^N$, $f \in L^{t}_{loc$

$$
u\in H^{1,t}_{\text{loc}}(\Omega)^N,\quad F\in L^q_{\text{loc}}(\Omega)^{nN},\quad f\in L^t_{\text{loc}}(\Omega)^N.
$$

Since q is the Sobolev exponent corresponding to t the assertion of the theorem follows as in case (i), replacing p by t and *s* by q. (ii) $n \leq p \leq q$: Define $t = nq/(n + q)$
 $u \in H_{loc}^{1,t}(\Omega)^N$, $F \in L_{loc}^q(\Omega)^{nN}$, joince q is the Sobolev exponent correspond

lows as in case (i), replacing p by t and s

(iii) $p \geq q$: This case is trivial \blacksquare

Proof of t

Proof of the corollary: Again assume $0\in\varOmega;$ according to (1:10) we find $R_\mathbf{0}$ such $\geq q$: This case is trivial \blacksquare

of the corollary: Again assume $0 \in \Omega$; according to (
 $k = 0, ..., n - 1$
 $\parallel T_{2r}^{s} \parallel \leq 1/2$ for $0 < 2r < \min (2R_0, \text{dist } (0, \partial \Omega)),$
 $\parallel C_{2r} \parallel P_{2r}^{s} \parallel \leq 1/\Omega$ resident $I_{2r} \rightarrow 0$ on the $u \in H_{loc}^{1,2}$ $p \leq q$: Define $t = nq/(n + q) \in (1, n)$ and observe
 $u \in H_{loc}^{1,4}(\Omega)^N$, $F \in L_{loc}^q(\Omega)^{nN}$, $f \in L_{loc}^l(\Omega)^N$.

the Sobolev exponent corresponding to t the assertion of the theorem fol-
 $\geq q$: This case is triv

$$
||T_{2r}^{s*}|| \leq 1/2 \text{ for } 0 < 2r < \min\big(2R_0, \text{ dist }(0, \partial \Omega)\big), \qquad s_k = np/(n - kp).
$$

Now let $u \in H^{1,p}(B_{2r}(0))$ ^N satisfy $Lu = 0$ on the ball $B_{2r}(0)$. Using the notations from
the proof of Theorem 2 we get according to (1.14) that
 $\tilde{u} - T_{2r}^{s} \tilde{u} = P_{2r}(\tilde{f}) - \tilde{P}_{2r}(\tilde{F})$ $(k = 0, ..., n - 1),$ (1.15)
w

$$
\tilde{u} - T_{27}^{s*} \tilde{u} = P_{27}(\tilde{f}) - \tilde{P}_{27}(\tilde{F}) \qquad (k = 0, ..., n-1), \qquad (1.15)
$$

where we suppose that *r* satisfies the above-stated smallness condition. Now choos-

$$
||u||_{1,s_1;3r/2} \leq Cr^{-1} ||u||_{1,p;2r},
$$

 $\begin{aligned}\n\text{where } \text{Cov}(X) &= \text{Cov}(X) + \text{Cov}(X)$ where *||. $\|_{k,p,r}$ denotes the *-norm in $H^{k,p}(B_r(0))$. Inequality (1.11) now follows by a simple iteration argument, (1.12) is an easy consequence of (1.11) (use Hölder's inequality). To prove the last statement of the corollary we proceed as above, the only difference is that we use cut-off functions η with compact supports on rings centered at 0 The Green Matrix for Strongly Ellip
 $= (Id - T_{27}^2)$ w. Using the fact that $Id - T_{27}^p$ is an isomorphism
 $\alpha \in \mathcal{U} = w$, since $T_p^p = T_p^*$, on the space $H^{1,s}$. Choosing $\eta = 1$
 $(ii) n \leq p \leq q$: Define $t = nq/(n + q) \in (1, n)$ *L*₁*u L₁ L*₂*i L* The Green Matrix for Strongly Elliptic Systems

= $(dA - T_{\phi}^2)$ to. Using the fact that $Id - T_{\phi}^2$ is an isomorphism of the space as a set of the space and d^2 . (I.b) as $\leq p \leq T_{\phi}^2$ on the space $H^{1/2}$, (i.e., a

2. Systems with vector-valued measures on the right-hand side

In this section we use the results of the preceding paragraph to prove existence and uniqueness of a weak solution to the boundary value' problem

$$
L_{ij}u^{j} = \mu^{i}, i = 1, ..., N, \text{ on } \Omega, u_{|\partial\Omega} = 0,
$$
\n(2.1)

whenever μ^{i} , $i = 1, ..., N$, are prescribed signed Radon measures with finite variation. We will show that (2.1) admits a unique weak solution in the space

 $\hat{H} = \{u : \Omega \to \mathbb{R}^N \mid u \in \hat{H}^{1,r}(\Omega)^N \text{ for all } 1 \leq r < n/(n-1)\}\dots$

The idea of the proof follows arguments of LITTMAN, STAMPACCHIA and WEINBERGER [14].

In the sequel we denote by $M(\Omega)$ the space of all signed Radon measures μ on Ω with finite total variation $|\mu| (\Omega)$; $M(\Omega)^N$ is the space of all $\mu = (\mu^1, \ldots, \mu^N)$ with components $\mu^i \in M(\Omega)$. Obviously $C_0^0(\Omega, \mathbb{R}^N)^*$ (dual space) is isomorphic to $M(\Omega)^N$ [4: Thm. 2.5.5]. From now on we will use the norms

$$
|u|_{1,r} = \|\nabla u\|_{L^r(\Omega)}, \quad \|T\|_{-1,p} = \inf \left\{\|X\|_{L^p(\Omega)} : X \in L^p(\Omega)^{n^N} \right\} \text{ represents } T\}
$$

on the spaces $\mathring{H}^{1,r}(\Omega)^N$ and $H^{-1,p}(\Omega)^N$, respectively. Then there exists a natural isometric isomorphism *k* $|u|_{1,r} = \|\nabla u\|_{L^r(\Omega)}, \quad \|T\|_{-1,p} = \inf \{ \|X\| \text{ spaces } \hat{H}^{1,r}(\Omega)^N \text{ and } H^{-1,p}(\Omega)^N \text{, respe} \}$
 K_p: $\hat{H}^{1,p'}(\Omega, \mathbb{R}^N)^* \to H^{-1,p}(\Omega)^N$,
 K_p: $\hat{H}^{1,p'}(\Omega, \mathbb{R}^N)^* \to H^{-1,p}(\Omega)^N$,

I

$$
K_p: \mathring{H}^{1,p'}(\Omega, \mathbf{R}^N)^* \to H^{-1,p}(\Omega)^N, \qquad p' = p/(p-1).
$$

33 Analysis Bd. 5, Heft 6 (1986)

514 M. Fucus

We therefore agree to identify the energy $\hat{H}^{1,p'}(O, \mathbf{R}^N)$ and $H^{-1,p}(O, N)$. In case is We therefore agree to identify the spaces $\mathring{H}^{1,p'}(\Omega,\mathbf{R}^N)^*$ and $H^{-1,p}(\Omega)^N$. In conclusion we can consider any function $u \in \hat{H}^{1,p'}(\Omega)^N$ as an element of $H^{-1,p}(\Omega, \mathbb{R}^N)^*$ by defin*ing* $\langle u, U \rangle = \langle U, u \rangle$ for $U \in H^{-1,p}(\Omega)^N$.
For solving the boundary value problem (2.1) we assume 4 M. Fuces

e therefore agree to identify the spaces $\hat{H}^{1,p'}(\Omega, \mathbf{R}^N)^*$ and $H^{-1,p}(\Omega)^{N}$. If

i can consider any function $u \in \hat{H}^{1,p'}(\Omega)^N$ as an element of $H^{-1,p}(\Omega, \mathbf{R}^N)$
 $g \langle u, U \rangle = \langle U, u \rangle$ for $U \in H^{-1,p}(\Omega)^$ fore agree to identify the s

onsider any function $u \in \mathring{H}$
 $\rangle = \langle U, u \rangle$ for $U \in H^{-1,p}(\Omega)$

ving the boundary value $\{(\mathbf{G}\mathbf{A})\}$ and $A_{a,\beta}^{ij} \in C^0(\overline{\Omega})$

the Law Milmany theory paces $\mathring{H}^{1,p'}(Q, \mathbf{R}^N)^*$ and $H^{-1,p}(Q)^N$! In conclusion
 $^{1,p'}(Q)^N$ as an element of $H^{-1,p}(Q, \mathbf{R}^N)^*$ by defin-
 $(2)^N$.

problem (2.1) we assume
 $(i, j = 1, ..., N; \alpha, \beta = 1, ..., n)$. (2.2)

q the solution operator $S: H^{-1,2$ We therefore agree to identify the spaces $\mathring{H}^{1,p'}(\Omega, \mathbb{R}^N)^*$ and $H^{-1,p}(\Omega)^N$. In
we can consider any function $u \in \mathring{H}^{1,p'}(\Omega)^N$ as an element of $H^{-1,p}(\Omega, \mathbb{R}^N)$
ing $\langle u, U \rangle = \langle U, u \rangle$ for $U \in H^{-1,p}(\Omega)^N$.
For s

(GA) and
$$
A_{\alpha\beta}^{ij} \in C^0(\bar{\Omega})
$$

 $(i, j = 1, ..., N; \alpha, \beta = 1, ..., n).$ (2.2)

Then by the Lax-Milgram theorem the solution operator $S: H^{-1,2}(\Omega)^N \to \dot{H}^{1,2}(\Omega)^N$, $v = S(T)$ being the unique $\dot{H}^{1,2}$ -solution to the adjoint system $L^t_i v^j = T^i$, $i = 1, ...,$ N , on Ω , is well-defined and continuous. The "duality method" introduced in [14] is based on the following principle: Corollary 2 to Theorem 1 implies that for values $n < p < \infty$ the solution operator S maps $H^{-1,p}(\Omega)^N$ into the space $C_0(0,\Omega)^N$ of functions $v : \Omega \to \mathbb{R}^N$ which are continuous on $\overline{\Omega}$ with boundary values zero. Therefore S^* (dual operator) maps linear functionals μ defined on a certain space of continuous functions on functions $u_{\mu} \in \mathring{H}^{1,p}(Q)^N$, and we will show that u_{μ} is the natural solution to the system (2.1) . $iZ \rightarrow \mathbf{R}^{n}$ which are continuous
operator) maps linear function
 i on functions $u_{\mu} \in \hat{H}^{1,p'}(\Omega)^{N},$ as
stem (2.1).
precise we consider for numbe
 $i_{rs}: \hat{H}^{1,r}(\Omega)^{N} \rightarrow \hat{H}^{1,s}(\Omega)^{N},$
 $h \mapsto \hat{H}^{1,p}(\Omega)^{N} \rightarrow C^{0}(\$ by the Lax-Milgram theorem the solution operator λ
 S(T) being the unique $\hat{H}^{1,2}$ -solution to the adjoint syst
 sQ, is well-defined and continuous. The "duality me

sed on the following principle: Corollary 2 *i* the following principle: Corollary 2 to Theorem 1 implies t
 i the solution operator *S* maps $H^{-1,p}(Q)^N$ into the space C_0
 $\rightarrow \mathbb{R}^N$ which are continuous on $\overline{\Omega}$ with boundary values zerator) maps linear

To be precise we consider for numbers $1 \leq s \leq r < \infty$, $p > n$ the embeddings

$$
i_{rs}\colon \hat{H}^{1,r}(\Omega)^N \to \hat{H}^{1,s}(\Omega)^N, \qquad j_{rs}\colon H^{-1,r}(\Omega)^N \to H^{-1,s}(\Omega)^N,
$$

$$
h_p\colon \hat{H}^{1,p}(\Omega)^N \to C_0^0(\Omega)^N
$$

and define the linear operators'

•

functions on functions
$$
u_{\mu} \in \tilde{H}^{1,p'}(\Omega)^N
$$
, and we will show that u_{μ}
to the system (2.1).
To be precise we consider for numbers $1 \leq s \leq r < \infty$, $p >$
 $i_{rs} \colon \tilde{H}^{1,r}(\Omega)^N \to \tilde{H}^{1,s}(\Omega)^N$, $j_{rs} \colon H^{-1,r}(\Omega)^N \to H^{-1,s}$.
 $h_p \colon \tilde{H}^{1,p}(\Omega)^N \to C_0^0(\Omega)^N$
and define the linear operators
 $\sum_p : H^{-1,p}(\Omega)^N \to \tilde{H}^{1,p}(\Omega)^N$, $\sum_p(T) = S \circ j_{p2}(T)$,
 $S_p : H^{-1,p}(\Omega)^N \to C_0^0(\Omega)^N$, $S_p = h_p \circ \sum_p$
with norms depending only on *n*, *N*, *p*, λ , *A*, *Q* and the modul
coefficients. For exponents $p > n$ we look at the dual operator
 $S_p^* : C_0^0(\Omega, \mathbb{R}^N)^* \to H^{-1,p}(\Omega, \mathbb{R}^N)^*$, $\mu \to \mu \circ S_p$.
Recalling the isomorphism mentioned above we have a cont
 S_p^* : $M(\Omega)^N \to \tilde{H}^{1,p'}(\Omega)^N$ which satisfies
 $|S_p^*(\mu)|_{1,p'} \leq C |\mu| (\Omega)$ for all $\mu \in M(\Omega)^N$
for some positive constant *C* which only depends on *n*, *N*, *p*, λ
of continuity of the coefficients.
It is now easy to show that $S_p^*(\mu)$, $\mu \in M(\Omega)^N$, induces the sa
for all values $p > n$. Let $q > p > n$ be arbitrary and observe
 $\sum_{\Omega} \circ i_{\Omega} = i_{\Omega} \circ \sum_{\Omega} h_{\Omega} \circ \sum_{\Omega$

with norms depending only on *n*, *N*, *p*, λ , Λ , Ω and the modulus of continuity of the coefficients. For exponents $p > n$ we look at the dual operator

 $u\rightarrow\mu\circ S_p^{\dagger}$.

Recalling the isomorphism mentioned above we have a continuous linear operator S_p^* : $M(\Omega)^N \to \hat{H}^{1,p'}(\Omega)^N$ which satisfies rents $p > n$ we look at the dua
 N^y $* \rightarrow H^{-1,p}(\Omega, \mathbb{R}^N)^*$, $\mu \rightarrow$

ism mentioned above we have
 N^{*N*} which satisfies
 C $|\mu|(\Omega)$ for all $\mu \in M(\Omega)^N$

ant *C* which only depends on *n*

$$
|S_p^{\ast}(\mu)|_{1,p'} \leq C |\mu| (Q) \quad \text{for all } \mu \in M(\Omega)^N
$$

for some positive constant *C* which only depends on *n*, *N*, *p*, λ , Λ , Ω and the modulus of continuity of the coefficients.

It is now easy to show that $S_p^*(\mu)$, $\mu \in M(\Omega)^N$, induces the same element in $\hat{H}^{1,1}(\Omega)^N$. for all values $p > n$. Let $q > p > n$ be arbitrary and observe the relations

$$
\Sigma_p \circ j_{qp} = i_{qp} \circ \Sigma_q, \quad h_p \circ i_{qp} = h_q, \quad S_p = h_p \circ \Sigma_p, \quad S_q = h_q \circ \Sigma_q,
$$

$$
j_{qp}^* = i_{p'q'},
$$

from which we get $S_q^* = i_{p'q'} \circ S_p^*$. This relation has the following interpretation.

Lemma 2.1: If (2.2) is satisfied, then for arbitrary real $p, q > n$ and measures Lemma 2.1: If (2.2) is satisfied, then for arbitrary real p, $q > n$ and measures $\mu \in M(\Omega)^N$ we have $S_q^*(\mu) = S_p^*(\mu)$ in $\hat{H}^{1,r}(\Omega)^N$ for $r = \min \{p/(p-1), q/(q-1)\}$. *Therefore* $S_p^*(\mu)$ *induces a Sobolev function* $u_\mu \in \hat{H}^{1,1}(\Omega)$ ^N which is contained in the $|S_p^*(\mu)|_{1,p'} \leq C |\mu| (\Omega)$ for all $\mu \in M$
for some positive constant C which only depends of
of continuity of the coefficients.
It is now easy to show that $S_p^*(\mu)$, $\mu \in M(\Omega)^N$, in
for all values $p > n$. Let $q > p > n$ be ar r some positive constant C which only depends on n, N, p, λ , A, Ω and the mo
continuity of the coefficients.
It is now easy to show that $S_p^*(\mu)$, $\mu \in M(\Omega)^N$, induces the same element in H^1 .

all values $p > n$. It is now easy to show that $S_p^*(\mu)$, $\mu \in M(\Omega)^N$, induces the same element in $H^{1,1}(\Omega)$

i. all values $p > n$. Let $q > p > n$ be arbitrary and observe the relations
 $\Sigma_p \circ j_{qp} = i_{qp} \circ \Sigma_q$, $h_p \circ i_{qp} = h_q$, $S_p = h_p \circ \Sigma_p$, S_q

The order 3: Suppose that (2.2) holds and that $\mu \in M(\Omega)^N$ is given.
(i) The function u_{μ} defined in Lemma 2.1 is a weak solution of problem (2.1).

 $\epsilon \in [1, n/(n-{\epsilon}$
continuity (ii) *For* $r \in [1; n/(n - 1))$ *there is a constant C depending on n, N, r, A,* λ *,* Ω *and the modulus of continuity of the coefficients such that* **in the Community of the coefficients of continuity of the coefficients such that
** $||u_{\mu}||_{H^{1,r}(\Omega)} \leq C |\mu| (\Omega).$ **
(iii)** *If v* **belongs to the space** $\hat{H}^{1,1+\epsilon}(\Omega)^N$ **for some** $\varepsilon > 0$ **and is a weak**

$$
||u_{\mu}||_{H^{1,r}(\Omega)} \leq C |\mu| (\Omega).
$$

(iii) If v belongs to the space $\hat{H}^{1,1+\epsilon}(\Omega)^N$ for some $\epsilon > 0$ and is a weak solution of problem (2.1) *, then* $v = u_u$ *.*

Definition: Under the assumption (2.2) we call the function u_u defined in the preceding theorem the weak solution of the boundary value problem (2.1).

As a simple consequence of Theorem 3 we have the following approximation lemma.

Lemma 2.2: If μ_m , μ , $m \in \mathbb{N}$, *belong to the space* $M(\Omega)^N$ with the property

theorem the *weak* solution of the boundary value
inple consequence of Theorem 3 we have the follow
a 2.2: If
$$
\mu_m
$$
, μ , $m \in \mathbb{N}$, belong to the space $M(\Omega)$
 $\lim_{m\to\infty} \int_{\Omega} f^{\alpha} d\mu_m^{\alpha} = \int_{\Omega} f^{\alpha} d\mu^{\alpha}$ for all $f \in C_0^0(\Omega)^N$

(iii) If v belongs to the space $\hat{H}^{1,1+\epsilon}(\Omega)^N$ for some $\varepsilon > 0$ and is a weak solution (2.2), then $v = u_\mu$.

Definition: Under the assumption (2.2) we call the function u_μ depreceding theorem the weak solution of $1 < p < n/(n-1)$, for the corresponding solutions of the boundary value problem (2.1).

3. Definition **and** first properties **of the Green** matrix

 $-$ First we apply Theorem 3 to show existence and uniqueness of a Green matrix G of the system under consideration. Elementary properties of G such as continuity on domains that do not meet the singular diagonal follow directly from the local regularity theory stated in Section 1. Moreover, we prove a representation formula for the weak solution u_{μ} to problem (2.1): u_{μ} equals the convolution $G \bullet \mu$ Lⁿ-a.e. on Ω (Lⁿ means the *n*-dimensional Lebesgue measure on \mathbf{R}^{n}). *G: Qx* **Q** *** **Ru',** *(x, y)* **-*** (Gk (x, *y))1i.k,^N; -* **Definition and first properties of the Green matrix**

st we apply Theorem 3 to show existence and uniqueness of a Green matrix

der consideration. Elementary properties of G such as continuity on doma

et the singular under consideration. Elementary properties of G such as continuity on exert the singular diagonal follow directly from the local regularity theory Mereover, we prove a representation formula for the weak solution u_{μ}

Definition: Assume that (2.2) holds. (i) For, $y \in \Omega$, $k = 1, ..., N$, denote by $G_k(.,y) \in \hat{H}$ the *unique solution of* $L_i v^j = \delta_{ik} \delta_{ij}, i = 1, ..., N$, on Ω , $v_{i\partial\Omega} = 0$.

$$
G\colon\varOmega\times\varOmega\to{\mathbf R}^{N^\bullet},\,(x,\,y)\to (G_{\boldsymbol{k}}{}^\boldsymbol{\iota}(x,\,y))_{1\leq\boldsymbol{\iota},\,\boldsymbol{k}\leq N^\perp}
$$

is called the *Green matrix for the operator* $(L_{ij})_{1 \le i,j \le N}$ on the domain Ω . on of $L_i, v^j = \partial_{ik}\partial_y$, $i = 1, ..., N$, on Ω ,
 $\mathcal{L}(g, y) \rightarrow (G_k(i(x, y))_{1 \le i, k \le N}$

the operator $(L_{ij})_{1 \le i, j \le N}$ on the domain is

fine the *Radon measures*
 $\partial_{ik} \cdot \overline{L}^n \perp B_e(y) \cap \Omega$, $i = 1, ..., N$,
 ∂ associated solutio

$$
\mu^i = \mathbf{L}^n(B_e(y) \cap \Omega)^{-1} \cdot \delta_{ik} \cdot \mathbf{L}^n \cup B_e(y) \cap \Omega, \qquad i = 1, \ldots, N,
$$

(cf. [4: p. 54]) and denote the *associated solution* of (2.1) by the symbol $G_k^e(., y)$. We call G^e -the *mollified Green matrix* to $(L_{ij})_{1 \le i,j \le N}$ on the domain Ω . $G: \Omega \times \Omega \to \mathbb{R}^{N^*}, (x, y) \to (G_k^i(x, y))_{1 \leq i}$
is called the *Green matrix for the operator* $(L_{ij})_{1 \leq i}$
(ii) For $y \in \Omega$, $\varrho > 0$ we define the *Radon me*
 $\mu^i = \mathbf{L}^n(B_e(y) \cap \Omega)^{-1} \cdot \delta_{ik} \cdot \mathbf{L}^n \cup B_e(y) \cap$
(cf. [4:

(iii) If the differential operator *L* is replaced by L^t , we use the notations ${}^tG, {}^tGe$.

Le m m a 3.1: If (2.2) is satisfied, then for all $y \in \Omega$, $\rho > 0$ the following statements are *true*: *-*

Definition:
 $G_k(.,y) \in \mathring{H}$ the
 $G: \mathcal{Q} \times$

is called the *Gre*

(ii) For $y \in \mathcal{Q}$
 $\mu^i = \mathbf{L}$

(cf. [4 : p. 54]) a

call G^e -the molli

(iii) If the diff

The results fr

Lemma 3.1 :
 true:

(i) $G(.,y) \in \math$ $\mu^i = \mathbf{L}^n(B_e(y) \cap \Omega)^{-1} \cdot \delta_i$

(cf. [4 : p. 54]) and denote the ass

call G^e -the mollified Green matrix

(iii) If the differential operator

The results from Section 2 are

Lemma 3.1 : If (2.2) is satisfied

true:
 The results from Section 2 are summarized in
 Lemma 3.1: If (2.2) is satisfied, then for all $y \in \Omega$, $\varrho > 0$ the following statements are
 true:

(i) $G(\cdot, y) \in H_{loc}^{1,r}(\Omega \setminus \{y\})^N$ for all $1 \le r < \infty$; in particular Lemma 3.1: If (2.2) is satisfied, then for all $y \in \Omega$, $\rho > 0$ the following statements are
 $\begin{array}{lll} \mathcal{L}(i) & G(\cdot, y) \in H_{\text{loc}}^{1,r}(\Omega \setminus \{y\})^{N^{\bullet}} & \text{for} \quad \text{all} \quad 1 \leq r < \infty; & \text{in} \quad \text{particular} \quad G(\cdot, y) \in C^{0}(\Omega \setminus \{y\})^{N^{\bullet}} \text{ ($ $\mu^i = \mathbf{L}^n(B_e(y) \cap \Omega)^{-1} \cdot \delta_{ik} \cdot \mathbf{L}^n \cup B_e(y) \cap \Omega$, $i = 1, ..., N$,

(cf. [4: p. 54]) and denote the associated solution of (2.1) by the symbol $G_k^o(\cdot, g)$

call G^o the mollified Green matrix to $(L_{ij})_{1 \le i,j \le N}$ on the III, and denote the associated solution of (2.1) by the symbol G_k^e

i.e mollified Green matrix to $(L_{ij})_{1 \le i,j \le N}$ on the domain Ω .

the differential operator L is replaced by L^t , we use the notations $\{t\}$

su

 $\times (Q \setminus \{y\})^N$.

(ii) $G^{\varrho}(\cdot, y) \in \hat{H}^{1,r}(\Omega)^{N^*}$ for all $1 \leq r < \infty$; in particular $G^{\varrho}(\cdot, y) \in C^{0,\alpha}(\Omega)^{N^*}$ for all $0 < \alpha < 1$.

(iii) For each $p \in [1, n/(n - 1))$ there exists a constant C depending on n, N, p,

(iii) For each $p \in [1, n/(n-1))$ there exists a constant C depending on n, N, p, λ , Λ , Ω *and the modulus of continuity of the coefficients such that*

33*

The integrability properties of $G(\cdot, y)$ as stated in part (i) of the lemma can be improved.

Lemma 3.2: Assume (2.2) and let $B_n(y)$ be a ball compactly contained in Ω . Then $G(\cdot, y) \in H^{1,r}(\Omega \setminus B_R(y))$ ^{N'} for any exponent $r < \infty$ and

$$
||G(\cdot, y)||_{H^{1,r}(\Omega \setminus B_R(y))} \leq C \tag{3.1}
$$

II ^G*(-, Y)IIn'.'(Q\BR(y)* :5 *C .. - '- (3.1) for some constant C depending on n, N, i, A, Q, r,* $B_R(y)$ *and the modulus of continuity of the coefficients.* (3.1) holds (with the same C) if $G(\cdot, y)$ is replaced by $G^{\varrho}(\cdot, y)$, provided $\rho < R$. Lemma 3.2: Assume (2.2) and let $B_R(y)$ be a ball $G(\cdot, y) \in H^{1,r}(\Omega \setminus B_R(y))^N$ for any exponent $r < \infty$ a
 $||G(\cdot, y)||_{H^{1,r}(\Omega \setminus B_R(y))} \leq C$

for some constant C depending on n, N, i, A, Ω , r , B_R

of the coefficients. (3 *to and*

(3.1)
 $B_R(y)$ *and the modulus of continuity*
 $\langle ', y \rangle$ *is replaced by* $G^{\varrho}(\cdot, y)$ *, provided*
 \int_a^x -norm of $G^{\varrho}(\cdot, y)$ on $\Omega \setminus B_R(y)$ is

corollary provides a useful tool in

matrix.
 tave for all po

Corollary: For $g < R$ and $0 < \alpha < 1$ the C^{0.4} norm of $G^{\varrho}(\cdot, y)$ on $\Omega \setminus B_R(y)$ is *estimated independent of o.*

We omit the simple proof of this lemma. The corollary provides a useful tool in

oving certain symmetry properties of Green's matrix.

Theorem 4: *Under the assumptions* (2.2) we have for all points $x, y \in \Omega$ and inter
 proving certain symmetry properties of Green's matrix.

Theorem 4: Under the assumptions (2.2) we have for all points $x, y \in \Omega$ and inteies of Green's *i*
tions (2.2) we h
i.e. $G_k^{\ t}(x,y) =$

$$
G(x, y) = {^tG(y, x)}^T, \qquad i.e. \ G_k^l(x, y) = {^tG_l}^k(y, x). \tag{3.2}
$$

Corollaries: 1. If the coefficients are symmetric, $A_{\alpha\beta}^{ij} = A_{\beta\alpha}^{ji}$, then $G(x, y) = G(y, x)^T$ *for all x, y* $\in \Omega$ *. 2. For any x, y* $\in \Omega$ *, 0* $\lt \varrho \lt$ *dist (y,* $\partial \Omega$ *),*

ficients. (3.1) holds (with the same C) if
$$
G(\cdot, y)
$$
 is replaced by $G^{\varrho}(\cdot, y)$, provided $\text{lary}:$ For $o < R$ and $0 < \alpha < 1$ the $C^{0, \alpha}$ -norm of $G^{\varrho}(\cdot, y)$ on $\Omega \setminus B_R(y)$ is independent of ϱ .

Since ${}^{t}G(\cdot, x)$, for fixed x, is a continuous function on $\Omega \setminus \{x\}$, formula (3.2) shows continuity of $G(x, \cdot)$ as a function of the second argument, which is not a direct consequence of the definition. Corollary 2 justifies the name *mollified Green's matrix* for Ge. $G(x, y) = {G(x, x)}^T$, i.e. $G_k^{-1}(x, y) = {G_i^k(y, x)}$. (3.2)

aries: 1. If the coefficients are symmetric, $A_{x\beta}^{ij} = A_{x\beta}^{ij}$, then $G(x, y) = G(y, x)^T$
 $y \in \Omega$. 2. For any $x, y \in \Omega$, $0 < \varrho <$ dist $(y, \partial\Omega)$,
 $G^e(x, y) = \int_{\varrho(y)} (G(z, x$ Bridge $G^g(x, y) = \int_{B_0(y)}^{B_0(x)} \{f(Q(z, x))\}^T dz = \int_{B_0(y)}^{B_0(x)} G(z, z) dz.$ (3.3)

Since ${}^tG(\cdot, x)$, for fixed x, is a continuous function on $\Omega \setminus \{x\}$, formula (3.2) shows continuity

of $G(x, \cdot)$ as a function of the second a *a* $g \in \Omega$. 2. *For any* $x, y \in \Omega$, $0 < \varrho <$ dist $(y, \partial \Omega)$,
 $G^e(x, y) = \int_{B_{\varrho}(y)} \{ (G(z, x))^T dz = \int_{B_{\varrho}(y)} G(x, z) dz.$ (3.3)
 $B_{\varrho}(y)$
 $G(\cdot, x)$, for fixed x , is a continuous function on $\Omega \setminus \{x\}$, formula (3.2) shows c

Proof of Theorem 4: Let $x \neq y \in \Omega$ and choose sequences (ϱ_{ν}) , (σ_{μ}) tending to zero such that

$$
G^{\varrho_{\nu}}(\cdot, y) \to G(\cdot, y), \quad t_G^{\sigma_{\mu}}(\cdot, x) \xrightarrow[\mu \to \infty]{} ^tG(\cdot, x) \text{ a.e. on } \Omega. \tag{3.4}
$$

from the definition of the mollified Green matrix $\sqrt{ }$ Abbreviating $G' = G^{\rho}(\cdot, y)$, ${}^tG^{\mu} = {}^tG^{\sigma}{}^{\mu}(\cdot, x)$, $B_{\nu} = B_{\rho}(\nu)$, $B_{\mu} = B_{\sigma}(\nu)$, we get

$$
a_{\nu\mu}^{kl} := \oint_{B_{\nu}} {}^{l}G_{l}^{\mu k'} dz = \oint_{B_{\mu}} G_{k}^{\nu l'} dz, \qquad k, l = 1, ..., N.
$$
 (3.5)

We know ${}^tG^{\mu} \to {}^tG(., x) = : {}^tG$ for $\mu \to \infty$ in $H^{1,p}(\Omega)^{N^2}$, $1 < p < n/(n-1)$, and

 $\int_R G' dz \xrightarrow[\mu \to \infty]{} G'(x)$ by the continuity of G' at x,

so that (3.5) implies the relation

such that
\nsuch that
\n
$$
G^{e_{\nu}}(t, y) \rightarrow G(\cdot, y), t_{G}{}^{a_{\mu}}(\cdot, x) \xrightarrow{\mu \to \infty} {}^{t}G(\cdot, x) \text{ are. on } \Omega.
$$
\n
$$
(3.4)
$$
\nreviating $G' = G^{e_{\nu}}(\cdot, y), t_{G}{}^{a_{\mu}}(x) \xrightarrow{k} G^{a_{\mu}}(x, x), B_{\nu} = B_{e_{\nu}}(y), B_{\mu} = B_{e_{\mu}}(x), \text{ we get}$ \nthe definition of the modified Green matrix
\n
$$
a_{\nu\mu}^{kl} := \oint_{B_{\nu}} {}^{t}G_{l}{}^{\mu\vec{k}} dz = \oint_{B_{\mu}} G_{k}{}^{t} dz, \qquad k, l = 1, ..., N.
$$
\n
$$
(3.5)
$$
\n
$$
a_{\nu\mu}^{kl} := \oint_{B_{\nu}} {}^{t}G_{l}{}^{\mu\vec{k}} dz = \oint_{B_{\mu}} G_{k}{}^{t} dz, \qquad k, l = 1, ..., N.
$$
\n
$$
(3.6)
$$
\n
$$
a_{\mu}^{L} = \frac{1}{2} G_{\mu}^{L} \left(\frac{1}{2} \right) \text{ and } a_{\mu}^{L} = \frac{1}{2} G_{\mu}^{L} \left(\frac{1}{2} \right) \text{ and } a_{\mu}^{L} = \frac{1}{2} G_{\mu}^{L} \left(\frac{1}{2} \right) \text{ and } a_{\mu}^{L} = \frac{1}{2} G_{\mu}^{L} \left(\frac{1}{2} \right).
$$
\n
$$
(3.6)
$$
\n
$$
G_{k}{}^{t}(x) = \oint_{B_{\nu}} {}^{t}G_{l}{}^{k} dz.
$$
\n
$$
G_{k}{}^{t}(x) = \lim_{B_{\nu}} \int_{B_{\nu}} \lim_{\alpha} G_{\alpha}{}^{l}(x) \text{ and } G_{\alpha}{}^{l}(x) = \lim_{\alpha} \int_{B_{\alpha}} \lim_{\alpha} G_{\alpha}{}^{l}(x) \text{ and } a_{\mu}^{l}(x) = \lim_{\alpha} \int_{B_{\alpha}} \lim_{\alpha} G_{\alpha}{}^{l}(x) \text{ and } a_{\mu}^{l}(
$$

Since ^tG is continuous at the point y we may pass to the limit $v \to \infty$ in (3.6) to get

$$
{}^{t}G_{l}{}^{k}(y) = \lim_{\nu \to \infty} \left(\lim_{\mu \to \infty} a_{\nu\mu}^{kl} \right) \qquad \text{and} \qquad G_{k}{}^{l}(x) = \lim_{\mu \to \infty} \left(\lim_{\nu \to \infty} a_{\nu\mu}^{kl} \right). \tag{3.7}
$$

the definition of the mollified Green matrix,
 $a_{\nu}^{kl} := \int_{\mu} G_{l}^{\mu} dx = \int_{\mu} G_{k}^{\mu} dz = \int_{\mu} G_{k}^{\mu} dz$, $k, l = 1, ..., N$. (3.5)
 $a_{\nu}^{kl} := \int_{\mu} G_{l}^{\mu} dx = \int_{\mu} G_{k}^{\mu} dz$, $k, l = 1, ..., N$. (3.5)
 $G_{l}^{\mu}G_{l}^{\mu} \rightarrow G_{l}^{\mu}G_{l$ Now (3.2) follows from (3.7) since a_{ν}^{kl} converges uniformly in ν as μ tends to infinity. To prove this fix $0 < R < |x - y|$. Then, by the corollary of Lemma 3.2, there is a constant *C* independent of *v* such that for any given $0 < \alpha < 1$ the $C^{0,a}$ norm of

 G^r on the ball $B_R(x)$ is estimated by *C*. Arzela's theorem in combination with (3.4) implies *JIG*
 JIG, Are Green Matrix for Strongly Elliptic System in the straight of $B_R(x)$ *is estimated by C. Arzela's theorem in combination* $||G' - G||_{L^{\infty}(B_{R/2}(x))} \longrightarrow 0$ *, therefore* $\left| \oint_{B_{\mu}} G' dz - \oint_{B_{\mu}} G dz \right| \longrightarrow 0$ *

<i>I* with Green Matrix for Strong
 C. Arzela's theorem

therefore $\left| \oint_{B_{\mu}} G^* dz - \int_{B_{\mu}} G(y, x)^T \right|$ is trivia

$$
||G^* - G||_{L^{\infty}(B_{R/2}(x))} \longrightarrow 0, \text{ therefore } \left| \oint\limits_{B_{1}} G^* \, dz - \oint\limits_{B_{1}} G \, dz \right| \longrightarrow 0
$$

The Green Matrix for Strongly
 G' on the ball $B_R(x)$ is estimated by C. Arzela's theorem in

implies
 $||G' - G||_{L^{\infty}(B_{R/2}(x))} \longrightarrow 0$, therefore $\left| \oint_{B_{\mu}} G' dz - \oint_{B_{\mu}} G'$

uniformly with respect to μ

Proof of the cor Proof of the corollaries: $G(x, y) = G(y, x)^T$ is trivial since $G = G$ in the symmetric case. (3.3) follows from (3.6) by replacing ρ , by $\rho <$ dist (y, $\partial\Omega$)

As already remarked, the symmetry relation (3.2) implies' certain continuity properties of Green's matrix. Now we are interested in the regularity of G as a function of two variables.

Theorem 5: If (2.2) holds, then G is locally Hölder continuous on $\Omega \times \Omega \setminus \{(x, x)\}$ $x \in \Omega$ with any exponent $0 < \alpha < 1$.

Proof: Let $0 < \alpha < 1$ and define $p = n/(n - \alpha)$, $q = n/(1 - \alpha)$. Consider two point two variables.

Theorem 5: If (2.2) holds, then G is locally Holder $x \in \Omega$ with any exponent $0 < \alpha < 1$.

Proof: Let $0 < \alpha < 1$ and define $p \le n/(n - \alpha)$,

points $x \neq y$ in Ω and choose $r > 0$ so small that

(i) r *(i) exponent* $0 < \alpha < 1$.
 Proof: Let $0 < \alpha < 1$ and define $p \neq n$

ints $x \neq y$ in Ω and choose $r > 0$ so smal

(i) $r \leq |x - y|/4$, $B_{2r}(x) \cup B_{2r}(y) \subset \Omega$.

According to Theorem 4 and the, corollaries, we have for points $(x', y') \in B_r(x) \times B_r(y)$, bettons $x + y$ in Ω and choose $r > 0$ so stimate (i) $r \leq |x - y|/4$, $B_{2r}(x) \cup B_{2r}(y) \subset \text{according to Theorem 4 and the, corollaric form}$
= max $(|x - x'|, |y - y'|)$ the estimate $\varrho = \max(|x - x'|, |y - y'|)$ the estimate

properties of Green's matrix. Now we are interested in the regularity of G as a function of two variables.
\nTheorem 5: If (2.2) holds, then G is locally Hölder continuous on
$$
\Omega \times \Omega \setminus \{(x : x \in \Omega) \text{ with any exponent } 0 < \alpha < 1\}
$$
.
\nProof: Let $0 < \alpha < 1$ and define $p = n/(n - \alpha)$, $q = n/(1 - \alpha)$. Consider points $x \neq y$ in Ω and choose $r > 0$ so small that
\n(i) $r \leq |x - y|/4$, $B_{2r}(x) \cup B_{2r}(y) \subset \Omega$.
\nAccording to Theorem 4 and the,corollaries, we have for points $(x', y') \in B_r(x) \times B_r$,
\n $\rho = \max(|x - x'|, |y - y'|)$ the estimate
\n
$$
\mathcal{L}^{(p)}(x', y') - G^{\rho}(x, y)| \leq |G^{\rho}(x', y') - G^{\rho}(x, y')| + |G^{\rho}(x, y') - G^{\rho}(x, y)|
$$
\n
$$
\leq \int_{B_{\rho}(y')} |G(u, x') - G(u, x)| du + \int_{B_{\rho}(y')} f^{*}G(u, x) du - \int_{B_{\rho}(y)} f^{*}G(u, x) du|
$$
\n
$$
\leq \int_{B_{\rho}(y')} |G(u, x') - G(x, u)| du + \int_{B_{\rho}(y')} f^{*}G(u, x) du \leq \int_{B_{\rho}(y')} f^{*}G(u, x) du
$$
\n
$$
+ \int_{B_{\rho}(y')} |G(u, x) - G(x, u)| du + \int_{B_{\rho}(y')} f^{*}G(u, x) du \leq \int_{B_{\rho}(y')} f^{*}G(u, x) du
$$
\n
$$
+ \int_{B_{\rho}(y')} |G(u, x) - G(y, x)| du = a + b + c.
$$
\nIn addition to (i) we require

\n(ii) $r \leq R_0, R_0$ defined in the corollary to Theorem 2.\nApplying estimate (1.11) to each function $G(\cdot, u)$, $u \in B_{\rho}(y')$, we get by the Sobole
\nEmbedding Theorem and Lemma 3.1 $(C_0 = C_0(n, N, \$

In addition to (i) we require

• 0

(ii) $r \leq R_0$, R_0 defined in the corollary to Theorem 2. Applying estimate (1.11) to each function $G(\cdot, u)$, $u \in B_{\rho}(y')$, we get by the Sobolev $\begin{aligned}\n &\mathcal{F}_{B_{\ell}}(u) \\
 &\mathcal{F}_{B_{\ell}}(u) \\
 &\text{on to (i) we require} \\
 &\mathcal{F}_{B_{\ell}}, R_{0} \text{ defined in the corollary to Theorem 2.} \\
 &\text{estimate (1.11) to each function } G(\cdot, u), u \in B_{\ell}(y'), \\
 &\text{mg Theorem and Lemma 3.1 } (C_{0} = C_{0}(n, N, \lambda, \Lambda, \alpha)) \\
 &\text{*} \|\mathcal{G}(\cdot, u)\|_{H^{1,q}(B_{\ell}(x))} \leq C_{0} r^{1-n} \|\mathcal{G}(\cdot, u)\|_{H^{1,p}(B_{\$ *r*_{II} $|du := a + b + c$.
 r corollary to Theorem 2.
 r function $G(\cdot, u)$, $u \in B_e(y')$, we get
 r **l**₁ $C_0 = C_0(n, N, \lambda, A, \alpha)$
 *r*¹⁻ⁿ $\|G(\cdot, u)\|_{H^{1,p}(B_n(x))}$
 *r*¹⁻ⁿ $\|G(\cdot, u)\|_{H^{1,p}(Q)} \le C_1 r^{1-n}$,
 *r*¹⁻ⁿ $\|G(\cdot, u)\|$

$$
||G(\cdot, u)||_{H^{1,q}(B_r(x))} \leq C_0 r^{1-n} ||G(\cdot, u)||_{H^{1,p}(B_{2r}(x))}
$$

$$
\leqq C_0 r^{1-n} \left\|G(\cdot,u)\right\|_{H^{1,p}(\Omega)} \leqq C_1 r^{1-n},
$$

where C_1 also depends on Ω and the modulus of continuity of the coefficients. This implies $a \leq C_1 r^{1-n} \varrho^{\alpha}$. Replacing $G(\cdot, u)$ by ${}^tG(\cdot, x)$ in the above argument we get the same bound, for *^b* and *C: G*_{*E*_{*e*}(*y*) π *f* π *(i)* π *c f (i)* π *c f c c*_{*c*} *f c c*_{*s*} *c*_{*c*} *f c c*_{*c*} *f c c*_{*c*} *f <i>c c c*_{*c*} *f <i>c c*_{*s*} *f c c*_{*c*} *f <i>c <i>c c*_**} In addition to (i) we require

(ii) $r \leq R_0$, R_0 defined in the corollary to Theorem 2.

Applying estimate (1.11) to each function $G(\cdot, u)$, $u \in B_6$

Embedding Theorem and Lemma 3.1 $(C_0 = C_0(n, N, \lambda, \lambda, \lambda, \lambda, \lambda, \lambda, \lambda, \$ ing Theorem and Lemma 3.1 $(C_0 = C_0(n, N, \lambda, A, \alpha))$
 $*||G(\cdot, u)||_{H^{1,q}(B_r(x))} \leq C_0 r^{1-n} * ||G(\cdot, u)||_{H^{1,p}(B_{ir}(x))}$
 $\leq C_0 r^{1-n} ||G(\cdot, u)||_{H^{1,p}(B_{ir}(x))}$
 $\leq C_0 r^{1-n} ||G(\cdot, u)||_{H^{1,p}(B)} \leq C_1 r^{1-n}$,
 C_1 also depends on Ω and the modul $*||G(\cdot, u)||_{H^{1,0}(B_r(x))} \leq C_0 r^{1-n} * ||G(\cdot, u)|$
 $\leq C_0 r^{1-n} ||G(\cdot, u)||$

where C_1 also depends on Ω and the moduli

implies $a \leq C_1 r^{1-n} \varrho^a$. Replacing $G(\cdot, u)$ by G

same bound for b and c :
 $|G^{\varrho}(x', y') - G^{\varrho}(x, y$ *G*(*, u*) by ${}^tG(\cdot, x)$ in the above argument we get the
 $C_1 r^{1-n} \rho^s$. (3.8)
 $|G(x', y') - G^e(x', y')| + |G^e(x', y') - G^e(x, y)|$
 $+ |G^e(x, y) - G(x, y)|$.

the right-hand side are estimated as above, for the
 $C_1 r^{1-n} \rho^s$ for all $(x', y') \$

$$
|G^{\varrho}(x', y') - G^{\varrho}(x, y)| \leq C_1 r^{1-n} \varrho^{\alpha}.
$$

and for b and c:
\n
$$
|G^{e}(x', y') - G^{e}(x, y)| \leq C_1 r^{1-n} \varrho^{\alpha}.
$$
\n
$$
|G(x', y') - G(x, y)| \leq |G(x', y') - G^{e}(x', y')| + |G^{e}(x', y') - G^{e}(x, y)|
$$
\n
$$
+ |G^{e}(x, y) - G(x, y)|.
$$

The-first and the last term on the right-hand side-are estimated as above, for the

second term we use (3.8), so that
 $|G(x, y) - G(x', y')| \leq C_1 r^{1-n} \varrho^{\bullet}$ for all $(x', y') \in B_r(x) \times B_r(y)$

As a first application of Green's matrix we derive a representation formula for the weak solution of the boundary value problem. M. Foces

Internation of Green's matrix we defive a representation formula for
 $L_i u^j = \mu^i$, $i = 1, ..., N$, on Ω , $u_{|\partial \Omega} = 0$ $(\mu \in M(\Omega)^N)$.
 e m 6: If (2.2) holds, $\mu \in M(\Omega)^N$ is a vector valued signed Radon measurely

$$
L_{ij}u^j=\mu^i, \quad i=1,\ldots,N, \quad \text{on } \Omega, \, u_{|\partial\Omega}=0 \qquad \left(\mu \in M(\Omega)^N\right). \tag{3.9}
$$

Theorem 6: If (2.2) holds, $\mu \in M(\Omega)^N$ is a vector valued signed Radon measure of *finite total variation and if* $u \in \mathring{H}$ denotes the unique weak solution of (3.9), then for 518 M. Focus

As a first application of Green's matrix we derive a

weak solution of the boundary value problem
 $L_i u^j = \mu^i$, $i = 1, ..., N$, on Ω , $u_{|\partial \Omega} = ($

Theorem 6: If (2.2) holds, $\mu \in M(\Omega)^N$ is a vector if

finite *ution of the boundary value problem*
 $L_{ij}u^j = \mu^i$, $i = 1, ..., N$, on Ω , $u_{|\partial\Omega} = 0$ $(\mu \in M(\Omega)^N)$. (3.9)
 e em 6: *If* (2.2) holds, $\mu \in M(\Omega)^N$ is a vector valued signed Radon measure of
 ul variation and if $u \in \hat$ **(iii)** $\begin{aligned}\n\mathbf{F}(\mathbf{z}) &= \mu^t, \quad i = 1, ..., N, \text{ on } \Omega, u_{|\partial \Omega} = 0 \quad (\mu \in M(\Omega)^N). \tag{3.9}\n\end{aligned}$

Theorem 6: *If* (2.2) holds, $\mu \in M(\Omega)^N$ is a vector valued signed Radon measure of tinite total variation and if $u \in \hat{H}$ denote

$$
u^{k}(y) = \int_{\Omega} {}^{t}G_{\mu}{}^{i}(x, y) d\mu^{i}(x) = \int_{\Omega} G_{i}{}^{k}(y, x) d\mu^{i}(x), \qquad k = 1, ..., N. \tag{3.10}
$$

For the proof we heed

Lemma 3.3: *Assume that* (2.2) is satisfied and that $v \ge 0$ is a finite Radon measure on Ω . Then

(i) *G* is $(\nu \times L^n)$ *-measurable on* $\Omega \times \Omega$.

 $w^*(y) = \int_a^b G_f^*(x, y) d\mu^*(x) = \int_a^b G_f^*(y, x) d\mu^*(x), \qquad k = 1, ...$

For the proof we need

Lemma 3.3: Assume that (2.2) is satisfied and that $v \ge 0$ is a finite Rc

2. Then

(i) G is $(v \times L^n)$ -measurable on $\Omega \times \Omega$.

(ii) $\int |G|$ *to the space* $L^1(\Omega)^{N^*}$.

Proof of the lemma: (i) By Theorem 5 the statement follows if we show that $(v \times L^n)$ *(D)* = 0 for the diagonal $D = \{(x, x): x \in \Omega\}$. To this purpose we choose a sequence of disjoint Borel sets $(A_i)_{i\in\mathbb{N}}$ such that $\Omega = A_1 \cup A_2 \cup \ldots, \mathbf{L}^n(A_i) \leq \varepsilon$, where $\epsilon > 0$ is given. We get Lemma 3.3: Assume that (2.2) is satisfied and that $v \ge 0$ is a finite Radon measure of

(i) G is $(v \times L^n)$ -measurable on $\Omega \times \Omega$.

(ii) $\int |G| d(v \times L^n)$ is finite; especially the function $\Omega \ni y \rightarrow \int G(x, y) d\nu(x)$ belong,
 ω *I*_I *I* *I*² *I*₂ *I*² *I*₂ *I*² *I*³ *I*² *I*²

$$
(\nu \times \mathbf{L}^n) (D) \leq \sum_{i=1}^{\infty} (\nu \times \mathbf{L}^n) (A_i \times A_i) = \sum_{i=1}^{\infty} \nu(A_i) \mathbf{L}^n(A_i) \leq \varepsilon \nu(\Omega),
$$

(ii) Since $|G|$ is a non-negative $(\nu \times \mathbf{L}^n)$ -measurable function, Fubini's theorem implies

$$
(V \times H)(D) \leq \sum_{i=1}^{n} (V \times H^{-1})(A_i \times A_i) = \sum_{i=1}^{n} V(A_i) \cdot H^{-1}(A_i) \leq \varepsilon
$$

\nand by the finiteness of v we conclude $(V \times H^{n})(D) = 0$.
\n(ii) Since |G| is a non-negative $(V \times H^{n})$ -measurable function, Ful
\nplies
\n
$$
\int_{Q \times Q} |G(x, y)| d(v \times H^{n}) (x, y) = \int_{Q} \left(\int_{Q} |G(x, y)| dL^{n}(y) \right) dv(x)
$$
\n
$$
= \int_{Q} \left(\int_{Q} |G(y, x)| dL^{n}(y) \right) dv(x),
$$
\nand by Lemma 3.1 the inner integral is bounded by a constant inde
\nProof of Theorem 6: We may assume that μ^{i} , $i = 1, ..., N$, a
\nmeasures of finite mass, otherwise we decompose $\mu^{i} = \mu_{+}{}^{i} - \mu_{+}$
\n $1 \leq k \leq N$ and $0 < \rho <$ dist $(y, \partial \Omega)$. Testing the weak form of
\n $\in C^{0} \cap H^{1,2}(\Omega)^{N}$ we get
\n
$$
\int_{B_{\rho}(y)} u^{k} dz = \int_{Q} {}^{t}G_{k}{}^{ei}(x, y) d\mu^{i}(x) = \int_{Q} \left(\int_{B_{\rho}(y)} {}^{t}G_{k}{}^{i}(x, z) dL^{n}(z) \right) d\mu
$$

, and by Lemma 3.1 the inner integral is bounded by a constant independent of x I

 $=\int_{\Omega} \left(\int_{\Omega} |{}^{t}G(y, x)| d\mathbf{L}^{n}(y) \right) dv(x),$
and by Lemma 3.1 the inner integral is bounded by a constant independent of $x \in \mathbb{R}$
Proof of Theorem 6: We may assume that μ^{i} , $i = 1, ..., N$, are positive Radon
measures and by Lemma 3.1 the inner integral is bounded by a constant independent of $x \in \mathbb{R}$
Proof of Theorem 6: We may assume that μ^i , $i = 1, ..., N$, are positive Radon
measures of finite mass, otherwise we decompose $\mu^i = \mu$ **• Figure 11.001** of **Finders** in the mass, otherwise we decomposition $1 \le k \le N$ and $0 < \rho <$ dist $(y, \partial \Omega)$. Testing $\in C^0 \cap H^{1,2}(\Omega)^N$ we get
 $\int_{B_{\rho}(y)} u^k dz = \int_{\Omega} {}^t G_k e^i(x, y) d\mu^i(x) = \int_{\Omega} \int_{B_{\rho}(y)} f_{\rho(y)}$
 $= \int_{\Omega$

plies\n
$$
\int_{9\times 9} |G(x, y)| d(\nu \times \mathbf{L}^n) (x, y) = \int_{a} \left(\int_{a} |G(x, y)| d\mathbf{L}^n(y) \right) d\nu(x)
$$
\n
$$
= \int_{a} \left(\int_{a} |^{\alpha}(y, x)| d\mathbf{L}^n(y) \right) d\nu(x),
$$
\nand by Lemma 3.1 the inner integral is bounded by a constant independent of x .\nProof of Theorem 6: We may assume that μ^i , $i = 1, ..., N$, are positive Radon measures of finite mass, otherwise we decompose $\mu^i = \mu_i^i - \mu_i^i$. Choose $y \in \Omega$, $i \in \mathbb{Z} \leq N$ and $0 < \varrho <$ distinct values, we decompose $\mu^i = \mu_i^i - \mu_i^i$. Choose $y \in \Omega$, $i \in \mathbb{C} \setminus \Omega$, we get\n
$$
\int_{B_0(\mu)} \mu^k dz = \int_{a}^{t} G_k e^{i(x, y)} d\mu^i(x) = \int_{a} \int_{B_0(\mu)} f^{\alpha}(x, z) d\mu^i(x)
$$
\n
$$
= \int_{B_0(\mu)} \int_{a}^{t} G_k e^{i(x, z)} d\mu^i(x) dx
$$
\nHere we used Lemma 3.3 with G replaced by G . Observing that \mathbf{L}^n -almost all points are Lebesgue points of u^k and the function\n
$$
f^i : Q \ni z \rightarrow \int_{a}^{t} G_k i(x, z) d\mu^i(x),
$$
\nwe define Q_0 to be the common set of Lebesgue points of u^k and f^k , $k = 1, ..., N$. Obviously, $\Gamma(Q \setminus Q_0) = 0$ and (3.10) is valid for all $y \in Q_0$.\n
$$
\int_{a}^{b} G_k e^{i(x, y)} d\mu^i(x) dx
$$

Here we used Lemma 3.3 with G replaced by G . Observing that $Lⁿ$ -almost all points are Lebesgue points of *u'* and the function

$$
f^{\boldsymbol{k}}: \Omega \ni z \to \int\limits_{\Omega} {}^{t}G_{k}{}^{\boldsymbol{i}}(x,z) \, d\mu^{\boldsymbol{i}}(x),
$$

we define Ω_0 to be the common set of Lebesgue points of u^k and f^k , $k = 1, ..., N$.
Obviously, $\mathbf{L}^n(\Omega \setminus \Omega_0) = 0$ and (3.10) is valid for all $y \in \Omega_0$.

4. .Growth properties or Green's matrix

•

 \mathbb{R}^{2}

In the scalar case $N = 1$ most applications of Green's function g depend essentially on the In the scalar case $N = 1$ most applications of Green's function g depend essentially on the growth properties of g near the singular diagonal, compare $[9-11, 14]$. The purpose of this section is to prove at least a local section is to prove at least a local version of the standard estimate, i.e. we want to show 4. Green
4. Green
growth
section
for point roperties of
to prove at
 $|G(x, y)| \le$ es of Green's matrix
 $= 1$ most applicat
 g near the singular

least a local versio
 C $|x - y|^{2-n}$.

$$
|G(x, y)| \leq C |x - y|^{2-n}.
$$
\n
$$
(4.1)
$$

for points x, y in small balls compactly contained in Ω . This behaviour of G is suggested by the • growth properties of the fundamental matrix *B* for systems with constant coefficients (see Lem. ma 1.2). For technical reasons (compare Lemma 4.3) the proof of (4.1) only works in the case of Hölder continuous coefficients so that we assume for the rest of this section (GA) and there exist constants $0 < \alpha < 1$, $L \ge 0$ such that $|G(x, y)| \le C |x - y|^{2-n}$.
 x, y in small balls compactly contained in Ω . This be

operties of the fundamental matrix E for systems with

or technical reasons (compare Lemma 4.3) the proof

ntinuous coefficients so that

\n At the values coefficients, so that we assume for the rest of this section, the rest of this section, the first constants:\n
$$
0 < \alpha < 1
$$
,\n $L \geq 0$ such that:\n $|A(x) - A(y)|/|x - y|^{\alpha} : x + y \in \overline{\Omega} \} \leq L.$ \n \n (4.2) \n

Under this assumption the basic local estimate (4.1) is proven in Theorem 7. As a consequence of this theorem we derive inequalities concerning the behaviour of the first derivatives of Green's matrix near the diagonal.

The method of our proof is based on a pertubation argument: We freeze the coefficients at an arbitrary point $y \in \Omega$ and write $G(x, y) = E(x - y) + H_y(x)$, where *E* denotes the fundamental matrix for the operator with constant coefficients $A(y)$. Using the Hölder condition (4.2) it is possible to control the size of the perturbation H_y at least locally near y. We hope to be able to extend our technique to derive global estimates for G up to the boundary. Under this assumption the basic local estimate (4.1) is proven in Theorem 7. As a consequence
of this incorem we derive inequalities concerning the behaviour of the first derivatives of
Green's matrix near the diagonal.
T

Theorem 7: *Suppose that* (4.2) *holds and let* $0 < \beta < 1$ *be given. Then there are constants C₁* depending on n, N, L, λ , Λ , α , R_0 also depending on β and C_2 also depending *on* β *and* Ω *such that*

$$
|G(x, y)| \leq C_1 |x - y|^{2-n} + C_2 R^{1+\beta-n}
$$
\n(4.3)

for all $x \in B_{2R}(y) \setminus \{y\}, y \in \Omega, 0 \lt R \leq \min(R_0, \text{dist}(y, \partial \Omega)/4).$

Corollary: *Let y* $\in \Omega$ *be given. Then there exists* R_y depending on *n, N, L, 2, A, a*
 $|G(x, z)| \leq C_1 |x - z|^{2-n} + C_2 R_y^{1-n+\alpha}$ (4.4) *and dist* $(y, \partial \Omega)$ *such that*

$$
|G(x, z)| \leq C_1 |x - z|^{2-n} + C_2 R_y^{1-n+\alpha}
$$
\n(4.4)

for all x, $\overline{z} \in B_{R_n}(y) \subset \Omega$. Here C_1 , C_2 are as in Theorem 7 with $\beta = \alpha$.

(4.4) is the precise formulation of the local estimate (4.1): Since $R_y \rightarrow 0$ when y approaches

the boundary $\partial\Omega$ we see that inequality. (4.4) essentially depends on the location of the ball $B_{R_{\nu}}(y)$.

Since the proof of the theorem is lengthy we found it useful to proceed in several

steps summarized as Lemm *BR(in) BR(in) Bg(in) C*) *E B_{R(in)})* \setminus *ig)*, *y* \in *Q.* $0 \lt R \leq \min(R$ Since the proof of the theorem is lengthy we found it useful to proceed in several steps summarized as Lemma 4.1—4.3. From now on we assume that the assumptions of Theorem 7 are satisfied. Define $p = n/(n - \beta)$, $q = n/(1 - \beta)$, $p_0 = n/(n - \alpha)$. and assume that $y := 0$ is contained in Ω . According to (1.10) there exists R_0 such $|G(x, z)| \leq C_1 |x - z|^{2-n} + C_2 R_y^{1-n+2}$
 for all $x, \overline{z} \in B_{R_y}(y) \subset \Omega$. Here C_1, C_2 are as in Theorem 7 with $\beta = \alpha$.

(4.4) is the precise formulation of the local estimate (4.1): Since $R_y \to 0$ when y approx

the bou he proof of the theorem is lengthy we found it useful to proceed in sever
imarized as Lemma 4.1¹-4.3. From now on we assume that the assumptio
em 7 are satisfied. Define $p = n/(n - \beta)$, $q = n/(1 - \beta)$, $p_0 = n/(n -$
me that $y :=$ for all $x, \overline{z} \in B_{R_y}(y) \subset \Omega$. Here C_1, C_2 are as in Theorem (4.4) is the precise formulation of the local estimate (4.1): S
the boundary $\partial \Omega$ we see that inequality (4.4) essentially depend
 $B_{R_y}(y)$.
Since the pr nd it useful to

we assume the
 $r, q = n/(1 - \mu)$
 $r - kp$
 $k - kp$
 k
 $r + kp$
 $r + kp$
 $r + cp$

$$
||T_{4R}^s|| \leq 1/2, \qquad s = p_0, \text{ and } s = np/(n - kp), \qquad k = 0, ..., n - 1,
$$
\n
$$
(4.5)
$$

that
 $*||T_{4R}^s|| \le 1/2$, $s = p_0$ and $s = np/(n - kp)$, $k = 0, ..., n - 1$,

for all $0 < R \le \min (R_0, \text{dist } (0, \partial \Omega)/4)$. Fix R with (4.5) and for $1 \le k \le N$ let
 $G := G_k(0, 0), E := E_k$ and $\iota = 1,$
 (4.5)
 $\leq N$ let
 (4.6)
 $\iota(0)$ ^N : 1

$$
w := G - T_{2R}^p G - E \in H^{1,p}(B_{2R}(0))^N.
$$
 (4.6)

(E is defined in Lemma 1.2b)). Obviously, *w* belongs to the space $\bigcap \{H^{1,r}(B_{2R}(0))^N:1\}$ $\leq r < n/(n-1)$ and from (4.6) we infer $L_{0i}w^{i} = 0$ on $B_{2R}(0)$. In conclusion, *w* is analytic on $B_{2R}(0)$, especially bounded on any ball $B_{r}(0)$ with radius $r < 2R$. The following lemma shows boundedness of w on the whole ball $B_{2R}(0)$.

Lemma 4.1: The function w is contained in the space $H^{1,q}(B_{2R}(0))^N$ and satisfies the estimate $||w||_{q,2R} \leq C_2 R^{1-n}$. 520 M. FUCHS

Lemma 4.1: The function w is contained in the space $H^{1,q}(B_2)$

estimate $*||w||_{q,2R} \leq C_2 R^{1-n}$.

Here and in the sequel we abbreviate $*||.||_{s,r} = *||.||_{H^{1,q}(B_r(0))}$.

Final $\epsilon^*||w||_{q,2R} \leq C_2 R^{1-n}$.

Here and in the sequel we abbreviate $||\cdot||_{s,r} = ||\cdot||_{H^{1,q}(B_r(0))}$.

Proof of the lemma: The corollary of Theor. 2 implies $||w||_{q,R} \leq C_2 R^{1-n} ||w||_{p,2R}$. To estimate the norm on the right-hand side we use the defining equation (4.6):
 $*$ $||G||_{p,2R} \le \frac{1}{R} \{ \mathbf{L}^n (B_{2R}(0))^s ||G||_{L^s(p)} (B_{1R}(0)) \} + ||\nabla G||_{L^p(\Omega)}$

520 M. FUCBS
\nLemma 4.1: The function w is contained in the space
$$
H^{1,q}(B_{2R}(0))^N
$$
 and satisfy
\nestimate *||w||_{2R} $\leq C_2 R^{1-n}$.
\nHere and in the sequel we abbreviate *||. $||_{s,r} =$ *||. $||_{H^{1,q}(B_{2R}(0))}$.
\nProof of the lemma: The corollary of Theor. 2 implies *||w||_{q,R} $\leq C_2 R^{1-n}$ *||w
\nTo estimate the norm on the right-hand side we use the defining equation (4.6
\n*||G||_{p,2R} $\leq \frac{1}{R}$ { $L^n(B_{2R}(0))^3$ ||G||_{L*(0)(B_{4R}(0))}} + ||\nabla G||_{L*(0)}
\n $\leq C_2$ || $\nabla G||_{L^p(\Omega)} \leq C_2$, $\delta := \frac{1}{p} - \frac{n-p}{np}$, $s(p) := \frac{np}{n-p}$,
\nby Lemma 3.1 (iii). From (4.5) and (1.7) we infer
\n*|| $T_{2R}^p G||_{p,2R} \leq \frac{1}{2}$ *|| $G||_{p,2R} \leq C_2$, *|| $E||_{p,2R} \leq C_2 R^{1-\beta}$,
\nin conclusion,
\n*||w||_{q,R} $\leq C_2 R^{1-n}$.
\nTo complete the proof of the lemma we have to show
\n $\frac{1}{R}$ ||u||_{L^q(T_{2R})} + || $\nabla u||_{L^q(T_{2R})} \leq C_2 R^{1-n}$
\nfor the functions $u := G$, $T_{2R}^p G$, E , where T_{2R} denotes the ring $B_{2R}(0) \setminus B_R(0)$.
\n $u := E$, U , U , for $u := G$ we use estimate (1.13). I

by Lemma 3.1 (iii). From (4.5) and (1.7) we infer

$$
^*\|T_{2R}^p G\|_{p,2R}\leqq \frac{1}{2}^*\|G\|_{p,2R}\leqq C_2^{},\qquad^*\|E\|_{p,2R}\leqq C_2R^{1-\beta},
$$

$$
^*||w||_{q,R} \leq C_2 R^{1-n}
$$

 $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$
\frac{1}{R} \|u\|_{L^{q}(T_{2R})} + \|\tilde{\nabla}u\|_{L^{q}(T_{2R})} \leq C_{2} R^{1-n}
$$
\n(4.8)

(4.7)

(4.10)

for the functions $u := G$, $T_{2R}^p G$, E , where T_{2R} denotes the ring $B_{2R}(0) \setminus B_R(0)$. For $u := E(4.8)$ is already contained in (1.7), for $u := G$ we use estimate (1.13). Dropping all indices we let $\omega(y) = A(y) - A(0)$ and write for $x \in B_{2R}(0)$ *•* $\|T_{2R}^p G\|_{p,2R} \leq \frac{1}{2}$ *•* $\|G\|_{p,2R} \leq C_2$, \qquad *•* $\|E\|_{p,2R} \leq C_2 R^{1-\beta}$,
 *i*n conclusion,
 • $\|w\|_{q,R} \leq C_2 R^{1-\eta}$.
 To complete the proof of the lemma we have to show
 $\frac{1}{R} \|w\|_{L^q(T_{1R})$ *- B*, $\lim_{R_1 \to 0} \frac{R_1}{R_2}$ *l*(*R*) \neq *i s* **a** i = *G*, $T_{2R}^p G$, *E*, where T_{2R} denotes the ring $B_{2R}(0) \setminus B_R(0)$. For $u := E$ (4.8) is already contained in (1.7), for $u := G$ we use estimate (1.13). Droppin

$$
\frac{1}{R} ||u||_{L^{q}(T_{1R})} + ||\tilde{V}u||_{L^{q}(T_{1R})} \leq C_{2}R^{1-n}
$$
\n
$$
u := E_{1R} \cdot \frac{1}{R} \cdot \frac{1
$$

Obviously, $\varphi_2 = \tilde{P}_{2R}(F)$ for $F(y) := 0$ on $B_{R/2}(0)$, $F(y) := \omega(y) \nabla G(y)$ on $B_{2R}(0)$
 $\sum B_R/(0)$. Lemma 1.3 implies

$$
^*||\varphi_2||_{q,2R} \leq C_2 ||F||_{L^q(D)} \leq C_2 ||\nabla G||_{L^q(D)} \leq C_2 R^{1-n}, \qquad D := B_{2R}(0) \setminus B_{R/2}(0)
$$

by a version of inequality (1.13). Since φ_1 is of class C^1 on T_{2R} we get (4.8) for φ_1 by direct calculation. Combining (4.7) and (4.8), the assertion of the lemma follows \blacksquare

Obviously,
$$
\varphi_2 = I_{2R}(T)
$$
 for $T(y) = 0$ on $B_{R/2}(0)$, $T(y) = \omega(y) \vee G(y)$ on $B_{2R}(0)$.

\n $\|\varphi_2\|_{q,2R} \leq C_2 \|F\|_{L^q(D)} \leq C_2 \|G\|_{L^q(D)} \leq C_2 R^{1-n}$, $D := B_{2R}(0) \setminus B_{R/2}(0)$, by a version of inequality (1.13). Since φ_1 is of class C^1 on T_{2R} we get (4.8) for φ_1 by direct calculation. Combining (4.7) and (4.8), the assertion of the lemma follows.

\nUsing $w \in H^{1,q}(B_{2R}(0))^N$, we can rewrite (4.6) in the form

\n $G = E + \sum_{l=1}^{\infty} (T_{2R}^p)^l E + \sum_{l=0}^{\infty} (T_{2R}^q)^l w$ on $B_{2R}(0)$,

\nwhere the last term on the right-hand side satisfies (use (4.5))

\n $\|\sum_{l=0}^{\infty} (T_{2R}^q)^l w\|_{q,2R} \leq \sum_{l=1}^{\infty} E_{2R}^{1-n}$, and by the Sobolev Embedding Theorem we get

\n $\sup_{B_{2R}(0)} \left| \sum_{l=0}^{\infty} (T_{2R}^q)^l w \right| \leq C_2 R^{1-n+\beta}$.

\nEstimate (1.7) implies $|E(x)| \leq C_1 |x|^{2-n}$ for all $x \neq 0$, so that (4.3) follows from (4.9) and (4.10), provided the sum over l of all $(T_{2R}^p)^l E$ has the correct growth. The calculus

where the last term on the right-hand side satisfies (use (4.5))
 $*||$ $\frac{1}{2}$ $\frac{1}{2}$

by a version of inequality (1.13). Since
$$
\varphi_1
$$
 is of clas
direct calculation. Combining (4.7) and (4.8), the as
Using $w \in H^{1,q}(B_{2R}(0))^N$, we can rewrite (4.6) in

$$
G = E + \sum_{l=1}^{\infty} (T_{2R}^{\rho})^l E + \sum_{l=0}^{\infty} (T_{2R}^q)^l w
$$
 on B_2
where the last term on the right-hand side satisfies

$$
\prod_{l=0}^{\infty} (T_{2R}^q)^l w \Big|_{q,2R} \leq \prod_{l=0}^{\infty} (T_{2R}^q)^l w
$$
and by the Sobolev Embedding Theorem we get

$$
\sup_{B_{2R}(0)} \left| \sum_{l=0}^{\infty} (T_{2R}^q)^l w \right| \leq C_2 R^{1-n+\beta}.
$$

Estimate (1.7) implies $|E(x)| \leq C_1 |x|^{2-n}$ for all $x \neq$
and (4.10), provided the sum over l of all $(T_2^p)^l E$

and by the Sobolev Embedding Theorem we get

-

$$
\sup_{\beta_{2R}(0)}\bigg|\sum_{l=0}^{\infty}\left(T_{2R}^{q}\right)^{l}w\bigg|\leq C_{2}R^{1-n+\beta}.
$$

Estimate (1.7) implies $|E(x)| \leq C_1 |x|^{2-n}$ for all $x \neq 0$, so that (4.3) follows from (4.9) and (4.10), provided the sum over *l* of all $(T_{2h}^{p})^t E$ has the correct growth. The cal-• culation of the growth order of this remaining term is contained in

Lemma 4.2: *Suppose that* σ_{γ} *z* are real numbers satisfying $0 < \tau$, $\sigma < n$, $\sigma + \tau > n$. *Then for all* $x, y \in \mathbb{R}^n$

The Green Matrix for Strongly Elliptic Systems
\n1 m.a 4.2: Suppose that
$$
\sigma_{y} \tau
$$
 are real numbers satisfying $0 < \tau$, $\sigma < n$, $\sigma + \tau > n$.
\nor all $x, y \in \mathbb{R}^n$
\n
$$
\int |x - z|^{-\sigma} |y - z|^{-\tau} dz \leq C(n, \tau, \sigma) |x - y|^{n-\tau-\sigma}.
$$
\n(4.11)
\n
$$
\lim_{\mathbb{R}^n} 4.3: \text{Let } u_i = (T_{2R}^{p_i})^t E, l \in \mathbb{N}_0. \text{ Then there exists a constant } C_1 = C_1(n, N,
$$

Lemma 4.3. Let $u_l = (T_{2R}^{p_0})^l E$, $l \in \mathbb{N}_0$. Then there exists a constant $C_1 = C_1(n, N, N)$ *L*, λ , Λ , α) such that for all balls $B_{4R}(0) \subset \Omega$ and all points $x, x_1, x_2 \in B_{2R}(0) \setminus \{0\}$, $R < 1$, the following estimates hold: The Green Matrix for

Lemma 4.2: Suppose that $\sigma_{y} \tau$ are real numbers satisfied $\sigma_{h} \tau$ are real numbers satisfied $\int |x - z|^{-\sigma} |y - z|^{-\tau} dz \leq C(n, \tau, \sigma) |x - y|$

Lemma 4.3: Let $u_i = (T_{2h}^{p})^l E, l \in N_0$. Then there $L, \lambda, \$ The Green Matrix for Strongly Elliptic

Lemma 4.2: Suppose that σ_{12} are real numbers satisfying $0 < \tau$, σ

lemma 4.2: σ_{21} σ_{22} σ_{23} σ_{24} are real numbers satisfying $0 < \tau$, σ
 $\int |x-z|^{-\sigma} |y-z|^{-\$ **I** Lemma 4.2: Suppose that $\sigma_{n,\tau}$ are real numbers satisfying $0 < \tau$, $\sigma < n$, $\sigma + \tau > n$.
 I hen for all $x, y \in \mathbb{R}^n$
 $\int |x - z|^{-\sigma} |y - z|^{-\tau} dz \leq C(n, \tau, \sigma) |x - y|^{n-r-\sigma}$ (4.11)
 I Lemma 4.3: Let $u_i = (T_{2R}^p)^t E$,

$$
(|\cdot| \nabla u_i(x)| \leq C_1^{l+1} R^{l\alpha} |x|^{1-n},
$$

•
•

 (4.12) (ii) $|\nabla u_i(x_1) - \nabla u_i(x_2)| \leq C_1^{l+1} R^{l\alpha} |x_1 - x_2|^{\alpha} \max (|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}).$

The proof of Lemma 4.2 is an easy calculation, whereas the proof of Lemma 4.3 is somewhat more involved. We therefore first finish the proof of Theorem 7: For (i) $|\nabla u_i(x)| \leq C_1^{l+1} R^{l\alpha} |x|^{1-n}$,

(ii) $|\nabla u_i(x_1) - \nabla u_i(x_2)| \leq C_1^{l+1} R^{l\alpha} |x_1 - x_2|^{\alpha} \max (|x_1|^{1-n-\alpha})$

The proof of Lemma 4.2 is an easy calculation, whereas

somewhat more involved. We therefore first finish t $l \in \mathbb{N}, x \in B_{2R}(0) \setminus \{0\}$ we have

$$
u_i(x)| \leq C_1^{l+1} R^{l\alpha} |x|^{1-n},
$$

\n
$$
u_i(x_1) - \nabla u_i(x_2)| \leq C_1^{l+1} R^{l\alpha} |x_1 - x_2|^{\alpha} \max (|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}).
$$

\n
$$
u_1(x_1) - \nabla u_i(x_2)| \leq C_1^{l+1} R^{l\alpha} |x_1 - x_2|^{\alpha} \max (|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}).
$$

\n
$$
u_1(x_1) - \nabla u_i(x_2)| \leq C_1^{l+1} R^{l\alpha} \sum_{i=1}^{\infty} |x_i - x_i|^{\alpha} \min\{1, 1\} \text{ for } i \in \mathbb{Z}
$$

\n
$$
|P(x_1, y_1)| \leq C_1 R^{\alpha} \int |x - y|^{1-n} C_1 R^{l-1} |x_1|^{1-n} dy \leq C_1^{l+1} R^{l\alpha} |x|^{2-n},
$$

\n
$$
|x_1| \leq C_1 R^{\alpha} \int |x - y|^{1-n} C_1 R^{l-1} |x_1|^{1-n} dy \leq C_1^{l+1} R^{l\alpha} |x|^{2-n},
$$

\n
$$
|x_1| \leq C_1 \sum_{i=1}^{\infty} |x_i - x_i|^{\alpha} \min\{1, 1\} \text{ for } i \in \mathbb{Z}
$$

\n
$$
|x_1| \leq C_1 \sum_{i=1}^{\infty} |x_i - x_i|^{\alpha} \min\{1, 1\} \text{ for } i \in \mathbb{Z}
$$

\n
$$
|x_1| \leq C_1 R^{\alpha} \leq 1/2 \text{ for } i \in \mathbb{Z}
$$

\n
$$
|x_1| \leq C_1 R^{\alpha} \text{ for } i \in \mathbb{Z}
$$

\n
$$
|x_1| \leq C_1 R^{\alpha} \text{ for } i \in \mathbb{Z}
$$

\n
$$
|x_1| \leq C_1 R^{\alpha} \text{ for } i \in \
$$

according to (4.11) , (4.12) . From this we infer

$$
\left|\sum_{l=1}^{\infty} (T_{2R}^{p_0})^l E(x)\right| \leq C_1 \sum_{l=1}^{\infty} (C_1 R^{\alpha})^l |x|^{2-\alpha}.
$$

Requiring $C_1R_0^* \leq 1/2$ (C_1 from Lemma 4.3) we arrive at (4.3) \blacksquare

• Before proceeding further let us give some comment on Lemma 4.3: In standard potential theory it is shown that Lebesgue, Sobolev and Holder classes are reproduced by certain singular integral operators (compare for exrnple [15]). Here we extend this reproduction property toa. class of functions having an isolated singularity with prescribed growth order. sory it is shown that Lebesgue, Sobolev and Hölder classes are reproduced by certain singular
egral operators (compare for example [15]). Here we extend this reproduction property to a
ss of functions having an isolated s *I*^{*n*}_{2*R*}(*I***^I)**, (4.12). From this we infer
 $\left| \sum_{i=1}^{\infty} (T_{2R}^{p})^{l} E(x) \right| \leq C_{1} \sum_{i=1}^{\infty} (C_{1}R^{a})^{l} |x|^{2-n}$.
 *I*² *IR*₀² $\leq 1/2$ (*C*₁ from Lemma 4.3) we arrive at (4.3) **I**
 *I*² *Proc*

. dropping all indices

$$
T_{2h}^{p_k}u(x) = \int_{B_{2h}(0)} \nabla^{2N-1} K(x-y) \omega(y) \nabla u(y) dy
$$

and proceed by induction. For $l = 0$ (4.12) immediately follows from (1.7). Now assume that l is a positive integer and that for some constant D_{l-1} the estimates

$$
T_{2n}^{p_i}u(x) = \int \nabla^{2N-1} K(x - y) \omega(y) \nabla u(y) dy
$$

proceed by induction. For $l = 0$ (4.12) immediately follows from (1.7). Now
ume that *l* is a positive integer and that for some constant D_{l-1} the estimates

$$
|\nabla u_{l-1}(x)| \leq D_{l-1} |x|^{1-n},
$$

$$
|\nabla u_{l-1}(x_1) - \nabla u_{l-1}(x_2)| \leq D_{l-1} |x_1 - x_2|^{\alpha} \max (|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha})
$$
(4.13)
1 for all points x, x_1, x_2 in $B_{2R}(0)$. Abbreviating $\Delta(y) = \nabla^{2N} K(y), y \in \mathbf{R}^n \setminus \{0\},$ we

'hold for all points x, x₁, x₂ in $B_{2R}(0)$. Abbreviating $\Delta(y) = \nabla^{2N} K(y)$, $y \in \mathbb{R}^n \setminus \{0\}$, we have the following formula for the derivative of u_i (compare [15: Thm. 3.4.2b)], Froot of Lemma 4.3: we write for $x \in B$
dropping all indices
 $T_{2n}^{p_i}u(x) = \int \nabla^{2N-1} K(x - y) \omega(y) \nabla u$
 $B_{2n}(0)$
and proceed by induction. For $l = 0$ (4.12)
assume that l is a positive integer and that for
 $|\nabla u_{l-1}(x)| \le$

$$
|\nabla u_{l-1}(x_1) - \nabla u_{l-1}(x_2)| \leq D_{l-1} |x_1 - x_2|^{\alpha} \max(|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}) \qquad (4.13)
$$

hold for all points x, x_1, x_2 in $B_{2R}(0)$. Abbreviating $\Delta(y) = \nabla^{2N} K(y), y \in \mathbb{R}^n \setminus \{0\}$, we
have the following formula for the derivative of u_l (compare [15: Thm. 3.4.2b)],
here and in the sequel C_1 denotes constants depending on the parameters stated in
Lemma 4.3)

$$
\nabla u_l(x) = C_1 \omega(x) \nabla u_{l-1}(x) + \lim_{\rho \to 0} \int \Delta(x - y) \omega(y) \nabla u_{l-1}(y) dy
$$

$$
\omega^{\alpha \to 0} B_{2R}(0) \Delta_{\rho}(x)
$$

$$
=: f(x) + \lim_{\rho \to 0} W_{\rho}(x), \qquad x \in B_{2R}(0) \setminus \{0\}.
$$
(4.14)

522 M. Fucus

a) We discuss $W(x) := \lim_{\rho \to 0} W_{\rho}(x)$ ($\rho \to 0$) for fixed $x \in B_{2R}(0) \setminus \{0\}$: Choose $0 < \sigma < \rho < |x|/2$ and apply [14: Thm. 2.6.5] to get. 622 M. Fucus

a) We discuss $W(x) := \lim_{x \to 0} W_e(x)$ $(\varrho \to 0)$ for fixed $x \in B_{2R}(0) \setminus \{0\}$: Cl
 $\varrho < |x|/2$ and apply [14: Thm. 2.6.5] to get.
 $W_e(x) - W_{\varrho}(x) = \int_{B_{\varrho}(x) \setminus B_{\varrho}(x)} \Delta(x - y) (\varphi(y) - \varphi(x)) dy$, $\varphi = \omega^{\mathsf{V}}$

(4. *M.* FUCHS
 Wether Fig. 7 f (x) π *f (x)* (π -> 0) for fixed $x \in B_{2R}(0) \setminus \{0\}$: Choose

and apply [14: Thm. 2.6.5] to get.
 $W_e(x) - W_o(x) = \int_{B_e(x) \setminus B_o(x)} \Delta(x - y) (\varphi(y) - \varphi(x)) dy, \qquad \varphi^* = \omega \nabla u_{i-1}.$

es the inequ

$$
W_e(x) - W_{\sigma}(x) = \int \Delta(x - y) (\varphi(y) - \varphi(x)) dy, \qquad \varphi^* = \omega \nabla u_{l-1}
$$

ves the inequality

$$
|\varphi(y) - \varphi(x)| \leq C_1 D_{l-1}(|x|^{\alpha} |x - y|^{\alpha} \max (|x|^{1-n-\alpha}, |y|^{1-n-\alpha})
$$

 (4.13) gives the inequality

discuss
$$
W(x) := \lim_{\rho(x) \to 0} W_{\rho}(x) (p \to 0)
$$
 for fixed $x \in B_{2R}(0) \setminus \{0\}$: Choose $0 < \sigma <$
\n*l* and apply [14: Thm. 2.6.5] to get.
\n $W_{\rho}(x) - W_{\sigma}(x) = \int_{B_{\rho}(x) \setminus B_{\sigma}(x)} \Delta(x - y) (\varphi(y) - \varphi(x)) dy$, $\varphi := \omega \nabla u_{l-1}$.
\n $\varphi(y) - \varphi(x) \leq C_1 D_{l-1} \{|x|^{\alpha} |x - y|^{\alpha} \max (|x|^{1-n-\alpha}, |y|^{1-n-\alpha})$
\n $+ |y|^{1-n} |x - y|^{\alpha}$.
\nWe can estimate the above integral as follows:

By this we can estimate the above integral as follows:

(4.13) gives the inequality
\n
$$
|\varphi(y) - \varphi(x)| \leq C_1 D_{l-1}(|x|^{\alpha}|x-y|^{\alpha} \max(|x|^{1-n-\alpha}, |y|^{1-n-\alpha}) + |y|^{1-n} |x-y|^{\alpha}).
$$
\n(4.15)
\nBy-this we can estimate the above integral as follows:
\n
$$
|W_{\rho}(x) - W_{\sigma}(x)| \leq C_1 D_{l-1} \left(\int_{B_{\rho}(x) \setminus B_{\sigma}(x)} |x|^{1-n} \max(|x|^{1-n-\alpha}, |y|^{1-n-\alpha}) dy |x|^{\alpha} \right)
$$
\n
$$
+ \int_{B_{\rho}(x) \setminus B_{\sigma}(x)} |x-y|^{\alpha-n} |y|^{1-n} dy
$$
\n
$$
\leq C_1 D_{l-1}(|x|^{\alpha}|x|^{1-n-\alpha} + |x|^{1-n}) \int |x-y|^{\alpha-n} dy
$$
\n
$$
\leq C_1 D_{l-1} |x|^{1-n} \varrho^{\alpha}.
$$
\nConsequently, $(W_r(x))_{r>0}$ is a Cauchy sequence, the limit $W(x)$ exists and satisfies for $x \in B_{2R}(0) \setminus \{0\}, \varrho < |x|/2, \varrho < 2R - |x|$ the inequality
\n
$$
|W_{\varrho}(x) - W(x)| \leq C_1 D_{l-1} \varrho^{\alpha}|x|^{1-n}. \qquad (4.16)
$$
\nFurthermore we have for x and ϱ as above, using (4.11) and (4.15).
\n
$$
|W_{\varrho}(x)| \leq \int_{B_{2R}(0) \setminus B_{\varrho}(x)} |\Delta(x-y)| |\varphi(y) - \varphi(x)| dy \leq C_1 D_{l-1} R^{\alpha} |x|^{1-n}.
$$
\n(4.16)
\nBy (4.13), estimate (4.17) holds for the function f defined in (4.14). We thus have proven
\n
$$
|W(x)| \leq C_1 D_{l-1} R^{\alpha} |x|^{1-n}, \qquad x \in B_{2R}(0) \setminus \{0\}.
$$
\n(4.17)
\nBy (4.13), estimate (4.17) holds for the function f defined in (4.14). We

is
equently, $(W_r(x))_{r>0}$ is a Cauchy sequence, the limit $W(x)$ exists and
 $B_{2R}(0) \setminus \{0\}$, $\varrho < |x|/2$, $\varrho < 2R - |x|$ the inequality
 $|W_{\varrho}(x) - W(x)| \leq C_1 D_{l-1} \varrho^{\alpha} |x|^{1-n}$.
thermore we have for x and ϱ as above,

$$
|W_e(x) - W(x)| \le C_1 D_{l-1} e^x |x|^{1-n}.
$$
\n(4.16)

Furthermore we have for x and ρ as above, using (4.11) and (4.15),

$$
|\Psi_{\varrho}(x)| \leqq \int_{B_{2R}(0) \setminus B_{\varrho}(x)} |\Delta(x-y)| |\varphi(y) - \varphi(x)| dy \leqq C_1 D_{l-1} R^{\alpha} |x|^{1-n}.
$$

If we combine this result with (4.16) we arrive at

By (4.13) , estimate (4.17) holds for the function f defined in (4.14) . We thus have proven lVuj(x)l *C*₁*D*₁₋₁*g* |*x*| \leq *C*₁*D*₁₋₁*g* |*x*| \leq *C*₁*D*₁-1*R*^c |*x*|¹⁻ⁿ.
 *C*₁*D*_{*x*(0*i*)*B*_{*g*(*x*)}
 *c*sult with (4.16) we arrive at
 *c*₁*D*_{*t*-1}*R*^{*x*} |*x*|¹⁻ⁿ}, \leq *x* \leq *B*

$$
|\nabla u_i(x)| \le C_1 D_{i-1} |x|^{1-n} R^{\alpha}
$$
 (4.18)

for all points x in the punctured ball $B_{2R}(0) \setminus \{0\}$.

b) We now derive the Hölder condition for ∇u_i : Let $x_1, x_2 \in B_{2R}(0) \setminus \{0\}$ be given and assume (4.17)

We thus have

(4.18)

(4.18)

(4.19)

(4.19)

(4.19)

(4.16)

(4.16)

$$
\varrho := |x_1 - x_2| \leq \min(|x_1|, |x_2|)/5.
$$

hore we have for x and ρ as above, using (4.11
 $|W_{\rho}(x)| \leq \int |\Delta(x - y)| |\varphi(y) - \varphi(x)| dy \leq$
 $B_{2R}(0) \setminus B_{\rho}(x)$

thin this result with (4.16) we arrive at
 $|W(x)| \leq C_1 D_{l-1} R^{\alpha} |x|^{1-n}, \quad x \in B_{2R}(0) \setminus \{0\}$
 $|W(u)| \leq C_1 D_{l-1$ For all points x in the punctured ball $B_{2R}(0) \setminus \{0\}$.

b) We now derive the Hölder condition for ∇u_i : Let $x_1, x_2 \in B_{2R}(0) \setminus \{0\}$ be given

and assume
 $\varrho := |x_1 - x_2| \leq \min(|x_1|, |x_2|)/5$.

One would like to argu $- W_{\varrho}(x_2)$ and the Holder condition for *W* is a consequence of (4.16). Unfortunately, (4.16) is restricted to the case $\varrho < 2R - |x_i|$, $i = 1, 2$, and this condition is obviously violated for $\rho := |x_1 - x_2| \leq \min(|x_1|, |x_2|)/5$.
One would like to argue as follows: By calculating ∇W_{ρ} one gets a bound for $|W_{\rho}(x_1)|$. x_1, x_2 near $\partial B_{2R}(0)$. To overcome this technical difficulty we extend the function ∇u_{l-1} to the punctured ball $B_{4R}(0) \setminus \{0\}$, assuming that (4.13) continues to hold for the extended function, which we also denote by ∇u_{l-1} . The constant D_{l-1} appearing in (4.13) has to be replaced by a constant of the form C_1D_{l-1} , but this does not change the argument. oven
 $|\nabla u_i(x)| \leq C_1 D_{l-1} |x|^{1-n} R^x$

All points x in the punctured ball $B_{2R}(0) \setminus \{0\}$.

b) We now derive the Hölder condition for ∇u_i : Let

d assume
 $\varrho := |x_1 - x_2| \leq \min (|x_1|, |x_2|)/5$.

One would like to argue **•** $\frac{r}{x}$
 • $\frac{x}{y}$
 • $\frac{r}{y}$

• $\frac{r}{y}$ *w*_{*e*}(*x*₂)| and the notate condition for *W* is a consequence of (4.10). C stricted to the case $\varrho < 2R - |x_i|$, $i = 1, 2$, and this condition is x_2 near $\partial B_{2R}(0)$. To overcome this technical difficulty we exten $\varrho := |x_1 - x_2| \leq \min (|x_1|, |x_2|)|5$.

One would like to argue as follows: By calculating ∇W_{ϱ} one gets
 $-V_{\varrho}(x_2)|$ and the Hölder condition for *W* is a consequence of (4.16). U

estricted to the case $\varrho < 2R - |x_i$

-

On
$$
B_{4R}(0) \setminus \{0\}
$$
 we define the functions

Then we also denote by
$$
vu_{l-1}
$$
. The constant D_{l-1} appears
\nthat of the form C_1D_{l-1} , but this does not change to
\n
$$
W_{1e}(x) = \int_{B_{4R}(0) \setminus B_{\varrho}(x)} \Delta(x-y) \omega(y) \nabla u_{l-1}(y) \, dy
$$
\n
$$
W_1(x) = \lim_{\varrho \to 0} W_{1\varrho}(x),
$$
\n
$$
W_2(x) = \int_{T_{2R}(0)} \Delta(x-y) \omega(y) \nabla u_{l-1}(y) \, dy
$$

Obviously, $W(x) = W_1(x) - W_2(x)$ on $B_{2R}(0)$, and the inequalities (4.16), (4.17) hold with *W* replaced by W_1 and $2R$ replaced by $4R$. Now a simple calculation shows The Green Matrix for Strongly Elliptic Systems 523

(bytiously, $W(x) = W_1(x) - W_2(x)$ on $B_{2R}(0)$, and the inequalities (4.16), (4.17)

hold with W replaced by W_1 and $2R$ replaced by $4R$. Now a simple calculation shows
 $(H^{n-1}$ denotes the $(n - 1)$ -dimensional Hausdorff measure) mgly Elliptic System

e inequalities (4.

w a simple calcula

ure)
 $d\mathbf{H}^n(y)$ Obviously,
hold with \vec{W}
 $(\mathbf{H}^{n-1} \text{ denote } \nabla \vec{W})$
for points x Obviousl
 hold with
 $(H^{n-1} \text{ der})$
 for point

VW¹ (x) **=** *_f.(x -) ()- (x))* Lo *BQ (X) -* - + *f* V(x - y) (^p(y) - q(x)) *dy - - B4R(0)\B(X)* for points x E *B2R(0) \ B2 e(0).* The resulting term' s satisfy the estimates . *j :.. dH''(y)*

-
- 1

Obviously,
$$
W(x) = W_1(x) - W_2(x)
$$
 on $B_{2R}(0)$, and the inequalities (4.16), (4
\nhold with W replaced by W_1 and 2R replaced by 4R. Now a simple calculation sh
\n(\mathbf{H}^{n-1} denotes the $(n-1)$ -dimensional Hausdorff measure)
\n
$$
\nabla W_{1e}(x) = -\int \Delta(x-y) (\varphi(y) - \varphi(x)) \frac{y-x}{e} d\mathbf{H}^n(y)
$$
\n
$$
\frac{\partial B_e(x)}{\partial B_e(x)}
$$
\n
$$
+ \int \nabla \Delta(x-y) (\varphi(y) - \varphi(x)) dy
$$
\nfor points $x \in B_{2R}(0) \setminus B_{2e}(0)$. The resulting terms satisfy the estimates
\n
$$
\begin{vmatrix}\n\int \dots d\mathbf{H}^{n-1}(y) \, dy \\
\int \dots d\mathbf{H}^{n-1}(y) \, dy\n\end{vmatrix} =
$$
\n
$$
\leq C_1 D_{l-1} \left\{ |x|^s \int |x-y|^{s-n} \max(|x|^{1-n-s}, |y|^{1-n-s}) d\mathbf{H}^{n-1}(y) \right\}
$$
\n
$$
+ \int_{\partial B_e(x)} |x-y|^{s-n} |y|^{1-n} d\mathbf{H}^{n-1}(y) \right\} \leq C_1 D_{l-1} R^s |x|^{1-n-s} \varphi^{-1},
$$
\nwhere we used (4.15) and the fact that $|y| \geq |x|/2$ on $\partial B_e(x)$. Let us write
\n
$$
B_{4R}(0) \setminus B_e(x) = \left\{ |B_{4R}(0) \setminus B_{|x|/2}(0) \right\} \setminus B_{e}(x) \right\} \cup B_{|x|/2}(0) =: \Omega_1 \cup \Omega_2.
$$
\nObserving $|y| \geq |x|/2$ on Ω_1 , $|x-y| \geq |x|/2$ on Ω_2 we get
\n
$$
\begin{vmatrix}\n\int \nabla \Delta(x-y) (\varphi(y) - \varphi(x)) dy \\
\vdots \\
\Omega_1 D_{l-1} \int |x-y|^{s-1-n} (|x|^s \max(|x|^{1-n-s}, |y
$$

where we used (4.15) and the fact that $|y| \ge |x|/2$ on $\partial B_{\rho}(x)$. Let us write

$$
B_{4R}(0) \setminus B_{\varrho}(x) = \{ (B_{4R}(0) \setminus B_{|x|/2}(0)) \setminus B_{\varrho}(x) \} \cup B_{|x|/2}(0) =: \Omega_1 \cup \Omega_2.
$$

Observing $|y| \geq |x|/2$ on Ω_1 , $|x - y| \geq |x|/2$ on Ω_2 we get *•*

where we used (4.15) and the fact that
$$
|y| \geq |x|/2
$$
 on $\partial B_e(x)$. Let us write\n
$$
B_{4R}(0) \setminus B_e(x) = \left\{ \left(B_{4R}(0) \setminus B_{|x|/2}(0) \right) \setminus B_e(x) \right\} \cup B_{|x|/2}(0) =: \Omega_1 \cup \Omega_2.
$$
\n
$$
\text{Observing } |y| \geq |x|/2 \text{ on } \Omega_1, |x - y| \geq |x|/2 \text{ on } \Omega_2 \text{ we get}
$$
\n
$$
\left| \int_{\Omega_1} \nabla \Delta(x - y) \left(\varphi(y) = \varphi(x) \right) dy \right|
$$
\n
$$
\leq C_1 D_{l-1} \int_{\Omega_1} |x - y|^{a-1-n} \left(|x|^a \max\left(|x|^{1-n-a}, |y|^{1-n-a} \right) + |y|^{1-n} \right) dy
$$
\n
$$
\leq C_1 D_{l-1} R^a |x|^{1-n-a} \int_{\mathbb{R}^n B_e(x)} |x - y|^{a-1-n} dy = C_1 D_{l-1} R^a |x|^{1-n-a} e^{a-1}
$$
\nand the same estimate holds for the integral over Ω_2 . We have thus shown for points\n
$$
x \in B_{2R}(0) \setminus B_{2e}(0)
$$
 that\n
$$
|\nabla w_{1e}(x)| \leq C_1 D_{l-1} R^a |x|^{1-n-a} e^{a-1}.
$$
\nBy integrating (4.20) over the path $x_1 x_2$ (which is contained in $B_{2R}(0) \setminus B_{2e}(0)$) we conclude

and the same estimate holds for the integral over Ω_2 . We have thus shown for points

$$
|\nabla w_{1\varrho}(x)| \leq C_1 D_{l-1} R^{\alpha} |x|^{1-n-\alpha} \varrho^{\alpha-1}.
$$
 (4.20)

By integrating (4.20) over the path $\overrightarrow{x_1x_2}$ (which is contained in $B_{2R}(0) \setminus B_{2\varrho}(0)$) we conclude \mathbf{B} , \mathbf{B} ,

$$
|W_{1\varrho}(x_1)-W_{1\varrho}(x_2)|\leq C_1D_{l-1}R^{\alpha}|x_1-x_2|^{\alpha}\max(|x_1|^{1-n-\alpha},|x_2|^{1-n-\alpha}). \quad (4.21)
$$

Applying (4.16) in the version for W_1 on the ball $B_{4R}(0)$, (4.21) holds for the function *W*₁ itself. It therefore remains to prove (4.21) for W_2 . For $x \in B_{2R}(0)$ we have by definition $\leq C_1 D_{l-1} R^a |x|^{1-n-a} \int_{\mathbb{R}^n \setminus B_\varrho(x)} |x-y|^{a-1-n} dy = C_1 D_{l-1} R^a |x|^{1-n-a} \varrho^{a-1}$

and the same estimate holds for the integral over Ω_2 . We have thus shown for poin
 $x \in B_{2R}(0) \setminus B_{2\varrho}(0)$ that
 $|\nabla w_{l\varrho}(x)| \le$ $R^n B_{\varrho}(x)$
 T $R_2(\theta) \setminus B_{2\varrho}(0)$ that
 $|\nabla w_{1\varrho}(x)| \leq C_1 D_{l-1} R^a |x|^{1-n-a} \varrho^{a-1}$.
 T ϱ^{a-1} *tegrating* (4.20) over the path x_1x_2 (which is contained
 $|W_{1\varrho}(x_1) - W_{1\varrho}(x_2)| \leq C_1 D_{l-1} R^a |x_1 - x_2|^$ $|Vw_{1e}(x)| \le C_1 D_{l-1} R^2 |x|^{1-n-2} e^{x-1}.$

integrating (4.20) over the path x_1x_2 (which is contained in $B_{2R}(0)$ bude
 $|W_{1e}(x_1) - W_{1e}(x_2)| \le C_1 D_{l-1} R^2 |x_1 - x_2|^2 \max(|x_1|^{1-n-s}, |x_2|^{1-n})$

lying (4.16) in the version f

$$
\nabla W_2(x) = \int_{T_{2R}(0)} \nabla \Delta(x - y) \omega(y) \nabla u_{l-1}(y) dy.
$$

From (4.13) we infer

•

•

$$
|\nabla W_2(x)| \leq C_1 D_{l-1} R^{1-n+\alpha} \delta^{-1}(x), \qquad \delta(x) := 2R - |x|.
$$

This estimate shows that ∇W_2 behaves well in the interior of the ball $B_{2R}(0)$. Intro-
ducing $\omega(x) \nabla u_{l-1}(x)$ in the expression for ∇W_2 and using (4.15), we see
 $|\nabla W_2(x)| \leq C_1 D_{l-1} R^s \delta(x)^{s-1} |x|^{1-n-s}$ (4 ducing $\omega(x)$ $\nabla u_{l-1}(x)$ in the expression for ∇W_2 and using (4.15), we see **•**
•
•
•
•
• $VW_2(x) = \int V\Delta(x - y) \omega(y) Vu_{l-1}(y) dy.$
 *T*_{2*n*(0)}
 J3) we infer
 $|\nabla W_2(x)| \leq C_1 D_{l-1} R^{1-n+a} \delta^{-1}(x), \qquad \delta(x) := 2$

mate shows that ∇W_2 behaves well in the interior $\sum_{i} W_{i-1}(x)$ in the expression for ∇W_2 and usi **erior of**
ng (4.1)

$$
\nabla W_2(x)| \leq C_1 D_{l-1} R^s \delta(x)^{s-1} |x|^{1-n-s} \tag{4.22}
$$

for $x\in B_{2R}(0)\diagdown\{0\}.$ (4.22) has the advantage that $\nabla\,W_2$ increases of lower order when $x \rightarrow \partial B_{2R}(0)$. As before let x_1, x_2 satisfy (4.19). We want to show $\mathbb{W}_{2}(\theta) \setminus \{0\}$. (4.22) has the advantage that ∇W_{2} increases of lower order 0). As before let x_{1} , x_{2} satisfy (4.19). We want to show $W_{2}(x_{1}) = W_{2}(x_{2}) \leq C_{1}D_{l-1}R^{2}|x_{1} - x_{2}|^{\alpha} \max(|x_{1}|^{1-n-\alpha}, |x_{2}|^{1$

$$
|W_2(x_1) - W_2(x_2)| \leq C_1 D_{l-1} R^{\alpha} |x_1 - x_2|^{\alpha} \max (|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}) \qquad (4.23).
$$

and consider the following cases (compare $[15: Thm. 2.6.6]$):

|W

d conside

1. $|x_i| \leq$

2. $|x_i| >$ $2R - \varrho$: Integration of (4.22) over the path $\overrightarrow{x_1 x_2}$ implies (4.23).

and consider the following cases (c
 $1. |x_i| \leq 2R - \varrho$: Integration of
 $2. |x_1| > 2R - \varrho$, $|x_2| \leq 2R - \varrho$
 $\overline{0x_1}$ with $|x_3| = 2R - \varrho$. Observing $2R - \varrho$: Consider the path $\overrightarrow{x_1 x_3 x_2}$, where x_3 is on the ray mplies (4.23).
where x_3 is on the ray
3 $|x_2|/5$, $0 \le t \le 1$,

 $|x_1 - x_3| \leq \varrho$, $|x_2 - x_3| \leq 2\varrho$, $|x_2 + t(x_3 - x_2)| \geq 3 |x_2|/5$, $0 \leq t \leq$ we get the estimates

$$
|W_2(x_1) - W_2(x_2)| \leq C_1 D_{l-1} R^{\alpha} |x_1 - x_2|^{\alpha} \max(|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}) \qquad (4.23)
$$
\nand consider the following cases (compare [15: Thm. 2.6.6]):
\n1. $|x_i| \leq 2R - \varrho$: Integration of (4.22) over the path $\overline{x_1 x_2}$ implies (4.23).
\n2. $|x_1| > 2R - \varrho$, $|x_2| \leq 2R - \varrho$: Consider the path $\overline{x_1 x_3 x_2}$, where x_3 is on the ra
\n $0x_1$ with $|x_3| = 2R - \varrho$. Observing
\n $|x_1 - x_3| \leq \varrho$, $|x_2 - x_3| \leq 2\varrho$, $|x_2 + t(x_3 - x_2)| \geq 3 |x_2|/5$, $0 \leq t \leq 1$,
\nwe get the estimates
\n $|W_2(x_2) - W_2(x_3)| \leq C_1 D_{l-1} R^{\alpha} |x_2|^{1-n-\alpha} \varrho^{\alpha}$,
\n $|W_2(x_1) - W_2(x_3)| \leq C_1 D_{l-1} R^{\alpha} |x_1 - x_3| \int_{0}^{1} (\varrho - t |x_1 - x_3|)^{\alpha-1} dt |x_3|^{1-n-\alpha}$
\n $\leq C_1 D_{l-1} R^{\alpha} |x_1 - x_3|^{\alpha} |x_2|^{1-n-\alpha}$.

This proves (4.23) in the second case.

3. $|x_1| > 2R - \varrho$, $|x_2| > 2R - \varrho$: Choose x_3 on $\overrightarrow{0x_1}, x_4$ on $\overrightarrow{0x_2}$ with $|x_3| = |x_4|$ $=2R-\varrho$. We have

$$
\leq C_1 D_{l-1} R^* |x_1 - x_3|^{\alpha} |x_2|^{1-n-\alpha}.
$$

This proves (4.23) in the second case.

$$
3 |x_1| > 2R - \varrho, |x_2| > 2R - \varrho: \text{ Choose } x_3 \text{ on } \overline{0x_1}, x_4 \text{ on } \overline{0x_2} \text{ with } |x_3| = |x_4|
$$

$$
= 2R - \varrho. \text{ We have}
$$

$$
|x_1 - x_3| \leq \varrho, |x_2 - x_4| \leq \varrho, |x_3 - x_4| \leq 3\varrho, |x_1 + t(x_3 - x_1)| \geq 4 |x_1|/5,
$$

$$
|x_2 + t(x_4 - x_2)| \geq 4 |x_2|/5, |x_3 + t(x_4 - x_3)| \geq 2R - 4\varrho \geq |x_1|/5
$$
for $0 \leq t \leq 1$.
Similar calculations as in the cases 1, 2 yield (4.23).
Collecting our results we have shown

$$
|W(x_1) - W(x_2)| \leq C_1 D_{l-1} R^* |x_1 - x_2|^{\alpha} \max (|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}) \qquad (4.24)
$$
for points x_1, x_2 with (4.19). By (4.15), inequality (4.24) holds for the function f defined

$$
|W(x_1) - W(x_2)| \leq C_1 D_{l-1} R^{\alpha} |x_1 - x_2|^{\alpha} \max (|x_1|^{1-n-\alpha}, |x_2|^{1-n-\alpha}) \qquad (4.24)
$$

in (4.14). So it remains to consider the case $|x_1 - x_2| \ge \min(|x_1|, |x_2|)/5$. But under this assumption the Hölder condition for ∇u_i is a trivial consequence of (4.18) 4 $|x_2|/5$, $|x_3 + t(x_4 - x_3)| \ge 2K - 4\varrho \le |x_1|/5$

e cases 1, 2 yield (4.23).

ave shown
 $C_1D_{t-1}R^2 |x_1 - x_2|^s \max(|x_1|^{1-n-s}, |x_2|^{1-n-s})$ (4.24)

y (4.15), inequality (4.24) holds for the function / defined

misider the case Similiar calculations as in the cases 1, 2 yield (4.23).

Collecting our results we have shown
 $|W(x_1) - W(x_2)| \leq C_1 D_{l-1} R^2 |x_1 - x_2|^2 \max(|x_1|^{1-n-\epsilon}, |x_2|^{1-n-\epsilon})$

for points x_1, x_2 with (4.19). By (4.15), inequality (4.2

In the scalar case Green's function g can be estimated from below in terms of $|x - y|^{2-n}$, mption the Holder condition for $\mathbf{v}u_l$ is a trivial consequence of \mathbf{v}_l
scalar case Green's function g can be estimated from below in terms.
[8]. We mention the following weaker result which is valid for $N > 1$

Proposition: *Under the assumptions of Theorem 7 we have for all* $y \in \Omega$ *and* $k = 1, ..., N$

In the scalar case Green's function
$$
y
$$
 can be estimated from below in terms of $|x - y|$, compare [8]. We mention the following weaker result which is valid for $N > 1$. Proposition: Under the assumptions of Theorem 7 we have for all $y \in \Omega$ and $k = 1, ..., N$ $\limsup_{x \to y} |G_k(x, y)| |x - y|^{n-2} > 0.$ (4.25) $x \to y$ (4.26) $2 - n$, $\limsup |E(x)| |x|^{n-2} = 0$ would imply $E = 0$. Therefore $x \to 0$ $|E(x)| |x|^{n-2} \geq C$, $v \in \mathbb{N}$, (4.26) for some constant $C > 0$ and a suitable sequence $x \to 0$. Recalling (4.9) we obtain the inequality

Proof: We use the notations from the proof of Theorem 7. Since *E* is homogeneous of degree $-n$, $\limsup |E(x)| |x|^{n-2} = 0$ would imply $E = 0$. Therefore

$$
|E(x_*)| |x_*|^{n-2} \geq C, \qquad v \in \mathbb{N},
$$

2 - n, lim sup $|E(x)| |x|^{n-2} = 0$ would imply $E = 0$. Therefore
 $|E(x_*)| |x_*|^{n-2} \ge C$, $v \in \mathbb{N}$, (4.26)

for some constant $C > 0$ and a suitable sequence $x_* \to 0$. Recalling (4.9) we obtain the in-

equality
 $|G(x_*)| \ge |$

for points
$$
x_1, x_2
$$
 with (4.19). By (4.15), the
quantity (4.24) holds for the function f
in (4.14). So it remains to consider the case $|x_1 - x_2| \ge \min(|x_1|, |x_2|)/5$. But
this assumption the Hölder condition for ∇u_i is a trivial consequence of (4.18)
In the scalar case Green's function g can be estimated from below in terms of $|x$
compare [8]. We mention the following weaker result which is valid for $N > 1$.
Proposition: Under the assumptions of Theorem 7 we have for all $y \in \Omega$ and $k = 1$,...
 $\limsup_{x\to y} |G_k(x, y)| |x - y|^{n-2} > 0$.
 $x \to y$
Proof: We use the notations from the proof of Theorem 7. Since E is homogeneous on
 $2 - n$, $\limsup_{x\to 0} |E(x)| |x|^{n-2} = 0$ would imply $E = 0$. Therefore
 $x \to 0$
 $|E(x_1)| |x_2|^{n-2} \ge C$, $v \in \mathbb{N}$,
for some constant $C > 0$ and a suitable sequence $x_i \to 0$. Recalling (4.9) we obtain
equality
 $|G(x_1)| \ge |E(x_1)| - \left|\sum_{i=1}^{\infty} (T_{i2i}^{p_i})^t E(x_i)\right| - \left|\sum_{i=1}^{\infty} (T_{i2i}^q)^t w(x_i)^r\right|$
 $\ge |E(x_2)| - C_1 \sum_{i=1}^{\infty} (C_1 R^{\alpha})^t |x_i|^{2-n} - C_2 R^{1+\beta-n}$,

i.

and by (4.26) we get

26) we get
\n
$$
\limsup_{t\to\infty} |x_t|^{n-2} |G(x_t)| \geq C - C_1 \sum_{l=1}^{\infty} (C_1 R^{\alpha})^l,
$$

The Green Matrix for Strongly Elliptic Systems 52

and by (4.26) we get
 $\limsup_{x \to 0} |x_i|^{n-2} |G(x_i)| \geq C - C_1 \sum_{i=1}^{\infty} (C_i R^a)^l$,

which proves the proposition if B is chosen small enough **I**

Next we use the well-known Ca Next we use the well-known Campanato technique (see [3, 7]) together with Theo-
 -1 Theorem 8: Suppose that (4.2) *holds and let* $0 < \beta < 1$ *be given. Then*
 $|\partial G(x, y)|\partial x| \leq C_1 |x - y|^{1-n} + C_2 |x - y|^{-1} R^{1+\beta-n}$ (4.27) The Green Matrix for Strongly

and by (4.26) we get
 $\limsup_{r\to\infty} |x_r|^{n-2} |G(x_r)| \geq C - C_1 \sum_{l=1}^{\infty} (C_1 R^{\alpha})^l$,

which proves the proposition if R is chosen small enough

Next we use the well-known Campanato technique (s

Theorem 8: *Suppose that* (4.2) holds and let $0 < \beta < 1$ be given. Then

$$
|\partial G(x, y)|\partial x| \leq C_1 |x - y|^{1-n} + C_2 |x - y|^{-1} R^{1+\beta-n}
$$
 (4.27)

Theorem 8: Suppose that (4.2) holds and let $0 < \beta < 1$ be given. Then
 $|\partial G(x, y)|\partial x| \leq C_1 |x - y|^{1-n} + C_2 |x - y|^{-1} R^{1+\beta-n}$ (4.27)

for all $x \in B_R(y) \setminus \{y\}, y \in \Omega, 0 < R \leq \min (R_0, \text{dist } (y, \partial \Omega)/4)$. Here C_1, C_2, R_0 are the

const

Proof: As before we may assume $y = 0 \in \Omega$ and use the notations from the proof of the preceding theorem. Let R satisfy the above hypothesis and choose $z \in B_R(0)$. Define $D = B_{|z|/8}(z)$; for $x_0 \in D$ and $0 < \varrho \le r \le \text{diam } (D) = |z|/4$ let the func-For all x_1e^{-x} $|\mathcal{G}(x_1)| \geq C - C_1 \sum_{i=1}^n (C_1R^a)^i$,
which proves the proposition if R is chosen small enough \blacksquare
Nèxt we use the well-known Campanato technique (see [3, 7]) them 7 to derive gradient bounds for *e* use the well-known Campanato technique (see [3, 7]) together with Theoderive gradient bounds for Green's matrix.
 e m 8: *Suppose that* (4.2) *holds and let* $0 < \beta < 1$ *be given. Then* $\partial G(x, y)/\partial x$ $\leq C_1 |x - y|^{1-n} + C$ Next we use the well-known Campanato technique (see [3, 7]) together with

rem 7 to derive gradient bounds for Green's matrix.

Theorem 8: Suppose that (4.2) holds and let $0 < \beta < 1$ be given. Then
 $|\partial G(x, y)|\partial x| \leq C_1 |x - y|$ for α *s*: Suppose that (4.2) holds and let $0 < \beta$
 $|\partial G(x, y)|\partial x| \leq C_1 |x - y|^{1-n} + C_2 |x - y|^{1-n}$
 $\vdots B_R(y) \setminus \{y\}, y \in \Omega, 0 < R \leq \min(R_0, \text{dist})$
 appearing in Theorem 7.
 \therefore As before we may assume $y = 0 \in \Omega$ and
 α roof: As before we may assume preceding theorem. Let R

ine $D = B_{|z|/8}(z)$; for $x_0 \in D$
 $v \in H^{1,2}(B_r(x_0))^N$ be the solut
 $-D_r(A_{r_0}^{ij}(x_0) D_b v^j) = 0$ of satisfies the Campanato es
 $\int |\nabla v|^2 dx \leq C_1(e/r)^n \int_{B_r(x_0)}$
 $C_1 = C_$

$$
-D_{\mathbf{y}}(A_{\mathbf{y}}^{ij}(x_0) D_{\delta} v^j) = 0 \text{ on } B_{\mathbf{r}}(x_0), \quad v_{\partial B_{\mathbf{r}}(x_0)} = G_{|\partial B_{\mathbf{r}}(x_0)},
$$

$$
\int\limits_{B_{\varrho}(x_0)} |\nabla v|^2\,dx \leqq C_1(\varrho/r)^n \int\limits_{B_r(x_0)} |\nabla v|^2\,dx
$$

which satisfies the Campanato estimate [(7: Thm. 2.1/p. 80])
\n
$$
\int_{B_{\rho}(x_0)} |\nabla v|^2 dx \leq C_1(\rho/r)^n \int_{B_r(x_0)} |\nabla v|^2 dx
$$
\nwith $C_1 = C_1(n, N, \lambda, A)$. The function $w = G - v$ is a solution of
\n
$$
\int_{B_r(x_0)} A_{\gamma\delta}^{ij}(x_0) D_{\gamma} \Phi^i D_{\delta} w^j dx = -\int_{B_r(x_0)} (A_{\gamma\delta}^{ij} - A_{\gamma\delta}^{ij}(x_0)) D_{\gamma} \Phi^i D_{\delta} G^j dx
$$

for all $\Phi \in \hat{H}^{1,2}(B_r(x_0))^N$. Inserting $\Phi = w$, a simple calculation shows

$$
-D_{\gamma}(A_{\gamma\delta}^{ij}(x_{0}) D_{\delta}v^{j}) = 0 \text{ on } B_{r}(x_{0}), \quad v_{\partial B_{r}(x_{0})} = G_{|\partial B_{r}(x_{0})},
$$

\nch satisfies the Campanato estimate [(7: Thm. 2.1/p. 80])
\n
$$
\int |\nabla v|^{2} dx \leq C_{1}(e/r)^{n} \int |\nabla v|^{2} dx
$$
\n(4.28)
\n
$$
B_{\rho}(x_{0})
$$

\n
$$
C_{1} = C_{1}(n, N, \lambda, \Lambda). \text{ The function } w = G - v \text{ is a solution of}
$$

\n
$$
\int A_{\gamma\delta}^{ij}(x_{0}) D_{\gamma} \Phi^{i} D_{\delta}w^{j} dx = -\int (A_{\gamma\delta}^{ij} - A_{\gamma\delta}^{ij}(x_{0})) D_{\gamma} \Phi^{i} D_{\delta}G^{j} dx
$$

\n
$$
B_{r}(x_{0})
$$

\nall $\Phi \in \mathring{H}^{1,2}(B_{r}(x_{0}))^{N}$. Inserting $\Phi = w$, a simple calculation shows
\n
$$
\int |\nabla w|^{2} dx \leq C_{1}r^{2\alpha} \int |\nabla G|^{2} dx.
$$
\n(4.19)
\n
$$
B_{r}(x_{0})
$$

\n

Here C_1 has the former meaning. Combining (4.28) and (4.29) we get the following growth condition for

has the former meaning. Combining (4.28) and
condition for

$$
\varphi(t) = \int_{B_t(x_0)} |\nabla G|^2 dx: \varphi(\varrho) \leq C_1((\varrho/r)^n + r^{2\alpha}) \varphi(r),
$$

and a well-known iteration lemma due to Giusti [7: Lemma *2.1/p.* 87] implies for all $0 \lt o \le r \le$ diam (D)

$$
\varphi(o) \leq C_1(o/r)^{n-\alpha} \varphi(r),
$$

 $\in \hat{H}^{1,2}(B_r(x_0))^N$. Inserting $\Phi = w$, a simple calculation sh
 $\int_{x_0}^{\infty} |\nabla w|^2 dx \leq C_1 r^{2\alpha} \int_{B_r(x_0)} |\nabla G|^2 dx$.

has the former meaning. Combining (4.28) and (4.29) v

ondition for
 $\varphi(t) = \int_{B_t(x_0)} |\nabla G|^2 dx$: $\varphi(\$ provided R_0 is sufficiently small. Since this smallness condition on R_0 involves the parameters $n, N, L, \lambda, \Lambda, \alpha$ we may assume that in the beginning R_0 has been chosen $\int_{d} |\nabla w|^2 dx \leq C_1 r^{2a} \int_{B_r(x_0)} |v_r(x_0)|^2 dx$

Here C_1 has the former meaning

growth condition for
 $\varphi(t) = \int_{B_t(x_0)} |\nabla G|^2 dx$: $\varphi(\varrho)$

and a well-known iteration lemm
 $0 < \varrho \leq r \leq \text{diam } (D)$
 $\varphi(\varrho) \leq C_1(\varrho/r)^{n$ $\varphi(\varrho) \leq C_1(\varrho/r)^{n-\alpha} \varphi(r)$,
vided R_0 is sufficiently small. Since this smallness condition
ameters *n*, *N*, *L*, *λ*, *A*, *α* we may assume that in the beginnin
he right way. Let
 $\psi(\varrho) := \int_{B_{\varrho}(x_0)} |\nabla G - (\nabla G)$ 2.1/p. 87] implies for all

(4.30)

ition on R_0 involves the

ning R_0 has been chosen

dx;

dx. (4.31)

+ $r^{n+a}d^{a-n}\varphi(d)$),

\n
$$
\text{ght way. Let}
$$
\n

\n\n $\psi(\varrho) := \int |\nabla G - (\nabla G)_{x_0, \varrho}|^2 \, dx,$ \n $(\nabla G)_{x_0, \varrho} := \int_{\varrho(x_0)} \nabla G \, dx;$ \n

\n\n $\text{g to } [7: \text{Thm. } 2.1/p. \text{ 80] the function } v \text{ satisfies}$ \n

\n\n $\int |\nabla v - (\nabla v)_{x_0, \varrho}|^2 \, dx \leq C_1(\varrho/r)^{n+2} \int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0, r}|^2 \, dx.$ \n

according to $[7: Thm. 2.1/p. 80]$ the function v satisfies

ng to [7: Thm. 2.1/p. 80] the function v satisfies
\n
$$
\int_{B_{\varrho}(x_0)} |\nabla v - (\nabla v)_{x_0,\varrho}|^2 dx \leq C_1(\varrho/r)^{n+2} \int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0,r}|^2 dx.
$$
\nring G and v as above, we get from (4.29)–(4.31)
\n
$$
\psi(\varrho) \leq C_1((\varrho/r)^{n+2} \psi(r) + r^{2\alpha}\varphi(r)) \leq C_1((\varrho/r)^{n+2} \psi(r) + r^{n+\alpha}d^{\alpha-n} \varphi(d)),
$$

Comparing G and v as above, we get from $(4.29) - (4.31)$

$$
w(a) \leq C_1((\rho/r)^{n+2} w(r) + r^{2\alpha} \varphi(r)) \leq C_1((\rho/r)^{n+2} w(r) + r^{n+\alpha} d^{\alpha-n} \varphi(d)),
$$

 (4.28)

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 $d:=$ diam (D) . From the iteration lemma cited above we infer

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\n
$$
d := \text{diam } (D). \text{ From the iteration lemma cited above we infer}
$$
\n
$$
\psi(e) \leq C_1 e^{n+\alpha} \{r^{-n-\alpha}\psi(r) + d^{\alpha-n}\varphi(d)\}, \qquad 0 < \varrho < r \leq d.
$$
\nChoosing $r := d$ we finally arrive at
\n
$$
e^{-n-\alpha}\psi(e) \leq C_1 |z|^{-n-\alpha} \int |\nabla G|^2 dx.
$$

/ ^V -

 $\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$ Choosing $r := d$ we finally arrive at

$$
e^{-n-\alpha}\psi(\varrho)\leqq C_1\,|z|^{-n-\alpha}\int\limits_{B_{|z|/2}(z)}|\nabla G|^2\,dx
$$

 $f(x) = \frac{1}{2} \pi \int_{0}^{\pi} |\nabla G|^2 dx$
 f $\int_{0}^{\pi} |\nabla G|^2 dx$
 f $\int_{$ 526 M. Focus

d: = diam (D). From the iteration $\psi(e) \leq C_1 e^{n+\alpha} \{r^{-n-\alpha}\psi(r) + C_1\}$

Choosing $r := d$ we finally arrive at
 $e^{-n-\alpha}\psi(e) \leq C_1 |z|^{-n-\alpha} \int_{B_{121/2}(z)}^{\infty}$

for all $x_0 \in D$ and $0 < e \leq d$. Thus

(compare [7]), a for all $x_0 \in D$ and $0 < \varrho \leq d$. Thus ∇G belongs to the Campanato space $L^{2,n+\alpha}(D)^{nN}$.
(compare [7]), and from (4.32) we get the bound (628) M. Focns

(d) $d := \text{diam}(D)$. From the iteration lemma cited above we infer
 $\psi(e) \leq C_1 e^{n+a} \{r^{-n-a}\psi(r) + d^{a-n}\psi(d)\}, \qquad 0 < \varrho < r \leq d.$

Choosing $r := d$ we finally arrive at
 $e^{-n-a}\psi(e) \leq C_1 |z|^{-n-a} \int |\nabla G|^2 dx$ (4.32)

(or al supposition in the iteration lemma cited above we infer
 $\psi(e) \leq C_1 e^{n+\alpha} \{r^{-n-\alpha}\psi(r) + d^{\alpha-n}\phi(d)\}, \qquad 0 < \varrho < r \leq d$
 $r := d$ we finally arrive at
 $e^{-n-\alpha}\psi(e) \leq C_1 |z|^{-n-\alpha} \int |\nabla G|^2 dx$
 $\varrho^{-n-\alpha}\psi(e) \leq C_1 |z|^{-n-\alpha} \int |\nabla G|^2 dx$
 \in $d := \text{diam } (D)$. From the iteration lemma cited above we infer
 $\psi(\varrho) \leq C_1 e^{n+a} \{r^{-n-a}\psi(r) + d^{a-n}\varphi(d)\}, \qquad 0 < \varrho < r \leq d.$

Choosing $r := d$ we finally arrive at
 $e^{-n-a}\psi(\varrho) \leq C_1 |z|^{-n-a} \int |\nabla G|^2 dx$

for all $x_0 \in D$ and $0 < \varrho$ **Example 18** $\psi(\theta) \geq C_1 |z|^{n-1} \int |\nabla G|^2$ **

for all** $x_0 \in D$ **and** $0 < \varrho \leq d$ **. Thus** ∇G **

(compare [7]), and from (4.32) we get the

sup** $|\nabla G| \leq C_1 |z|^{-n/2} ||\nabla G||_{L^1(B_{12})}$ **

The Dirichlet integral of G over B_{|z|/2}(** for all $x_0 \in D$ and $0 < \varrho \leq d$. Thus ∇G belongs to the Campanato space $L^{2,n+\alpha}$ (

(compare [7]), and from (4.32) we get the bound
 $\sup | \nabla G | \leq C_1 |z|^{-n/2} ||\nabla G||_{L^1(B_{12}||2(z))}$.

The Dirichlet integral of G over

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$$
\sup_{D} |\nabla G| \leqq C_1 |z|^{-n/2} ||\nabla G||_{L^4(B_{121/2}(z))}
$$

$$
C_{1} |z|^{-1} ||G||_{L^{4}(B_{3}|z|/4(\bar{z}))} \ldots
$$

The Dirichlet integral of G over $B_{|z|/2}(z)$ is controlled by
 $C_1 |z|^{-1} ||G||_{L^1(B_{3|z|/4}(z))}$.
 Estimating the L^2 **-norm of** G with the help of Theorem 7 we get (4.27) \blacksquare

We just showed that ∇G belongs to the the proof of Theorem 8 contains more information: Take two points $z, \overline{z} \in B_{R/2}(0) \setminus \{0\},$ The Dirichlet integral of G over $B_{|z|/2}(z)$ is controlled by
 $C_1 |z|^{-1} ||G||_{L^4(B_{3|z|/4}(z))}$.

Estimating the L^2 -norm of G with the help of Theorem 7 we get (4.27) **I**

We just showed that ∇G belongs to the space The Dirichlet integral of G over $B_{|z|/2}(z)$ is contained in the integral of G over $B_{|z|/2}(z)$ is contained by C_1 . $|z|^{-1}$ $||G||_{L^4(B_{3|z|/4}(z))}$.

Estimating the L^2 -norm of G with the help of W e just showed th gs to the Ca

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trolled by

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ace $L^{2,n+\alpha}(D)$

ion: Take tv
 $\langle L_2|_{2}(z)\rangle$ we
 $C_1e^{n+2\alpha}d^{-2\alpha}$ *p() C1 2 {d' ² p(d) -1- M ² } . .C1 +2ad-2M2 .* **^V** The Dirichlet integral of G over $B_{|z|/2}(z)$ is controlled by
 $C_1 |z|^{-1} ||G||_{L^4(B_{31d1d10})}$.

Estimating the L^2 -norm of G with the help of Theorem 7 we get (4.27)

We just showed that VG belongs to the space $L^{2,n+4$ $+r^{n+2\alpha}M^2$ and by the iteration lemma Estimating the L^2 -norm of G with the help of Theorem 7 we μ
 We just showed that ∇G belongs to the space $L^{2,n+4}(D)$, i.e

the proof of Theorem 8 contains more information: Take two po
 $|z| \leq |\overline{z}|$. Defin We just showed that ∇G belongs to the space
the proof of Theorem 8 contains more information $|z| \leq |\overline{z}|$. Defining $M = \sup \{|\nabla G(x)| : x \in B_{|x|}\} + r^{n+2s} M^2\}$ and by the iteration lemma
 $\psi(\varrho) \leq C_1 \varrho^{n+2s} \{d^{-n-2s}\psi$ We just showed that VG belongs to the space $L^{2,n+4}(D)$, i.e. $\nabla G \in C^{0,n/2}(D)^{nN}$, but
 $\leq \text{proot of Theorem 8 containing more information: Take two points } z, \bar{z} \in B_{R/2}(0) \setminus \{0\},$
 $\leq |\bar{z}|$. Defining $M = \text{sup } |\nabla G(x)| : x \in B_{|z||2}(z)|$ we get $\psi(\varrho) \leq C_1((\varrho/r)^{n+2$ According to the isotropic form of the a-Holder constant for *a*(*x*) θ and θ an

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$$
w^{2} \text{ and by the iteration lemma}
$$
\n
$$
\psi(\varrho) \leq C_{1}\varrho^{n+2\alpha}\{d^{-n-2\alpha}\psi(d) + M^{2}\} \leq C_{1}\varrho^{n+2\alpha}d^{-2\alpha}M^{2}.
$$
\nplies

V ' V ^V *Case* 1: ED. Then (4.27) and **(4.33)-give V ^V**

 $\begin{aligned} \omega^2 \psi(a) + M^2 \leq C_1 \ |\langle a - b |^a : a + b \in \end{aligned}$
 $\begin{aligned} \frac{1}{a} &\to 0^a : a + b \in \end{aligned}$
 $\begin{aligned} \frac{1}{a} &\to \bar{z} \ |\langle a | z |^{1-n-a} \rangle = \bar{z} \ |\geq |z|/8, \text{ the } i \end{aligned}$ $\sup{\{\vert \nabla G(a) - \nabla G(b) \}}$
 Case 1: $\overline{z} \in \overline{D}$. Then (4.27)
 $\vert \nabla G(z) - \nabla G(\overline{z}) \vert \leq \vert z \vert$
 Case 2: $\overline{z} \notin \overline{D}$. Observing $\vert z \vert$

quence of (4.27). *z*/8, the above estimate is a trivial conse ^{*n*}}.
a tr *Case* $1: \overline{z} \in \overline{D}$. Then (4.27) and (4.33) give
 $|\nabla G(z) - \nabla G(\overline{z})| \leq |z - \overline{z}|^{\alpha} \{G_1 |z|^{1-\alpha-\alpha} + C_2 |z|^{-1-\alpha} R^{1+\beta-\eta} \}$.
 Case $2: \overline{z} \notin \overline{D}$. Observing $|z - \overline{z}| \geq |z|/8$, the above estimate is a trivi

Theorem 9: If (4.2) is satisfied and if $0 < \beta < 1$ is given, then *r* The

Case 1:
$$
\bar{z} \in D
$$
. Then (4.27) and (4.33) give
\n
$$
|\nabla G(z) - \nabla G(\bar{z})| \leq |z - \bar{z}|^{\alpha} \{C_1 |z|^{1-n-\alpha} + C_2 |z|^{-1-\alpha} R^{1+\beta-n}\}.
$$
\nCase 2: $\bar{z} \notin \bar{D}$. Observing $|z - \bar{z}| \geq |z|/8$, the above estimate is a trivis
\nquence of (4.27).
\nWe state these facts in
\nTheorem 9: If (4.2) is satisfied and if $0 < \beta < 1$ is given, then
\n
$$
|\nabla_z G(x, y) - \nabla_z G(\bar{x}, y)| \leq |C_1 \max (|x - y|^{1-n-\alpha}, |\bar{x} - y|^{1-n-\alpha}) + C_2 \max (|x - y|^{-1-\beta}, |\bar{x} - y|^{-1-\alpha}) R^{1+\beta-n}\} |x - \bar{x}|^{\alpha}
$$
\nfor all $x, \bar{x} \in B_{R/2}(y) \setminus \{y\}, y \in \Omega, 0 < R \leq \min (R_0, \text{dist } (y, \partial \Omega)/4)$. Here C
\n R_0 are as in Theorem 7.
\nAccording to this estimate the α -Hölder constant for $\partial G(\cdot, y)/\partial x$ on rin
\n $\forall B_r(y)$ with sufficiently small radius grows of order $r_r^{1-n-\alpha}$ when r becomes
\n5. Some applications

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According to this estimate the α *-Hölder constant for* $\partial G(\cdot, y)/\partial x$ *on rings* $B_{2r}(y) \searrow B_r(y)$ *with sufficiently small radius grows of order* $r^{1-n-\alpha}$ *when <i>r* becomes smaller.

5. Some applications

As Green's function for a single elliptic operator has become a useful tool in various fields (compare [9-11, 14]) we want to give two applications of Green's matrix. We start with the description of the behaviour of a week solution to a homogeneous elliptic system having an isolated singularity of prescribed growth. Then either the singularity is removable or of order $2 - n$.
Our result corresponds to a well-known fact for harmonic functions. $\begin{aligned}\n &+ C_2 \max\left(|x-y|^{-1-\frac{\alpha}{r}}, |\bar{x}-y|^{-1-\alpha}\right) R^{1+\beta-n}\right| |x-\bar{x}|^s\nonumber\ \end{aligned}$

for all $x, \bar{x} \in B_{R/2}(y) \setminus \{y\}, y \in \Omega, 0 < R \leq \min\left(R_0, \text{dist}(y, \partial \Omega)/4\right)$. Here
 R_0 are as in Theorem 7.

According to this estimate the α -Hölder **p**-
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n.

(4.32)

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• Theorem 10: *Suppose that (GA) holds andt hat the coefficients 4ij satisfy a Lipschitz condition on* $\overline{\Omega}$. Let $\partial\Omega$ be of class C^2 and let $u \in H_{loc}^{1,1}(\Omega \setminus \{y\})^N$, $y \in \Omega$, be a.weak solu*tion to the system* $L_i u^j = 0$ *on* $\Omega \setminus \{y\}$ *having the following properties:* **Theorem 10:** *Suppose that* (GA) *holds and that the coefficients* A_{s}^{ij} *satisf adition on* $\overline{\Omega}$. Let $\partial\Omega$ be of class C^2 and let $u \in H_{\text{loc}}^{1,1}(\Omega \setminus \{y\})^N$, $y \in \Omega$, be *n* to the system $L_{ij}u^j = 0$

points x near y.

(ii) For some $\delta > 0$ and all balls $B_{\rho}(y)$ with sufficiently small radius u *belongs to the space H*^{$1.1 + \delta$} $(\varOmega \setminus B_{\varrho}(y))$ ^N having boundary values zero on $\partial \varOmega$. *Then is a points* $\lim_{t \to \infty} L_{ij}u^j = 0$ *on* $\Omega \setminus \{y\}$ *having the following pr*

(i) *There are constants* $C, \varepsilon \geq 0$ *such that* $|u(x)| \leq C$ $|x -$

points x *near* y .

(ii) *For some* $\delta > 0$ *and all balls* B_{\v

Proof: We may assume $y = 0 \in \Omega$. It is easy to see (compare Lemma 3.2) that *Then* $u = C_i G_i(\cdot, y)$ on Ω with suitable constants $C_i \in \mathbb{R}$.
 Proof: We may assume $y = 0 \in \Omega$. It is easy to see (compare Lemma 3.2) that $u \in H^{1,r}(\Omega \setminus B_{\varrho}(0))^N$ for all $1 \leq r < \infty$ and small radii ϱ , and the method gives $u \in H_{loc}^{2,2}(\Omega \setminus \{0\})^{\overline{N}}$. Since *u* vanishes at the boundary we have the stronger result $u \in H^{2,2}(\Omega \setminus B_{\rho}(0))^N$ (compare [8: proof of Thm. 8.12]; the technique described there also applies to elliptic systems). Moreover, (i) implies the gradient bound (see Theorem 10: Suppose that (GA) holds and that the coefficients A
condition on $\overline{\Omega}$. Let $\partial\Omega$ be of class C^2 and let $u \in H_{10}^{1.5}(\Omega \setminus \{y\})^N$, y
tion to the system $L_{ij}u^j = 0$ on $\Omega \setminus \{y\}$ having the followi *For some 6* > 0 and all balls $B_{\rho}(y)$ with sufficiently small radius u
 ℓ *C is to the space* $H^{1,1+\delta}(Q \setminus B_{\rho}(y))$ ^N having boundary values zero on $\partial\Omega$.
 $u = C_iG_i(\cdot, y)$ on Ω with suitable constants $C_i \in \mathbb{$ Proof: We may assume $y = 0 \in \Omega$. It is easy to set
 $H^{1,r}(\Omega \setminus B_{\rho}(0))^N$ for all $1 \leq r < \infty$ and small radii ϱ
 ℓ thod gives $u \in H^{2,2}(\Omega \setminus \{0\})^N$. Since u vanishes at the bult
 $u \in H^{2,2}(\Omega \setminus B_{\rho}(0))^N$ (compare

$$
|\nabla u(x)| \leq C |x|^{\epsilon - n}
$$

for all points x near the origin, where C is independent of x. Now choose $0 < R_0$ $<$ dist (0, $\partial\Omega$), $0 < \varrho < R < R_0$ and $z \in B_{R_0}(0) \setminus B_R(0)$. Let $w = {}^{t}G_k(\cdot, z)$. Then the relation $\int A_{ab}^{ij} D_a w^i D_b \phi^j dx = \Phi^k(z)$ holds for all $\Phi \in C_0^{\infty}(\Omega)^N$. Take a cut-off function for all points x near the origin, where
 \langle dist $(0, \partial\Omega), 0 \langle \varrho \rangle \langle R \rangle \langle R_0 \rangle$ and z $\langle R_1 \rangle$
 relation $\int A_{a\beta}^{ij} D_a w^i D_\beta \Phi^j dx = \Phi^k(z)$ holds
 $\eta \in C_0^{\infty}(\Omega)$ such that $\eta = 1$ on $T = \Phi = \eta u$ in the above identi *i* there also applies to elliptic systems). Moreover, (i) implies the gradient k proof of Theorem 8)
 $| \nabla u(x) | \leq C |x|^{t-n}$

for all points x near the origin, where C is independent of x . Now choos
 $\langle x|$ dist

 $\eta \in C_0^{\infty}(\Omega)$ such that $\eta = 1$ on $T = B_{R_0}(0) \setminus B_R(0), \eta = 0$ on $B_{R/2}(0)$ and insert $\Phi = \eta u$ in the above identity to get

$$
\Phi = \eta u
$$
 in the above identity to get
\n
$$
u^k(z) = \int_{T} A^{\mathbf{ij}}_{\alpha\beta} D_a w^i D_\beta u^j dx + \int_{\Omega \setminus T} A^{\mathbf{ij}}_{\alpha\beta} D_a w^i D_\beta (\eta u^j) dx =: (1) + (2).
$$
\nSince $L_{ij} u^j = 0$ on T we see that $(v$ is the outer normal to ∂T)
\n
$$
(1) = \int_{\partial T} A^{\mathbf{ij}}_{\alpha\beta} w^i D_\beta u^j v_a d\mathbf{H}^{n-1}.
$$
\nObserving $w \in H^{2,2}(\Omega \setminus T)^N$ and $L^i_{ij} w^j = 0$ on $\Omega \setminus T$ we get
\n
$$
(2) = -\int_{\partial T} A^{\mathbf{ij}}_{\alpha\beta} D_a w^i u^j v_\beta d\mathbf{H}^{n-1}.
$$
\nFor functions $f, g: \partial B_r(0) \to \mathbf{R}^N$ we let
\n
$$
M(f, g) = A^{\mathbf{ij}}_{\alpha\beta} (g^i D_\beta f^j v_a - D_\alpha g^i f^j v_\beta), \qquad v(x) = x/r.
$$
\nObviously, the above relations can be rewritten as
\n
$$
u^k(z) = \int M(u, w) d\mathbf{H}^{n-1} - \int M(u, w) d\mathbf{H}^{n-1}.
$$

Since $L_i u^j = 0$ on *T* we see that (*v* is the outer normal to ∂T)

$$
(1) = \int\limits_{\partial T} A^{ij}_{\alpha\beta} w^{i} D_{\beta} u^{j} v_{\alpha} d\mathbf{H}^{n-1}.
$$

$$
(1) = \int_{\partial T} A_{\alpha\beta}^{ij} w^i D_{\beta} w^j v_{\alpha} d\mathbf{H}^{n-1}.
$$

Observing $w \in H^{2,2}(\Omega \setminus T)^N$ and $L_{ij}^i w^j = 0$

$$
(2) = -\int_{\partial T} A_{\alpha\beta}^{ij} D_{\alpha} w^i u^j v_{\beta} d\mathbf{H}^{n-1}.
$$

For functions $f, g : \partial B_r(0) \to \mathbf{R}^N$ we let

$$
M(f,g)=A_{\alpha\beta}^{ij}(g^iD_\beta f^j\nu_\alpha-D_\alpha g^i f^j\nu_\beta),\qquad \nu(x)=x/r.
$$

$$
u^{k}(z) = \int_{\partial B_{R_{0}}(0)} M(u, w) dH^{n-1} - \int_{\partial B_{R}(0)} M(u, w) dH^{n-1}
$$

 $u_i u^j = 0$ on *T* we see that (*v* is the outer normal to ∂T)
 $(1) = \int_{\partial T} A_{\alpha\beta}^{ij} w^i D_\beta u^j v_\alpha d\mathbf{H}^{n-1}.$
 $\log w \in H^{2,2}(\Omega \setminus T)^N$ and $L_{ij}^i w^j = 0$ on $\Omega \setminus T$ we get
 $(2) = -\int_{\partial T} A_{\alpha\beta}^{ij} D_\alpha w^i u^j v_\beta d\mathbf{H}^{n-$ Since $u_{|\partial \Omega} = w_{|\partial \Omega} = 0$ the first integral vanishes, and integration by parts shows the identity
 $\int_{\partial B_R(0)} M(u, w) d\mathbf{H}^{n-1} = \int_{\partial B_Q(0)} M(u, w) d\mathbf{H}^{n-1}$,

and we arrive at the formula Observing $w \in H^{2,2}(\Omega \setminus T)^N$ and $L'_i w^j = 0$ on $\Omega \setminus T$ we get
 $(2) = -\int_{\partial T} A^i_{\alpha\beta} D_{\alpha} w^i u^j v_{\beta} d\mathbf{H}^{n-1}$

For functions $f, g : \partial B_r(0) \to \mathbf{R}^N$ we let
 $M(f, g) = A^i_{\alpha\beta}(g^t D_{\beta} f^j v_{\alpha} - D_{\alpha} g^i f^j v_{\beta}), \qquad v(x) =$ Observing $w \in H^{2,2}(\Omega \setminus T)^N$ and $L^t_{ij}w^j = 0$ on $\Omega \setminus T$ we get
 $(2) = -\int_{\partial T} A^{\text{ij}}_{\alpha\beta} D_a w^i w^j v_\beta d\mathbf{H}^{n-1}$.

For functions $f, g : \partial B_r(0) \to \mathbf{R}^N$ we let
 $M(f, g) = A^{\text{ij}}_{\alpha\beta} (g^t D_\beta f^j v_a - D_a g^t f^j v_\beta)$, $v(x) =$ tions.*f*, $g: \partial B_r(0) \to \mathbf{R}^n$ we let
 $M(f, g) = A_{2\beta}^{ij} (g^i D_\beta f^j \nu_a - D_a g^i f^j \nu_\beta), \qquad \nu(x) = x/r$.

y, the above relations can be rewritten as
 $u^k(z) = \int M(u, w) d\mathbf{H}^{n-1} - \int M(u, w) d\mathbf{H}^{n-1}$.
 $\frac{\partial B_R(0)}{\partial B_{R_0}(0)}$ $\frac{\partial B_R$ *;- -*

- .'

$$
\int_{B_R(0)} M(u, w) d\mathbf{H}^{n-1} = \int_{\partial B_Q(0)} M(u, w) d\mathbf{H}^{n-1},
$$

arrive at the formula

$$
u^k(z) = \int_{\partial B_Q(0)} M(u, w) d\mathbf{H}^{n-1}
$$

and we arrive at the formula

$$
u^k(z) = \iint\limits_{\partial B_0(0)} M(u, w) d\mathbf{H}^{n-1}
$$

for all sufficiently small values of ρ . By definition we-have

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\nfor all sufficiently small values of
$$
\varrho
$$
. By definition we have
\n
$$
\int M(u, w) dH^{n-1} = w^t(0) \int A_{\alpha\beta}^{ij} D_{\beta} u^{j} v_{\alpha} dH^{n-1}
$$
\n
$$
\frac{\partial B_{\varrho(0)}}{\partial B_{\varrho(0)}} + \int A_{\alpha\beta}^{ij} (w^i - w^i(0)) D_{\beta} u^{j} v_{\alpha} dH^{n-1} - \int A_{\alpha\beta}^{ij} D_{\alpha} w^{i} u^{j} v_{\beta} dH^{n-1}
$$
\n
$$
= w^t(0) \cdot a_i + b - c.
$$
\nSince $L_{ij}^{t} w^{j} = 0$ on $B_R(0)$, $m = \sup \{ |\nabla w(x)| : x \in B_{R/2}(0) \}$ is finite and we infer from
\n(5.1), (5.2) $|b| \leq CA m \varrho e^{n-1} e^{t-n} = \text{const } \varrho^{\epsilon}, |c| \leq \text{const } \varrho^{\epsilon}$. Moreover,
\n $a_i = \int A_{\alpha\beta}^{ij} D_{\beta} u^{j} v_{\alpha} dH^{n-1} =: C_i$.
\nNow passing to the limit $\varrho \to 0$, (5.3) becomes $u^{k}(z) = C_i G_i^{k}(z, 0)$ for all $z \in B_{R_0}(0)$.
\nCorollary 1 of Theorem 1 implies $y = C_i G_i(t, 0)$ all over O .

•

$$
a_i = \int_{\partial B_{R_0}(0)} A_{\alpha\beta}^{ij} D_{\beta} u^j v_{\alpha} d\mathbf{H}^{n-1} =: C_i.
$$

Now passing to the limit $\varrho \to 0$, (5.3) becomes $u^k(z) = C_i G_i^k(z, 0)$ for all $z \in B_{R_0}(0)$. Corollary 1 of Theorem 1 implies $u = C_i G_i(\cdot, 0)$ all over Ω

1. According to Theorem 10 the column vectors of Green's matrix $G(\cdot, y)$ form a basis of the space X of all non-trivial solutions with zero boundary values to the system $L_i \dot{u} = 0$ on $\Omega \setminus \{y\}$ satisfying the growth condition (5.1) which requires a growth order less than $|x - y|^{1-n}$, when $x \in \Omega$ approaches the point y. After Theorem 7 we remarked lim sup $|G(x, y)| |x - y|^{n-2}$

Allegation $\mathcal{U} = w^t(0)$ $D_\beta u^j v_a d\mathbf{H}^{n-1} - \int A_{\alpha\beta}^{ij} D_a w^i u^j v_\beta d\mathbf{H}^{n-1}$
 $\partial B_{\alpha}(\mathbf{0})$
 $\partial B_{\alpha}(\mathbf{0})$
 $\partial B_{\alpha}(\mathbf{0})$
 $\partial B_{\alpha}(\mathbf{0}) = 0$ on $B_R(\mathbf{0})$, $m = \sup \{|\nabla w(x)| : x \in B_{RR}(\mathbf{0})\}$ is finite and w >0 and in Theorem 7 we proved an inequality of the form $|G(x, y)| \leq C |x - y|^{2-n}$. On the other hand, $u \in X \setminus \{0\}$ cannot satisfy a local growth condition of the form $|u(x)| \leq C |x-y|^{2+\epsilon-n}$ for some positive ε (this would imply $\nabla u \in L^{n/(n-1)}$ near y and therefore $u = 0$, compare the > 0 and in Theorem 7 we proved an inequality of the form $|G(x, y)| \le C |x - y|^{2-n}$. On the other hand, $u \in X \setminus \{0\}$ cannot satisfy a local growth condition of the form $|u(x)| \le C |x - y|^{2+\epsilon-n}$ for some positive ε (this wou following remark), so that the statement of Theorem 10 can be reformulated as follows: If *u* is
a non trivial solution of the system $L_{ij}u^j = 0$ on $\Omega\{y\}$ with zero boundary values which increases of order less than $|x-y|^{1-n}$, when $x \to y$, then the growth order of *u* is exactly $|x-y|^{2-n}$. **Example 1** Integral 1 implies $u = C_iG_i(\cdot, 0)$ all over Ω **1**
 1. According to Theorem 10 the column vectors of Green's matrix $G(\cdot$ ice X of all non-trivial solutions with zero boundary values to the $\setminus \{y\}$ satis of all non-trivial solutions with, zero boundary values
of all non-trivial solutions with, zero boundary values
tisfying the growth condition (5.1) which requires a grow
2 approaches the point y. After Theorem 7 we remark

S

- - **S**

$$
u \in \tilde{H}^{1,1+\delta}(\Omega)^N
$$
 for some $\delta > 0$.

2. Let us replace condition (5.1) by
 $u \in \mathring{H}^{1,1+\delta}(\Omega)^N$ for some $\delta > 0$.

For $\delta \ge 1/(n-1)$ an easy calculation shows that u is a weak solution on the whole domain and therefore vanishes identically. Consider the ciently small values of $|x-y|$ we get the inequalities (assume $y = 0$) $u \in \mathring{H}^{1,1+\delta}(\Omega)^N$ for some $\delta > 0$.
 $\delta \geq 1/(n-1)$ an easy calculation shows that u is a weak solut

therefore vanishes identically. Consider the case $p := 1 + \delta \in$

ly small values of $|x - y|$ we get the inequalities α **s** β **s** β **s** β **s** β **s** β **s** β **s** α **s s** α **s s**

$$
|\nabla u(x)| \leq C |x|^{-n/2} |\nabla u||_{L^2(B_{\lfloor r\rfloor/2}(x))}
$$
 (proof of Theorem 8),

$$
||u||_{H^{1,1}(B_{|x|/2}(x))} \leq C |x|^{n/2} |x|^{-n/p} ||u||_{H^{1,p}(B_{|x|}(x))},
$$

from which we get $|\nabla u(x)| \leq C |x|^{-n/p} ||u||_{H^1(\mathcal{P}(\Omega))}$. Thus u satisfies a local growth condition of the form (5.2) and by integration we get (5.1) (i). Let us state this observation as

Corollary: The statement of Theorem 10 continues to hold if the local growth condition in (5.1) is replaced by $(5.1)'$.

2. Let us replace condition of the system

creases of order less than $|x - y|^1$.

2. Let us replace condition (5.
 $u \in \mathring{H}^{1,1+\delta}(\Omega)^N$ for som

For $\delta \geq 1/(n-1)$ an easy calcu

and therefore vanishes identical

ciently We finally return to the representation formula proved in Theorem 6 and want to show that the solution to the boundary value problem (2.1) has certain regularity properties if the measure μ does not behave too bad. and therefore vanishes identically. Consider the case $p := 1 + \delta \in$

ciently small values of $|z - y|$ we get the inequalities (assume $y = 0$)
 $|\nabla u(x)| \leq C |x|^{-n/2} |\nabla u||_{L^2(B|x|/2^{(x)})}$ (proof of Theorem 8),
 $||u||_{H^{1,1}(B|x|/2$ fix $\|E\| \leq C |x|^{-n/2} \|\nabla u\|_{L^4(B_{\lfloor x \rfloor/2}(x))}$ (proof of Theorem 8),
 $H^{1,1}(B_{\lfloor x \rfloor/2}(x)) \leq C |x|^{-n/p} \, \|u\|_{H^{1,p}(B_{\lfloor x \rfloor}(x))}$,
 $\|g\|_{L^4(B_{\lfloor x \rfloor}(x))} \leq C |x|^{-n/p} \, \|u\|_{H^{1,p}(B_{\rfloor})}$. Thus a satisfies a local growt *Then the following statements hold:*
 Then the form the following statement of Theorem 10 continues to hold if the local group is replaced by (5.1).
 The statement of Theorem 10 continues to hold if the local group is We finally return to the representation formula proved in Theore

ow that the solution to the boundary value problem (2.1) has coperties if the measure μ does not behave too bad.

Theorem 11: Assume that (4.2) holds an

Theorem 11: *Assume that* (4.2) *holds and that* $u \in \hat{H}$ *is the unique weak solution of* (2.1) *with* $\mu \in M(\Omega)^N$, $\mu^i = \mu_+^i - \mu_-^i$, *salisfying*

$$
u(t, \Delta t) = \mu_1 t + \mu_2 t + \mu_3
$$
\n
$$
u(t, \Delta t) = \mu_1 t + \mu_2 t + \mu_3
$$
\n
$$
u(t, \Delta t) = \mu_1 t + \mu_2 t + \mu_3
$$
\n
$$
\sum_{i=1}^{N} \int_{\Omega} |x - y|^{2-n} d(\mu_1 t + \mu_2 t) (x) < \infty \text{ for all } y \in \Omega.
$$
\n(5.5)\n
$$
u(t, \Delta t) = \sum_{i=1}^{N} \int_{\Omega} |x - y|^{2-n} d(\mu_1 t + \mu_2 t) (x) < \infty \text{ for all } y \in \Omega.
$$
\n(5.6)\n
$$
u(t, \Delta t) = \sum_{i=1}^{N} \int_{\Omega} |x - y|^{2-n} d(\mu_1 t + \mu_2 t) (x) < \infty \text{ for all } y \in \Omega.
$$
\n(5.7)\n
$$
u(t, \Delta t) = \sum_{i=1}^{N} \int_{\Omega} |x - y|^{2-n} d(\mu_1 t + \mu_2 t) (x) < \infty \text{ for all } y \in \Omega.
$$

$$
\overline{u}(y) := \lim_{\rho \to 0} \int_{B_{\rho}(y)} u \, dx
$$

(5.1)'

The Green Matrix for Strongly Elliptic Systems 529

ts for all y. Identifying u with the representative \bar{u} we get *exists for all y. Identifying u with the representative* \bar{u} we get

The Green Matrix for Strongly I
is for all y. Identifying u with the representative
$$
\bar{u}
$$
 we get

$$
u^k(y) = \int_{\Omega} G_i^k(y, x) d\mu^i(x) \qquad (y \in \Omega, k = 1, ..., N).
$$

(ii) For all $\varepsilon > 0$ and all balls $B_R(y_0) \subset \Omega$ there exists a ball $B_r(y) \subset B_R(y_0)$ such that *the oscillation of u on* $B_r(y)$ *is controlled by* ε *. exists for all y. Identifying u with the representative* \bar{u} we get
 $u^k(y) = \int_a^b G_i^k(y, x) d\mu^i(x)$ $(y \in \Omega, k = 1, ..., N$

(ii) *For all* $\varepsilon > 0$ *and all balls* $B_R(y_0) \subset \Omega$ *there exists a b*
 the oscillation of u on B *y* Elliptic Systems 529
 i.
 oo, (5.6)

(iii) 11(5.5) is replaced by the stronger condition

$$
\sup \left\{ \sum_{i=1}^N \int_{\Omega} |x-y|^{2-n} d(\mu_+^i + \mu_-^i)(x) : y \in \Omega \right\} < \infty, \tag{5.6}
$$

then u is locally bounded.

(ii) $F \circ r$ all $\varepsilon > 0$ and all balls $B_R(y_0) \subseteq \Omega$ there exists a ball $B_r(y) \subseteq B_R(y_0)$ such that
 α : α coscillation of u on $B_r(y)$ is controlled by ε .

(iii) If (5.5) is replaced by the stronger condition
 α Let $\mathcal{L} \subset \Omega$ denote a compact $(n - 1)$ -dimensional manifold and define, for $A \subset \Omega$, $\mu^{i}(A)$
= $\mathbf{H}^{n-1}(A \cap \mathcal{L})$, $i = 1, ..., N$. Obviously, (5.6) is satisfied, so that all statements of Theorem 11,hold. Such properties as local boundedness and generic continuity are not contained in Section 1: Theorem 2 for example only describes the behaviour of *u* on $\Omega \setminus \Sigma$, so that $u(x)$ could behave most irregularly in the limit $\Omega \setminus \Sigma$ $\Rightarrow x \rightarrow x_0 \in \Sigma$.

The proof of Theorem 11 is based on ideas from [11: Thm. 2.3, 2.6], which we combine with our previous results. For further details we refer to *[5:* Section *5,* Thm. *12].* Theorem 11 immediately applies to elliptic systems with quadratic growth, which are studied for example in [7, 10, 11]. *of the system L_i* $u^j = f^i(t, u, \nabla u)$, $i = 1, ..., N$, where f is a Caratheodicy function $f(x, y, z) = \frac{1}{2}$, $\frac{1}{2}$, The proof of Theorem 11 is based on ideas from [11: Thm. 2.3, 2,
bine with our previous results. For further details we refer to [5: Se
Theorem 11 immediately applies to elliptic systems with quadrat
are studied for examp

Theorem 12: Assume that (4.2) holds and let $u \in H^{1,2} \cap L^{\infty}(\Omega)^N$ be a weak solution $a |p|^2 + b$ with positive constants a, b. If

 \int _{*j*} $|\nabla u(x)|^2 |x-y|^{2-n} dx$ is finite for all $y \in \Omega$,

then each point is a Lebesgue point and u is of class $C¹$ *on a dense open subset of* Ω *.*

In [6] we had described another application of Green's matrix. Suppose that u is a weak

 $Fv = \int\limits_{\Omega} A^{\prime\prime}_{\alpha\beta} D_{\alpha} v^i D_{\beta} v^j dx$

in a class-of functions described by a side condition of the form $v \cdot e \geq \psi$ for a fixed vector *e* and a smooth real-valued function ψ . If the coefficients satisfy (4.2), we had shown that u is regular in Ω . The arguments rest on a careful analysis of the sign properties of G combined with potenof the system $L_{ij}u^j = f^i(\cdot, u, \nabla u)$, $i = 1, ..., N$, w
satisfying $|f(x, y, p)| \le a |p|^2 + b$ with positive cons
 $\int |\nabla u(x)|^2 |x - y|^{2-n} dx$ is finite for all $y \in$
then each point is a Lebesgue point and u is of class $\{$ in [6] we ha then each point is a Lebesgue point and u is of class C^1

in [6] we had described another application of Green's

minimum of the quadratic functional
 $Fv = \int_A A_{\alpha\beta}^{\alpha} D_{\alpha} v^i D_{\beta} v^j dx$

in a class of functions descr

Up to now we only considered the case $n \geq 3$ and assumed the coefficients of the system to be continuous functions. In two dimensions the existence of Green's matrix-can be proved for boundedmeasurable coefficients satisfying the strong ellipticity condition. This follows from the in a class of functions described by a side condition of the form $v \cdot e \geq \psi$ for a fixed vector e and
a smooth real-valued function ψ . If the coefficients satisfy (4.2), we had shown that u is regular
in Ω . The fact that for $n = 2$ the unique $H^{1,2}$ -solution to the system $L_{ij}u^j = -D_a F_a^i$, $i = 1, ..., N$, on Ω with $F \in L^p(\Omega)^{nN}$ for some $p > 2$ is continuous with vanishing boundary values, so that the method of Section 2 applies (compare [5] for details). In general it is possible to construct Green's matrix by duality whenever regularity theorems are available. In higher dimensions •
•
•
•
• such regularity results for systems with L^{∞} -coefficients are only true under additional smallness conditions. Thus for proving the existence of G the continuity hypothesis can be dropped in

34 Analysis Bd. 5, Heft 0 (1986)

two dimensions and has to be replaced by a smallness condition in higher dimensions, respectively. But it seems to be impossible to derive the standard estimate for $G(x, y)$ in the more general situation: the technique used in this piper does not apply since Lemma 4.3 essentially rests on the Holder continuity of the coefficient matrix. For continuous coefficients the corollary of Theorem 2 gives the information $|G(x,y)| \leq C(\varepsilon) |x-y|^{\varepsilon+1-n}$ for all $0 < \varepsilon < 1$, but the right growth order can not be achieved by this simple argument. A. FUCHS

Sions and has to be replaced by a smallness condition in l

t it seems to be impossible to derive the standard estima

uation: the technique used in this paper does not apply si

e Hölder continuity of the coeff

We wish to remark that in the case $n = 2$ Theorem 7 has the analogue

$$
|G(x, y)| \leq C_1 \log (1/|x - y|) + C_2 R^{\beta - 1} \log (R^{-1})
$$

for all $x \in B_{2R}(y)$, $y \in \Omega$, $0 < R \leq \min(R_0, \text{dist}(y, \partial \Omega)/4)$ (notations as in Theorem 7). The proof uses ideas from Section 4 combined with an appropriate modification of the local estimates in Section 1. From (*) one easily obtains gradient bounds and estimates for the Hölder norm of the first derivatives. For details we again refer to [5].

()*

In this paper we studied Green's matrix for strongly elliptic operators of the form L $= (-D_a(A_{a}^{\prime\prime}B_a))$. By a suitable extension of the method it is also possible to prove the existence of a Green matrix with the correct growth order for more general operators $\vec{L} = L$ $+ (B_{\alpha}^{ij}D_{\alpha} + a^{ij})$, provided the leading part *L* satisfies the Legendre-Hadamard condition and some mild regularity properties are imposed on the lower order terms. Since the details are somewhat technical, this generalization is discussed in a-separate paper.

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Buchbespreehung

G. HEINIG and K. ROSY: **Algebraic Methods for Toeplitz-like Matrices and Operators** (Math. Forschung: Bd. 19). Berlin: Akadenie-VerIag 1984, p. 212.

The book under review is divided into two parts: $Part I - "Toeplitz$ and Hankel matrices" contains 9-chapters (sections, according to the authors' terminology.) and Part II - "Toeplitzlike operators" - contains 7 chapters. The book is supplied with a list of references (117 titles), subject and notation indices.

Many characteristic properties of Toeplitz matrices can be deduced from the fact that *AU*_n – U_mA has at most rank two; here *A* is an $m \times n$. Toeplitz matrix and U_n is the matrix rank $AU - VA$ is small compared to rank A ; U and V being some fixed operators. The authors call such operators A Toeplitz-like operators. Let us explain the contents of the book under
review in more detail. G. HEINIG and K. KOST: Algebraic Me
(Math. Forschung: Bd. 19). Berlin: Ak
The book under review is divided into two
contains 9 chapters (sections, according to
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like operators" – subject and notation indices.

A $U_n - U_mA$ has at most rank two; here *A* is an $m \times n$. Toeplitz matrix and U_n is the matrix $A U_n - U_mA$ has at most rank are *A* is an $m \times n$. Toeplitz matrix and U_n is the matrix rank A Figure 1.1 The space of the authors in the problem of the space of the function of the distribution of the space of the space of the distribution of $A:U$ and V being some fixed of call such operators A Toeplitz-like

Part I is devoted to the algebraic theory of finite Toeplitz and Hankel matrices (T. and H -matrices). The main problems are the following:

1. Fast inversion algorithms:
2. Structure of T - and H -matrices, their rank and signature and the relation between T -

3. Application to some-problems of Wiener-Hopf equations theory.

Chapter 0 contains some facts which are utilized on the full length of the book; in particular the notions of *T*- and *H*-matrices are formulated and some special matrices of these classes are described.

In Chapter 1 the problem of inversion of T - and H -matrices is investigated. It is shown that one can reduce the last problem to the problem of the solution of two special linear algebraic. systems, which the authors call "fundamental equations". Tf the fundamental equations are solvable then the given matrix is invertible and one receives a simple inversion formula The right-hand side of the second fundamental equation depends on the given matrix. If this matrix satisfies some complementary conditions then it is possible to change the righthand side of the equation by a certain fixed vector. Further, operators Δ and ∇ are introclasses are described.

In Chapter 1 the problem of inversion of T. and H matrices is investigated. It is shown

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In Chapter 1 the problem that one can reduce the last praise systems, which the auth are solvable then the giv