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### A Priori Estimates for Elliptic Systems

### H. BEGEHR<sup>1</sup>) and G. C. HSIAO<sup>2</sup>)

Es werden A-priori-Abschätzungen für die allgemeine komplexe Beltrami-Gleichung im Zusammenhang mit Riemann-Hilbertschen Randbedingungen hergeleitet, die für Existenz- und Eindeutigkeitsaussagen von zugehörigen nichtlinearen Problemen herangezogen werden können. Dazu wird die Gleichung zusammen mit den Randbedingungen in die kanonische Form transformiert und wesentlich eine Darstellungsformel von Haack-Wendland benutzt.

Выводятся оценки для общего комплексного уравнения Бельтрами в связи с краевыми условиями Римана-Гильберта и привлекаемые к утверждениям о существовании и единственности соответствующих нелинейных проблем. Для этого уравнение и краевые условия преобразуются в каноническую форму и существенно используется одна формула представления Хаака-Вендланда.

A priori estimates for the general complex Beltrami equation in connection with Riemann-Hilbert boundary conditions are developed, which can be used for existence as well as uniqueness statements for related nonlinear problems. For this reason the equation together with the boundary conditions are transformed into the canonical form and essentially a representation formula originally given by Haack-Wendland is used.

### 1. Introduction

In this paper a priori estimates will be derived for solutions of the general Beltrami<sup>6</sup> equation

$$
w_{\overline{z}} + \mu_1 w_{\overline{z}} + \mu_2 \overline{w_z} = aw + b\overline{w} + c
$$

under the ellipticity condition

$$
|\mu_1(z)| + |\mu_2(z)| \leq q < 1.
$$

In particular, we are concerned with the boundary value problem consisting of  $(1)$ together with certain boundary and side conditions which is known in complex analysis as the Hilbert or the Riemann-Hilbert boundary value problem (see, e.g. BEGEHR [2, 3], BEGEHR and GILBERT [5-7, 9], GAKHOV [19], GILBERT [20], MUSHKELISHVILI [26], and WENDLAND [30]).

One of the purposes for obtaining a priori estimates for solutions of linear equations is that they may be used to establish existence as well as uniqueness theorems for the related nonlinear problems. Indeed, in several recent papers by the authors

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 $(1)$ 

 $(2)$ 

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<sup>2</sup><br>2 H. BEGERR and G. C. HSIAO<br>[12-15], these a priori estimates [12-15], these a priori estimates are utilized' to investigate boundary value problems for nonlinear elliptic equations of the type

$$
w_{\bar{z}}=H(z,w,w_z)
$$

BEGEHR and G. C. HSIAO<br>
ese a priori estimates and<br>
ese a priori estimates and<br>
linear elliptic equations<br>  $= H(z, w, w_z)$ <br>
ponding nonlinear bound<br>
quation (1) have also b with corresponding nonlinear boundary and side conditions. Similar a priori estimates for equation (1) have also been employed to study equations of type  $(3)$ with respect to the so-called Riemann boundary value problem [10, 11]. As indicated in a celebrated paper by WENDLAND [28] (see also WENDLAND [30]), because of the specific form of these estimates, they are particularly suitable for .the constructive existence proof of solutions for the nonlinear elliptic system in connection with the Newton embedding procedure.

Our main results concerning a priori estimates for (1) are presented in Section 5. The derivations of these estimates are based on the representation formulas in Section 4 and the reduction of the relevant boundary value problem for (1) to a canonical'problem in Section,3. In this reduction, as will be seen, by introducing appropriate transformations the Beltrami operator in (1) is reduced to the Cauchy Riemann operator according to BERS and NIRENBERG [16] (see also KUNZI [23] and MONAHOV [25]). Section 2 contains some basic properties concerning the homeomorphisms of the Beltrami equation which will be needed later for establishing Riemann operator according to BERS and NIRENBERG [16] (see a<br>and MONAHOV [25]). Section 2 contains some basic properties concern<br>morphisms of the Beltrami equation which will be needed later inccessary bounds of the transf **EXECUTE 10.1** is understood that by a complete homeomorphism of the Beltrani equation which will be needed later for establishing morphisms of the Beltrami equation which will be needed later for establishing necessary b  $\chi$ <sub>2</sub> is each of the Beltrami equation which will be needsary bounds of the Heltrami equation which will be needsary bounds of the transformations introduced in Sections of the transformations introduced in Sections are

### 2. The Beltrami equation

 $\zeta_{\bar{z}} = \mu \zeta_z$ 

It is understood that by a complete homeomorphism of the Beltrami equation

bijective mapping of the z-plane onto the  $\zeta$ -plane (see VEKUA [27]). If Figure 1.4. The selectram equation<br>
(4)<br>
ling on  $\mu$  which defines a<br>
sua [27]). If<br>
(5)<br>  $\frac{1}{1-x}$  (5)

$$
|\mu(z)|\leqq q<1
$$

then  $\zeta$  is a, K-quasiconformal mapping with  $K = (1 + q)/(1 - q)$ . The solution of (4) can be made unique by imposing different additional conditions; for instance; one may require that zero and infinity are fixed points of the mapping  $\zeta$ . informal mapping with  $K = (1 +$ <br>
ue by imposing different addition<br>
zero and infinity are fixed points<br>
a measurable function in C satisfy<br>
(C),  $p' < 2$ . Then there exists a c<br>
to a class  $C^*(C), 0 < \alpha < 1$  with  $\zeta$ <br>
mma see *d* mapping with  $K = (1$ <br>imposing different additid<br>*d* infinity are fixed point<br>*arable function* in C satis<br> $\langle 2$ . Then there exists a<br>*is*  $C^{\alpha}(C), 0 \langle \alpha \langle 1 \rangle$ <br>is  $C^{\alpha}(C), 0 \langle \alpha \rangle$ <br>**d** and  $C^{\alpha}(C)$  and  $C^{\alpha}(C)$ <br>*d*

Lemma 1: Let  $\mu$  be a *measurable function in* C satisfying (5) for some non-negative *constant q and*  $\mu \in L_p(C)$ *,*  $p' < 2$ *. Then there exists a complete homeomorphism*  $\zeta$  of equation (4) belonging to a class  $C^{\alpha}(C)$ ,  $0 < \alpha < 1$  with  $\zeta(\infty) = \infty$ .<br>
For a proof of this lemma see AHLFORS [1], BOJARSKI [17 *equation (4) belonging to a class*  $C^{\alpha}(\mathbb{C})$ *,*  $0 < \alpha < 1$  *with*  $\zeta(\infty) = \infty$ *.* 

For a proof of this lemma see AHLFORS [1], BOJARSKI [17], LEHTO and VIRTANEN [24], MONAHOV [25], and VEKUA [27]. If in this proof instead of the operator  $\mathfrak{T}$ ,

$$
\mathfrak{X}w(z) = -\frac{1}{\pi}\int\limits_{\mathbb{C}} w(\zeta)\frac{d\xi\,d\eta}{\zeta-z} \quad (\zeta=\xi+i\eta),
$$

one uses the operator  $\tilde{x}$  defined by  $\tilde{x}w(z) = \tilde{x}w(z) - \tilde{x}w(0)$ , then the solution  $\zeta$  has the representation  $\zeta = z + \mathfrak{X}w(z)$  and fulfils the additional condition  $\zeta(0) = 0$ .

We note that from the assumption on  $\mu$  we have  $\mu \in L_p(\mathbb{C})$  for  $p' \leq p$ . In what follows, we will assume  $\mu$  to have compact support. As will be seen, with this assumption the Jacobian of  $\zeta$  may be estimated. For the  $L_p^{\circ}(\mathbb{C})$  space,  $1 < p \leq +\infty$ , we will denote by  $\|\cdot\|_p$  the usual  $L_p$ -norm. Furthermore if in (6), C is replaced by a bounded domain *D*, we then write  $\mathfrak{T}_b$  instead of  $\mathfrak{T}$ . From VEKUA [27: p. 38], we.

(3)

*(4)* 

 $(6)$ 

have for  $w \in L_p(\hat{D}),\, 2 < p,\, \hat{D} = D$   $\cup$  $w \in L_p(\hat{D})$ <br> $\|\mathfrak{X}_Dw\|_{\mathfrak{a}} \le$ 

 

$$
\|\mathfrak{X}_D w\|_{\alpha} \leq M(p, D) \|w\|_{p}, \qquad \alpha = (p-2)/p \tag{7}
$$

*A* Priori Estimates fo<br> *.*  $2 < p$ ,  $\hat{D} = D \cup \partial D$ ,<br> *M*(*p*, *D*)  $||w||_p$ ,  $\alpha = (p - 2)/p$ <br>
the Hölder norm in  $C^{\alpha}(\mathbb{C})$ , and for *w*<br>  $||p||_p^2|^\alpha$ ,  $\alpha = (p - 2)/p$ , where *M*(*p*) is<br>
oreover the operator *II* defined by A Priori Estimates for Elliptic Systems<br>
.<br>
have for  $w \in L_p(\hat{D}), 2 < p, \hat{D} = D \cup \partial D$ ,<br>  $\|\mathfrak{T}_D w\|_{\alpha} \le M(p, D) \|w\|_p$ ,  $\alpha = (p - 2)/p$  (7)<br>
where  $\|\cdot\|_{\alpha}$  denotes the Hölder norm in  $C^{\circ}(\mathbb{C})$ , and for  $w \in L_p(\mathbb{C}), 2 < p$ , A Priori Estimates for Elliptic Systems<br>
have for  $w \in L_p(D)$ ,  $2 < p$ ,  $D = D \cup \partial D$ ,<br>  $||\mathfrak{D}_D w||_a \leq M(p, D) ||w||_p$ ,  $\alpha = (p - 2)/p$ <br>
where  $||\cdot||_a$  denotes the Hölder norm in  $C^a(C)$ , and for  $w \in L_p(C)$ ,  $2 < p$ , we<br>  $|\mathfrak{D}w(z)| \leq M$ 

A Priori Estimates for Elliptic Systems  
\n
$$
w \in L_p(\hat{D}), 2 < p, \hat{D} = D \cup \partial D,
$$
\n
$$
\|\mathfrak{T}_D w\|_{\alpha} \leq M(p, D) \|w\|_p, \qquad \alpha = (p - 2)/p \tag{7}
$$
\n
$$
\|a\|_p \leq M(p) \|w\|_p |z|^{\alpha}, \qquad \alpha = (p - 2)/p \tag{7}
$$
\n
$$
\leq M(p) \|w\|_p |z|^{\alpha}, \qquad \alpha = (p - 2)/p, \text{ where } M(p) \text{ is a non-negative constant}
$$
\n
$$
\text{and on } p. \text{ Moreover, the operator } \Pi \text{ defined by}
$$
\n
$$
\text{If } w(z) = -\frac{1}{\pi} \int_{C} w(\zeta) \frac{d\zeta}{(\zeta - z)^2} \tag{8}
$$
\n
$$
\text{and so } L_p(\mathbb{C}), 1 < p < +\infty, \text{ and satisfies the estimate (see VERUA [27:17])}
$$
\n
$$
\text{Thus, } \mathbb{Z}_p \text{ is a positive continuous function of } p \text
$$

is bounded on  $L_p(C)$ ,  $1 < p < +\infty$ , and satisfies the estimate (see VEKUA [27: p. 71))  $||Tw||_p \leq A_p ||w||_p$ ,  $w \in L_p(\mathbb{C})$ , where  $A_p$  is a positive continuous function of p and  $A_2 = 1$ . Thus, if q is a fixed constant,  $0 < q < 1$ , then for p sufficiently close to 2, we have  $\mathcal{D}(\mathcal{D}) = \mathcal{D}(\mathcal{D})$ ,  $\mathcal{D}(\mathcal{D}) = \mathcal{D}(\mathcal{D})$ ,<br>  $\|\mathcal{D}_D w\|_{\mathbf{a}} \leq M(p, D) \|w\|_{p}$ ,  $\alpha = (p - 2)/p$ <br>  $\mathcal{D}_\mathbf{a}$  denotes the Hölder norm in  $C^{\mathbf{a}}(\mathbf{C})$ , and for  $M(p) \|w\|_{p} \|z|^{\mathbf{a}}, \alpha = (p - 2)/p$ , where  $M(p)$ where  $||\cdot||_a$  denotes the Hölder norm in  $C^s(\mathbb{C})$ , and for  $w \in L$ <br>  $|\mathfrak{T}w(z)| \leq M(p) ||w||_p |z|^a$ ,  $\alpha = (p-2)/p$ , where  $M(p)$  is a nidepending on  $p$ . Moreover the operator  $\Pi$  defined by<br>  $\Pi w(z) = -\frac{1}{\pi} \int w(\zeta) \frac{d\zeta}{(\z$ 

$$
0
$$

Similarly, for a bounded domain *D*, we define  $\Pi_D$  by (8) with C replaced by *D*.

Obviously  $\Pi_D$  has the same norm as  $\Pi$ .<br>Now if we denote by  $W_p^{\bullet m}$  the Sobolev space consisting of functions with generalized

m-th order derivatives in  $L_p(C)$ , we have the following result.<br>
Lemma 2: Let  $\mu \in W_p^2$  with  $p > 2$  and have compact suppor<br>
domain D. In addition suppose  $\mu$  fulfils (5), and (9) is valid.<br>
homeomorphism  $\zeta$  of (4) w Lemma 2: Let  $\mu \in W_p^2$  with  $p > 2$  and have compact support K lying in a bounded<br>
main D. In addition suppose  $\mu$  fulfils (5), and (9) is valid. Then for the complete<br>
meomorphism  $\zeta$  of (4) with  $\zeta(0) = 0$  and  $\zeta(\$  $domain\ D.$  In addition suppose  $\mu$  fulfils (5), and (9) is valid. Then for the complete *abmain D. In didution suppose*  $\mu$  *futilits* (c), *and* (5) *is odd.* I *here for the complete homeomorphism*  $\zeta$  of (4) *with*  $\zeta(0) = 0$  *and*  $\zeta(\infty) = \infty$ , *there exists a constant M* depending only on p, q and K *dependin4 only on p, q and K such that z* of functions with generalized<br>
result.<br> *support K lying in a bounded*<br> *i* valid. Then for the complete<br>  $\circ$ , there exists a constant M<br>  $(z \in C)$ . (10) 2, we have<br>  $0 < qA_p < 1$ .<br>
The singular integral equation of  $\mathbf{y} \leq \mathbf{y}$ , we have<br>  $0 < qA_p < 1$ .<br>
The singular integral equation of  $\mathbf{y} \leq \mathbf{y}$  and  $\mathbf{y} \leq \mathbf{y}$ <br>
Now if we denote by  $W_p^m$  the Sobolev space c q is a fixed con<br>
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e same norm as<br>
y  $W_p^m$  the Sobol<br>
ss in  $L_p(C)$ , we h<br>  $W_p^2$  with  $p > 2$ <br>
on suppose  $\mu$  fu<br>
(4) with  $\zeta(0) =$ <br>
q and K such the<br>
q in K such the<br>  $\mu_x||_p$   $\leq |\zeta_z(z)| \leq$ <br>
ar integ *z* derivatives in  $L_p(C)$ , we have the following result.<br> *i* 2: Let  $\mu \in W_p^2$  *with*  $p > 2$  *and have compact support K lying in a bounded*<br> *In addition suppose*  $\mu$  *fulfils* (5), *and* (9) *is valid. Then for the co* 

$$
\exp \{-M \|\mu_z\|_p\} \leq |\zeta_z(z)| \leq \exp \{M \|\mu_z\|_p\} \qquad (z \in \mathbb{C}).
$$

 $\cdot$ 

$$
\phi = \mu \Pi_p \phi = \mu_i
$$

has a unique solution  $\Phi \in W_p^1(\hat{D})$  such that  $\|\Phi\|_p \leq \|\mu_z\|_p/(1-q\Lambda_p)$ . Because of (11), we see that

$$
\hat{\zeta}(z) := \int\limits_{0}^{z} \exp \mathfrak{T}_{D} \Phi(\zeta) \left[ d\zeta + \mu \, d\bar{\zeta} \right] \qquad (z \in \mathbb{C}) \tag{12}
$$

is independent of the path of integration. As is shown in Monahov [25:  $V § 3$ ],  $\hat{\zeta}$  is then a complete homeomorphism of (4) such that  $\hat{\zeta}(z) - z \in C^{1+\alpha}(\mathbb{C})$  and  $\hat{\zeta}(z)$  $z = z + O(|z|^{-1})$  as  $z \to \infty$ . Since  $\hat{\zeta}(0) = 0$ ,  $\hat{\zeta}$  must coincide with  $\zeta$ . Thus, from (12) we have  $\zeta_z(z) = \exp{\{\mathfrak{D}_0 \Phi(z)\}}$  and (10) follows immediately from (7) and the esti-<br>mate for  $||\Phi||_p$   $\blacksquare'$ .<br>From (4) and (10), it is easy to see that the Jacobian *J* of  $\zeta$  defined by  $J = |\zeta_z|^2 - |\zeta_{\overline{z}}|^2$  is<br>bo mate for  $\|\Phi\|_p$  $\hat{\zeta}(z) := \int \exp \mathfrak{D}_{\rho} \Phi(\zeta) \left[ d\zeta + \mu d\bar{\zeta} \right]$  (z<br>
is independent of the path of integration. As<br>  $\hat{\zeta}$  is then a complete homeomorphism of (4) su<br>  $z = z + O(|z|^{-1})$  as  $z \to \infty$ . Since  $\hat{\zeta}(0) = 0$ ,  $\hat{\zeta}$  mus<br>
we

mate for  $\|\Phi\|_p$  **a**<br>From (4) and (10), it is easy to see that the Jacobian *J* of  $\zeta$  defined by  $J = |\zeta_z|^2 - |\zeta_{\overline{z}}|^2$  is ,<br>bounded below from zero. In fact, we have  $J \ge (1 - q^2) \exp(-2M ||\mu_z||_p)$ .

Lemma 2 remains valid, if the conditions on  $\mu$  are slightly weakened. More precisely, we have the following result.

Lemma 3: *Inequalities* (10) *remain valid if*  $\mu$  *is only in*  $W_p^1$  *while all other as-sumptions from Lemma* 2 are satisfied.  $f \nleq \text{defined by } 0 \cdot (-2M ||\mu_x||_p)$ <br> *.* slightly wea<br> *. mly in*  $W_p$ <sup>1</sup> *i*<br> *. b* that  $qA_{nn}$ . *-* -

**Proof:** Choose  $p_1 > 1$  but sufficiently close to 1 so that  $qA_{pp_1} < 1$  and define  $p_2$ mate for  $||\Psi||_p$ <br>
From (4) and (10), it is easy to see that the Jacobian J of  $\xi$  defined by  $J = |\zeta_z|^2 - |\zeta_z|^2$  is<br>
bounded below from zero. In fact, we have  $J \geq (1 - q^2) \exp(-2M ||\mu_z||_p)$ .<br>
Lemma 2 remains valid, if the con

*•(9)* 

(11)

**4** *P. p. p.* H. BEGERR and G. C. HSIAO.<br>  $= 0$  in  $C \setminus \hat{D}$  such that  $||\mu_n - \mu||_{W_{pp_1}^1} \to 0$  for  $n \to +\infty$ . Then, because<br>  $p_2$ , we also have  $\lim_{n \to +\infty} ||\mu_n - \mu||_{W_{pp_1}^1} = 0$  and  $\lim_{n \to +\infty} ||\mu_n - \mu||_{W_{p_1}^1} = 0$ .<br>
w let  $\zeta$ **1.** BEGEHR and G. C.<br> **c**  $\Diamond$  *D* such that<br>
also have<br>  $\lim_{n \to +\infty} ||\mu_n - \mu||_{W^1_{pp_1}}$ <br>  $\rightarrow +\infty$ <br>  $\Box_n$  be the complet H. BEGEHR<br>in  $C \setminus \hat{D}$  s<br>we also have<br> $\lim_{n \to +\infty} ||\mu_n -$ <br>t  $\zeta_n$  be the **4** H. BEGER a<br>  $\mu_n = 0$  in  $C \setminus \hat{D}$  su<br>  $\leq p_2$ , we also have  $\left(\lim_{n \to +\infty} ||\mu_n - \lim_{n \to +\infty} \mu_n\right)$ <br>
Now let  $\zeta_n$  be the a<br>
given by<br>  $\zeta_n(z) = \int_0^z e^{-iz}$ *j*,  $\|\mu_n - \mu\|_{W^1_{pp_1}} \to 0$  for  $n \to +\infty$ . Then, because  $1 < p_1$ <br>  $= 0$  and  $\lim_{n \to +\infty} \|\mu_n - \mu\|_{W_p} = 0$ .<br> *p*,  $\lim_{n \to +\infty} \|\mu_n - \mu\|_{W_p} \to 0$ .<br> *p*,  $\int_{p_n} f(\xi) \left[ d\xi + \mu_n d\xi \right]$ <br>  $\mu_n H_p \Phi_n = \mu_n$ . Then from  $\Phi_n - \Phi_m - \mu_m$ **H H.** BEGER and G. C. HSTAO .<br>  $\mu_n = 0$  in  $C \setminus \hat{D}$  such that  $\|\mu_n - \mu\|_{W^1_{pp_1}} \to 0$  for  $n \to +\infty$ . Then,<br>  $\leq p_2$ , we also have  $\lim_{n \to +\infty} ||\mu_n - \mu||_{W^1_{pp_1}} = 0$  and  $\lim_{n \to +\infty} ||\mu_n - \mu||_{W_p^1} = 0$ .<br>
Now let  $\zeta$ 0 in C  $\setminus D$  such that  $\|\mu_n - \mu\|_{W_{pp_1}} \to 0$  for  $n \to +\infty$ . Then, becaus<br>we also have<br> $\lim_{n \to +\infty} ||\mu_n - \mu||_{W_{pp_1}^1} = 0$  and  $\lim_{n \to +\infty} ||\mu_n - \mu||_{W_{p^1}} = 0$ .<br>let  $\zeta_n$  be the complete homeomorphism from Lemma 2 corresp IIF The Society and G. C. Hstao<br>  $\mu_n = 0$  in  $C \setminus \hat{D}$  such that  $||\mu_n - \mu||_{W_{p,p_1}^1} \to 0$  for  $n \to +$ <br>  $\leq p_2$ , we also have<br>  $\lim_{n \to +\infty} ||\mu_n - \mu||_{W_{p,p_1}^1} = 0$  and  $\lim_{n \to +\infty} ||\mu_n - \mu||_{W_p^1} =$ <br>
Now let  $\zeta_n$  be the

$$
\lim_{n \to +\infty} ||\mu_n - \mu||_{W^1_{pp_1}} = 0 \text{ and } \lim_{n \to +\infty} ||\mu_n - \mu||_{W^{-1}_{p}} = 0.
$$

Now let  $\zeta_n$  be the complete homeomorphism from Lemma 2 corresponding to  $-\mu$ <sub>*lw*</sub>,<br>from L

$$
\zeta_n(z) = \int\limits_0^z \exp \mathfrak{T}_D \Phi_n(\zeta) \left[ d\zeta + \mu_n d\bar{\zeta} \right]
$$

 $\leq p_2$ , we also have<br>  $\lim_{n \to +\infty} ||\mu_n - \mu||_{W_p^1} = 0$  and  $\lim_{n \to +\infty} ||\mu_n - \mu||_{W_p^1} = 0$ .<br>
Now let  $\zeta_n$  be the complete homeomorphism from Lemma 2 corresponding to  $\mu_n$ <br>
given by<br>  $\zeta_n(z) = \int_0^z \exp \mathfrak{T}_0 \Phi_n(\zeta) [d\zeta + \$  $\leq p_2$ , we also have<br>  $\lim_{n \to +\infty} ||\mu_n - \mu||_{W_{pp_1}^1} = 0$  and  $\lim_{n \to +\infty} ||\mu_n - \text{Now let } \zeta_n$  be the complete homeomorphism from<br>
given by<br>  $\zeta_n(z) = \int_0^z \exp \mathfrak{T}_D \Phi_n(\zeta) [d\zeta + \mu_n d\bar{\zeta}]$ <br>
with  $\Phi_n$  satisfying  $\Phi_n - \mu_n H_D \Phi$ Now let  $\zeta_n$  be the complete hom<br>given by<br> $\zeta_n(z) = \int_0^z \exp \mathfrak{T}_D \Phi_n(\zeta) \left[ d\zeta \right]$ <br>with  $\Phi_n$  satisfying  $\Phi_n - \mu_n \Pi_D \Phi_n$ <br> $= (\mu_n - \mu_m)_z + (\mu_n - \mu_m) \Pi_D \Phi_n$ ,<br> $\|\Phi_n - \Phi_m\|_p \leq \frac{1}{1 - qA_p}$ <br>Since<br> $\|[\Pi_D \Phi_n\|_{pp_1} \leq A_{pp_1} \|\Phi_n\|_{pp$ 

$$
\|\Phi_n - \Phi_m\|_p \leq \frac{1}{1 - qA_p} \left\{ ||(\mu_n - \mu_m)_z||_p + ||\mu_n - \mu_m||_{pp_1} ||\Pi_D \Phi_n||_{pp_1} \right\}.
$$

•1

$$
\|\Phi_n - \Phi_m\|_p \le \frac{1}{1 - qA_p} \{ ||(\mu_n - \mu_m)_z||_p + ||\mu_n \|
$$
  

$$
||H_D \Phi_n||_{pp_1} \le A_{pp_1} \|\Phi_n||_{pp_1} \le \frac{A_{pp_1}}{1 - qA_{pp_1}} \|\mu_{n_2}\|_{pp_1}
$$

$$
\|\mu_{nz}\|_{pp_1}\leq \|(\mu_n-\mu)_z\|_{pp_1}+\|\mu_z\|_{pp_1}\leq 1+\|\mu_z\|_{pp_1},
$$

we thus have for *n* sufficiently large

with 
$$
\Phi_n
$$
 satisfying  $\Phi_n - \mu_n H_D \Phi_n = \mu_{n2}$ . Then from  $\Phi_n - \Phi_m - \mu_m H_D (\Phi_n - \Phi_m)$   
\n
$$
= (\mu_n - \mu_m)_z + (\mu_n - \mu_m) H_D \Phi_n
$$
, it follows that  
\n
$$
||\Phi_n - \Phi_m||_p \le \frac{1}{1 - qA_p} \{||(\mu_n - \mu_m)_z||_p + ||\mu_n - \mu_m||_{pp_1} ||H_D \Phi_n||_{pp_1}\}.
$$
\nSince  
\n
$$
||H_D \Phi_n||_{pp_1} \le A_{pp_1} ||\Phi_n||_{pp_1} \le \frac{A_{pp_1}}{1 - qA_{pp_1}} ||\mu_{n2}||_{pp_1}
$$
\nand for *n* large enough  
\n
$$
||\mu_{n2}||_{pp_1} \le ||(\mu_n - \mu)_{2}||_{pp_1} + [|\mu_{2}||_{pp_1} \le 1 + ||\mu_{2}||_{pp_1},
$$
\nwe thus have for *n* sufficiently large  
\n
$$
||\Phi_n - \Phi_m||_p \le \frac{1}{1 - qA_p} \left[ ||\mu_n - \mu_m||_p + \frac{A_{pp_1}(1 + ||\mu_{2}||_{pp_1})}{1 - qA_{pp_1}} ||\mu_n - \mu_m||_{pp_1} \right].
$$
\nThis proves convergence of  $\{\Phi_n\}$  in  $L_p(C)$ . Let  $\Phi$  be the limit of  $\{\Phi_n\}$  in  $L_p(C)$ . The

 $\begin{align*}\n p &\leq \frac{1}{1-q} \frac{1}{\mu_p} \left\{ \left\| (\mu_n - \mu) \right\|_{pp_1} \right\} \\
 &\leq \frac{1}{1-q} \frac{1}{\mu_p} \left\{ \left\| (\mu_n - \mu) \right\|_{pp_1} \right\} \\
 &\leq \frac{1}{1-q} \frac{1}{\mu_p} \left\{ \left\| \mu_n - \mu \right\|_{pp_1} \right\} \\
 &\leq \frac{1}{1-q} \frac{1}{\mu_p} \left\{ \left\| \mu_n - \mu \right\|_{pp_1} \right\} \\
 &\leq \frac{1}{1-q} \frac$ This proves convergence of  $\{\Phi_n\}$  in  $L_p(C)$ . Let  $\Phi$  be the limit of  $\{\Phi_n\}$  in  $L_p(C)$ . Then and for *n* large enough<br>  $\|\mu_{n^2}\|_{pp_1} \leq \|(\mu_n - \mu)_z\|_{pp_1} + \|\mu_z\|_{pp_1} \leq 1 + \|\mu_z\|_{pp_1},$ <br>
we thus have for *n* sufficiently large<br>  $\|\Phi_n - \Phi_m\|_p \leq \frac{1}{1 - qA_p} \left[ \|\mu_n - \mu_m\|_p + \frac{A_{pp_1}(1 + \|\mu_z\|_{pp_1})}{1 - qA_{pp_1}} \|\mu_n - \mu_m\|_{pp_$  $\|\Phi_n - \Phi_m\|_p \leq \frac{1}{1 - qA_p} \left[ \|\mu_n - \mu_m\|_p + \frac{A_{pp_1}(1 + \|\mu_x\|_{pp_1})}{1 - qA_{pp_1}} \|\mu_n - \mu_m\|_{pp_1} \right].$ <br>This proves convergence of  $\{\Phi_n\}$  in  $L_p(\mathbb{C})$ . Let  $\Phi$  be the limit of  $\{\Phi_n\}$  in  $L_p(\mathbb{C})$ . Then clearly,  $\Phi = 0$  in  $||H_D \Phi_n||_{pp_1} \leq A_{pp_1} ||\Phi_n||_{pp_1} \leq \frac{A_{pp_1}}{1-qA_{pp_1}} ||\mu_{n_2}||_p$ <br>and for *n* large enough<br> $||\mu_{n_2}||_{pp_1} \leq ||(\mu_n - \mu)_2||_{pp_1} + ||\mu_2||_{pp_1} \leq 1 + ||\mu_2||_p$ <br>we thus have for *n* sufficiently large<br> $||\Phi_n - \Phi_m||_p \leq \frac{1}{1-qA_p} \left[ ||$  $\|\mu_n - \mu_m\|_{pp_1}\Big\}.$ <br>  $\phi_n$  in  $L_p(\mathbb{C})$ . Then<br>  $\Phi = \mu_2$ . If now  $\zeta$ <br> *N* of (4) in  $W_p^{-1}(\mathbb{C})$ <br> *C*<sub>n</sub>, by taking  $\|\mu_{n_2}\|_{pp_1} \leq \|\mu_n - \mu_2\|_{pp_1} + \|\mu_2\|_{pp_1} \leq$ <br>
we thus have for *n* sufficiently large<br>  $\|\Phi_n - \Phi_m\|_p \leq \frac{1}{1 - qA_p} \left[\|\mu_n - \mu_m\|\right]$ <br>
This proves convergence of  $\{\Phi_n\}$  in  $L_p(\mathbb{C})$ . Let<br>
clearly,  $\Phi = 0$  in  $\mathbb{C}$  $\begin{aligned} \text{L}_p(\mathbb{C}). \ \text{L}^{\alpha}(\mathbb{C}), \ \text{an} \ \text{is a c}, \ \text{Reca} \ \text{to} \quad \mathbb{R}. \end{aligned}$ <br>  $\text{s}$ <br>  $\text{L}_p(\mathbb{C})$ ,  $p \geq 2$ ,  $\text{L}_p(\mathbb{C})$ ,  $p \geq 2$ ,  $\text{L}_p(\mathbb{C})$  $\begin{aligned} \n\lim_{m} \|p + \frac{A_{pp,1}}{1} \n\end{aligned}$ <br>
Let  $\widetilde{\Phi}$  be the defect of  $\Phi$  satisfies<br>
omplete hor<br>
use (10) is<br>
satisfy<br>
satisfy<br>  $\leq M_1$ . This proves defined as<br>
clearly,  $\Phi = 0$ <br>
is given by  $\Phi$ <br>
satisfying  $\zeta(0)$ <br>
the limit we s<br>
3. Reduction<br>  $\ln \Phi$  materials<br>  $\|\mu_1\|_0$ <br>
and<br>  $\|\mu_{1z}\|_r$ <br>
Define via (12), this fi<br>  $y = 0$  and  $\zeta(\infty)$ <br>
ee that it holds<br>
to canonical pro<br>
vs, let  $\mu_1, \mu_2 \in V$ <br>  $+ ||\mu_2||_0 \leq q <$ <br>  $+ ||\mu_1||_p + ||\mu_2||_p$  $\frac{1}{q\Lambda_p} \left[ \|\mu_n - \mu_m\|_p + \frac{\Lambda_{pp,1}(1 + \|\mu_z\|_{pp,1})}{1 - q\Lambda_{pp,1}} \right] \|\mu_p\|_p$ <br>  $\Phi_n$  in  $L_p(\mathbb{C})$ . Let  $\Phi$  be the limit of  $\{\Phi_n\}$ <br>  $\Phi \in C^*(\mathbb{C})$ , and  $\Phi$  satisfies  $\Phi - \mu \Pi_p \Phi$ <br>
inction is a complete homeomorphism<br>  $\psi =$ This proves convergence of  $(\Phi_n)$  in  $L_p(0)$ . Let  $\Phi$  be the limit of  $(\Phi_n)$  in  $L_p(0)$ . Then<br>clearly,  $\Phi = 0$  in  $C \setminus D$ ,  $\mathfrak{X}_D \Phi \in C^{\alpha}(0)$ , and  $\Phi$  satisfies  $\Phi - \mu I J_0 \Phi = \mu_2$ . If now  $\zeta$ <br>is given by  $\Phi$  via ( Solve the proton contribution of  $\langle \Phi_n | n \rangle$  Let  $\Psi$  be the first or  $\langle \Phi_n | n \rangle$  in  $L_p(v)$ . Let  $\Psi$  be  $\Phi$  in  $D_p(v)$ ,  $\Phi$  and  $\Phi$  satisfies  $\Phi - \mu I_0 \Phi = \mu$ . If n is given by  $\Phi$  via (12), this function is a complete

**3.** Reduction to canonical  $\ln \ln \theta$  what follows, let  $\mu_1$ ,  $\mu_2$ In what follows, let  $\mu_1, \mu_2 \in W_p^{-1}(\mathbb{C}), p \geq 2$ , satisfy  $\frac{1}{1}$ <br> *V*  $\frac{1}{1}$ <br> **V**  $\frac{1}{1}$ 3. Redu $\ln$  what<br>and Define<br>and  $\therefore$ <br>where • 

tion to canonical problems  
follows, let 
$$
\mu_1, \mu_2 \in W_p^1(\mathbb{C}), p \ge 2
$$
, satisfy  

$$
\|\mu_1\|_0 + \|\mu_2\|_0 \le q < 1
$$
 (13)

$$
\|\mu_{1z}\|_p + \|\mu_{1\overline{z}}\|_p + \|\mu_{2z}\|_p + \|\mu_{2\overline{z}}\|_p \leq M_1.
$$
 (14)

satisfying 
$$
\zeta(0) = 0
$$
 and  $\zeta(\infty) = \infty$ . Because (10) is valid for each  $\zeta_n$ , by taking  
the limit we see that it holds for  $\zeta$  too  
  
3. Reduction to canonical problems  
  
  
1n what follows, let  $\mu_1, \mu_2 \in W_p^1(\mathbb{C}), p \ge 2$ , satisfy  
  
 $||\mu_1||_0 + ||\mu_2||_0 \le q < 1$   
and  
  
 $||\mu_1||_p + ||\mu_1||_2 + ||\mu_2||_p + ||\mu_2||_p + ||\mu_2||_p \le M_1$ .  
  
Define  
  
 $\mu = \frac{1 + |\mu_1|^2 - |\mu_2|^2 - \sqrt{\Delta}}{2\overline{\mu}_1} = \frac{2\mu_1}{1 + |\mu_1|^2 - |\mu_2|^2 + \sqrt{\Delta}}$   
and  
  
 $a = \frac{1 - |\mu_1|^2 + |\mu_2|^2 - \sqrt{\Delta}}{2\overline{\mu}_2} = \frac{2\mu_2}{1 - |\mu_1|^2 + |\mu_2|^2 + \sqrt{\Delta}}$  (16)  
where  
 $\Delta = (1 - |\mu_1|^2 - |\mu_2|^2)^2 - 4 |\mu_1 \mu_2|^2 = [1 - (|\mu_1| + |\mu_2|)^2][1 - (|\mu_1| - |\mu_2|)^2].$ 

 $\mathcal{L}^{(1)}$ 

$$
a = \frac{1 - |\mu_1|^2 + |\mu_2|^2 - \sqrt{A}}{2\overline{\mu}_2} = \frac{2\mu_2}{1 - |\mu_1|^2 + |\mu_2|^2 + \sqrt{A}}
$$
(16)

 $\mathcal{F}$ 

$$
\Delta = (1 - |\mu_1|^2 - |\mu_2|^2)^2 - 4 |\mu_1 \mu_2|^2 = [1 - (|\mu_1| + |\mu_2|)^2][1 - (|\mu_1| - |\mu_2|)^2].
$$

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It is easy to see that  $\Delta > 0$ . In fact,  $\Delta \geq [1 - (|\mu_1| + |\mu_2|)^2]^2 > 0$ , from which we obtain

$$
|\mu| \leq |\mu_1| + |\mu_2|
$$
 and  $|a| \leq |\mu_1| + |\mu_2|$ . (17)

A Priori Estimates for E<br>  $\mu_1$  to see that  $\Delta > 0$ . In fact,  $\Delta \geq [1 - (|\mu_1| + |\mu_2|)^2]$ <br>  $|\mu| \leq |\mu_1| + |\mu_2|$  and  $|a| \leq |\mu_1| + |\mu_2|$ .<br>  $\mu_1$ , a simple computation shows that  $a = \mu_2/(|\mu_2|^2 -$ <br>
with the identities Moreover, a simple computation shows that  $a = \mu_2/(|\mu_2|^2 - |\mu_1|^2 + \mu_1\mu^{-1})$ . This together with the identities

A Priori Estimates for Elliptic  
\ny to see that 
$$
\Delta > 0
$$
. In fact,  $\Delta \geq [1 - (|\mu_1| + |\mu_2|)^2]^2 > 0$   
\n $|\mu| \leq |\mu_1| + |\mu_2|$  and  $|a| \leq |\mu_1| + |\mu_2|$ .  
\nr, a simple computation shows that  $a = \mu_2/(|\mu_2|^2 - |\mu_1|^2)$   
\nwith the identities  
\n $|\mu_1|^2 (1 - |\mu|^2) = \sqrt{\Delta} \bar{\mu}_1 \mu$ ,  
\n $(|\mu_2|^2 - |\mu_1|^2) (1 - |\mu|^2) + \sqrt{\Delta} = |1 - \bar{\mu}_1 \mu|^2 - |\mu_2 \mu|^2$   
\nan alternative form  
\n $a = \frac{\mu_2 (1 - |\mu|^2)}{|1 - \bar{\mu}_1 \mu|^2 - |\mu_2 \mu|^2}$ .  
\nna 4: Let  $w \in W_p^{-1}(0)$  fulfill the equation  
\n $w_{\overline{z}} - \mu_1 w_z - \mu_2 \overline{w_z} = F$   
\nbe a complete homeomorphism of  
\n $\zeta_{\overline{z}} = \mu \zeta_z$ 

leads to an alternative form

$$
|\mu_1|^2 (1 - |\mu_1|^2) = \sqrt{A} \bar{\mu}_1 \mu,
$$
  
\n
$$
(|\mu_2|^2 - |\mu_1|^2) (1 - |\mu_1|^2) + \sqrt{A} = |1 - \bar{\mu}_1 \mu|^2 - |\mu_2 \mu|^2
$$
  
\nIn alternative form  
\n
$$
a = \frac{1}{|1 - \bar{\mu}_1 \mu|^2 - |\mu_2 \mu|^2}.
$$
  
\n(a 4: Let  $w \in W_p^{-1}(0)$  fulfill the equation  
\n $w_{\bar{x}} - \mu_1 w_z - \mu_2 \overline{w_z} = F$  (19)  
\nbe a complete homeomorphism of  
\n $\zeta_{\bar{x}} = \mu \zeta_z$  (20)  
\n $\psi_{\bar{x}} = \mu \zeta_z$  (20)  
\n $\psi_{\bar{x}} = \mu \zeta_z$  (21)  
\n $\psi_{\bar{x}} = w - a \overline{w},$  (21)  
\n $\omega = w - a \overline{w},$  (22)  
\n $\psi_{\bar{x}} = A\omega + B\overline{\omega} + C$  (22)

Lemma 4: Let  $w \in W_n^1(\mathbb{C})$  fulfil the equation

$$
w_{\bar{z}} - \mu_1 w_z - \mu_2 \overline{w_z} = F \tag{19}
$$

*and let C be a complete horneomorphism* **of**

$$
\zeta_{\bar{z}} = \mu \zeta_z \tag{20}
$$

with  $\mu$  given by (15). Then by changing the variable from z to  $\zeta$  and the corresponding *unknown from w to co according to.* 

$$
\omega = w - a\overline{w},\tag{21}
$$

*with a given by (16), equation* (19) *can be transformed into the canonical form*

$$
\omega_{\tilde{\epsilon}} = A\omega + B\overline{\omega} + C \tag{22}
$$

*with*

$$
w_{\overline{z}} - \mu_1 w_z - \mu_2 \overline{w_z} = F
$$
\nand let  $\zeta$  be a complete homeomorphism of

\n
$$
\zeta_{\overline{z}} = \mu \zeta_z
$$
\nwith  $\mu$  given by (15). Then by changing the variable from  $z$  to  $\zeta$  and the corresponding unknown from  $w$  to  $\omega$  according to.

\n
$$
\omega = w - a\overline{w},
$$
\nwith a given by (16), equation (19) can be transformed into the canonical form

\n
$$
\omega_{\overline{z}} = A\omega + B\overline{\omega} + C
$$
\nwith

\n
$$
A = \frac{-\overline{a}a}{1-|a|^2}, \quad B = \frac{-a\overline{z}}{1-|a|^2} \quad \text{and} \quad C = \frac{(1-\overline{\mu_1}\mu)F + \mu\mu_2\overline{F}}{[|1-\overline{\mu_1}\mu|^2 - |\mu_2\mu|^2]\overline{\zeta_z}}.
$$
\n(The inverse transformation transforms (22) back into (19) accordingly.)

\nProof: In view of (20) we note that

\n
$$
w_z = \zeta_z w_z + \overline{\mu}\overline{\zeta_z}w_{\overline{z}}
$$
\nSubstituting these expressions into (19) and its conjugate equation, we obtain two

*(The inverse transformation transforms (22) back into (19) accordingly.)* 

Proof: In view of (20) we note that

$$
w_z = \zeta_z w_\zeta + \bar{\mu} \overline{\zeta_z} w_{\overline{\zeta}}
$$
 and  $w_{\overline{z}} = \mu \zeta_z w_{\zeta} + \overline{\zeta_z} w_{\overline{\zeta}}$ .

equations in four unknowns  $w_{\bar{i}}$ ,  $\overline{w}_{\bar{i}}$ ,  $\overline{w}_{\bar{i}}$  and  $w_{\bar{i}}$ . However, because of the choice of  $\mu$ in (15), it is not difficult to see that both  $\overline{w_t}$  and  $w_t$  can be eliminated. Thus, we arrive at the equation  $A = \frac{-\bar{a}a\bar{\zeta}}{1-|a|^2}, \quad B = \frac{-a\bar{\zeta}}{1-|a|^2} \quad and$ <br> *rse transformation transforms* (22) back<br>  $\omega_z = \zeta_z w_\zeta + \bar{\mu}\overline{\zeta_z}w_\zeta$  and  $w_z = \mu\zeta_z w_\zeta$ <br>
ing these expressions into (19) and it;<br>
in four unknowns  $w\bar{\zeta}, \overline{w$ and let  $\zeta$  be a complete homeomorph.<br>  $\zeta_{\overline{z}} = \mu \zeta_z$ <br>
with  $\mu$  given by (15). Then by chan<br>
whenown from w to  $\omega$  according to.<br>
with a given by (16), equation (19) c<br>  $\omega = w - a\overline{w}$ ,<br>
with<br>  $\omega_i = A\omega + B\overline{\omega} + C$ <br> Proof: In view of (20) we note that<br>  $w_z = \zeta_z w_i + \overline{\mu}\overline{\zeta_z}w_i$  and  $w_z = \mu\zeta_z w_i + \overline{\zeta_z}w_i$ .<br>
bstituting these expressions into (19) and its conjugate equation, we obtain<br>
uations in four unknowns  $w_i$ ,  $\overline{w}_i$ ,  $\overline{$ in four unknowns  $w_{\zeta}$ ,  $\overline{w_{\zeta}}$ ,  $\overline{w_{\zeta}}$  and  $w_{\zeta}$ . Here is not difficult to see that both  $\overline{w_{\zeta}}$  and<br>the equation<br> $|1 - \mu_1 \overline{\mu}|^2 - |\mu_2 \mu|^2 \overline{\zeta_2} w_{\zeta} + [(\overline{\mu} - \overline{\mu}_1) \mu_2 \mu]$ <br>=  $(1 - \overline{\mu}_1 \mu) F +$ 

$$
[|1 - \mu_1 \bar{\mu}|^2 - |\mu_2 \mu|^2] \bar{\zeta}_1 w_{\bar{\zeta}} + [(\bar{\mu} - \bar{\mu}_1) \mu_2 \mu - (1 - \mu \bar{\mu}_1) \mu_2] \bar{\zeta}_2 \overline{w_{\zeta}}
$$
  
=  $(1 - \bar{\mu}_1 \mu) F + \mu_2 \mu \bar{F}$ ,

from which formula (22) follows immediately from (21) and the definition of  $\boldsymbol{a}$ 

•'

$$
= (1 - \mu_1 \mu) r + \mu_2 \mu r,
$$
  
which formula (22) follows immediately from  
the that  

$$
|\overline{C_{\zeta_2}}| \le \frac{|F|}{1 - q^2} \text{ and } \frac{|\overline{a}\omega + \overline{\omega}|}{1 - |a|^2} \le \frac{|\omega|}{1 - q}.
$$

# 6 H. **BEOERR** and G. C. **HsIA0**

6 H. BEGERE and G. C. HISTAO<br>Moreover, the function  $\zeta_z$  can be estimated by using (10) if  $\mu_1$ ,  $\mu_2$  are required to yanish outside a finite domain *D.* 

For later calculations, estimates of the derivatives of  $\mu$  and  $\alpha$  are also needed. Because of the symmetric forms from the definitions (15) and (16), it is sufficient to consider only one of them, say  $\mu$ . However, the lemma below holds for a too. *D*.<br>ates of the density  $\mu$ . However<br>*e* assumptions<br>*defined by* (18<br>and  $|\mu_i|, |\mu_i|$ <br>ave  $||\mu||, \leq M$ oreover, the function  $\zeta$ <sub>z</sub> can be estimated by using (10) if  $\mu_1$ ,  $\mu_2$  are required to<br>mish outside a finite domain *D*.<br>For later calculations, estimates of the derivatives of  $\mu$  and *a* are also needed.<br>cea *M<sub>1</sub>,*  $\mu_2$  are requand *a* are also<br>
and *a* are also<br>
and (16), it is sumed in the set of the *M<sub>1</sub>,*  $\mu_2$  *vanish*<br> *M<sub>1</sub>, <i>A*<sub>1</sub>, *M<sub>1</sub>*, *M<sub>1*</sub>

Lemma 5: In addition to the assumptions  $(13)$  and  $(14)$  *let*  $\mu_1$ ,  $\mu_2$  vanish outside

to consider only one of them, say 
$$
\mu
$$
. However, the lemma below holds for a too.  
\nLemma 5: In addition to the assumptions (13) and (14) let  $\mu_1$ ,  $\mu_2$  vanish outside  
\nof a finite domain D. Then for  $\mu$  defined by (15), there hold the estimates  
\n $|\mu_1|, |\mu_{\bar{z}}| \le \frac{2M_1}{(1-q)^2}$  and  $|\mu_{\bar{z}}|, |\mu_{\bar{z}}| \le \frac{2M_1}{(1-q)^3} e^{M(p,q,D)M_1}$ . (23)

 $\mu_1, \mu_2 \in C^{1+\alpha}(\hat{D}), 0 < \alpha < 1$ , then similarly  $\|\mu_2\|_{\alpha}, \|\mu_1\|_{\alpha} \leq M(p, q, D, M_1)$ . Here, instead of<br>the p-norm as in (14),  $M_1$  is a bound for the sum of  $\alpha$ -norms of first order derivatives of the  $\mu_i$  's. In the same manner, it can be shown that  $||\zeta_z||_{\alpha} \leq M(p,q,\alpha, D, M_1)$ . From VEKUA [27: p. 38] we have  $||\mu_i|, ||\mu_i|| \ge \frac{1}{(1-q)^3} e^{\mu_i \mu_i}$ .<br>
From VEKUA [27: p. 38] we have  $||\mu_i||_{\alpha} \le M(p, q, D, M_1), \alpha = (p - \mu_2 \in C^{1+\alpha}(\hat{D}), 0 < \alpha < 1$ , then similarly  $||\mu_2||_{\alpha}, ||\mu^2||_{\alpha} \le M(p, q, D, M_1)$ .<br>  $p$ -norm as and  $|\mu_{\zeta}|, |\mu_{\zeta}| \le \frac{2M}{(1-\zeta)^{2}}$ <br>
have  $||\mu||_{a} \le M(p, q, D, M)$ <br>
an similarly  $||\mu_{z}||_{a}$ ,  $||\mu^{z}||_{a} \le \zeta$ <br>
bund for the sum of  $\alpha$ -norms<br>
be shown that  $||\zeta_{z}||_{a} \le M(p)$ <br>
have from (15)<br>  $[|\mu_{1}|^{2} - |\mu_{2}|^{2}]_{z} + \frac{$ 

Proof of the lemma: We have from (15)  
\n
$$
\mu_z = -\frac{\overline{\mu_{1\bar{z}}}}{\mu_1} \mu - \frac{\mu}{\sqrt{d}} \left[ |\mu_1|^2 - |\mu_2|^2 \right]_z + \frac{(|\mu_1|^2)_z}{\sqrt{d} \overline{\mu}_1} \text{ and } \mu_\zeta = (\overline{\zeta}_z \mu_z - \overline{\zeta}_z \mu_{\bar{z}})/J.
$$

As we indicated before, the Jacobian  $J = |\xi_1|^2 - |\xi_2|^2$  is positive. Now with the potential is positive. The integral of the integral of the *x*-s. In the same manner, it can be shown that  $||\xi_1||_2 \leq M(p, q, D, M_1)$ ,  $\alpha = (p$ results for  $|\mu_z|$  and  $|\mu_{\zeta}|$  follow immediately by direct computations. The estimates As we indicated before, the Jahrland help of the inequalities  $|\mu| \le$ <br>results for  $|\mu_z|$  and  $|\mu_{\zeta}|$  follow if for  $\mu_{\bar{z}}$  and  $\mu_{\bar{t}}$  follow similarly **Proof of the lemma:** We have from (15)<br>  $\mu_z = -\frac{\overline{\mu_1 z}}{\mu_1} \mu - \frac{\mu}{\sqrt{d}} [\mu_1]^2 - |\mu_2|^2]_z + \frac{(|\mu_1|^2)_z}{\sqrt{d} \overline{\mu_1}}$  and  $\mu_{\zeta} = (\overline{\zeta_z \mu_z} - \overline{\zeta_z \mu_z})$ <br>
s we indicated before, the Jacobian  $J = |\zeta_z|^2 - |\zeta_{\overline{z}}|^2$  i As we indicated before, the Jacobia<br>help of the inequalities  $|\mu| \leq |\mu_1|/(\n$ results for  $|\mu_z|$  and  $|\mu_{\zeta}|$  follow immet<br>for  $\mu_{\bar{z}}$  and  $\mu_{\bar{t}}$  follow similarly  $\blacksquare$ <br>We now derive the corresponding l<br>to the tra

We now derive the corresponding Hilbert boundary and side conditions according to the transformation  $(z, w) \rightarrow (\zeta, \omega)$  defined by (20) and (21). First we need some simple property concerning the boundary  $\partial D = \Gamma$  of *D*. results for  $|\mu_z|$  and  $|\mu_{\zeta}|$  follow immediately by direct computations. The estimates<br>for  $\mu_{\overline{z}}$  and  $\mu_{\overline{t}}$  follow similarly  $\blacksquare$ <br>We now derive the corresponding Hilbert boundary and side conditions accord

*Lemma 6: If*  $\Gamma \in C^{1+\alpha}$  then  $\tilde{\Gamma} := \zeta(\Gamma) \in C^{1+\alpha}$ .

*Proof:* **Let** *s* **be the arc length parameter and**  $L$  **be the total length of**  $\Gamma$ **. Then we have from (10) and (23)**  $\prime$ 

we now derive the corresponding Hilbert boundary and side conditions according  
to the transformation 
$$
(z, w) \rightarrow (\zeta, \omega)
$$
 defined by (20) and (21). First we need some  
simple property concerning the boundary  $\partial D = \Gamma$  of  $D$ .  
Lemma 6: If  $\Gamma \in C^{1+\alpha}$  then  $\tilde{\Gamma} := \zeta(\Gamma) \in C^{1+\alpha}$ .  
Proof: Let *s* be the arc length parameter and *L* be the total length of  $\Gamma$ . Then  
we have from (10) and (23)  $\frac{1}{\Gamma}$   

$$
\iint_{\tilde{\Gamma}} |d\zeta| = \int_{0}^{L} |d\zeta(z(s))| = \int_{0}^{L} |\zeta_z| |z'(s) + \mu z'(s)| ds \leq M(p, q, D, M_1) L,
$$
  
to that  $\tilde{\Gamma}$  is rectifiable with a total length, say  $\tilde{L}$ . Moreover if  $\tilde{s}$  denotes the arc  
length parameter of  $\tilde{\Gamma}$ , then  

$$
\frac{d\zeta}{d\tilde{s}} = (z'(s) \zeta_z + \overline{z'(s)} \zeta_{\tilde{s}}) \frac{ds}{d\tilde{s}} = \frac{\zeta_z(z'(s) + \mu \overline{z'(s)})}{|\zeta_z||z'(s) + \mu \overline{z'(s)}|}.
$$
  
In the case of  $\mu \in W_p^1(\mathbb{C})$ , all the functions involved here are in  $C^*([0, L])$  and the

length parameter of  $\tilde{\Gamma}$ , then

$$
\tilde{C} \text{ is rectifiable with a total length, say } \tilde{L}. \text{ Moreo}
$$
\n
$$
\frac{d\zeta}{d\tilde{s}} = \left(z'(s)\zeta_z + \overline{z'(s)}\zeta_{\tilde{z}}\right)\frac{ds}{d\tilde{s}} = \frac{\zeta_z(z'(s) + \mu\overline{z'(s)})}{|\zeta_z||z'(s) + \mu\overline{z'(s)}|}.
$$
\n
$$
\text{use of } \mu \in W_p \cdot (C), \text{ all the functions involved here a}
$$
\n
$$
\text{ator is bounded away from zero. In particular, we have}
$$
\n
$$
\frac{|\zeta_z(z(s)) - \zeta_z(z(s'))|}{|s - s'|^{\alpha}} = \frac{|\zeta_z(z(s)) - \zeta_z(z(s'))|}{|z(s) - z(s')|^{\alpha}} \cdot \frac{z(s) - z(s)}{s - s'}
$$
\n
$$
\text{ed, since } \zeta_z \in C^{\alpha}(C) \text{ and } z \in C^{1+\alpha}([0, L]), \text{ hence } \zeta_z(z(s')) = \zeta_z(z(s')) \text{ and } z \in C^{1+\alpha}([0, L]) \text{ and } z
$$

In the case of  $\mu \in W_p^1(\mathbb{C})$ , all the functions involved here are in  $C^1([0, L])$  and the denominator is bounded away from zero. In particular, we note that here that  $\tilde{\Gamma}$  is rectifiable<br>
that  $\tilde{\Gamma}$  is rectifiable<br>
gth parameter of  $\tilde{\Gamma}$ , then<br>  $\frac{d\zeta}{d\tilde{s}} = (z'(s) \zeta_z +$ <br>
the case of  $\mu \in W_p^{-1}(\mathbb{C})$ <br>
nominator is bounded<br>  $\frac{|\zeta_z(z(s)) - \zeta_z(z(s')|}{|s - s'|^{\alpha}}$ <br>
bounded, sinc

$$
\frac{|\zeta_{z}(z(s)) - \zeta_{z}(z(s'))|}{|s - s'|^{a}} = \frac{|\zeta_{z}(z(s)) - \zeta_{z}(z(s'))|}{|z(s) - z(s')|^{a}} \left| \frac{z(s) - z(s')}{s - s'} \right|^{a}
$$

 $\frac{|\zeta_z(z(s)) - \zeta_z(z(s'))|}{|s - s'|^{\alpha}} = \frac{|\zeta_z(z(s)) - \zeta_z(z(s'))|}{|z(s) - z(s')|^{\alpha}} \left| \frac{z(s) - z(s')}{s - s'} \right|^{\alpha}$ <br>is bounded, since  $\zeta_z \in C^s(\mathbb{C})$  and  $z \in C^{1+\alpha}([0, L])$ ; hence  $\zeta_z(z(\cdot)) \in C^s([0, L])$ . Similarly<br>one can show that  $\mu(z(\cdot)) \in C^{\alpha}([0, L$ 

Lemma 7: Let  $w$  fulfil the boundary and side conditions

$$
\text{Re}\left\{\mathrm{e}^{i\tau}w\right\} = \psi \quad \text{on} \quad \Gamma, \, \mathrm{e}^{i\tau}, \, \psi \in C^{\alpha}(\Gamma), \, n := \frac{-1}{2\pi} \int_{\Gamma} d\tau \geq 0
$$
\n
$$
\frac{1}{\Sigma} \int_{\Gamma} \text{Im}\left\{\mathrm{e}^{i\tau}w\right\} \sigma \, ds = \varkappa, \, \sigma \in C(\Gamma), \, 0 \leq \sigma, \, 0 < \Sigma := \int_{\Gamma} \sigma \, ds, \, \varkappa \in \mathbb{R}
$$
\n(24)

$$
w(z_k)=a_k, z_k\in D, a_k\in C\ (1\leq k\leq n).
$$

Then  $\omega$  fulfils analogous conditions of the form

$$
\begin{aligned}\n\text{Re}\left\{e^{i\mathbf{r}_{1}}\omega\right\} &= \frac{1-|a|^{2}}{\varrho} \quad \text{on} \quad \tilde{\Gamma} \\
\frac{1}{\Sigma_{1}} \int_{\tilde{\Gamma}} \text{Im}\left\{e^{i\mathbf{r}_{1}}\omega\right\} \sigma_{1} \, d\tilde{s} &= \varkappa_{1} \\
\omega(\zeta_{k}) &= a_{k} + a\bar{a}_{k} \ (1 \leq k \leq n).\n\end{aligned}
$$
\n
$$
(25)
$$

Here  $\tau_1 := \tau + \varphi$ 

$$
\sigma_1 := \frac{1}{\varrho} \cdot \frac{ds}{d\tilde{s}} \sigma, \quad \Sigma_1 := \int\limits_{\tilde{\Gamma}} \sigma_1 d\tilde{s}, \quad \varkappa_1 := \frac{\Sigma}{\Sigma_1} \varkappa - \frac{2}{\Sigma_1} \int\limits_{\Gamma} \text{Im} \{a e^{2it} e^{-2} \psi\} \sigma ds
$$

and  $\zeta_k = \zeta(z_k)$  while  $\varrho$  and  $e^{i\varphi}$  are  $C^*(\Gamma)$  functions defined by  $\varrho e^{i\varphi} = 1 + \bar{a} e^{-2i\varphi}$ Furthermore, all the z in (25) are replaced by  $z(\zeta)$ .

The proof follows by direct computations. Obviously the data functions in (25) fulfil with respect to  $\tilde{D} = \zeta(D)$ , the same conditions as those in (24) with respect to D. In particular, we see that because  $|a| < 1$  (cf. (17)), it follows that

$$
\int_{\Gamma} d \log \left(1 + \bar{a} e^{-2i\tau}\right) = 0 \text{ and hence } \int_{\Gamma} d\varphi = 0.
$$

Thus, the index in (25) is also equal to  $n$  as in (24). Thus, the index in (25) is also equal to *n* as in (24).<br>If  $\mu_1, \mu_2 \in C^{1+\alpha}(\hat{D})$ , then e<sup>i</sup>,  $\psi \in C^{1+\alpha}(\tilde{D})$  would imply e<sup>i</sup><sup>n</sup>,  $\frac{1-|a|^2}{a} \psi \in C^{1+\alpha}(\hat{D})$  as well.

From Lemma 4 and Lemma 7, the boundary value problem (19), (24) is transformed into the canonical problem (22), (25).

### 4. Representation formulas

A representation formula for  $C^1(\hat{D})$  functions in terms of Green's and Neumann's functions  $G^{I}(z, \zeta)$  and  $G^{II}(z, \zeta)$  of a  $C^{1+\alpha}$  domain D are given by HAACK and WEND-LAND [21: Formula 10.43/p. 271]. If  $\Phi$  denotes the conformal mapping of  $D$  onto the unit disc D, then Green's function is given by

$$
G^{\textrm{I}}(z,\zeta) = -\frac{1}{2\pi}\,\log\left|\frac{\varPhi(\zeta)-\varPhi(z)}{1-\overline{\varPhi(\zeta)}\,\varPhi(z)}\right|
$$

while the Neumann function is of the form (see HAACK and WENDLAND  $[21: § 4.7]$ 

$$
G_{\rm II}(z,\zeta)=\hat{G}_{\rm II}(z,\zeta)+V(z,\zeta)
$$

where

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$$
\hat{G}^{\Pi}(z,\zeta) = -\frac{1}{2\pi} \log \left| (\Phi(\zeta) - \Phi(z)) \left( 1 - \overline{\Phi(\zeta)} \ \Phi(z) \right) \right|
$$

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\nwhere  
\n
$$
\hat{G}^{\Pi}(z, \zeta) = -\frac{1}{2\pi} \log |(\Phi(\zeta) - \Phi(z)) (1 - \overline{\Phi(\zeta)} \Phi(z))|
$$
\nand V is harmonic in D given by  
\n
$$
V(z, \zeta) = \frac{1}{\pi \Sigma} \int_{\Gamma} \sigma(s) \log |[\Phi(\zeta(s)) - \Phi(z)] [\Phi(\zeta(s)) - \Phi(\zeta)]| ds
$$
\n
$$
- \frac{1}{\pi \Sigma^2} \int_{\Gamma} \int_{\Gamma} \sigma(s) \sigma(t) \log |\Phi(\zeta(s)) - \Phi(\zeta(t))| ds dt \quad (z, \zeta \in D).
$$
\nHere  $\sigma$  is a non-negative continuous function on  $\Gamma$  with  
\n
$$
\Sigma := \int_{\Gamma} \sigma(s) ds = 0.
$$
\nMoreover  
\n
$$
d_n V(z, \zeta) = \frac{1}{2\pi} |d\Phi(\zeta)| - \frac{\sigma(s)}{\Sigma} ds
$$
\nfor all  $z \in D, \zeta \in \Gamma$  and if  $\sigma(s) := \left| \frac{d\Phi(\zeta(s))}{ds} \right|$ , then V vanishes identically.  
\nLemma 8: Each  $w \in C^1(\hat{D})$  can be represented by

Here  $\sigma$  is a non-negative continuous function on  $\Gamma$  with

$$
\Sigma:=\int\limits_{\Gamma}\sigma(s)\,ds\,+\,0\,.
$$

$$
d_n V(z,\zeta) = \frac{1}{2\pi} |d\Phi(\zeta)| - \frac{\sigma(s)}{\Sigma} ds
$$

for all  $z \in D$ ,  $\zeta \in \Gamma$  and if  $\sigma(s) :=$  $\frac{d}{ds}$   $\frac{d}{ds}$   $\frac{d}{ds}$ *,* then *V* vanishes identically.

**Lemma 8:** Each  $w \in C^1(\hat{D})$  can be represented by

$$
d_n V(z, \zeta) = \frac{1}{2\pi} |d\Phi(\zeta)| - \frac{\partial \zeta}{\zeta} ds
$$
  
\n
$$
\in D, \zeta \in \Gamma \text{ and if } \sigma(s) := \left| \frac{d\Phi(\zeta(s))}{ds} \right|, \text{ then } V \text{ vanishes identically.}
$$
  
\n
$$
\text{ma 8:} \text{ } \text{Each } w \in C^1(\hat{D}) \text{ } \text{can be represented by}
$$
  
\n
$$
w(z) = -\theta(z) + i \int \{w(\zeta) [G_{\zeta}I + G_{\zeta}II](\zeta, z) + \overline{w(\zeta)} [G_{\zeta}I - G_{\zeta}II](\zeta, z)\} d\zeta d\bar{\zeta},
$$
  
\n
$$
\theta(z) := \int_{\Gamma} \text{Re } w(\zeta) [d_n G^I - idG^{II}](\zeta, z) + iC,
$$
  
\n
$$
C := \int_{\Gamma} \text{Im } w(\zeta) d_n G^{II}(\zeta, z) = -\frac{1}{\zeta} \int_{\Gamma} \text{Im } w(\zeta(s)) \sigma(s) ds.
$$
  
\nEXAMPLE 130: p. 22] this representation formula is reformulated for } C^1(\hat{D}) \text{ functions}  
\ntis for the homogeneous boundary and side conditions given by (24) with all belongs to

In WENDLAND [30: p. 22] this representation formula is reformulated for  $C^1(\hat{D})$  functions which satisfy the homogeneous boundary and side conditions given by  $(24)$  with  $e^{it}$  belonging to  $C^{1+\alpha}(\Gamma)$  and for  $\psi$ ,  $\alpha$  and  $a_k$  being equal to zero. It is not difficult to see that similar representatión formulas can be derived from Lemma 8 for non-homogeneous boundary and side conditions such as (24) with  $e^{i\tau}$ ,  $\psi \in C^{1+\alpha}(\Gamma)$ . to  $C^{1+\alpha}(\Gamma)$  and for  $\psi$ ,  $z$  and  $a_k$  being equal to zero. It is not differentiation formulas can be derived from Lemma 8 for non-homoglitions such as (24) with  $e^{it}$ ,  $\psi \in C^{1+\alpha}(\Gamma)$ .<br>Following WENDLAND [30], let  $P$ *(z)*  $\lim_{k \to \infty} w(k, 0) = -\frac{1}{2} \int_{0}^{2\pi} \lim_{k \to \infty} w(k, 0) = \frac{1}{2} \int_{0}^{2\pi} \lim_{k \to \infty$ 

Following WENDLAND [30], let  $P_k$  denote the polynomials in  $z$  of degree  $2n$  defined

*WI W - W2•1 W2 = k1 W(Zk) Pk(z),* 

and define  $\tilde{w}$  by

$$
\tilde{w}(z) = w_1(z) e^{-\hat{\Phi}(z)} \prod_{k=1}^n (z - z_k)^{-1}
$$

 $\bf{where}$ 

define 
$$
\tilde{w}
$$
 by  
\n
$$
\tilde{w}(z) = w_1(z) e^{-\tilde{\phi}(z)} \prod_{k=1}^n (z - z_k)^{-1}
$$
\n
$$
\tilde{\phi}(z) := i \int_{\Gamma} \tilde{\tau}[d_n G^{\mathbf{I}} - idG^{\mathbf{II}}](\zeta, z), \qquad e^{i\tilde{\tau}} := e^{i\tau} \prod_{k=1}^n \frac{z - z_k}{|z - z_k|}.
$$

• Then clearly  $\tilde{w} \in C^1(\hat{D})$  and satisfies the boundary condition

$$
\begin{aligned}\n\text{arly }\tilde{w} \in C^1(\hat{D}) \text{ and satisfies the boundary condition} \\
\text{Re }\tilde{w}|_{\Gamma} = \tilde{\psi} := [\psi - \text{Re}\left\{e^{i\tau}w_2\right\}] e^{-\text{Re}\tilde{\phi}(z)} \prod_{k=1}^n |z - z_k|^{-1} \quad \text{on } \Gamma,\n\end{aligned}
$$

A Prior Estimates for Elliptic Syst  
\nThen clearly 
$$
\tilde{w} \in C^1(\tilde{D})
$$
 and satisfies the boundary condition  
\n
$$
\text{Re } \tilde{w}|_{\Gamma} = \tilde{v} := [\psi - \text{Re} \{e^{i\tau}w_2\}] e^{-\text{Re}\tilde{\phi}(z)} \prod_{k=1}^n |z - z_k|^{-1} \quad \text{on } \Gamma,
$$
\nand the side condition  
\n
$$
-\frac{1}{\tilde{\Sigma}} \int_{\Gamma} \text{Im } \tilde{w}\tilde{\sigma} ds = \tilde{x} := \frac{\Sigma}{\tilde{\Sigma}} \times + \frac{1}{\tilde{\Sigma}} \int_{\Gamma} \text{Im} \{e^{i\tau}w_2\} \sigma ds
$$
\nwith  
\n
$$
\tilde{\sigma}(z) := \sigma(z) e^{\text{Re}\tilde{\phi}(z)} \prod_{k=1}^n |z - z_k|, \qquad z \in \Gamma, \qquad \text{and } \tilde{\Sigma} := \int_{\Gamma'} \tilde{\sigma} ds.
$$
\nNow let  $\tilde{G}^{\Pi}$  denote the Neumann function corresponding to  $\tilde{\sigma}$ . Then a

with

 

$$
\tilde{\sigma}(z) := \sigma(z) e^{\operatorname{Re} \tilde{\sigma}(z)} \prod_{k=1}^n |z - z_k|, \quad z \in \Gamma, \quad \text{and} \quad \tilde{\Sigma} := \int_{\Gamma} \tilde{\sigma} ds.
$$

Now let  $\tilde{G}^{\Pi}$  denote the Neumann function corresponding to  $\tilde{\sigma}$ . Then an application

Then clearly 
$$
w \in C^1(D)
$$
 and satisfies the boundary condition  
\n
$$
\text{Re } \tilde{w}|_{\Gamma} = \tilde{\psi} := [\psi - \text{Re } \{e^{i\tau}w_2\}] e^{-\text{Re}\tilde{\phi}(z)} \prod_{k=1}^n |z - z_k|^{-1} \text{ on } \Gamma,
$$
\nand the side condition,  
\n
$$
-\frac{1}{\tilde{\Sigma}} \int_{\Gamma} \text{Im } \tilde{w}\tilde{\sigma} ds = \tilde{x} := \frac{\tilde{\Sigma}}{\tilde{\Sigma}} \times + \frac{1}{\tilde{\Sigma}} \int_{\Gamma} \text{Im } \{e^{i\tau}w_2\} \sigma ds
$$
\nwith  
\n
$$
\tilde{\sigma}(z) := \sigma(z) e^{\text{Re}\tilde{\phi}(z)} \prod_{k=1}^n |z - z_k|, \qquad z \in \Gamma, \qquad \text{and } \tilde{\Sigma} := \int_{\Gamma} \tilde{\sigma} ds.
$$
\nNow let  $\tilde{G}^{\Pi}$  denote the Neumann function corresponding to  $\tilde{\sigma}$ . Then an application  
\nof Lemma 8 to  $\tilde{w}$  yields the following lemma.  
\nLemma 9: Each  $w \in C^1(\tilde{D})$  which fulfils (24) with  $a_k = 0$  can be represented by  
\n
$$
w(z) = \sum_{k=1}^n w_2(z_k) \overline{P_k(z)} - \theta_n(z) e^{\tilde{\phi}(z)} \prod_{k=1}^n (z - z_k)
$$
\n
$$
+ i \int_{\tilde{D}} \left\{ e^{\tilde{\phi}(z) - \tilde{\phi}(z)} \prod_{k=1}^n \frac{z - z_k}{\zeta - z_k} \left[ w_{\tilde{L}}(\zeta) - \sum_{k=1}^n w_{\tilde{L}}(z_k) \overline{P_k(\zeta)} \right] \right\}
$$
\n
$$
\times [G_{\zeta}^{\dagger} + \tilde{G}_{\zeta}^{\dagger}1] (\zeta, z) + e^{\overline{\tilde{\phi}(z)} - \overline{\tilde{\phi}(z)}} \prod_{k=1}^n \frac{z - z_k}{\overline{\zeta - \tilde{z}_k}} \left[ \overline{w_{\tilde{L}}(\zeta)} - \sum_{k=1}^n \overline{w_{\tilde{L}}(z_k)} P
$$

*for*  $z \in D$ , where

$$
\times [G_{\zeta}I + \tilde{G}_{\zeta}II](\zeta, z) + e^{\overline{\phi}(z) - \overline{\phi}(\zeta)} \prod_{k=1}^{n} \frac{z - z_{k}}{\overline{\zeta} - \overline{z}_{k}} \left[ \overline{w_{\zeta}(\zeta)} - \sum_{k=1}^{n} \overline{w_{\overline{z}}(z_{k})} P_{k}(\zeta) \right]
$$
  
\n
$$
\times [G_{\zeta}^{L}I - \tilde{G}_{\zeta}II](\zeta, z) \, d\zeta \, d\overline{\zeta},
$$
  
\nfor  $z \in \hat{D}$ , where  
\n
$$
\theta_{n}(z) = \theta(z) - \int_{\Gamma} \text{Re} \left\{ e^{i\tau} \sum_{k=1}^{n} w_{\overline{z}}(z_{k}) \overline{P_{k}(\zeta)} \right\} e^{-\tilde{\phi}(\zeta)} \prod_{k=1}^{n} |\zeta - z_{k}|^{-1}
$$
  
\n
$$
\times [d_{n}G^{1} - id\tilde{G}^{II}](\zeta, z) + \frac{i}{\overline{z}} \int_{\Gamma}^{z} \text{Im} \left\{ e^{i\tau} \sum_{k=1}^{n} w_{\overline{z}}(z_{k}) \overline{P_{k}(\zeta)} \right\} \sigma \, ds,
$$
  
\nwith  
\n
$$
\theta(z) = \int_{\Gamma} \psi(\zeta) e^{-\tilde{\phi}(\zeta)} \prod_{k=1}^{n} |\zeta - z_{k}|^{-1} [d_{n}G^{I} - id\tilde{G}^{II}](\zeta, z) + i \frac{\Sigma}{\overline{\Sigma}} \times.
$$
  
\nThis representation formula is also valid for  $w \in W_{p}^{1}(\hat{D})$ . In this case the regularity assumptions on all and  $m$  can be we be

$$
\times [d_n G^1 - id \tilde{G}^{II}] (\zeta, z) + \frac{i}{\tilde{\Sigma}} \int \limits_{\Gamma} \text{Im} \left\{ e^{it} \sum_{k=1}^n w_{\tilde{z}}(z_k) \overline{P_k(\zeta)} \right\}
$$

$$
\theta(z) = \int \limits_{\Gamma} \psi(\zeta) e^{-\tilde{\Phi}(\zeta)} \prod_{k=1}^n |\zeta - z_k|^{-1} [d_n G^I - id \tilde{G}^{II}] (\zeta, z) + i \frac{\tilde{\Sigma}}{\tilde{\Sigma}} \times
$$

tions on e<sup>it</sup> and  $\psi$  can be weakened to e<sup>it</sup>,  $\psi \in C^{\alpha}(\Gamma)$  as in (24).

The representation formula in the lemma can be made more explicit in terms of the conformal mapping function  $\Phi$ . In particular, we note that

$$
\times [a_n G^T - i a G^H](\zeta, z) + \frac{1}{\sum \limits_{i=1}^n \int dm \left\{e^x \sum_{k=1}^n w_{\bar{z}}(z_k) P_k(\zeta)\right\}
$$
  
\n*th*  
\n
$$
\cdot \qquad \theta(z) = \int_{\Gamma} \psi(\zeta) e^{-\tilde{\phi}(\zeta)} \prod_{k=1}^n |\zeta - z_k|^{-1} [d_n G^I - i d \tilde{G}^H](\zeta, z) + i \frac{\sum \limits_{i=1}^n \sum_{k=1}^n \zeta_k}{\sum \limits_{i=1}^n \zeta_k}
$$
  
\nThis representation formula is also valid for  $w \in W_p^1(\hat{D})$ . In this case the reg  
\nThis representation formula in the lemma can be made more explicit  
\nthe conformal mapping function  $\Phi$ . In particular, we note that  
\n
$$
[G_{\zeta}I + \tilde{G}_{\zeta}H](\zeta, z) = -\frac{1}{2\pi} \frac{\Phi'(\zeta)}{\Phi(\zeta) - \Phi(z)} - \frac{1}{2\pi} \lambda(\zeta),
$$
\n
$$
[G_{\zeta}I - \tilde{G}_{\zeta}H](\zeta, z) = -\frac{1}{2\pi} \frac{\Phi(z) \overline{\Phi'(\zeta)}}{1 - \Phi(z) \overline{\Phi(\zeta)}} + \frac{1}{2\pi} \overline{\lambda(\zeta)}
$$

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\nand  
\n
$$
[d_nG^1 - id\tilde{G}^{11}](\zeta, z) = -\frac{1}{2\pi i} \left[ \frac{\Phi'(\zeta) d\zeta}{\Phi(\zeta) - \Phi(z)} - \frac{\Phi(z) \overline{\Phi'(\zeta)} d\zeta}{1 - \overline{\Phi(\zeta)} \Phi(z)} \right]
$$
\n
$$
- \frac{1}{2\pi i} \left[ \lambda(\zeta) d\zeta + \overline{\lambda(\zeta)} d\bar{\zeta} \right]
$$
\nwhere  
\n
$$
\lambda(z) = \frac{1}{\tilde{\Sigma}} \int_{\Gamma} \tilde{\sigma}(\zeta(s)) \frac{\Phi'(z) ds}{\Phi(\zeta(s) - \Phi(z))}
$$
\nThese relations will be utilized for obtaining estimates for  $w$  as well a  
\nlater use, we also need a similar representation formula for  $w_z$ . This can  
\nby differentiating the representation formulas for  $w$ . Explicitly we have  
\n
$$
w_z(z) = \left[ \tilde{\Phi}'(z) + \sum_{k=1}^{n} (z - z_k)^{-1} \right] \left[ w(z) - \sum_{k=1}^{n} w_{\bar{z}}(z_k) \overline{P_k(z)} \right]
$$

where

$$
\lambda(z) = \frac{1}{\tilde{\Sigma}} \int\limits_{\Gamma} \tilde{\sigma}(\zeta(s)) \, \frac{\boldsymbol{\phi}'(z) \, ds}{\boldsymbol{\phi}(\zeta(s)) - \boldsymbol{\phi}(z)}
$$

These relations will be utilized for obtaining estimates for *w* as well as for *w.* For

Here we also need a similar representations of *w* as well as for *w*, For  
later use, we also need a similar representation formula for *w*. Explicitly we have  
by differentiating the representation formulas for *w*. Explicitly we have  

$$
w_z(z) = \left[\tilde{\Phi}'(z) + \sum_{k=1}^n (z - z_k)^{-1}\right] \left[w(z) - \sum_{k=1}^n w_{\bar{z}}(z_k) \overline{P_k(z)}\right] + \frac{1}{\pi i} \int \left[\psi(\zeta) - \text{Re}\left\{e^{-i\frac{r}{\zeta}}\sum_{k=1}^n w_{\bar{z}}(z_k) \overline{P_k(z)}\right\}\right] e^{\tilde{\phi}(z) - \tilde{\phi}(z)}
$$

$$
+ \frac{1}{2\pi i} \int \left[\psi(\zeta) - \text{Re}\left\{e^{-i\frac{r}{\zeta}}\sum_{k=1}^n w_{\bar{z}}(z_k) \overline{P_k(\zeta)}\right\}\right] e^{\tilde{\phi}(z) - \tilde{\phi}(\zeta)}
$$

$$
+ \frac{1}{2\pi i} \int \left[e^{\tilde{\phi}(z) - \tilde{\phi}(z)} \prod_{k=1}^n \frac{z - z_k}{\zeta - z_k} \left[w_{\bar{z}}(\zeta) - \sum_{k=1}^n w_{\bar{z}}(z_k) \overline{P_k(\zeta)}\right]\right]
$$

$$
\times \frac{\phi'(z) \phi'(\zeta)}{(\phi(\zeta) - \phi(z))^2} d\zeta d\bar{\zeta}
$$

$$
+ \frac{1}{2\pi i} \int e^{\overline{\phi}(z) - \overline{\phi}(z)} \prod_{k=1}^n \frac{z - z_k}{\zeta - \overline{z}_k} \left[\overline{w_{\bar{z}}(\zeta)} - \sum_{k=1}^n \overline{w_{\bar{z}}(z_k)} P_k(\zeta)\right]
$$

$$
\times \frac{\phi'(z) \overline{\phi'(\zeta)}}{(\overline{1 - \phi(\zeta)} \phi(z))^2} d\zeta d\bar{\zeta}.
$$
With the representation formulas for *w* and *w*, may be derived.  
We begin with the following crucial estimates in connection with *w*.  
Lemma 10:  $\tilde{I}f z_k$  ( $1 \leq k \leq n$ ) are distinct points in

With the representation formulas available bounds for  $w$  and  $w<sub>z</sub>$  may be derived. We begin with the following crucial estimates in connection with  $w_z$ .  $I(1 - \Phi(\zeta) \Phi(z))^2$ <br> *If*  $z_k$  (1  $\leq k \leq n$ ) are distinct points in the unit disc **D**, then *•* 

Lemma 10:

$$
\times \frac{1}{(1 - \overline{\phi(\zeta)} \Phi(z))^2} d\zeta d\zeta.
$$
  
the representation formulas available bounds for  $w$  and  $w_z$  me  
in with the following crucial estimates in connection with  $w_z$ .  
na 10: If  $z_k$  ( $1 \le k \le n$ ) are distinct points in the unit disc **D**,  

$$
\frac{1}{\pi} \int_{\mathbf{D}} \prod_{k=1}^{n} \frac{z - z_k}{\zeta - z_k} \frac{d\zeta d\eta}{(\zeta - z)^2} \quad \text{and} \quad \frac{1}{\pi} \int_{\mathbf{E}} \prod_{k=1}^{n} \frac{z - z_k}{\overline{\zeta} - z_k} \frac{d\zeta d\eta}{(1 - \overline{\zeta}z)^2}
$$
  
ded in **D**.

*•* 

$$
2\pi i \int_{D} \hat{k-1} \zeta - \bar{z}_k \left[ \frac{\sqrt{2\pi i} \zeta}{\sqrt{2\pi i}} \right] \frac{\partial'(z)}{\partial(z)} \frac{\partial'(z)}{\partial(z)}
$$
\nWith the representation formulas available bounds for  $w$  and  $w_z$  ma  
\ne begin with the following crucial estimates in connection with  $w_z$ .  
\nLemma 10:  $\int_{\tilde{z}} z_k (1 \le k \le n)$  are distinct points in the unit disc **D**, i  
\n
$$
\frac{1}{\pi} \int_{k=1}^{n} \prod_{\zeta=-z_k}^{z^2 - z_k} \frac{d\xi d\eta}{(\zeta-z)^2} \quad \text{and} \quad \frac{1}{\pi} \int_{k=1}^{n} \prod_{\zeta=-\bar{z}_k}^{z^2 - z_k} \frac{d\xi d\eta}{(1-\bar{\zeta}z)^2}
$$
\ne bounded in **D**.  
\nProof: Let  $g(z) = \prod_{k=1}^{n} (z - z_k)^{-1}$ . We note that  
\n
$$
\frac{1}{2\pi i} \int_{k=1}^{z} \bar{\zeta} g(\zeta) \frac{d\zeta}{\zeta-z} = \frac{1}{2\pi i} \int_{\zeta(\zeta)-z}^{z} \frac{d\zeta'}{\zeta(\zeta-z)} = 0
$$
\n
$$
\frac{1}{|z|=1} \int_{|z|=1}^{z} \bar{\zeta} g(\zeta) \frac{d\zeta}{\zeta-z} = \frac{1}{2\pi i} \int_{|z|=1}^{z^2} \frac{d\zeta'}{\zeta(\zeta-z)} = 0
$$

for  $|z|$  < 1. Hence an application of Green's formula

$$
\frac{1}{\pi}\int\limits_D w_{\zeta}\,d\xi\,d\eta=\frac{1}{2\pi i}\int\limits_{\delta D} w\,d\zeta
$$

*fwdd=fwdC* 2 to  $\bar{\zeta}g(\zeta)/(\zeta -z)$  in a domain *D* obtained from **D** by removing circles with sufficiently  $\frac{1}{\pi} \int w_{\bar{\zeta}} d\xi d\eta = \frac{1}{2\pi i} \int w d\zeta$ <br>to  $\bar{\zeta} g(\zeta)/(\zeta - z)$  in a domain *D* obtained from **D** by removing circl<br>small radii around the points *z* and *z*<sub>k</sub> (1  $\leq k \leq n$ ) from **D** yields

1. Hence an approximation of Green's formula:  
\n
$$
\frac{1}{\pi} \int_{D} w_{\bar{\zeta}} d\xi d\eta = \frac{1}{2\pi i} \int_{\delta D} w d\zeta
$$
\n
$$
(\zeta - z) \text{ in a domain } D \text{ obtained from } D \text{ by removing}
$$
\n
$$
\text{div} \text{ around the points } z \text{ and } z_k \text{ (1 } \leq k \leq n) \text{ from } I
$$
\n
$$
\frac{1}{\pi} \int_{D} g(\zeta) \frac{d\xi d\eta}{\zeta - z} = \left[ -\bar{z} + \sum_{k=1}^{n} \bar{z}_k \prod_{l+k} \frac{z - z_l}{z_k - z_l} \right] g(z).
$$
\n
$$
z \text{ both sides by } g(z) \text{ and then differentiating with}
$$

Dividing both sides by  $g(z)$  and then differentiating with respect to  $z$ , we obtain

$$
\frac{1}{\pi} \int_{\mathbf{p}} \left\{ \prod_{k=1}^{n} \frac{z-z_k}{\zeta-z_k} \frac{1}{(\zeta-z)^2} + \sum_{k=1}^{n} \prod_{l+k} \frac{z-z_l}{\zeta-z_l} \frac{1}{(\zeta-z_k)(\zeta-z)} \right\} d\xi d\eta
$$
\n
$$
= \sum_{k=1}^{n} \overline{z_k} \sum_{l+k} \prod_{r+k,l} \frac{z-z_r}{z_k-z_r} \frac{1}{z_k-z_l}.
$$
\nand integral again can be integrated explicitly by the preceding  
\n
$$
\frac{1}{\pi} \int_{\mathbf{p}} \sum_{k=1}^{n} \prod_{l+k} \frac{z-z_l}{\zeta-z_l} \frac{1}{(\zeta-z_k)(\zeta-z)} d\xi d\eta
$$

The second integral again can be integrated explicitly by the preceding formula.. Indeed,

$$
\frac{1}{\pi} \int \left\{ \prod_{k=1}^{n} \frac{z-z_k}{\zeta-z_k} \frac{1}{(\zeta-z)^2} + \sum_{k=1}^{n} \prod_{l+k} \frac{z-z_l}{\zeta-z_l} \frac{1}{(\zeta-z_k)(\zeta-z)} \right\} d\xi d\eta
$$
\n
$$
= \sum_{k=1}^{n} \overline{z_k} \sum_{l+k} \prod_{r+k,l} \frac{z-z_r}{z_k-z_r} \frac{1}{z_k-z_l}.
$$
\nThe second integral again can be integrated explicitly by the preceding  
\nIndeed,  
\n
$$
\frac{1}{\pi} \int \sum_{k=1}^{n} \prod_{l+k} \frac{z-z_l}{\zeta-z_l} \frac{1}{(\zeta-z_k)(\zeta-z)} d\xi d\eta
$$
\n
$$
= \sum_{k=1}^{n} \frac{-\overline{z}}{z-z_k} + \sum_{k=1}^{n} \frac{1}{z-z_k} \sum_{r=1}^{n} \overline{z_r} \prod_{l+r} \frac{(z-z_l)}{(z_r-z_l)}.
$$
\nConsequently, after some simplifications, we arrive at the relation  
\n
$$
\frac{1}{\pi} \int \prod_{k=1}^{n} \frac{z-z_k}{\zeta-z_k} \frac{d\xi d\eta}{(\zeta-z)^2} = \sum_{k=1}^{n} \frac{\overline{z-z_k}}{z-z_k} + \sum_{k=1}^{n} \frac{\overline{z_k}}{z-z_k} \left[1 - \prod_{l+k} \frac{z-l}{z_k} \right]
$$
\nHere the right-hand side is bounded in **D**.  
\nSimilarly, we apply Green's formula  
\n
$$
\frac{1}{\pi} \int w_{\zeta} d\xi d\eta = -\frac{1}{2\pi i} \int w d\overline{\zeta}
$$
\nto  $\zeta \frac{\overline{z}}{\overline{z}} \int (1 - \overline{\zeta} z) \frac{1}{\overline{z}} \int w_{\zeta} d\xi d\eta = -\frac{1}{2\pi i} \int w d\overline{\zeta}$ 

$$
= \sum_{k=1}^{\infty} \frac{1}{z - z_k} + \sum_{k=1}^{\infty} \frac{1}{z - z_k} \sum_{\nu=1}^{z} \frac{z \cdot \prod_{i+\nu} (z_i - z_i)}{(z - z_i)}.
$$
  
\nently, after some simplifications, we arrive at the relation  
\n
$$
\frac{1}{\pi} \int_{0}^{\pi} \prod_{k=1}^{n} \frac{z - z_k}{\zeta - z_k} \frac{d\xi \, d\eta}{(\zeta - z)^2} = \sum_{k=1}^{\pi} \frac{\overline{z} - \overline{z}_k}{z - z_k} + \sum_{k=1}^{\pi} \frac{\overline{z}_k}{z - z_k} \left[1 - \prod_{i+k} \frac{z - z_i}{z_i - z_i}\right]
$$

Similarly, we apply Green's formula Consequence Conseq

Here the right-hand side is bounded in **D**.  
\nSimilarly, we apply Green's formula  
\n
$$
\frac{1}{\pi} \int_{D} w_{\zeta} d\xi d\eta = -\frac{1}{2\pi i} \int_{\partial D} w d\overline{\zeta}
$$
\nto  $\zeta \overline{g(\zeta)}/(1 - \overline{\zeta}z)$  in a proper subdomain *D* of **D**, and obtain

A Priori Estimates for Elliptic's  
\nfor 
$$
|z| < 1
$$
. Hence an application of Green's formula  
\n
$$
\frac{1}{\pi} \int_{B} w_{\xi} d\xi d\eta = \frac{1}{2\pi i} \int_{\partial D} w d\zeta
$$
\nto  $\bar{\zeta}g(\zeta)/(\zeta - z)$  in a domain *D* obtained from *D* by removing circles  
\nsmall radii around the points *z* and  $z_{k}$  ( $1 \leq k \leq n$ ) from *D* yields  
\n
$$
\frac{1}{\pi} \int_{D} \tilde{g}(\zeta) \frac{d\xi}{\zeta - z} d\zeta = \left[ -\bar{z} + \sum_{k=1}^{n} \overline{z}_{k} \prod_{l+k} \frac{z - z_{l}}{z_{l} - z_{l}} \right] g(z).
$$
\nDividing both sides by  $g(z)$  and then differentiating with respect to  
\n
$$
\frac{1}{\pi} \int_{D} \left\{ \prod_{l=1}^{n} \frac{z - z_{l}}{\zeta - z_{l}} \frac{1}{(\zeta - z_{l})^{2}} + \sum_{k=1}^{n} \prod_{l+k} \frac{z - z_{l}}{\zeta - z_{l}} \frac{1}{(\zeta - z_{k})(\zeta - z)} \right\}
$$
\n
$$
= \sum_{k=1}^{n} \overline{z}_{k} \sum_{l+k} \frac{z - z_{l}}{\zeta - z_{l}} \frac{1}{(\zeta - z_{l})^{2}} + \sum_{k=1}^{n} \prod_{l+k} \frac{z - z_{l}}{\zeta - z_{l}} \frac{1}{(\zeta - z_{l})(\zeta - z_{l})}
$$
\n
$$
\frac{1}{\pi} \int_{D} \int_{E=1}^{D} \sum_{l=k}^{n} \frac{z - z_{l}}{\zeta - z_{l}} \frac{1}{\zeta - z_{l}} \frac{1}{\zeta - z_{l}} \frac{1}{\zeta - z_{l}} \frac{1}{\zeta - z_{l}}
$$
\nConsequently, after some simplifications, we arrive at the relation  
\n
$$
\frac{1}{\pi} \int_{D} \int_{k=1}^{n} \frac{z - z_{k}}{\zeta - z_{k}} \frac{d\xi}{\zeta - z_{l}} \frac{1}{z - z_{k}} \frac{1}{\
$$

$$
k = 1 \le k \qquad k = 1 \le k \le k \qquad \angle (\vee k(1 - \angle k \angle j)
$$
\ng use of

\n
$$
\frac{1}{2\pi i} \int_{|z|=1} \zeta \overline{g(\zeta)} \frac{d\overline{\zeta}}{1 - \overline{\zeta}z} = -\prod_{k=1}^{n} \frac{-1}{z_k} - \sum_{k=1}^{n} \frac{1}{\overline{z}_k(1 - \overline{z}_k z)} \prod_{l \ne k} \frac{1}{\overline{z}_k - \overline{z}_l}.
$$

Multiplying by *z* and differentiating with respect to *z* then lead to

H. BEGEHR and G. C. HSTAO  
\nng by z and differentiating with respect t  
\n
$$
\frac{1}{\pi} \int \prod_{k=1}^{n} \frac{1}{\xi - \overline{z}_k} \frac{d\xi \, d\eta}{(1 - \overline{\zeta}z)^2}
$$
\n
$$
= \prod_{k=1}^{n} \frac{-1}{\overline{z}_k} + \sum_{k=1}^{n} \frac{1 + |z_k|^2}{(1 + \overline{z}_k z) \overline{z}_k} \prod_{\ell \neq k} \frac{1}{\overline{z}_k - \overline{z}_\ell}
$$

Because this as well as  $\prod_{k=1}^{n} (z - z_k)$  is bounded in **D**, the lemma is proved

As a consequence of Lemma 10, we now have the following result.

*inequality*

$$
\begin{aligned}\n\mathbf{B} &= \prod_{k=1}^{n} \frac{-1}{\bar{z}_k} + \sum_{k=1}^{n} \frac{1+|z_k|^2}{(1+\bar{z}_k z)\bar{z}_k} \prod_{l+k} \frac{1}{\bar{z}_k - \bar{z}_l} \\
\text{Because this as well as } \prod_{k=1}^{n} (z - z_k) \text{ is bounded in } \mathbf{D}, \text{ the lemma is proved} \\
\text{As a consequence of Lemma 10, we now have the following result.} \\
\text{Lemma 11: } Let \ z_k \ (1 \leq k \leq n) \ be \ distinct \ points \ in \ the \ unit \ disc \ \mathbf{D}. \text{ Then the inequality} \\
&\quad \left| \frac{1}{\pi} \int_{\mathbf{D}} \left[ \frac{f(\zeta)}{(\zeta - z)^2} \prod_{k=1}^{n} \frac{z - z_k}{\zeta - z_k} + \frac{\overline{f(\zeta)}}{(1 - \overline{\zeta}z)^2} \prod_{k=1}^{n} \frac{z - z_k}{\overline{\zeta} - \overline{z}_k} \right] d\zeta \ d\eta \right| \leq M \||f||_a \\
\text{holds for every } f \in C^{\alpha}(\mathbf{D}) \text{ with } 0 < \alpha < 1, \text{ where } \mathbf{D} := \mathbf{D} \cup \partial \mathbf{D} \text{ and } M \text{ is a non-negative constant depending on } z_k \text{ and } \alpha.\n\end{aligned}
$$
\nProof: Let  $g(\zeta) = \prod_{k=1}^{n} \frac{z - z_k}{\zeta - z_k}$  and for  $f \in C^{\alpha}(\mathbf{D})$ , we write

*holds for every*  $f \in C^{\alpha}(\mathbf{D})$  *with*  $0 < \alpha < 1$ , where  $\mathbf{D} := \mathbf{D} \cup \partial \mathbf{D}$  and M is a non-negative

$$
\begin{aligned}\n\boxed{\pi} \int \left[ \frac{\zeta}{(\zeta - z)^2} \prod_{k=1}^{l} \frac{\zeta - z_k}{\zeta - z_k} + \frac{\zeta - \zeta}{(1 - \zeta z)^2} \prod_{k=1}^{l} \frac{\zeta - z_k}{\zeta - z_k} \right] dz \, d\eta \end{aligned}
$$
\n
$$
\begin{aligned}\n\text{and, for every } f \in C^{\alpha}(\mathbf{D}) \text{ with } 0 < \alpha < 1, \text{ where } \mathbf{D} := \mathbf{D} \cup \partial \mathbf{D} \text{ and } M \text{ is}\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\text{Proof: Let } g(\zeta) = \prod_{k=1}^{n} \frac{z - z_k}{\zeta - z_k} \text{ and for } f \in C^{\alpha}(\mathbf{D}), \text{ we write} \\
\int f(\zeta) g(\zeta) \frac{d\xi \, d\eta}{(\zeta - z)^2} \\
= \int \left[ f(\zeta + z) - f(z) \right] g(\zeta + z) \frac{d\xi \, d\eta}{\zeta^2} + f(z) \int g(\zeta) \frac{d\xi \, d\eta}{(\zeta - z)^2} \\
\text{and we see that} \\
\int \left| f(\zeta + z) - f(z) \right| g(\zeta + z) \frac{d\xi \, d\eta}{|\zeta|^2} \leq H_{\alpha}(f) \int \left| g(\zeta) \right| \frac{d\xi}{|\zeta - z|^2} \\
\text{and we see that} \\
\int \left| f(\zeta + z) - f(z) \right| g(\zeta + z) \frac{d\xi \, d\eta}{|\zeta|^2} \leq H_{\alpha}(f) \int \left| g(\zeta) \right| \frac{d\xi}{|\zeta - z|^2} \\
\text{and the integral on the product of } f, \text{ and the integral on the product of } f.\n\end{aligned}
$$

Then we see that

$$
\int\limits_{|z|<1} |f(\zeta+z)-f(z)| |g(\zeta+z)| \frac{d\xi\,d\eta}{|\zeta|^2} \leq H_{\alpha}(f) \int\limits_{\mathbf{D}} |g(\zeta)| \frac{d\xi\,d\eta}{|\zeta-z|^{2-\alpha}}.
$$

Here  $H_a(f)$  denotes the Hölder coefficient of *f*, and the integral on the right-hand side is bounded. This follows from the inequalities

$$
\int_{\zeta_1}^{1/\sqrt{3}} \frac{1}{|z|^2} \, dz
$$
\n
$$
=
$$
\n
$$
|z|^2
$$
\n(f) denotes the Hölder coefficient of *f*, and  
\ned. This follows from the inequalities  
\n
$$
\int_{\mathbb{D}} \frac{d\xi \, d\eta}{|\zeta - z_k| \, |\zeta - z|^{2-\alpha}} \leq M(\alpha) |z - z_k|^{\alpha - 1}
$$

(see VEKUA [27: p. 39]) by decomposing  $g$  into partial fractions. Similarly, we find

$$
\left| \int_{\mathbf{D}} f(\zeta) g(\zeta) \frac{d\xi \, d\eta}{(1 - \zeta \overline{z})^2} \right|
$$
  
\n
$$
\leq H_a(f) \int_{\mathbf{D}} |g(\zeta)| \frac{|\zeta - z|^{\alpha}}{|1 - \zeta \overline{z}|^2} d\xi \, d\eta + ||f||_0 \left| \int_{\mathbf{D}} g(\zeta) \frac{d\xi \, d\eta}{(1 - \overline{\zeta} z)^2} \right|
$$

and also

• 

$$
\int\limits_{\mathbf{D}_s}|g(\zeta)|\,\frac{|\zeta-z|^{\mathfrak{a}}}{|1-\zeta\overline{z}|^2}\,d\xi\,d\eta\leq \int\limits_{\mathbf{D}}|g(\zeta)|\,\frac{d\xi\,d\eta}{|\zeta-z|^{2-\mathfrak{a}}}.
$$

By using Lemma 10, this will complete the proof

A Priori Estimates for Elliptic Systems 13<br>
] it is shown that for  $n > 0$  the integral operator from Lemma 11 In **BEOEHR** and **HSIAO** [15] it is shown that for  $n > 0$  the integral operator from Lemma 11 fails to be a unitary operator in  $L_2(\hat{\mathbf{D}})$ ; its norm in  $L_p$   $(p > 1)$  is, in general, greater than one. For  $n = 0$  this operator reduces to the  $\tilde{\Pi}$ -operator (see VEKUA [27: p. 210]). **nd Hsrao** [15] it is shotary operator in  $L_2(\hat{\mathbf{D}})$ <br>
operator reduces to the<br>  $\therefore If \varrho \in C^{1+\alpha}(\Gamma)$ , the<br>  $= \frac{1}{2\pi i} \int \varrho(\zeta) \frac{\varphi(\zeta)}{\varphi(\zeta)}$ 

*Lemma 12: If*  $\varrho \in C^{1+\mathfrak{a}}( \Gamma)$ *, then*  $P$ 

$$
P(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \varrho(\zeta) \, \frac{\varPhi(\zeta) + \varPhi(z)}{\varPhi(\zeta) - \varPhi(z)} \, \frac{d\varPhi(\zeta)}{\varPhi(\zeta)} \qquad (z \in D),
$$

*belongs to C*<sup> $1+\alpha$ </sup>(*D*) and satisfies the inequality  $||P||_{1+\alpha} \leq M ||Q||_{1+\alpha}$ , for some constant M *depending only on*  $\alpha$  *and D. Here*  $\Phi$  *denotes the conformal mapping of D onto the unit* 

A Priori Estimates for the BET of the set of Proof: Let  $z_0 \in D$  be the zero of  $\Phi$ . Then from Privalov's theorem (see COURANT)  $kH_{\alpha,\Gamma}(0) |z-z_0|^{\alpha}$  so that Hepenality only on  $\alpha$  and  $D$ . Here  $\Phi$  denotes the conformal mapping of  $D$  onto the unit disc **D**.<br> **Proof:** Let  $z_0 \in D$  be the zero of  $\Phi$ . Then from Privalov's theorem (see COURANT<br>
and HILBERT [18:  $\bar{p}$ , 380] by part yields ' perator reduces to the *I*1-operator (see VERUA [27: p. 210]).<br> *If*  $\varrho \in C^{1+\alpha}(\Gamma)$ , then *P*,<br>  $=\frac{1}{2\pi i} \int \varrho(\zeta) \frac{\varphi(\zeta) + \varphi(z)}{\varphi(\zeta) - \varphi(z)} \frac{d\varphi(\zeta)}{\varphi(\zeta)}$  ( $z \in D$ ),<br> *D*) and satisfies the inequality  $||P||_{1+\alpha} \$ *z*: Let  $z_0 \in D$  be the zero of  $\Phi$ . Then from<br> *d***entr** [18: p. 380]), we have  $|P(z) - ||\phi||_{\alpha, \Gamma} \leq \bar{k} ||d\phi/ds||_{0, \Gamma} \leq \bar{k} ||\phi||_{1, \Gamma}$ . Now  $P$ <br> *d*(*z*)<br> *d*(*z*) =  $\frac{1}{2\pi i} \int \frac{d\phi(\zeta)}{d\Phi(\zeta)} \frac{\phi(\zeta) + \phi(z)}{\phi(\zeta) -$ **Lemma 12:** If  $\varrho \in C^{1+s}(\Gamma)$ , then  $P$ ,<br>  $P(z) = \frac{1}{2\pi i} \int_{\Gamma} \varrho(\zeta) \frac{\varphi(\zeta) + \varphi(z)}{\varphi(\zeta) - \varphi(z)} \frac{d\varphi(\zeta)}{\varphi(\zeta)}$   $(z \in D)$ ,<br>
belongs to  $C^{1+s}(D)$  and satisfies the inequality  $||P||_{1+s} \leq M ||\varrho||_{1+s}$ , for some constant

$$
\frac{dP(z)}{d\Phi(z)} = \frac{1}{2\pi i} \int \frac{d\varrho(\zeta)}{d\Phi(\zeta)} \frac{\Phi(\zeta) + \Phi(z)}{\Phi(\zeta) - \Phi(z)} \frac{d\Phi(\zeta)}{\Phi(\zeta)} + \frac{1}{2\pi i} \int \frac{d\varrho(\zeta)}{d\Phi(\zeta)} \frac{d\Phi(\zeta)}{\Phi(\zeta)}.
$$

Again using Privalov's theorem, we get  $||P'||_a \leq k ||q||_{1+a, r}$ , where k is a constant independent of  $\rho$ 

### 5. Main results

 $\label{eq:2} \begin{split} \mathcal{L}_{\text{eff}}(\mathbf{r}) = \frac{1}{2} \mathcal{L}_{\text{eff}}(\mathbf{r}) \\ \mathcal{L}_{\text{eff}}(\mathbf{r}) = \frac{1}{2} \mathcal{L}_{\text{eff}}(\mathbf{r}) \end{split}$ 

This section contains various a priori estimates. These estimates are particularly useful, from the constructive point of view, for studying non-linear Riemann-Hubert boundary value problems consisting of non-linear boundary and side conditions.

Theorem 1: Let D be a  $C^{1+\alpha}$  domain and let  $a, b \in C^{1}(\hat{D})$  with  $||a||_{\alpha} + ||b||_{\alpha} \leq K$ . *Theorem 1: Let D be a*  $C^{1+\alpha}$  *domain and let a, b*  $\in C^2(\hat{D})$  *with*  $||a||_a + ||b||_a \leq K$ .<br> *Theorem 1: Let D be a*  $C^{1+\alpha}$  *domain and let a, b*  $\in C^2(\hat{D})$  *with*  $||a||_a + ||b||_a \leq K$ .<br> *Theorem 1: Let D be a*  $C^{1+\alpha}$  $(1 \leq k \leq n)$  but not on a and b such that for each  $w \in C^{1+\alpha}(\hat{D})$  satisfying (24) *with*  $e^{it}$ ,  $\psi \in C^{1+\alpha}(\Gamma)$ , the following estimate holds: oundary value problems consisting  $\left\{ \begin{array}{ll} \epsilon & \epsilon \end{array} \right\}$ <br>  $\epsilon$  in 1: Let D be a  $C^{1+\alpha}$  domain and<br>  $\epsilon$  exist constants  $\tilde{\gamma}_r$ ,  $1 \leq \nu \leq 4$ , dep<br>  $\left\{ \begin{array}{ll} n \\ n \end{array} \right\}$  but not on a and b such that<br>  $\psi \in C^{$ *a*, for studying non-linear Kiema<br>
of non-linear boundary and side  $\alpha$ <br> *at a, b*  $\in$   $C^s(\hat{D})$  with  $||a||_a + ||b||_a \le$ <br> *aending only on*  $\alpha$ ,  $D$ ,  $K$ ,  $\tau$ ,  $\sigma$ , and<br> *for each*  $w \in C^{1+\alpha}(\hat{D})$  satisfying<br> *as*:<br>

$$
||w||_{1+\alpha} \leq \tilde{\gamma}_1 ||\psi||_{1+\alpha, \Gamma} + \tilde{\gamma}_2 |x| + \tilde{\gamma}_3 \sum_{k=1}^n |a_k| + \tilde{\gamma}_4 ||w_{\bar{z}} - aw - b\overline{w}||_{\alpha}. \qquad (26)
$$

The estimate (26)is stated in **WENDLAND** [30: p. 20] based op *the* closed graph theorem where  $||w||_{1+\alpha} \leq \tilde{\gamma}_1 ||\psi||_{1+\alpha, \Gamma} + \tilde{\gamma}_2 |x| + \tilde{\gamma}_3 \sum_{k=1}^n |a_k|$ <br>
The estimate (26) is stated in WENDLAND [30: p. 20] bathe constants  $\tilde{\gamma}_r$ ,  $1 \leq v \leq 4$ , depend on a and *b* as well.<br>  $\therefore$  Proof of the theorem: Let

Proof of the theorem: Let us begin with the homogeneous data,  $\dot{v} = 0$ ,  $x = 0$ , Froot of the theorem: Let us begin with the nomogeneous data,  $\psi = 0$ ,  $\varkappa = 0$ ,  $a_k = 0$  (1  $\leq k \leq n$ ) and derive (26) for  $a = b = 0$ . From the representation formula for *w* in Lemma 9, it is easy to see that  $||w||_0$ ,  $||w$ for *w* in Lemma 9, it is easy to see that  $||w||_0$ ,  $||w||_0 \leq M ||w_z||_0$  (similarly as in BEGEHR and HSIAO [12]). Now from the representation formula for *w*<sub>z</sub>, we shall establish the inequality Theorem 1: Let D be a  $C^{1+\alpha}$  defined there exist constants  $\tilde{\gamma}_r$ ,  $1 \leq$ <br>  $(1 \leq k \leq n)$  but not on a and b<br>
with  $e^{i r}$ ,  $\psi \in C^{1+\alpha}(\Gamma)$ , the following e<br>  $||w||_{1+\alpha} \leq \tilde{\gamma}_1 ||\psi||_{1+\alpha,\Gamma} + \tilde{\gamma}_2 |x|$ <br>
The estimate (2 *not* on *a* and *b* such that for each  $w \in C^{1+\alpha}(\hat{D})$  *i*<br>(*I*), the following estimate holds:<br> $\tilde{\gamma}_1 ||\psi||_{1+\alpha, \Gamma} + \tilde{\gamma}_2 |z| + \tilde{\gamma}_3 \sum_{k=1}^n |a_k| + \tilde{\gamma}_4 ||w_2 - aw - b\overline{w}$ <br>is stated in WENDLAND [30: p. 20] based on the The estimate (26) is stated in WENDLAND [30: p. 20] based on the closed<br>the constants  $\tilde{p}_r$ ,  $1 \leq r \leq 4$ , depend on *a* and *b* as well.<br>
Proof of the theorem: Let us begin with the homogeneous  $a_k = 0$  ( $1 \leq k \leq n$ )

$$
||w_{\mathbf{z}}||_{\mathbf{0}} \leq M ||w_{\mathbf{z}}||_{\mathbf{a}}.
$$

*I*

Here the proof is more involved, and some clarifications are needed. Indeed, in view of the representation formula for *w,* let us first make the following observations.

a) For the conformal mapping  $\varPhi$  in Section 4, there exists a positive constant  $c_{\mathbf{0}}$ 

proof is more involved, and some clarifica presentation formula for 
$$
w_z
$$
, let us first must be conformal mapping  $\Phi$  in Section 4, the general mapping  $\Phi$  in Section 4, the general form  $c_0^{-1} \leq \left| \frac{\Phi(\zeta) - \Phi(z)}{\zeta - z} \right| \leq c_0$   $(z, \zeta \in \hat{D})$ .

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In addition we have the estimate  $\left|\frac{\Phi(\zeta) - \Phi(z)}{1 - \overline{\Phi(\zeta)}}\right| \le 1$  for all  $z, \zeta \in \hat{D}$ .<br>
b) The function  $\tilde{\Phi}$  defined by<br>  $\tilde{\Phi}(z) = i \int \tilde{r} d\omega G^T - i dG^T(r, z)$   $\tilde{\tau} = \tau + \sum_{n=1}^{n}$  $\frac{\Phi(\zeta) - \Phi(z)}{1 - \overline{\Phi(\zeta)}} \phi(z) \Big| \leq 1$ H. BEGERR and G. C. HSTAO<br>
addition we have the estimate  $\left|\frac{\Phi(\zeta) - \Phi(z)}{1 - \overline{\Phi(\zeta)} \Phi(z)}\right| \leq 1$  for all  $z, \zeta \in \mathcal{E}$ <br>
b) The function  $\tilde{\Phi}$  defined by<br>  $\tilde{\Phi}(z) = i \int_{\zeta} \tilde{\tau}[d_n G^1 - i dG^{II}](\zeta, z), \quad \tilde{\tau} = \tau + \sum_{k=$ 

$$
\tilde{\Phi}(z) = i \int_{\Gamma} \tilde{\tau}[d_n G^{\dagger} - i dG^{\dagger}] (\zeta, z), \quad \tilde{\tau} = \tau + \sum_{k=1}^{n} \arg (z - z_k)
$$
  
presentation formula belongs to  $C^{1+\alpha}(\tilde{D})$  and is analytic in  
at of  $\tilde{\Phi}'$  depends on that of  $d\tau/ds$ .  
given  $\psi$  and  $\varkappa$ , not necessarily equal to zero, there holds the  

$$
\left\| \frac{w(\cdot)}{\cdot - z_k} \right\|_0 \leq M[\|\psi\|_{\alpha, \Gamma} + |\varkappa| + \|\psi_z\|_{\alpha}].
$$
  
ows from the representation formula for  $w$  by making use

in the representation formula belongs to  $C^{1+\alpha}(\hat{D})$  and is analytic in *D*. The Hölder in the representation formula belongs to  $C^{1+\alpha}$ .<br> *coefficient* of  $\tilde{\Phi}'$  depends on that of  $d\tau/ds$ .

c) For given  $\psi$  and  $\infty$ , not necessarily equal to zero, there holds the estimate

e function 
$$
\tilde{\Phi}
$$
 defined by  
\n
$$
\tilde{\Phi}(z) = i \int_{\Gamma} \tilde{\tau}[d_n G^{\Pi} - i \, dG^{\Pi}](\zeta, z), \quad \tilde{\tau}
$$
\npresentation formula belongs to  $C^{1+\alpha}$  (it of  $\tilde{\Phi}'$  depends on that of  $d\tau/ds$ .)  
\ngiven  $\psi$  and  $\kappa$ , not necessarily equal to  
\n
$$
\left\| \frac{w(\cdot)}{\cdot - z_k} \right\|_0 \leqq M[\|\psi\|_{a,\Gamma} + |\kappa| + \|\omega_z\|_a].
$$
\n  
\nows from the representation formula  
\nthe estimates

This follows from the representation formula for  $w$  by making use of Privalov's in addition we have the estimate theorem.<br>  $\Phi(z) = i \int_{\Gamma} \tilde{\tau}[d_n G]$ <br>
in the representation formulation formulation of  $\Phi'$  depends or<br>
c) For given  $\psi$  and  $z$ , no<br>  $\left\| \frac{w(\cdot)}{\cdot - z_k} \right\|_0 \leq M[\|\psi\|]$ <br>
This follows from

*J* 

r given 
$$
\psi
$$
 and  $\varkappa$ , not necessarily equal to zero, there holds the estimate  
\n
$$
\left\|\frac{w(\cdot)}{\cdot - z_k}\right\|_0 \leq M[\|\psi\|_{\alpha,\Gamma} + |\varkappa| + \|\omega_{\tilde{z}}\|_0].
$$
\n  
\nlows from the representation formula for  $w$  by making use of Privalo,  
\n, the estimates\n
$$
\left|\frac{1}{\zeta - z_k} \left[w_{\tilde{\zeta}}(\zeta) - \sum_{i=1}^n w_{\tilde{z}}(z_i) \overline{P_i'(\zeta)}\right]\right|
$$
\n
$$
\leq \frac{H_a(\omega_{\tilde{z}})}{|\zeta - z_k|^{1-\alpha}} + |w_{\tilde{z}}(z_k)| \left|\frac{\overline{P_k'(\zeta)} - 1}{\zeta - z_k}\right| + \sum_{i=k}^n |w_{\tilde{z}}(z_i)| \left|\frac{\overline{P_i'(\zeta)}}{\zeta - z_k}\right|
$$
\ns typical inequalities of the form (see VERUA [27: p. 39])

as well as typical inequalities of the form (see VEKUA [27: p.39])

as well as typical inequalities of the form (see Vexua [27: p. 39])  
\n
$$
\int_{D} \frac{d\xi d\eta}{|\zeta - z_k|^{1-\alpha} |\zeta - z|} \leq M(\alpha, D),
$$
\n
$$
\int_{D} \frac{d\xi d\eta}{|\zeta - z_l| |\zeta - z| |\zeta - z_k|^{1-\alpha}} \leq \frac{M(\alpha, D, z_l, z_k)}{|z - z_l|^{1-\alpha}} \quad (z_l + z_k),
$$
\n
$$
\int_{D} \frac{d\xi d\eta}{|\zeta - z_l| |\zeta - z_k|^{1-\alpha}} \leq \frac{M(\alpha, D, z_l, z_k)}{|z - z_l|^{1-\alpha}} \quad (z_l + z_k),
$$
\n
$$
\int_{D} \prod_{i \neq k} \left| \frac{z - z_i}{\zeta - z_i} \right| \frac{d\xi d\eta}{|\zeta - z_k|^{1-\alpha} |\zeta - z|} \leq M(\alpha, D, z_k \quad (1 \leq k \leq n)).
$$
\nThe last inequality follows from the second one by applying decompartial fractions to the product.  
\nIt remains now to estimate the supremum norm of the area integral.  
\nachieved by transforming it into an integral over the unit disc D to why  
\napply Lemma 11 in view of a). The result here will be needed for the gen-

The last inequality follows from the second one by applying decomposition into

It remains now to estimate the supremum norm of the area integral. This can be achieved by transforming it into an integral over the unit disc **D** to which we may achieved by transforming it into an integral over the unit disc **D** to which we may apply Lemma 11 in view of a). The result here will be needed for the general case for nonvanishing *a* and *b*.<br> *We* next consider the c  $\int_{D} \frac{1}{|\zeta - z_l|} \frac{z-z_l}{|\zeta - z_l|} \frac{z}{|\zeta - z_l|}$   $\int_{D} \prod_{i \neq k} \frac{z-z_i}{|\zeta - z_i|} \frac{z}{|\zeta - z_l|}$ The last inequality follows from<br>
partial fractions to the product<br>
It remains now to estimate achieved by transforming it in<br> he last inequality follows from the second one by applying decomposition into ritial fractions to the product.<br>It remains now to estimate the supremum norm of the area integral. This can be hieved by transforming it into  $\equiv M(x, t)$ <br>and one by<br>all over the<br>all over the<br>or w. To fact<br>or w. To fact<br>representation

nonvanishing a and b.<br>
We next consider the case  $\psi = 0$ ,  $\varkappa = 0$ . Again we assume that a vanishes and proceed from the representation formulas for *w.* To facilitate the presentation, let *o*<br>ent<br>*c* 

$$
J_1(z) := \theta_n(z) e^{\tilde{\Phi}(z)} \prod_{k=1}^n (z - z_k)
$$

and denote by  $J_2$  the area integral in the representation formula. We would like to  $J_1(z) := \theta_n(z) e^{\tilde{\Phi}(z)} \prod_{k=1}^n (z - z_k)$ <br>
and denote by  $J_2$  the area integral in the representation formula. We would<br>
show that the Hölder coefficients of  $J_1$  and  $J_2$  satisfy the estimates<br>  $H_1(J_1) \leq M[H_1(u) + ||u||_1 + ||u$ te by  $J_2$  the area integral in the representation formula. We would<br>t the Hölder coefficients of  $J_1$  and  $J_2$  satisfy the estimates<br> $H_a(J_1) \leq M[H_a(\psi) + ||\psi||_0 + |\mathbf{x}| + ||\mathbf{w}_2||_0]$  and  $H_a(J_2) \leq M ||\mathbf{w}_2||_0$ .<br>estimate fo

$$
H_{\alpha}(J_1) \leq M[H_{\alpha}(\psi) + ||\psi||_0 + |\psi| + ||\psi_{\bar{z}}||_0] \text{ and } H_{\alpha}(J_2) \leq M ||\psi_{\bar{z}}||_0
$$

The first estimate follows from Privalov's theorem. To establish the second one, it suffices to examine only the integral containing  $G_{\zeta}^{\text{I}} + \tilde{G}_{\zeta}^{\text{II}}$ , since the other integral suffices to examine only the integral containing  $G_t^I + \tilde{G}_t^{II}$ , since the other integral

**0** 

can be treated similarly. We note that  $G_{\zeta}I + \tilde{G}_{\zeta}II$  has a singularity of the form A Priori Estimates for Elliptic Systems 15<br>
can be treated similarly. We note that  $G_c^I + \tilde{G}_c^{II}$  has a singularity of the form<br>  $(\zeta - z)^{-1}$  and that  $(\cdot - z)/(\Phi(\cdot) - \Phi(z))$  is Hölder-continuous in *D*. Let us first con-<br>
si can be treated similarly. W<br>  $(\zeta - z)^{-1}$  and that  $(\cdot - z)/[\Phi]$ <br>
sider the product<br>  $S(\zeta, z) = \left(\prod_{k=1}^{n} \frac{z}{\zeta - z}\right)$ 

$$
S(\zeta,z)=\left(\prod_{k=1}^n\frac{z-z_k}{\zeta-z_k}\right)\frac{1}{\zeta-z}
$$

can be treated similarly. We note that 
$$
G_{\xi}I + \tilde{G}_{\xi}II
$$
 has a singularity of the form  
\n $(\zeta - z)^{-1}$  and that  $(\cdot - z)/(\Phi(\cdot) - \Phi(z))$  is Hölder-continuous in  $\hat{D}$ . Let us first con  
\nsider the product  
\n
$$
S(\zeta, z) = \left(\prod_{k=1}^{n} \frac{z - z_k}{\zeta - z_k}\right) \frac{1}{\zeta - z}
$$
\nin  $J_2$  for  $z = \zeta_1$  and  $\zeta_2$ . We observe that  
\n
$$
S(\zeta, \zeta_1) - S(\zeta, \zeta_2) = \left\{\prod_{k=1}^{n} \frac{\zeta_1 - z_k}{\zeta - z_k} \frac{1}{(\zeta - \zeta_1)(\zeta - \zeta_2)} + \sum_{l=1}^{n} \prod_{k=1}^{l-1} \frac{\zeta_1 - z_k}{\zeta - z_k} \prod_{k=l+1}^{n} \frac{\zeta_2 - z_k}{\zeta - z_k} \frac{1}{(\zeta - z_l)(\zeta - \zeta_2)}\right\} (\zeta_1 - \zeta_2);
$$
\nwhere  $\zeta$  is a set of  $z$  and  $z$  is a set of  $z$ .

moreover, since  $(\zeta - z_i)^{-1} = (\zeta - \zeta_i)^{-1} - (\zeta_i - z_i)/[(\zeta - z_i)(\zeta - \zeta_i)]$  this difference  $S(\zeta, \zeta_1) = S(\zeta, \zeta_2)$  can be rewritten in the form

$$
+\sum_{l=1}^{n} \prod_{k=1}^{r} \frac{z_{1}-z_{k}}{\zeta-z_{k}} \prod_{k=l+1}^{r} \frac{z_{2}-z_{k}}{\zeta-z_{k}} \frac{z_{1}}{(\zeta-z_{l})(\zeta-\zeta_{2})};
$$
\nmoreover, since  $(\zeta-z_{l})^{-1} = (\zeta-\zeta_{1})^{-1} \sim (\zeta_{1}-z_{l})/[(\zeta-z_{l})(\zeta-\zeta_{1})]$  this differ  
\n
$$
S(\zeta,\zeta_{1})-S(\zeta,\zeta_{2})
$$
 can be rewritten in the form\n
$$
\left\{\prod_{k=1}^{n} \frac{\zeta_{1}-z_{k}}{\zeta-z_{k}}+\sum_{l=1}^{n} \prod_{k=1}^{l-1} \frac{\zeta_{1}-z_{k}}{\zeta-z_{k}} \prod_{k=l+1}^{n} \frac{\zeta_{2}-z_{k}}{\zeta-z_{k}}
$$
\n
$$
-\sum_{l=1}^{n} \prod_{k=1}^{l} \frac{\zeta_{1}-z_{k}}{\zeta-z_{k}} \prod_{k=l+1}^{n} \frac{\zeta_{2}-z_{k}}{\zeta-z_{k}} \frac{(\zeta_{1}-\zeta_{2})}{(\zeta-\zeta_{1})(\zeta-\zeta_{2})},
$$
\nor\n
$$
S(\zeta,\zeta_{1})-S(\zeta,\zeta_{2})=\left\{\sum_{l=1}^{n} \prod_{k=1}^{l-1} \frac{\zeta_{1}-z_{k}}{\zeta-z_{k}} \prod_{k=l+1}^{n} \frac{\zeta_{2}-z_{k}}{\zeta-z_{k}}
$$
\n
$$
-\sum_{l=1}^{n-1} \prod_{k=1}^{l} \frac{\zeta_{1}-z_{k}}{\zeta-z_{k}} \prod_{k=l+1}^{n} \frac{\zeta_{2}-z_{k}}{\zeta-z_{k}}
$$
\nNow from the identity\n
$$
\prod_{k=1}^{n} \frac{\zeta_{1}-z_{k}}{\zeta-z_{k}} \frac{1}{\zeta-\zeta_{1}}=\frac{1}{\zeta-\zeta_{1}}-\sum_{k=1}^{m} \left(\prod_{k=k}^{l} \frac{\zeta_{1}-z_{2}}{z-z_{k}}\right) \frac{1}{(\zeta-\zeta_{1})(\zeta-\zeta_{2})}.
$$
\nNow from the identity\n
$$
\
$$

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$$
\prod_{k=1}^m \frac{\zeta_i - z_k}{\zeta - z_k} \frac{1}{\zeta - \zeta_i} = \frac{1}{\zeta - \zeta_i} - \sum_{k=1}^m \left( \prod_{i+k} \frac{\zeta_i - z_i}{z_k - z_k} \right) \frac{1}{\zeta - z_k}
$$

 $(i = 1, 2)$  each term in the above sum for the difference  $S(\zeta, \zeta_1) - S(\zeta, \zeta_2)$  may be

$$
-\sum_{i=1}^{n-1} \prod_{k=1}^{l} \frac{\zeta_{1} - z_{k}}{\zeta - z_{k}} \prod_{i=k+l+1}^{n} \frac{\zeta_{2} - z_{k}}{\zeta - z_{k}} \frac{\zeta_{1} - \zeta_{2}}{\zeta - \zeta_{1}}.
$$
\nNow from the identity\n
$$
\prod_{k=1}^{m} \frac{\zeta_{1} - z_{k}}{\zeta - z_{k}} \frac{1}{\zeta - \zeta_{1}} = \frac{1}{\zeta - \zeta_{1}} - \sum_{k=1}^{m} \left( \prod_{i=k}^{n} \frac{\zeta_{i} - z_{i}}{z_{k} - z_{i}} \right) \frac{1}{\zeta - z_{k}}
$$
\n(*i* = 1, 2) each term in the above sum for the difference  $S(\zeta, \zeta_{1}) - S(\zeta, \zeta_{2})$  may be reformulated, e.g.\n
$$
\left( \prod_{k=1}^{l} \frac{\zeta_{1} - z_{k}}{\zeta - z_{k}} \frac{1}{\zeta - \zeta_{1}} \right) \left( \prod_{k=l+1}^{n} \frac{\zeta_{2} - z_{k}}{\zeta - z_{k}} \frac{1}{\zeta - \zeta_{2}} \right)
$$
\n
$$
= \frac{1}{(\zeta - \zeta_{1})(\zeta - \zeta_{2})} - \sum_{k=1}^{l} \left( \prod_{i=k}^{n} \frac{\zeta_{1} - z_{i}}{z_{k} - z_{i}} \right) \frac{1}{(\zeta - z_{k})(\zeta - \zeta_{2})}
$$
\n
$$
+ \sum_{k=l+1}^{n} \left( \prod_{i=k}^{n} \frac{\zeta_{2} - z_{i}}{z_{k} - z_{i}} \right) \frac{1}{(\zeta - z_{k})(\zeta - \zeta_{1})}
$$
\n
$$
+ \sum_{k=1}^{l} \sum_{x=l+1}^{n} \left( \prod_{i=k}^{n} \frac{\zeta_{1} - z_{i}}{z_{k} - z_{i}} \right) \left( \prod_{i=k}^{n} \frac{\zeta_{2} - z_{i}}{z_{k} - z_{i}} \right) \frac{1}{(\zeta - z_{k})(\zeta - z_{i})}.
$$
\nHence it is not difficult to see that the difference  $J_{2}(\zeta_{1})$ 

$$
\frac{1}{(\zeta-\zeta_1)(\zeta-\zeta_2)}, \quad \frac{1}{(\zeta-z_k)(\zeta-\zeta_1)}, \quad \frac{1}{(\zeta-z_k)(\zeta-\zeta_2)}, \quad \text{and} \quad \frac{1}{(\zeta-z_k)(\zeta-z_2)}.
$$

All these integrals can be handled. In particular we see that  $(VERUA [27: p. 39])$ 

$$
\int\limits_{D} \frac{d\xi\,d\eta}{|\zeta-\zeta_1||\zeta-\zeta_2|}\leq M(1+|\log|\zeta_1-\zeta_2|).
$$

In the same manner, we find

This leads to the desired result.  
In the same manner, we find  

$$
H_a\left(\frac{w}{z-z_k}\right) \leq M\left(H_a(\psi) + ||\psi||_0 + |x| + ||w_{\bar{z}}||_0\right)
$$
and

and

$$
H_{a}(w) \leq M(H_{a}(\psi) + ||\psi||_{0} + |x| + ||w_{\bar{z}}||_{0}).
$$

For the estimates of the Hölder coefficients of the z-derivatives of  $J_1$  and  $J_2$ , we use Lemmas 11 and 12, and find (see  $VERUA$  [27: p. 63])

$$
\int_{B} |\zeta - \zeta_{1}| |\zeta - \zeta_{2}|
$$
\nThis leads to the desired result.  
\nIn the same manner, we find\n
$$
H_{a} \left( \frac{w}{z - z_{k}} \right) \leq M \left( H_{a}(\psi) + ||\psi||_{0} + |x| + ||\psi_{\overline{z}}||_{0} \right)
$$
\nand\n
$$
H_{a}(\psi) \leq M \left( H_{a}(\psi) + ||\psi||_{0} + |x| + ||\psi_{\overline{z}}||_{0} \right).
$$
\nFor the estimates of the Hölder coefficients of the z-derivatives of  $J_{1}$  and  $J_{2}$ , we use  
\nLemma 11 and 12, and find (see VEXUA [27: p. 63])\n
$$
H_{a} \left( \frac{\partial}{\partial z} J_{1} \right) \leq M (||\psi||_{a}r + ||\psi'||_{a} + ||\psi_{\overline{z}}||_{0}) \text{ and } H_{a} \left( \frac{\partial}{\partial z} J_{2} \right) \leq M ||\psi_{\overline{z}}||_{a}.
$$
\nThus, we obtain  $H_{a}(\psi_{z}) \leq M (||\psi||_{1+a}, r + |x| + ||\psi_{\overline{z}}||_{a})$  and hence (26) follows in the special case  $a_{k} = 0$  ( $1 \leq k \leq n$ ) with a and b equal to zero.  
\nIf now w does not vanish at the  $z_{k}$ , then we may use the transformation\n
$$
\Omega = w - f, \qquad f(z) = \sum_{k=1}^{n} a_{k} \prod_{l+k} \frac{z - z_{l}}{z_{k} - z_{l}}
$$
\nand reduce it to the previous case for  $\Omega$  satisfying

special case  $a_k = 0$  ( $1 \leq k \leq n$ ) with a and b equal to zero.<br>If now *w* does not vanish at the  $z_k$ , then we may use the transformation

$$
\begin{aligned}\n\langle \, cz & \, f \rangle \\
\text{the obtain} & H_a(w_z) \leq M(||\psi||_{1+a,1} + |x| + \|\cdot\| \\
\text{as} & e_k = 0 \ (1 \leq k \leq n) \text{ with } a \text{ and } b \text{ eq} \\
\text{with } w \text{ does not vanish at the } z_k \text{, then we find} \\
\Omega &= w - f, \qquad f(z) = \sum_{k=1}^n a_k \prod_{l+k} \frac{z - z_l}{z_k - z_l}\n\end{aligned}
$$

and reduce it to the previous case for  $\Omega$  satisfying

e obtain 
$$
H_a(w_z) \leq M(||\psi||_{1+a, r} + |x| + ||w_z||_a)
$$
 and  
\nuse  $a_k = 0$  ( $1 \leq k \leq n$ ) with a and b equal to zero  
\nw does not vanish at the  $z_k$ , then we may use the  
\n
$$
\Omega = w - f, \qquad f(z) = \sum_{k=1}^n a_k \prod_{i+k} \frac{z - z_i}{z_k - z_i}
$$
\n\nce it to the previous case for  $\Omega$  satisfying  
\n $\text{Re}\{e^{i\tau}Q\} = \psi - \text{Re}\{e^{i\tau}f\}$  on  $\Gamma$   
\n $\frac{1}{\Sigma} \int_{\Gamma} \text{Im}\{e^{i\tau}Q\} \sigma ds = \chi - \frac{1}{\Sigma} \int_{\Gamma} \text{Im}\{e^{i\tau}f\} \sigma ds$   
\n $\Omega(z_k) = 0$  ( $1 \leq k \leq n$ ).  
\n(26) is proved for the general case but with v  
\ne now in a position to establish (26) for the most  
\na with homogeneous boundary and side conditions.  
\nder the function  $\omega$  uniquely given by  
\n $\omega_{\bar{z}} = \begin{cases} a + b \frac{\overline{w}}{w}, & w \neq 0, \\ a, & w = 0, \end{cases}$ Im  $\omega|_{\Gamma} = 0$ , and  $\int_{\Gamma} \text{R}$   
\n $\omega_{\bar{z}} = \begin{cases} a + b \frac{\overline{w}}{w}, & w \neq 0, \\ a, & w = 0, \end{cases}$ 

In this way (26) is proved for the general ease but with vanishing a.and *b.* 

We are now in a position to establish (26) for the most general case. Again .we will begin with homogeneous boundary and side conditions. For arbitrary  $a, b \in C^*(\hat{D}),$ we consider the function  $\omega$  uniquely given by

$$
\Omega(z_k) = 0 \qquad (1 \le k \le n).
$$
\n(26) is proved for the general case but with vanishing  $a$  and  $b$  be now in a position to establish (26) for the most general case.

\nwith homogeneous boundary and side conditions. For arbitrary  $a$ ,  $b$  are the function  $\omega$  uniquely given by

\n
$$
\omega_z = \begin{cases}\n a + b \frac{\overline{w}}{w}, & w \ne 0, \\
 a, & w = 0, \\
 a, & w = 0,\n\end{cases}
$$
\nthat  $\tilde{\omega} = i\omega$  satisfies the homogeneous conditions (24) in the following theorem.

We note that  $\tilde{\omega} = i\omega$  satisfies the homogeneous conditions (24) in the special case  $n = 0$ ,  $\tau = 0$ . Although  $\tilde{\omega} \notin C^{1+\alpha}(\hat{D})$  in general, from the representation formula, as in the case for the homogeneous data, it can be shown that the inequality in the case for the homogeneous data, it can be shown that the inequality  $\|\tilde{\omega}\|_{\alpha} \leq M \|\tilde{\omega}_{\bar{x}}\|_{0}$  remains valid. Thus we have  $\|\omega\|_{\alpha} \leq M(\|a\|_{0} + \|b\|_{0}) \leq MK$ , and hence  $\|e^{\omega}\|_{\alpha} \leq e^{MK}(1 + MK)$ .<br>Next, let hence  $||e^{\omega}||_{\alpha} \leq e^{MK}(1 + MK).$ We note that  $\tilde{\omega} = \begin{cases} a + b \frac{\overline{w}}{w}, & w \neq 0, \\ a, & w = 0, \end{cases}$ <br>
We note that  $\tilde{\omega} = i\omega$  satisfies the homogeneous conditions. For an<br>
Consider the function  $\omega$  uniquely given by<br>  $\omega_i = \begin{cases} a + b \frac{\overline{w}}{w}, & w \neq 0, \\ a, & w =$ 

Next, let  $f_0$  be analytic in  $D$  and uniquely defined by

$$
\operatorname{Re}\, \{\mathrm{e}^{i\tau}f_0\}|_r=0,\quad \int\limits_{\Gamma} \operatorname{Im}\, \{\mathrm{e}^{i\tau}f_0\} \, \sigma \, ds=\Sigma,\quad f_0(z_k)=0 \quad (1\leq k\leq n).
$$

$$
f = w e^{-w} - Af_0
$$
 with  $A = \int_I Im \{e^{iv}w e^{-w}\} \sigma ds$ ,

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(27)

then *f* satisfies the equation  $f_z = (w_z - aw - b\overline{w}) e^{-w}$  in *D* together with the homogeneous boundary and side conditions

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stisfies the equation 
$$
f_z = (w_{\bar{z}} - aw - b\bar{w}) e^{-w}
$$
 in *D* together with the  
boundary and side conditions  
Re  $\{e^{i\tau}f\}_{|\tau} = 0$ ,  $\int_{\tau} Im \{e^{i\tau}f\} \sigma ds = 0$ ,  $f(z_k) = 0$   $(1 \le k \le n)$ ,  
vanishing data  $\psi = 0$ ,  $\kappa = 0$ ,  $a_k = 0$   $(1 \le k \le n)$ . Because  $f_{\bar{z}} \in C^{\alpha}$ .

Estimates<sub>*i*</sub> for<br>  $\overline{v}$ )  $e^{-w}$  in *D*<br>  $\overline{v}$ ,  $f(z_k) =$ <br>  $1 \leq k \leq n$ )<br> *f* and obtain<br>  $e^w$  and hence then *f* satisfies the equation  $f_z = (w_z - uw - bw) e^{-\ln D}$  sogether what the helip geneous boundary and side conditions<br>  $\text{Re } \{e^{i\tau}f\} |_{\Gamma} = 0, \quad \int \text{Im } \{e^{i\tau}f\} \sigma ds = 0, \quad f(z_k) = 0 \quad (1 \leq k \leq n),$ <br>
if w has vanishing data  $\psi =$ may apply the previously established estimates to f and obtain  $||f||_{a} \leq M ||f_{\bar{z}}||_{0}$ . Further, from the definition of f, we have  $w = (f + Af_0) e^{\omega}$  and hence

$$
A \int_{\Gamma} \text{Im} \{e^{i\tau}f_0 e^{\omega}\} \sigma ds + \int_{\Gamma} \text{Im} \{e^{i\tau}f e^{\omega}\} \sigma ds = 0
$$

for vanishing x. Observing the conditions satisfied by  $f_0$  on  $\Gamma$  and especially the consequence  $0 < \text{Im } \{e^{it}f_c\}$  on  $\Gamma$  leads to

nce 
$$
0 < \text{Im } \{e^{it}f_0\}
$$
 on  $\Gamma$  leads to\n
$$
\int \text{Im } \{e^{it}f_0 e^{\omega}\} \sigma \, ds = \int \text{Im } \{e^{it}f_0\} \text{ Re } e^{\omega}\sigma \, ds \geq \Sigma e^{-MK}.
$$
\nently we have\n
$$
|A| \leq e^{2MK} ||f||_0
$$
, and  $||w||_a \leq [||f||_a + e^{2MK} ||f||_0 ||f_0||_a]$   $||e^{\omega}||_a \leq \tilde{M}$ \nconstant  $\tilde{M}$ . Moreover, in view of the analyticity of  $f_0$ , we see that\n
$$
f_{\tilde{z}} e^{\omega} + aw + b\overline{w}.
$$
 Hence,\n
$$
||w_z||_a \leq M ||v_z||_a \leq \tilde{M}[||f||_a + ||f_2||_a] \leq \tilde{M}[||f||_a].
$$
\ntherefore, the area of the homogeneous data is:\n
$$
||w||_{1+a} \leq \gamma_4 ||w_{\tilde{z}} - aw - b\overline{w}||_a.
$$
\nHere the constant  $\gamma_4$  depends on  $||a||_a + ||b||_a$  but not on  $a$  and  $b$ ,\nsee now  $a_k = 0$  ( $1 \leq k \leq n$ ) but  $\psi + 0$  and  $\chi + 0$ . Then let  $\theta$  function from the representation formula (Lemma 9). Set

Consequently we have<br>  $|A| \leq e^{2MK} ||f||_0$  <sup>1</sup>

$$
|A| \leq e^{2MK} \|f\|_0
$$
 and  $\|w\|_a \leq (\|f\|_a + e^{2MK} \|f\|_b \|f_0\|_a) \|e^{\omega}\|_a \leq \tilde{M} \|f\|_a$ 

for some constant  $\tilde{M}$ . Moreover, in view of the analyticity of  $f_0$ , we see that  $w_{\tilde{z}} = f_{\tilde{z}} e^{\omega}$  $+ w\omega_{\bar{z}} = f_{\bar{z}} e^{\omega} + aw + b\overline{w}$ . Hence, A  $\int_{\Gamma}$  Im  $\{e^{it}f_0 e^{i\omega}\}\sigma ds + \int_{\Gamma}$  Im  $\{e^{it}f e^{i\omega}\}\sigma ds = 0$ <br>  $\int_{\Gamma}$ <br>
hing x. Observing the conditions satisfied by  $f_0$  on  $\Gamma$  and espector  $0 <$  Im  $\{e^{it}f_0\}$  on  $\Gamma$  leads to<br>  $\int \text{Im } \{e^{it}f_0 e^{i\omega}\}\sigma ds = \int \$ 

$$
||w_z||_{\alpha} \leq M ||w_{\bar{z}}||_{\alpha} \leq \hat{M}[||f||_{\alpha} + ||f_{\bar{z}}||_{\alpha}] \leq \tilde{M} ||f_{\bar{z}}||_{\alpha}.
$$

Thus, we arrive at (26) in the special case for the homogeneous data

$$
||w||_{1+\alpha} \leq \gamma_{\mathbf{A}} ||w_{\bar{z}} - \dot{a}w - b\overline{w}||_{\alpha}.
$$

Clearly here the constant  $\gamma_4$  depends on  $||a||_4 + ||b||_4$  but not on a and b.

Suppose now  $a_k = 0$  ( $1 \leq k \leq n$ ) but  $\psi + 0$  and  $\kappa + 0$ . Then let  $\theta$  denote the analytic function from the representation formula (Lemma 9). Set

$$
\omega = w + \tilde{\theta} \quad \text{with} \quad \tilde{\theta}(z) = \theta(z) e^{\tilde{\phi}(z)} \prod_{k=1}^{n} (z - z_k).
$$

Then it can be verified that

$$
||w_z||_a \leq M ||w\bar{z}||_a \geq M ||\bar{y}||_a + ||\bar{y}||_a + ||\bar{y}||_a
$$
\nThus, we arrive at (26) in the special case for the homogeneous dat\n
$$
||w||_{1+a} \leq \gamma_4 ||w_{\bar{z}} - aw - b\bar{w}||_a.
$$
\nClearly here the constant  $\gamma_4$  depends on  $||a||_a + ||b||_a$  but not on a an\nSuppose now  $a_k = 0$  ( $1 \leq k \leq n$ ) but  $\psi + 0$  and  $\chi + 0$ . Then\nanalytic function from the representation formula (Lemma 9). Set\n
$$
\omega = w + \bar{\theta} \quad \text{with} \quad \bar{\theta}(z) = \theta(z) e^{\bar{\phi}(z)} \prod_{k=1}^n (z - z_k).
$$
\nThen it can be verified that\n
$$
\omega_{\bar{z}} = w_{\bar{z}}, \qquad \omega(z_k) = 0 \quad (1 \leq k \leq n), \qquad \text{Re } \{e^{it}\omega\}|_{\bar{r}} = 0,
$$
\n
$$
\int \text{Im } \{e^{it}\omega\} \sigma \, ds = 0,
$$
\nand hence  $\omega$  satisfies (27). Thus\n
$$
||w||_{1+a} \leq ||\omega||_{1+a} + ||\bar{\theta}||_{1+a}, \qquad ||\omega||_{1+a} \leq \gamma_4 ||w_{\bar{z}} - aw - b\bar{w}||_a
$$
\nThis together with the already established estimates of  $||\bar{\theta}||_{1+a}$  is\n
$$
a_k = 0.
$$
\nFinally if the similar transformation\n
$$
Q = w - \int \int \text{with} \quad f(z) = \sum_{k=1}^n a_k \prod_{\ell \neq k} \frac{z - z_l}{z_k - z_l}
$$
\nis utilized, the general inequality for the non-homogeneous data that

• and hence  $\omega$  satisfies (27). Thus

$$
||w||_{1+\alpha} \leq ||\omega||_{1+\alpha} + ||\tilde{\theta}||_{1+\alpha}, \quad ||\omega||_{1+\alpha} \leq \gamma_4 ||w_{\bar{x}} - aw - b\overline{w}||_{\alpha} + \gamma_4 ||a\tilde{\theta} + b\tilde{\theta}||_{\alpha}.
$$

This together with the already established estimates of  $\|\tilde{\theta}\|_{1+\alpha}$  implies (26) with ed estimates of  $\|\tilde{\theta}\|_{1+\alpha}$ 

$$
\Omega = w - f^{'} \text{ with } f(z) = \sum_{k=1}^{n} a_k \prod_{i \neq k} \frac{z - z_i}{z_k - z_i}
$$

is utilized, the general inequality for the non-homogeneous data then follows immediately

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We emphasize that here (26) is found in a constructive way, and it is more suitable for numerical procedures such as the Newton embedding type (see WENDLAND [28], BEGEHR and HSLAO (14).

The considerations in Section 3 enable us now to extend Theorem 1 to the following result for the general Beltrami equation.

Theorem 2: Let  $w \in C^{1+\alpha}(\hat{D})$  satisfy (24) with  $e^{it}$ ,  $\psi \in C^{\alpha}(\Gamma)$ . Let  $\mu_1, \mu_2 \in C^{1+\alpha}(\mathbb{C})$ and  $v_1, v_2 \in C^{\alpha}(\hat{D})$  be given fulfilling the assumptions

$$
a)\hspace{1cm}\mu_1=\mu_2=0\hspace{0.3cm}in\hspace{0.3cm} \mathbb{C}\setminus D
$$

 $\|\mu_1\|_0 + \|\mu_2\|_0 \leq q < 1$ ,

and

 $||v_1||_a + ||v_2||_a \leq K$ ,  $\mathbf{b}$ 

respectively. Then there holds the estimate

$$
||w||_{1+\alpha} \leq \gamma_1 ||\psi||_{1+\alpha, r} + \gamma_2 |x| + \gamma_3 \sum_{k=1}^n |a_k|
$$

 $\|\mu_{1z}\|_{\alpha} + \|\mu_{1\overline{z}}\|_{\alpha} + \|\mu_{2z}\|_{\alpha} + \|\mu_{2\overline{z}}\|_{\alpha} \leq M_1,$ 

$$
+ \gamma_4 ||w_{\bar{z}} - \mu_1 w_z - \mu_2 w_{\bar{z}} - \nu_1 w - \nu_2 \overline{w}||_{\alpha},
$$

where  $\gamma_k$  (1  $\leq$  k  $\leq$  4) are constants depending on D,  $z_k$  (1  $\leq$  k  $\leq$  n),  $\sigma$ ,  $\tau$ ,  $\alpha$ ,  $q$ ,  $M_1$ , K but not on w,  $\mu_1$ ,  $\mu_2$ ,  $\nu_1$ ,  $\nu_2$ ,  $\nu_2$ ,  $\nu_3$ ,  $a_k$   $(1 \leq k \leq n)$ .

Proof: The transformation (20), (21) reduces the differential equation

$$
w_{\bar{z}} = \mu_1 w_z + \mu_2 \overline{w}_z + v_1 w + v_2 \overline{w} + \lambda, \qquad \lambda \in C^{\alpha}(\hat{D}),
$$

together with the boundary and side conditions (24) to the canonical problem defined by the differential equation

$$
\omega_{\bar\ell}=A\omega+B\overline{\omega}+C\ \ \text{in}\ \ \bar D
$$

and the boundary and side conditions (25). Here the coefficients  $A$ ,  $B$ , and  $C$  are given explicitly by

$$
A = \frac{(1 - \bar{\mu}_1 \mu) (v_1 + \bar{a} v_2) + \mu_2 \mu (\bar{a} \bar{v}_1 + \bar{v}_2)}{(1 - |a|^2) (|1 - \bar{\mu}_1 \mu|^2 - |\mu_2 \mu|^2) \bar{\zeta}_2} - \frac{a_{\bar{\zeta}} \bar{a}}{1 - |a|^2},
$$
  
\n
$$
B = \frac{(1 - \bar{\mu}_1 \mu) (a v_1 + v_2) + \mu_2 \mu (\bar{v}_1 + a \bar{v}_2)}{(1 - |a|^2) (|1 - \bar{\mu}_1 \mu|^2 - |\mu_2 \mu|^2) \bar{\zeta}_2} - \frac{a_{\bar{\zeta}}}{1 - |a|^2},
$$
  
\n
$$
C = \frac{(1 - \bar{\mu}_1 \mu) \lambda + \mu_2 \mu \bar{\lambda}}{(|1 - \bar{\mu}_1 \mu|^2 - |\mu_2 \mu|^2) \bar{\zeta}_2}.
$$

Obviously, these coefficients belong to  $C^*(\hat{D}), \hat{D} := \tilde{D} \cup \partial \tilde{D}$ , and are bounded according to

 $||A||_{\alpha,\tilde{D}}+||B||_{\alpha,\tilde{D}}\leq \tilde{K} \quad \text{and} \quad ||C||_{\alpha,\tilde{D}}\leq M \, ||\lambda||_{\alpha,D}$ 

with constants  $\tilde{K}$  and  $M$  depending only on q,  $M_1$ ,  $K$ , and  $D$ . Hence the estimate (26) is valid for  $\omega$  in  $\tilde{D}$ . Now from the transformation (21), we see that

$$
w=\frac{\omega+a\overline{\omega}}{1-|a|^2}\quad\text{ and hence }\|w\|_{1+a,D}\leq M_0\,\|\omega\|_{1+a,D}.
$$

### A' Priori Estimates for Elliptic Systems (19)

However, in order to apply (26) to  $\omega$ , we first show that the right-hand side is dominated by  $\|\omega\|_{1+\alpha,\tilde{D}}$ . To this end, let us consider the relations

A Priori Estimates for Elliptic Systems 19  
\nHowever, in order to apply (26) to 
$$
\omega
$$
, we first show that the right-hand side is  
\ndominated by  $||\omega||_{1+\alpha,\tilde{D}}$ . To this end, let us consider the relations  
\n $|\zeta_1 - \zeta_2| = \int_{\zeta_1} d\tilde{s} = \int_{\zeta_1} \frac{d\tilde{s}}{ds} ds = \int_{\zeta_2} |\zeta_2| |z'(s) + \overline{\mu z'(s)}| ds.$   
\nHere  $\overline{z_1z_2}$  is the line segment between  $z_1$  and  $z_2$ , and lies in *D* for  $|z_1 - z_2|$  sufficiently  
\nsmall while  $\zeta[\overline{z_1z_2}]$  is the image of  $\overline{z_1z_2}$  under the mapping  $\zeta$ . Hence by using the esti-

 $|\zeta_1 - \zeta_2| = \int d\tilde{s} = \int d\tilde{s} \frac{d\tilde{s}}{ds} ds = \int_{\tilde{z}_1 \tilde{z}_2} |\zeta_2| |z'(s) + \overline{\mu z'(s)}| ds.$ <br>
Here  $\overline{z_1 z_2}$  is the line segment between  $z_1$  and  $z_2$ , and lies in *D* for  $|z_1 - z_2|$  sufficiently small while  $\langle \overline{z_1$ Here  $\frac{1}{z_1z_2}$  is the line segment between  $z_1$  and  $z_2$ , and lies in D for  $|z_1 - z_2|$ <br>
small while  $\langle [z_1z_2] \rangle$  is the image of  $\overline{z_1z_2}$  under the mapping  $\zeta$ . Hence by us<br>
mate  $||\zeta_2||_0 \leq e^{MM_1}$  from mate  $\|\zeta_2\|_0 \leq e^{MM_1}$  from (10), we obtain the inequality  $|\zeta_1 - \zeta_2| \leq e^{MM_1}(1 + q)$  $\times$   $|z_1 - z_2|$ . This together with the identities  $\omega_z = \omega_c \zeta_z + \omega_{\bar{\ell}} \bar{\mu} \bar{\zeta}_z$  and  $\omega_{\bar{z}} = \omega_c \mu \zeta_z$ <br>+  $\omega_{\bar{\ell}} \bar{\zeta}_z$  implies that *A* Priori Estimates for Ellip<br> *A* Priori Estimates for Ellip<br> *A* Priori Estimates for Ellip<br> *A* cominated by  $||\omega||_{1+\alpha,\tilde{D}}$ . To this end, let us consider the relations<br>  $|\zeta_1 - \zeta_2| = \int d\tilde{s} = \int \frac{d\tilde{s}}{ds} ds = \int_{z_1\$ in order to apply (26) to  $\omega$ , we first show that the right-hand<br>
d by  $||\omega||_{1+e,\bar{D}}$ . To this end, let us consider the relations<br>  $|\zeta_1 - \zeta_2| = \int d\bar{s} = \int \frac{d\bar{s}}{ds} ds = \int |\zeta_2| |z'(s) + \overline{\mu z'(s)}| ds$ .<br>
is the line segment be However, in order to apply (26) to  $\omega$ , we first show that the right<br>dominated by  $\|\omega\|_{1+\alpha,\bar{D}}$ . To this end, let us consider the relations<br> $|\zeta_1 - \zeta_2| = \int d\tilde{s} = \int d\tilde{s} = \int d\tilde{s}$ <br> $\frac{d\tilde{s}}{ds} ds = \int |\zeta_2| |z'(s) + \overline{\mu z$ A Priori Estimat<br>
However, in order to apply (26) to  $\omega$ , we first show<br>
dominated by  $||\omega||_{1+\delta,b}$ . To this end, let us consider the<br>  $|\zeta_1 - \zeta_2| = \int_{\zeta_1} d\tilde{s} = \int_{\zeta_1} \frac{d\tilde{s}}{ds} ds = \int_{\zeta_1} |\zeta_2| |z'(s)|$ <br>
Here  $\overline{$ 

$$
\|\omega_z\|_{\alpha,D} + \|\omega_{\bar{z}}\|_{\alpha,D} \leq M e^{\alpha MM_1} (1+q)^{\alpha} \left[ \|\omega_{\zeta}\|_{\alpha,\bar{D}} + \|\omega_{\bar{\zeta}}\|_{\alpha,\bar{D}} \right].
$$
  
uently we have  

$$
\|\omega\|_{1+\alpha,D} \leq M e^{\alpha MM_1} (1+q)^{\alpha} \|\omega\|_{1+\alpha,\bar{D}},
$$

$$
\|\omega\|_{1+\alpha,D} \leq M e^{\alpha M M_1} (1+q)^{\alpha} \|\omega\|_{1+\alpha,D},
$$

and from (26),  $||\omega||_{1+\alpha,\bar{D}}$  is bounded by the appropriate terms with respect to norms on  $\tilde{D}$  and also on  $\tilde{P}$ . In particular, to recover those bounds on  $\tilde{P}$  with respect to the  $\|\omega_z\|_{\alpha,D} + \|\omega_{\tilde{\ell}}\|_{\alpha,D} \leq M e^{\alpha MM_1} (1+q)^{\alpha} [\|\omega_{\tilde{\ell}}\|_{\alpha,\tilde{D}} + \|\omega_{\tilde{\ell}}\|_{\alpha,\tilde{D}}].$ <br>
Consequently we have<br>  $\|\omega\|_{1+\alpha,D} \leq M e^{\alpha MM_1} (1+q)^{\alpha} \|\omega\|_{1+\alpha,\tilde{D}},$ <br>
and from (26),  $\|\omega\|_{1+\alpha,\tilde{D}}$  is bounded by the a 10), we obtain the inequality<br>with the identities  $\omega_z = \omega_{\zeta} \zeta_z$ <br> $\in M e^{a M M_1} (1 + q)^{\alpha} [\|\omega_{\zeta}\|_{\alpha, \bar{D}} + \|\zeta\|$ <br> $\cdot (1 + q)^{\alpha} \|\omega\|_{1 + \alpha, \bar{D}},$ <br>ounded by the appropriate te:<br>icular, to recover those bound<br> $f \in C^{\alpha}(\bar{\Gamma})$ <br>  $\times |z_1 - z_2|$ . This together with the identities  $\omega_z = \omega_z \zeta_z + \omega_{\tilde{\ell}} \overline{\mu} \overline{\zeta_z}$  and  $\omega_{\tilde{z}}$ <br>  $+ \omega_{\tilde{\ell}} \overline{\zeta_z}$  implies that<br>  $\|\omega_z\|_{\bullet, D} + \|\omega_{\tilde{z}}\|_{\bullet, D} \leq M e^{\alpha M M_1} (1 + q)^{\alpha} [\|\omega_{\zeta}\|_{\bullet, \tilde{D}} + \|\omega_{\tilde{\ell$ 

$$
||f||_{\alpha,\tilde{\Gamma}} \leq \frac{e^{sMM_1}}{(1-q)^{\alpha}} \, ||f||_{\alpha,\Gamma},
$$

where we have used the similar estimate as before

$$
+ \omega_{\xi\xi} \text{ implies that}
$$
\n
$$
\|\omega_z\|_{a,D} + \|\omega_{\tilde{z}}\|_{a,D} \leq M e^{\alpha MN_1}(1+q)^{\alpha} [\|\omega_{\zeta}\|_{a,\tilde{D}} + \|\omega_{\tilde{t}}\|_{a,\tilde{D}}].
$$
\nConsequently we have\n
$$
\|\omega\|_{1+\alpha,\tilde{D}} \leq M e^{\alpha MN_1}(1+q)^{\alpha} \|\omega\|_{1+\alpha,\tilde{D}},
$$
\nand from (26),  $\|\omega\|_{1+\alpha,\tilde{D}}$  is bounded by the appropriate terms with respect on  $\tilde{D}$  and also on  $\tilde{T}$ . In particular, to recover those bounds on  $\tilde{T}$  with respect norm on  $\Gamma$ , we see that for  $f \in C^{\alpha}(\tilde{\Gamma})$ \n
$$
\|f\|_{a,\tilde{\Gamma}} \leq \frac{e^{\alpha MM_1}}{(1-q)^{\alpha}} \cdot \|f\|_{a,\Gamma},
$$
\nwhere we have used the similar estimate as before\n
$$
|\zeta_1 - \zeta_2| = \int d\tilde{s} = \int |\zeta_2| |z'(s) + \mu z'(s)| |ds \geq e^{-MM_1}(1-q) \int \cdot ds
$$
\n
$$
\geq e^{-MM_1} (1-q) |z(\zeta_1) - z(\zeta_2)|.
$$
\nSimilarly from (25) we have, in view of Lemma (6),\n
$$
\left\| \frac{1-|a|^2}{q} \psi \right\|_{1+\alpha,\tilde{\Gamma}} \leq \left\| \frac{ds}{d\tilde{s}} \right\|_{a,\tilde{\Gamma}} \frac{e^{\alpha MM_1}}{(1-q)^{\alpha}} \left\| \frac{1-|a|^2}{q} \psi \right\|_{1+\alpha,\Gamma}.
$$
\nMoreover, it is not difficult to see that\n
$$
|\varkappa_1| \leq (1+q) \left[ |\varkappa| + \frac{2q}{(1-q)^2} ||\psi||_{0,\Gamma} \right]
$$
 and  $|a_k + a\bar{a}_k| \leq (1+q)$ 

$$
\left\|\frac{1-|a|^2}{2}\,\psi\right\|_{1+\alpha,\tilde{t}}\,\leq\,\left\|\frac{ds}{d\tilde{s}}\,\right\|_{\alpha,\tilde{t}}\,\frac{\mathrm{e}^{aMM_1}}{(1-q)^{\alpha}}\,\left\|\frac{1-|a|^2}{2}\,\psi\right\|_{1+\alpha,\tilde{t}}
$$

Moreover, it is not difficult to see that

Similarly from (25) we have, in view of Lemma (6),  
\n
$$
\left\| \frac{1-|a|^2}{2} \psi \right\|_{1+\alpha,\tilde{t}} \leqq \left\| \frac{ds}{d\tilde{s}} \right\|_{\alpha,\tilde{t}} \frac{e^{\alpha M n_1}}{(1-q)^{\alpha}} \left\| \frac{1-|a|^2}{2} \psi \right\|_{1+\alpha,\tilde{t}}.
$$
\nMoreover, it is not difficult to see that  
\n
$$
|\kappa_1| \leqq (1+q) \left[ |\kappa| + \frac{2q}{(1-q)^2} ||\psi||_{0,\tilde{t}} \right] \text{ and } |a_k + a\bar{a}_k| \leqq (1+q) |a_k|.
$$
\nThese estimates give  
\n
$$
||w||_{1+\alpha,D} \leqq \gamma_1 ||\psi||_{1+\alpha,\tilde{t}} + \gamma_2 |x| + \gamma_3 \sum_{k=1}^n |a_k| + \gamma_4 ||\lambda||_{\alpha,D},
$$
\nwhich proves the desired a priori estimate  
\nIf  $\mu_1, \mu_2$  are only Hölder continuous and  $e^{i\tau}, \psi \in C^{\alpha}(\Gamma)$ , one can still derive an a p-  
\nestimate using a subnorm of w:  
\n
$$
||w||_{\alpha} \leqq \gamma_1 ||\psi||_{\alpha} + \gamma_2 |x| + \gamma_3 \sum_{k=1}^n |a_k| + \gamma_4 ||w_{\tilde{s}} - \mu_1 w_{z} - \mu_2 \bar{w}_{z} - \nu_1 w - \nu_2 \bar{w}||_{0}.
$$

These estimates give

$$
|w||_{1+a,D} \le \gamma_1 \|w\|_{1+a,\Gamma} + \gamma_2 |x| + \gamma_3 \sum_{k=1}^n |a_k| + \gamma_4 \|z\|_{a,D},
$$
  
implies give

If  $\mu_1$ ,  $\mu_2$  are only Hölder continuous and e<sup>it</sup>,  $\psi \in C^{\mathfrak{a}}(F)$ , one can still derive an a priori If  $\mu_1$ ,  $\mu_2$  are only Hölder continuous and  $e^{it}$ ,  $\psi \in C^a(\Gamma)$ , one can<br>estimate using a subnorm of w:<br> $||w||_a \leq \hat{\gamma}_1 ||\psi||_a + \hat{\gamma}_2 |x| + \hat{\gamma}_3 \sum_{k=1}^n |a_k| + \hat{\gamma}_4 ||w_z - \mu_1 w_z - \mu_2 \overline{u}$ <br>We emphasize again that a prio

jwII **<sup>i</sup>**III + P2 H + P3EI akI + P4 **II <sup>W</sup> —** *P1Wz -* **P2WZ -** *VW V2JIo.* 

We emphasize again that a priori estimates such as (28) are most desirable for tablishing existence and uniqueness results for the nonlinear Hilbert problem establishing existence and uniqueness results for the nonlinear Hubert problem

**<sup>V</sup> -** 

consisting of nonlinear boundary and side conditions. In this regard, we refer to BEGEHR and HSIAO [14] where (28) is utilized to treat such problems.

To conclude the paper, we now state a similar result concerning an a priori estimate for functions with generalized derivatives.

*Theorem 3: Let*  $\mu_1$ *,*  $\mu_2$  *be two measurable functions in*  $\hat{D}$  *fulfilling (2), and let*  $v_1, v_2 \in L_p(D)$  for  $p > 2$  *but sufficiently close to 2. Then thère exist constants y*  $(1 \leq k \leq 4)$  *such that for*  $w \in W_p^{-1}(\hat{D})$  *satisfying* (24), the inequality  $||w||_0 + ||w_z||_p + ||w_{\bar{z}}||_p$ 

$$
\leq \gamma_1 \| \psi \|_{a,r} + \gamma_2 |x| + \gamma_3 \sum_{k=1}^n |a_k| + \gamma_4 \| w_{\overline{z}} - \mu_1 w_{\overline{z}} - \mu_2 \overline{w_{\overline{z}}} - \nu_1 w - \nu_2 \overline{w} \|_p
$$

 $\cdot$ holds.

The proof of this theorem is given in BEGEHR and HSLAO [15]<sup>3</sup>). Again this a priori estimate can be employed to establish existence and uniqueness theorems for the Hilbert boundary value problem on nonlinear equations of the form (3) with nonlinear boundary and side conditions. For details, we refer to BEGEHR and HSIAO [15]. ne proof of this theorem is given in BEGEHR and HSLAO [15]<sup>3</sup>). Againate can be employed to establish existence and uniqueness the<br>ert boundary value problem on nonlinear equations of the form (3) v<br>dary and side condition

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