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# A Priori Estimates for Elliptic Systems

H. BEGEHR[1]) and G. C. HSIAO[2])

Es werden A-priori-Abschätzungen für die allgemeine komplexe Beltrami-Gleichung im Zusammenhang mit Riemann-Hilbertschen Randbedingungen hergeleitet, die für Existenz- und Eindeutigkeitsaussagen von zugehörigen nichtlinearen Problemen herangezogen werden können. Dazu wird die Gleichung zusammen mit den Randbedingungen in die kanonische Form transformiert und wesentlich eine Darstellungsformel von Haack-Wendland benutzt.

Выводятся оценки для общего комплексного уравнения Бельтрами в связи с краевыми условиями Римана-Гильберта и привлекаемые к утверждениям о существовании и единственности соответствующих нелинейных проблем. Для этого уравнение и краевые условия преобразуются в каноническую форму и существенно используется одна формула представления Хаака-Вендланда.

A priori estimates for the general complex Beltrami equation in connection with Riemann-Hilbert boundary conditions are developed, which can be used for existence as well as uniqueness statements for related nonlinear problems. For this reason the equation together with the boundary conditions are transformed into the canonical form and essentially a representation formula originally given by Haack-Wendland is used.

## 1. Introduction

In this paper a priori estimates will be derived for solutions of the general Beltrami equation

$$w_{\bar z} + \mu_1 w_z + \mu_2 \overline{w_z} = aw + b\overline{w} + c \tag{1}$$

under the ellipticity condition

$$|\mu_1(z)| + |\mu_2(z)| \leqq q < 1. \tag{2}$$

In particular, we are concerned with the boundary value problem consisting of (1) together with certain boundary and side conditions which is known in complex analysis as the Hilbert or the Riemann-Hilbert boundary value problem (see, e.g. BEGEHR [2, 3], BEGEHR and GILBERT [5–7, 9], GAKHOV [19], GILBERT [20], MUSHKELISHVILI [26], and WENDLAND [30]).

One of the purposes for obtaining a priori estimates for solutions of linear equations is that they may be used to establish existence as well as uniqueness theorems for the related nonlinear problems. Indeed, in several recent papers by the authors

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[12—15], these a priori estimates are utilized to investigate boundary value problems for nonlinear elliptic equations of the type

$$w_{\bar z} = H(z, w, w_z) \tag{3}$$

with corresponding nonlinear boundary and side conditions. Similar a priori estimates for equation (1) have also been employed to study equations of type (3) with respect to the so-called Riemann boundary value problem [10, 11]. As indicated in a celebrated paper by WENDLAND [28] (see also WENDLAND [30]), because of the specific form of these estimates, they are particularly suitable for the constructive existence proof of solutions for the nonlinear elliptic system in connection with the Newton embedding procedure.

Our main results concerning a priori estimates for (1) are presented in Section 5. The derivations of these estimates are based on the representation formulas in Section 4 and the reduction of the relevant boundary value problem for (1) to a canonical problem in Section 3. In this reduction, as will be seen, by introducing appropriate transformations the Beltrami operator in (1) is reduced to the Cauchy Riemann operator according to BERS and NIRENBERG [16] (see also KÜNZI [23] and MONAHOV [25]). Section 2 contains some basic properties concerning the homeomorphisms of the Beltrami equation which will be needed later for establishing necessary bounds of the transformations introduced in Section 3.

## 2. The Beltrami equation

It is understood that by a complete homeomorphism of the Beltrami equation

$$\zeta_{\bar z} = \mu \zeta_z \tag{4}$$

we mean a solution of (4) in an appropriate class depending on $\mu$ which defines a bijective mapping of the $z$-plane onto the $\zeta$-plane (see VEKUA [27]). If

$$|\mu(z)| \leqq q < 1 \tag{5}$$

then $\zeta$ is a $K$-quasiconformal mapping with $K = (1 + q)/(1 - q)$. The solution of (4) can be made unique by imposing different additional conditions; for instance, one may require that zero and infinity are fixed points of the mapping $\zeta$.

Lemma 1: *Let $\mu$ be a measurable function in $\mathbb{C}$ satisfying* (5) *for some non-negative constant $q$ and $\mu \in L_{p'}(\mathbb{C})$, $p' < 2$. Then there exists a complete homeomorphism $\zeta$ of equation* (4) *belonging to a class $C^\alpha(\mathbb{C})$, $0 < \alpha < 1$ with $\zeta(\infty) = \infty$.*

For a proof of this lemma see AHLFORS [1], BOJARSKI [17], LEHTO and VIRTANEN [24], MONAHOV [25], and VEKUA [27]. If in this proof instead of the operator $\mathfrak{T}$,

$$\mathfrak{T}w(z) = -\frac{1}{\pi} \int_{\mathbb{C}} w(\zeta) \frac{d\xi\, d\eta}{\zeta - z} \quad (\zeta = \xi + i\eta), \tag{6}$$

one uses the operator $\mathring{\mathfrak{T}}$ defined by $\mathring{\mathfrak{T}}w(z) = \mathfrak{T}w(z) - \mathfrak{T}w(0)$, then the solution $\zeta$ has the representation $\zeta = z + \mathring{\mathfrak{T}}w(z)$ and fulfils the additional condition $\zeta(0) = 0$.

We note that from the assumption on $\mu$ we have $\mu \in L_p(\mathbb{C})$ for $p' \leqq p$. In what follows, we will assume $\mu$ to have compact support. As will be seen, with this assumption the Jacobian of $\zeta$ may be estimated. For the $L_p(\mathbb{C})$ space, $1 < p \leqq +\infty$, we will denote by $\|\cdot\|_p$ the usual $L_p$-norm. Furthermore if in (6), $\mathbb{C}$ is replaced by a bounded domain $D$, we then write $\mathfrak{T}_D$ instead of $\mathfrak{T}$. From VEKUA [27: p. 38], we have for $w \in L_p(\hat{D})$, $2 < p$, $\hat{D} = D \cup \partial D$,

$$\|\mathfrak{T}_D w\|_\alpha \leqq M(p, D) \|w\|_p, \qquad \alpha = (p-2)/p \tag{7}$$

where $\|\cdot\|_\alpha$ denotes the Hölder norm in $C^\alpha(\mathbf{C})$, and for $w \in L_p(\mathbf{C})$, $2 < p$, we have $|\mathfrak{T}w(z)| \leqq M(p) \|w\|_p |z|^\alpha$, $\alpha = (p-2)/p$, where $M(p)$ is a non-negative constant depending on $p$. Moreover the operator $\Pi$ defined by

$$\Pi w(z) = -\frac{1}{\pi} \int_{\mathbf{C}} w(\zeta) \frac{d\xi\, d\eta}{(\zeta - z)^2} \tag{8}$$

is bounded on $L_p(\mathbf{C})$, $1 < p < +\infty$, and satisfies the estimate (see VEKUA [27: p. 71]) $\|\Pi w\|_p \leqq \Lambda_p \|w\|_p$, $w \in L_p(\mathbf{C})$, where $\Lambda_p$ is a positive continuous function of $p$ and $\Lambda_2 = 1$. Thus, if $q$ is a fixed constant, $0 < q < 1$, then for $p$ sufficiently close to 2, we have

$$0 < q\Lambda_p < 1. \tag{9}$$

Similarly, for a bounded domain $D$, we define $\Pi_D$ by (8) with $\mathbf{C}$ replaced by $D$. Obviously $\Pi_D$ has the same norm as $\Pi$.

Now if we denote by $W_p{}^m$ the Sobolev space consisting of functions with generalized $m$-th order derivatives in $L_p(\mathbf{C})$, we have the following result.

Lemma 2: *Let $\mu \in W_p{}^2$ with $p > 2$ and have compact support $K$ lying in a bounded domain $D$. In addition suppose $\mu$ fulfils* (5), *and* (9) *is valid. Then for the complete homeomorphism $\zeta$ of* (4) *with $\zeta(0) = 0$ and $\zeta(\infty) = \infty$, there exists a constant $M$ depending only on $p$, $q$ and $K$ such that*

$$\exp\{-M \|\mu_z\|_p\} \leqq |\zeta_z(z)| \leqq \exp\{M \|\mu_z\|_p\} \qquad (z \in \mathbf{C}). \tag{10}$$

Proof: The singular integral equation

$$\Phi - \mu \Pi_D \Phi = \mu_z \tag{11}$$

has a unique solution $\Phi \in W_p{}^1(\hat{D})$ such that $\|\Phi\|_p \leqq \|\mu_z\|_p/(1 - q\Lambda_p)$. Because of (11), we see that

$$\hat{\zeta}(z) := \int_0^z \exp \mathfrak{T}_D \Phi(\zeta)\, [d\zeta + \mu\, d\bar{\zeta}] \qquad (z \in \mathbf{C}) \tag{12}$$

is independent of the path of integration. As is shown in MONAHOV [25: V § 3], $\hat{\zeta}$ is then a complete homeomorphism of (4) such that $\hat{\zeta}(z) - z \in C^{1+\alpha}(\mathbf{C})$ and $\hat{\zeta}(z) = z + O(|z|^{-1})$ as $z \to \infty$. Since $\hat{\zeta}(0) = 0$, $\hat{\zeta}$ must coincide with $\zeta$. Thus, from (12) we have $\zeta_z(z) = \exp\{\mathfrak{T}_D \Phi(z)\}$ and (10) follows immediately from (7) and the estimate for $\|\Phi\|_p$ ∎

From (4) and (10), it is easy to see that the Jacobian $J$ of $\zeta$ defined by $J = |\zeta_z|^2 - |\zeta_{\bar{z}}|^2$ is bounded below from zero. In fact, we have $J \geqq (1 - q^2) \exp\{-2M \|\mu_z\|_p\}$.

Lemma 2 remains valid, if the conditions on $\mu$ are slightly weakened. More precisely, we have the following result.

Lemma 3: *Inequalities* (10) *remain valid if $\mu$ is only in $W_p{}^1$ while all other assumptions from Lemma 2 are satisfied.*

Proof: Choose $p_1 > 1$ but sufficiently close to 1 so that $q\Lambda_{pp_1} < 1$ and define $p_2$ by $1/p_1 + 1/p_2 = 1$. Consider a sequence $\{\mu_n\} \subset W^2_{pp_2}(D)$ with $|\mu_n| \leqq q$ in $D$ and

$\mu_n = 0$ in $\mathbf{C} \setminus \hat{D}$ such that $\|\mu_n - \mu\|_{W^1_{pp_2}} \to 0$ for $n \to +\infty$. Then, because $1 < p_1 \leqq p_2$, we also have

$$\lim_{n\to+\infty} \|\mu_n - \mu\|_{W^1_{pp_1}} = 0 \quad \text{and} \quad \lim_{n\to+\infty} \|\mu_n - \mu\|_{W_p^1} = 0.$$

Now let $\zeta_n$ be the complete homeomorphism from Lemma 2 corresponding to $\mu_n$ given by

$$\zeta_n(z) = \int_0^z \exp \mathfrak{T}_D\Phi_n(\zeta) \, [d\zeta + \mu_n \, d\bar{\zeta}]$$

with $\Phi_n$ satisfying $\Phi_n - \mu_n \Pi_D \Phi_n = \mu_{nz}$. Then from $\Phi_n - \Phi_m - \mu_m \Pi_D(\Phi_n - \Phi_m) = (\mu_n - \mu_m)_z + (\mu_n - \mu_m) \Pi_D\Phi_n$, it follows that

$$\|\Phi_n - \Phi_m\|_p \leqq \frac{1}{1 - q\Lambda_p} \{\|(\mu_n - \mu_m)_z\|_p + \|\mu_n - \mu_m\|_{pp_2} \|\Pi_D\Phi_n\|_{pp_1}\}.$$

Since

$$\|\Pi_D\Phi_n\|_{pp_1} \leqq \Lambda_{pp_1} \|\Phi_n\|_{pp_1} \leqq \frac{\Lambda_{pp_1}}{1 - q\Lambda_{pp_1}} \|\mu_{nz}\|_{pp_1}$$

and for $n$ large enough

$$\|\mu_{nz}\|_{pp_1} \leqq \|(\mu_n - \mu)_z\|_{pp_1} + \|\mu_z\|_{pp_1} \leqq 1 + \|\mu_z\|_{pp_1},$$

we thus have for $n$ sufficiently large

$$\|\Phi_n - \Phi_m\|_p \leqq \frac{1}{1 - q\Lambda_p} \left[\|\mu_n - \mu_m\|_p + \frac{\Lambda_{pp_1}(1 + \|\mu_z\|_{pp_1})}{1 - q\Lambda_{pp_1}} \|\mu_n - \mu_m\|_{pp_2}\right].$$

This proves convergence of $\{\Phi_n\}$ in $L_p(\mathbf{C})$. Let $\Phi$ be the limit of $\{\Phi_n\}$ in $L_p(\mathbf{C})$. Then clearly, $\Phi = 0$ in $\mathbf{C} \setminus \hat{D}$, $\mathfrak{T}_{\hat{D}}\Phi \in C^\alpha(\mathbf{C})$, and $\Phi$ satisfies $\Phi - \mu\Pi_D\Phi = \mu_z$. If now $\zeta$ is given by $\Phi$ via (12), this function is a complete homeomorphism of (4) in $W_p^1(\mathbf{C})$ satisfying $\zeta(0) = 0$ and $\zeta(\infty) = \infty$. Because (10) is valid for each $\zeta_n$, by taking the limit we see that it holds for $\zeta$ too ∎

## 3. Reduction to canonical problems

In what follows, let $\mu_1, \mu_2 \in W_p^1(\mathbf{C})$, $p \geqq 2$, satisfy

$$\|\mu_1\|_0 + \|\mu_2\|_0 \leqq q < 1 \tag{13}$$

and

$$\|\mu_{1z}\|_p + \|\mu_{1\bar{z}}\|_p + \|\mu_{2z}\|_p + \|\mu_{2\bar{z}}\|_p \leqq M_1. \tag{14}$$

Define

$$\mu = \frac{1 + |\mu_1|^2 - |\mu_2|^2 - \sqrt{\Delta}}{2\bar{\mu}_1} = \frac{2\mu_1}{1 + |\mu_1|^2 - |\mu_2|^2 + \sqrt{\Delta}} \tag{15}$$

and

$$a = \frac{1 - |\mu_1|^2 + |\mu_2|^2 - \sqrt{\Delta}}{2\bar{\mu}_2} = \frac{2\mu_2}{1 - |\mu_1|^2 + |\mu_2|^2 + \sqrt{\Delta}} \tag{16}$$

where

$$\Delta = (1 - |\mu_1|^2 - |\mu_2|^2)^2 - 4\,|\mu_1\mu_2|^2 = [1 - (|\mu_1| + |\mu_2|)^2]\,[1 - (|\mu_1| - |\mu_2|)^2].$$

It is easy to see that $\Delta > 0$. In fact, $\Delta \geqq [1 - (|\mu_1| + |\mu_2|)^2]^2 > 0$, from which we obtain

$$|\mu| \leqq |\mu_1| + |\mu_2| \quad \text{and} \quad |a| \leqq |\mu_1| + |\mu_2|. \tag{17}$$

Moreover, a simple computation shows that $a = \mu_2/(|\mu_2|^2 - |\mu_1|^2 + \mu_1\mu^{-1})$. This together with the identities

$$|\mu_1|^2 (1 - |\mu|^2) = \sqrt{\Delta}\, \bar{\mu}_1\mu,$$

$$(|\mu_2|^2 - |\mu_1|^2)(1 - |\mu|^2) + \sqrt{\Delta} = |1 - \bar{\mu}_1\mu|^2 - |\mu_2\mu|^2$$

leads to an alternative form

$$a = \frac{\mu_2(1 - |\mu|^2)}{|1 - \bar{\mu}_1\mu|^2 - |\mu_2\mu|^2}. \tag{18}$$

Lemma 4: *Let $w \in W_p^1(\mathbf{C})$ fulfil the equation*

$$w_{\bar{z}} - \mu_1 w_z - \mu_2 \overline{w_z} = F \tag{19}$$

*and let $\zeta$ be a complete homeomorphism of*

$$\zeta_{\bar{z}} = \mu\zeta_z \tag{20}$$

*with $\mu$ given by* (15). *Then by changing the variable from $z$ to $\zeta$ and the corresponding unknown from $w$ to $\omega$ according to*

$$\omega = w - a\overline{w}, \tag{21}$$

*with $a$ given by* (16), *equation* (19) *can be transformed into the canonical form*

$$\omega_{\bar{\zeta}} = A\omega + B\overline{\omega} + C \tag{22}$$

*with*

$$A = \frac{-\bar{a}a_{\bar{\zeta}}}{1 - |a|^2}, \quad B = \frac{-a_{\bar{\zeta}}}{1 - |a|^2} \quad \textit{and} \quad C = \frac{(1 - \bar{\mu}_1\mu)\, F + \mu\mu_2\overline{F}}{[|1 - \bar{\mu}_1\mu|^2 - |\mu_2\mu|^2]\, \overline{\zeta_z}}.$$

*(The inverse transformation transforms* (22) *back into* (19) *accordingly.)*

Proof: In view of (20) we note that

$$w_z = \zeta_z w_\zeta + \bar{\mu}\overline{\zeta_z} w_{\bar{\zeta}} \quad \text{and} \quad w_{\bar{z}} = \mu\zeta_z w_\zeta + \overline{\zeta_z} w_{\bar{\zeta}}.$$

Substituting these expressions into (19) and its conjugate equation, we obtain two equations in four unknowns $w_{\bar{\zeta}}$, $\overline{w_\zeta}$, $\overline{w_{\bar{\zeta}}}$ and $w_\zeta$. However, because of the choice of $\mu$ in (15), it is not difficult to see that both $\overline{w_{\bar{\zeta}}}$ and $w_\zeta$ can be eliminated. Thus, we arrive at the equation

$$[|1 - \mu_1\bar{\mu}|^2 - |\mu_2\mu|^2]\, \overline{\zeta_z} w_{\bar{\zeta}} + [(\bar{\mu} - \bar{\mu}_1)\, \mu_2\mu - (1 - \mu\bar{\mu}_1)\, \mu_2]\, \overline{\zeta_z}\overline{w_\zeta}$$
$$= (1 - \bar{\mu}_1\mu)\, F + \mu_2\mu\overline{F},$$

from which formula (22) follows immediately from (21) and the definition of $a$ in (18) ∎

We note that

$$|C\overline{\zeta_z}| \leqq \frac{|F|}{1 - q^2} \quad \text{and} \quad \frac{|\bar{a}\omega + \overline{\omega}|}{1 - |a|^2} \leqq \frac{|\omega|}{1 - q}.$$

Moreover, the function $\zeta_z$ can be estimated by using (10) if $\mu_1, \mu_2$ are required to vanish outside a finite domain $D$.

For later calculations, estimates of the derivatives of $\mu$ and $a$ are also needed. Because of the symmetric forms from the definitions (15) and (16), it is sufficient to consider only one of them, say $\mu$. However, the lemma below holds for $a$ too.

Lemma 5: *In addition to the assumptions* (13) *and* (14) *let* $\mu_1, \mu_2$ *vanish outside of a finite domain* $D$. *Then for* $\mu$ *defined by* (15), *there hold the estimates*

$$|\mu_z|, |\mu_{\bar z}| \leqq \frac{2M_1}{(1-q)^2} \quad \text{and} \quad |\mu_\zeta|, |\mu_{\bar\zeta}| \leqq \frac{2M_1}{(1-q)^3} e^{M(p,q,D)M_1}. \tag{23}$$

From VEKUA [27: p. 38] we have $\|\mu\|_\alpha \leqq M(p, q, D, M_1)$, $\alpha = (p-2)/p$. Moreover, if $\mu_1, \mu_2 \in C^{1+\alpha}(\hat D)$, $0 < \alpha < 1$, then similarly $\|\mu_z\|_\alpha, \|\mu_{\bar z}\|_\alpha \leqq M(p, q, D, M_1)$. Here, instead of the $p$-norm as in (14), $M_1$ is a bound for the sum of $\alpha$-norms of first order derivatives of the $\mu_i$ 's. In the same manner, it can be shown that $\|\zeta_z\|_\alpha \leqq M(p, q, \alpha, D, M_1)$.

Proof of the lemma: We have from (15)

$$\mu_z = -\frac{\overline{\mu_{1\bar z}}}{\mu_1}\mu - \frac{\mu}{\sqrt{\Delta}}[|\mu_1|^2 - |\mu_2|^2]_z + \frac{(|\mu_1|^2)_z}{\sqrt{\Delta}\,\bar\mu_1} \quad \text{and} \quad \mu_\zeta = (\bar\zeta_z\mu_z - \bar\zeta_{\bar z}\mu_{\bar z})/J.$$

As we indicated before, the Jacobian $J = |\zeta_z|^2 - |\zeta_{\bar z}|^2$ is positive. Now with the help of the inequalities $|\mu| \leqq |\mu_1|/(1-q^2)$ and $|\mu| \leqq \sqrt{\Delta}\, q/(1-q^2)^2$ the desired results for $|\mu_z|$ and $|\mu_\zeta|$ follow immediately by direct computations. The estimates for $\mu_{\bar z}$ and $\mu_{\bar\zeta}$ follow similarly ∎

We now derive the corresponding Hilbert boundary and side conditions according to the transformation $(z, w) \to (\zeta, \omega)$ defined by (20) and (21). First we need some simple property concerning the boundary $\partial D = \Gamma$ of $D$.

Lemma 6: *If* $\Gamma \in C^{1+\alpha}$ *then* $\tilde\Gamma := \zeta(\Gamma) \in C^{1+\alpha}$.

Proof: Let $s$ be the arc length parameter and $L$ be the total length of $\Gamma$. Then we have from (10) and (23)

$$\int_{\tilde\Gamma} |d\zeta| = \int_0^L |d\zeta(z(s))| = \int_0^L |\zeta_z|\, |z'(s) + \mu\overline{z'(s)}|\, ds \leqq M(p, q, D, M_1)\, L,$$

to that $\tilde\Gamma$ is rectifiable with a total length, say $\tilde L$. Moreover if $\tilde s$ denotes the arc length parameter of $\tilde\Gamma$, then

$$\frac{d\zeta}{d\tilde s} = \left(z'(s)\,\zeta_z + \overline{z'(s)}\,\zeta_{\bar z}\right)\frac{ds}{d\tilde s} = \frac{\zeta_z\left(z'(s) + \mu\overline{z'(s)}\right)}{|\zeta_z|\,|z'(s) + \mu\overline{z'(s)}|}.$$

In the case of $\mu \in W_p^1(\mathbb{C})$, all the functions involved here are in $C^\alpha([0, L])$ and the denominator is bounded away from zero. In particular, we note that here

$$\frac{|\zeta_z(z(s)) - \zeta_z(z(s'))|}{|s - s'|^\alpha} = \frac{|\zeta_z(z(s)) - \zeta_z(z(s'))|}{|z(s) - z(s')|^\alpha}\left|\frac{z(s) - z(s')}{s - s'}\right|^\alpha$$

is bounded, since $\zeta_z \in C^\alpha(\mathbb{C})$ and $z \in C^{1+\alpha}([0, L])$; hence $\zeta_z(z(\cdot)) \in C^\alpha([0, L])$. Similarly one can show that $\mu(z(\cdot)) \in C^\alpha([0, L])$. Therefore it follows that $\zeta \in C^{1+\alpha}([0, \tilde L])$ ∎

Lemma 7: *Let w fulfil the boundary and side conditions*

$$\left.\begin{aligned} &\operatorname{Re}\{e^{i\tau}w\} = \psi \quad on \quad \Gamma,\ e^{i\tau}, \psi \in C^\alpha(\Gamma),\ n := \frac{-1}{2\pi}\int_\Gamma d\tau \geqq 0 \\ &\frac{1}{\Sigma}\int_\Gamma \operatorname{Im}\{e^{i\tau}w\}\,\sigma\,ds = \varkappa,\ \sigma \in C(\Gamma),\ 0 \leqq \sigma,\ 0 < \Sigma := \int_\Gamma \sigma\,ds,\ \varkappa \in \mathbf{R} \\ &w(z_k) = a_k,\ z_k \in D,\ a_k \in \mathbf{C}\ (1 \leqq k \leqq n). \end{aligned}\right\} \tag{24}$$

*Then $\omega$ fulfils analogous conditions of the form*

$$\left.\begin{aligned} &\operatorname{Re}\{e^{i\tau_1}\omega\} = \frac{1-|a|^2}{\varrho}\,\psi \ on\ \tilde\Gamma \\ &\frac{1}{\Sigma_1}\int_{\tilde\Gamma} \operatorname{Im}\{e^{i\tau_1}\omega\}\,\sigma_1\,d\tilde s = \varkappa_1 \\ &\omega(\zeta_k) = a_k + a\bar a_k\ (1 \leqq k \leqq n). \end{aligned}\right\} \tag{25}$$

*Here* $\tau_1 := \tau + \varphi$,

$$\sigma_1 := \frac{1}{\varrho}\frac{ds}{d\tilde s}\,\sigma, \quad \Sigma_1 := \int_{\tilde\Gamma} \sigma_1\,d\tilde s, \quad \varkappa_1 := \frac{\Sigma}{\Sigma_1}\varkappa - \frac{2}{\Sigma_1}\int_\Gamma \operatorname{Im}\{a\,e^{2i\tau}\varrho^{-2}\psi\}\,\sigma\,ds$$

*and $\zeta_k = \zeta(z_k)$ while $\varrho$ and $e^{i\varphi}$ are $C^\alpha(\Gamma)$ functions defined by $\varrho\,e^{i\varphi} = 1 + \bar a\,e^{-2i\tau}$. Furthermore, all the z in* (25) *are replaced by* $z(\zeta)$.

The proof follows by direct computations. Obviously the data functions in (25) fulfil with respect to $\tilde D = \zeta(D)$ the same conditions as those in (24) with respect to $D$. In particular, we see that because $|a| < 1$ (cf. (17)), it follows that

$$\int_\Gamma d\log(1 + \bar a\,e^{-2i\tau}) = 0 \quad \text{and hence} \int_{\tilde\Gamma} d\varphi = 0.$$

Thus, the index in (25) is also equal to $n$ as in (24).
If $\mu_1, \mu_2 \in C^{1+\alpha}(\hat D)$, then $e^{i\tau}, \psi \in C^{1+\alpha}(\tilde\Gamma)$ would imply $e^{i\tau_1}, \dfrac{1-|a|^2}{\varrho}\psi \in C^{1+\alpha}(\Gamma)$ as well.

From Lemma 4 and Lemma 7, the boundary value problem (19), (24) is transformed into the canonical problem (22), (25).

## 4. Representation formulas

A representation formula for $C^1(\hat D)$ functions in terms of Green's and Neumann's functions $G^{\mathrm{I}}(z,\zeta)$ and $G^{\mathrm{II}}(z,\zeta)$ of a $C^{1+\alpha}$ domain $D$ are given by Haack and Wendland [21: Formula 10.43/p. 271]. If $\Phi$ denotes the conformal mapping of $D$ onto the unit disc $\mathbf{D}$, then Green's function is given by

$$G^{\mathrm{I}}(z,\zeta) = -\frac{1}{2\pi}\log\left|\frac{\Phi(\zeta) - \Phi(z)}{1 - \overline{\Phi(\zeta)}\,\Phi(z)}\right|$$

while the Neumann function is of the form (see Haack and Wendland [21: § 4.7])

$$G^{\mathrm{II}}(z,\zeta) = \hat G^{\mathrm{II}}(z,\zeta) + V(z,\zeta)$$

where

$$\hat{G}^{\mathrm{II}}(z,\zeta) = -\frac{1}{2\pi}\log\left|\left(\Phi(\zeta)-\Phi(z)\right)\left(1-\overline{\Phi(\zeta)}\,\Phi(z)\right)\right|$$

and $V$ is harmonic in $D$ given by

$$V(z,\zeta) = \frac{1}{\pi\Sigma}\int_\Gamma \sigma(s)\log\left|\left[\Phi(\zeta(s))-\Phi(z)\right]\left[\Phi(\zeta(s))-\Phi(\zeta)\right]\right| ds$$
$$-\frac{1}{\pi\Sigma^2}\int_\Gamma\int_\Gamma \sigma(s)\,\sigma(t)\log\left|\Phi(\zeta(s))-\Phi(\zeta(t))\right| ds\,dt \quad (z,\zeta\in D).$$

Here $\sigma$ is a non-negative continuous function on $\Gamma$ with

$$\Sigma := \int_\Gamma \sigma(s)\,ds \neq 0.$$

Moreover

$$d_n V(z,\zeta) = \frac{1}{2\pi}\,|d\Phi(\zeta)| - \frac{\sigma(s)}{\Sigma}\,ds$$

for all $z\in D$, $\zeta\in\Gamma$ and if $\sigma(s) := \left|\frac{d\Phi(\zeta(s))}{ds}\right|$, then $V$ vanishes identically.

Lemma 8: *Each $w\in C^1(\hat{D})$ can be represented by*

$$w(z) = -\theta(z) + i\int_D \{w_{\bar\zeta}(\zeta)\,[G_\zeta{}^{\mathrm{I}} + G_\zeta{}^{\mathrm{II}}](\zeta,z) + \overline{w_{\bar\zeta}(\zeta)}\,[G_{\bar\zeta}{}^{\mathrm{I}} - G_{\bar\zeta}{}^{\mathrm{II}}](\zeta,z)\}\,d\zeta\,d\bar\zeta\,,$$

$$\theta(z) := \int_\Gamma \operatorname{Re} w(\zeta)\,[d_nG^{\mathrm{I}} - i\,dG^{\mathrm{II}}](\zeta,z) + iC\,,$$

$$C := \int_\Gamma \operatorname{Im} w(\zeta)\,d_nG^{\mathrm{II}}(\zeta,z) = -\frac{1}{\Sigma}\int_\Gamma \operatorname{Im} w(\zeta(s))\,\sigma(s)\,ds.$$

In Wendland [30: p. 22] this representation formula is reformulated for $C^1(\hat{D})$ functions which satisfy the homogeneous boundary and side conditions given by (24) with $e^{i\tau}$ belonging to $C^{1+\alpha}(\Gamma)$ and for $\psi$, $\varkappa$ and $a_k$ being equal to zero. It is not difficult to see that similar representation formulas can be derived from Lemma 8 for non-homogeneous boundary and side conditions such as (24) with $e^{i\tau}$, $\psi\in C^{1+\alpha}(\Gamma)$.

Following Wendland [30], let $P_k$ denote the polynomials in $z$ of degree $2n$ defined by $P_k(z_l) = 0$ and $P_k'(z_l) = \delta_{kl}$ $(1\leq k, l\leq n)$. We set

$$w_1 = w - w_2, \qquad w_2 = \sum_{k=1}^{n} w_{\bar z}(z_k)\,\overline{P_k(z)},$$

and define $\tilde{w}$ by

$$\tilde{w}(z) = w_1(z)\,e^{-\tilde\Phi(z)}\prod_{k=1}^{n}(z-z_k)^{-1}$$

where

$$\tilde\Phi(z) := i\int_\Gamma \tilde\tau[d_nG^{\mathrm{I}} - i\,dG^{\mathrm{II}}](\zeta,z), \qquad e^{i\tilde\tau} := e^{i\tau}\prod_{k=1}^{n}\frac{z-z_k}{|z-z_k|}.$$

Then clearly $\tilde{w} \in C^1(\hat{D})$ and satisfies the boundary condition

$$\operatorname{Re} \tilde{w}|_\Gamma = \tilde{\psi} := [\psi - \operatorname{Re} \{e^{i\tau} w_2\}] \, e^{-\operatorname{Re}\tilde{\Phi}(z)} \prod_{k=1}^{n} |z - z_k|^{-1} \quad \text{on } \Gamma,$$

and the side condition

$$-\frac{1}{\tilde{\Sigma}} \int_\Gamma \operatorname{Im} \tilde{w}\tilde{\sigma} \, ds = \tilde{\varkappa} := \frac{\Sigma}{\tilde{\Sigma}} \varkappa + \frac{1}{\tilde{\Sigma}} \int_\Gamma \operatorname{Im} \{e^{i\tau} w_2\} \, \sigma \, ds$$

with

$$\tilde{\sigma}(z) := \sigma(z) \, e^{\operatorname{Re}\tilde{\Phi}(z)} \prod_{k=1}^{n} |z - z_k|, \qquad z \in \Gamma, \qquad \text{and } \tilde{\Sigma} := \int_\Gamma \tilde{\sigma} \, ds.$$

Now let $\tilde{G}^{\mathrm{II}}$ denote the Neumann function corresponding to $\tilde{\sigma}$. Then an application of Lemma 8 to $\tilde{w}$ yields the following lemma.

Lemma 9: *Each $w \in C^1(\hat{D})$ which fulfils (24) with $a_k = 0$ can be represented by*

$$\begin{aligned} w(z) = {} & \sum_{k=1}^{n} w_{\bar{z}}(z_k) \overline{P_k(z)} - \theta_n(z) \, e^{\tilde{\Phi}(z)} \prod_{k=1}^{n} (z - z_k) \\ & + i \int_D \Bigg\{ e^{\tilde{\Phi}(z) - \tilde{\Phi}(\zeta)} \prod_{k=1}^{n} \frac{z - z_k}{\zeta - z_k} \left[ w_{\bar{\zeta}}(\zeta) - \sum_{k=1}^{n} w_{\bar{z}}(z_k) \overline{P_k'(\zeta)} \right] \\ & \times [G_\zeta^{\mathrm{I}} + \tilde{G}_\zeta^{\mathrm{II}}] (\zeta, z) + e^{\overline{\tilde{\Phi}(z)} - \overline{\tilde{\Phi}(\zeta)}} \prod_{k=1}^{n} \frac{\bar{z} - \bar{z}_k}{\bar{\zeta} - \bar{z}_k} \left[ \overline{w_{\bar{\zeta}}(\zeta)} - \sum_{k=1}^{n} \overline{w_{\bar{z}}(z_k)} \, P_k'(\zeta) \right] \\ & \times [G_{\bar{\zeta}}^{\mathrm{I}} - \tilde{G}_{\bar{\zeta}}^{\mathrm{II}}] [\zeta, z) \Bigg\} \, d\zeta \, d\bar{\zeta}, \end{aligned}$$

*for $z \in \hat{D}$, where*

$$\begin{aligned} \theta_n(z) = {} & \theta(z) - \int_\Gamma \operatorname{Re} \left\{ e^{i\tau} \sum_{k=1}^{n} w_{\bar{z}}(z_k) \overline{P_k(\zeta)} \right\} e^{-\tilde{\Phi}(\zeta)} \prod_{k=1}^{n} |\zeta - z_k|^{-1} \\ & \times [d_n G^{\mathrm{I}} - i d\tilde{G}^{\mathrm{II}}] (\zeta, z) + \frac{i}{\tilde{\Sigma}} \int_\Gamma \operatorname{Im} \left\{ e^{i\tau} \sum_{k=1}^{n} w_{\bar{z}}(z_k) \overline{P_k(\zeta)} \right\} \sigma \, ds, \end{aligned}$$

*with*

$$\theta(z) = \int_\Gamma \psi(\zeta) \, e^{-\tilde{\Phi}(\zeta)} \prod_{k=1}^{n} |\zeta - z_k|^{-1} \, [d_n G^{\mathrm{I}} - i d\tilde{G}^{\mathrm{II}}] (\zeta, z) + i \frac{\Sigma}{\tilde{\Sigma}} \varkappa.$$

This representation formula is also valid for $w \in W_p^1(\hat{D})$. In this case the regularity assumptions on $e^{i\tau}$ and $\psi$ can be weakened to $e^{i\tau}, \psi \in C^\alpha(\Gamma)$ as in (24).

The representation formula in the lemma can be made more explicit in terms of the conformal mapping function $\Phi$. In particular, we note that

$$[G_\zeta^{\mathrm{I}} + \tilde{G}_\zeta^{\mathrm{II}}] (\zeta, z) = -\frac{1}{2\pi} \frac{\Phi'(\zeta)}{\Phi(\zeta) - \Phi(z)} - \frac{1}{2\pi} \lambda(\zeta),$$

$$[G_{\bar{\zeta}}^{\mathrm{I}} - \tilde{G}_{\bar{\zeta}}^{\mathrm{II}}] (\zeta, z) = -\frac{1}{2\pi} \frac{\Phi(z) \overline{\Phi'(\zeta)}}{1 - \Phi(z) \overline{\Phi(\zeta)}} + \frac{1}{2\pi} \overline{\lambda(\zeta)}$$

and

$$[d_nG^{\mathrm{I}} - id\tilde{G}^{\mathrm{II}}](\zeta, z) = -\frac{1}{2\pi i}\left[\frac{\Phi'(\zeta)\,d\zeta}{\Phi(\zeta) - \Phi(z)} - \frac{\Phi(z)\,\overline{\Phi'(\zeta)}\,d\bar{\zeta}}{1 - \overline{\Phi(\zeta)}\,\Phi(z)}\right] - \frac{1}{2\pi i}\left[\lambda(\zeta)\,d\zeta + \overline{\lambda(\zeta)}\,d\bar{\zeta}\right]$$

where

$$\lambda(z) = \frac{1}{\tilde{\Sigma}} \int_\Gamma \tilde{\sigma}(\zeta(s)) \frac{\Phi'(z)\,ds}{\Phi(\zeta(s)) - \Phi(z)}.$$

These relations will be utilized for obtaining estimates for $w$ as well as for $w_z$. For later use, we also need a similar representation formula for $w_z$. This can be achieved by differentiating the representation formulas for $w$. Explicitly we have

$$\begin{aligned}
w_z(z) = {} & \left[\tilde{\Phi}'(z) + \sum_{k=1}^n (z - z_k)^{-1}\right]\left[w(z) - \sum_{k=1}^n w_{\bar{z}}(z_k)\,\overline{P_k(z)}\right] \\
& + \frac{1}{\pi i}\int_\Gamma \left[\psi(\zeta) - \mathrm{Re}\left\{e^{-i\tau}\sum_{k=1}^n w_{\bar{z}}(z_k)\,\overline{P_k(\zeta)}\right\}\right] e^{\tilde{\Phi}(z)-\tilde{\Phi}(\zeta)} \\
& \times \prod_{k=1}^n \frac{z - z_k}{|\zeta - z_k|}\,\frac{\Phi'(z)\,d\Phi(\zeta)}{(\Phi(\zeta) - \Phi(z))^2} \\
& + \frac{1}{2\pi i}\int_D e^{\tilde{\Phi}(z)-\tilde{\Phi}(\zeta)} \prod_{k=1}^n \frac{z - z_k}{\zeta - z_k}\left[w_{\bar{\zeta}}(\zeta) - \sum_{k=1}^n w_{\bar{z}}(z_k)\,\overline{P_k'(\zeta)}\right] \\
& \times \frac{\Phi'(z)\,\Phi'(\zeta)}{(\Phi(\zeta) - \Phi(z))^2}\,d\zeta\,d\bar{\zeta} \\
& + \frac{1}{2\pi i}\int_D e^{\overline{\tilde{\Phi}(z)-\tilde{\Phi}(\zeta)}} \prod_{k=1}^n \frac{z - z_k}{\bar{\zeta} - \bar{z}_k}\left[\overline{w_{\bar{\zeta}}(\zeta)} - \sum_{k=1}^n \overline{w_{\bar{z}}(z_k)}\,P_k'(\zeta)\right] \\
& \times \frac{\Phi'(z)\,\overline{\Phi'(\zeta)}}{(1 - \overline{\Phi(\zeta)}\,\Phi(z))^2}\,d\zeta\,d\bar{\zeta}.
\end{aligned}$$

With the representation formulas available bounds for $w$ and $w_z$ may be derived. We begin with the following crucial estimates in connection with $w_z$.

Lemma 10: *If $z_k$ $(1 \leq k \leq n)$ are distinct points in the unit disc* $\mathbf{D}$, *then*

$$\frac{1}{\pi}\int_{\mathbf{D}} \prod_{k=1}^n \frac{z - z_k}{\zeta - z_k}\,\frac{d\xi\,d\eta}{(\zeta - z)^2} \quad \text{and} \quad \frac{1}{\pi}\int_{\mathbf{D}} \prod_{k=1}^n \frac{z - z_k}{\bar{\zeta} - \bar{z}_k}\,\frac{d\xi\,d\eta}{(1 - \bar{\zeta}z)^2}$$

*are bounded in* $\mathbf{D}$.

Proof: Let $g(z) = \prod_{k=1}^n (z - z_k)^{-1}$. We note that

$$\frac{1}{2\pi i}\int_{|\zeta|=1} \bar{\zeta} g(\zeta)\,\frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{\zeta(\zeta - z)\prod_{k=1}^n (\zeta - z_k)} = 0$$

for $|z| < 1$. Hence an application of Green's formula

$$\frac{1}{\pi}\int_D w_{\bar\zeta}\, d\xi\, d\eta = \frac{1}{2\pi i}\int_{\partial D} w\, d\zeta\,,$$

to $\bar\zeta g(\zeta)/(\zeta - z)$ in a domain $D$ obtained from $\mathbf{D}$ by removing circles with sufficiently small radii around the points $z$ and $z_k$ $(1 \leqq k \leqq n)$ from $\mathbf{D}$ yields

$$\frac{1}{\pi}\int_{\mathbf{D}} g(\zeta)\,\frac{d\xi\, d\eta}{\zeta - z} = \left[-\bar z + \sum_{k=1}^{n} \bar z_k \prod_{l\neq k} \frac{z - z_l}{z_k - z_l}\right] g(z).$$

Dividing both sides by $g(z)$ and then differentiating with respect to $z$, we obtain

$$\frac{1}{\pi}\int_{\mathbf{D}} \left\{\prod_{k=1}^{n} \frac{z - z_k}{\zeta - z_k}\,\frac{1}{(\zeta - z)^2} + \sum_{k=1}^{n}\prod_{l\neq k}\frac{z - z_l}{\zeta - z_l}\,\frac{1}{(\zeta - z_k)\,(\zeta - z)}\right\} d\xi\, d\eta$$

$$= \sum_{k=1}^{n} \bar z_k \sum_{l\neq k}\ \prod_{\nu\neq k,l} \frac{z - z_\nu}{z_k - z_\nu}\,\frac{1}{z_k - z_l}.$$

The second integral again can be integrated explicitly by the preceding formula. Indeed,

$$\frac{1}{\pi}\int_{\mathbf{D}} \sum_{k=1}^{n}\prod_{l\neq k}\frac{z - z_l}{\zeta - z_l}\,\frac{1}{(\zeta - z_k)\,(\zeta - z)}\, d\xi\, d\eta$$

$$= \sum_{k=1}^{n}\frac{-\bar z}{z - z_k} + \sum_{k=1}^{n}\frac{1}{z - z_k}\sum_{\nu=1}^{n} \bar z_\nu \prod_{l\neq\nu}\frac{(z - z_l)}{(z_\nu - z_l)}.$$

Consequently, after some simplifications, we arrive at the relation

$$\frac{1}{\pi}\int_{\mathbf{D}} \prod_{k=1}^{n}\frac{z - z_k}{\zeta - z_k}\,\frac{d\xi\, d\eta}{(\zeta - z)^2} = \sum_{k=1}^{n}\frac{\bar z - \bar z_k}{z - z_k} + \sum_{k=1}^{n}\frac{\bar z_k}{z - z_k}\left[1 - \prod_{l\neq k}\frac{z - z_l}{z_k - z_l}\right].$$

Here the right-hand side is bounded in $\mathbf{D}$.

Similarly, we apply Green's formula

$$\frac{1}{\pi}\int_D w_\zeta\, d\xi\, d\eta = -\frac{1}{2\pi i}\int_{\partial D} w\, d\bar\zeta$$

to $\zeta\overline{g(\zeta)}/(1 - \bar\zeta z)$ in a proper subdomain $D$ of $\mathbf{D}$, and obtain

$$\frac{1}{\pi}\int_{\mathbf{D}} \overline{g(\zeta)}\,\frac{d\xi\, d\eta}{1 - \bar\zeta z} = \sum_{k=1}^{n}\frac{z_k}{1 - \bar z_k z}\prod_{l\neq k}\frac{1}{\bar z_k - \bar z_l}$$

$$+\prod_{k=1}^{n}\frac{-1}{\bar z_k} + \sum_{k=1}^{n}\prod_{l\neq k}\frac{1}{\bar z_k - \bar z_l}\,\frac{1}{\bar z_k(1 - \bar z_k z)}$$

by making use of

$$\frac{1}{2\pi i}\int_{|\zeta|=1} \zeta\overline{g(\zeta)}\,\frac{d\bar\zeta}{1 - \bar\zeta z} = -\prod_{k=1}^{n}\frac{-1}{z_k} - \sum_{k=1}^{n}\frac{1}{\bar z_k(1 - \bar z_k z)}\prod_{l\neq k}\frac{1}{\bar z_k - \bar z_l}.$$

Multiplying by $z$ and differentiating with respect to $z$ then lead to

$$\frac{1}{\pi}\int_{\mathbf{D}}\prod_{k=1}^{n}\frac{1}{\bar\zeta-\bar z_k}\frac{d\xi\,d\eta}{(1-\bar\zeta z)^2}$$

$$=\prod_{k=1}^{n}\frac{-1}{\bar z_k}+\sum_{k=1}^{n}\frac{1+|z_k|^2}{(1+\bar z_k z)\,\bar z_k}\prod_{l\neq k}\frac{1}{\bar z_k-\bar z_l}.$$

Because this as well as $\prod_{k=1}^{n}(z-z_k)$ is bounded in $\mathbf{D}$, the lemma is proved ∎

As a consequence of Lemma 10, we now have the following result.

Lemma 11: *Let $z_k$ $(1\leq k\leq n)$ be distinct points in the unit disc $\mathbf{D}$. Then the inequality*

$$\left|\frac{1}{\pi}\int_{\mathbf{D}}\left[\frac{f(\zeta)}{(\zeta-z)^2}\prod_{k=1}^{n}\frac{z-z_k}{\zeta-z_k}+\frac{\overline{f(\zeta)}}{(1-\bar\zeta z)^2}\prod_{k=1}^{n}\frac{z-z_k}{\bar\zeta-\bar z_k}\right]d\xi\,d\eta\right|\leq M\,\|f\|_\alpha$$

*holds for every $f\in C^\alpha(\hat{\mathbf{D}})$ with $0<\alpha<1$, where $\hat{\mathbf{D}}:=\mathbf{D}\cup\partial\mathbf{D}$ and $M$ is a non-negative constant depending on $z_k$ and $\alpha$.*

Proof: Let $g(\zeta)=\prod_{k=1}^{n}\frac{z-z_k}{\zeta-z_k}$ and for $f\in C^\alpha(\mathbf{D})$, we write

$$\int_{\mathbf{D}}f(\zeta)\,g(\zeta)\frac{d\xi\,d\eta}{(\zeta-z)^2}$$

$$=\int_{|\zeta+z|<1}[f(\zeta+z)-f(z)]\,g(\zeta+z)\frac{d\xi\,d\eta}{\zeta^2}+f(z)\int_{\mathbf{D}}g(\zeta)\frac{d\xi\,d\eta}{(\zeta-z)^2}.$$

Then we see that

$$\int_{|\zeta+z|<1}|f(\zeta+z)-f(z)|\,|g(\zeta+z)|\frac{d\xi\,d\eta}{|\zeta|^2}\leq H_\alpha(f)\int_{\mathbf{D}}|g(\zeta)|\frac{d\xi\,d\eta}{|\zeta-z|^{2-\alpha}}.$$

Here $H_\alpha(f)$ denotes the Hölder coefficient of $f$, and the integral on the right-hand side is bounded. This follows from the inequalities

$$\int_{\mathbf{D}}\frac{d\xi\,d\eta}{|\zeta-z_k|\,|\zeta-z|^{2-\alpha}}\leq M(\alpha)\,|z-z_k|^{\alpha-1}$$

(see VEKUA [27: p. 39]) by decomposing $g$ into partial fractions. Similarly, we find

$$\left|\int_{\mathbf{D}}f(\zeta)\,g(\zeta)\frac{d\xi\,d\eta}{(1-\zeta\bar z)^2}\right|$$

$$\leq H_\alpha(f)\int_{\mathbf{D}}|g(\zeta)|\frac{|\zeta-z|^\alpha}{|1-\zeta\bar z|^2}\,d\xi\,d\eta+\|f\|_0\left|\int_{\mathbf{D}}g(\zeta)\frac{d\xi\,d\eta}{(1-\bar\zeta z)^2}\right|$$

and also

$$\int_{\mathbf{D}}|g(\zeta)|\frac{|\zeta-z|^\alpha}{|1-\zeta\bar z|^2}\,d\xi\,d\eta\leq\int_{\mathbf{D}}|g(\zeta)|\frac{d\xi\,d\eta}{|\zeta-z|^{2-\alpha}}.$$

By using Lemma 10, this will complete the proof ∎

In BEGEHR and HSIAO [15] it is shown that for $n > 0$ the integral operator from Lemma 11 fails to be a unitary operator in $L_2(\hat{\mathbf{D}})$; its norm in $L_p$ $(p > 1)$ is, in general, greater than one. For $n = 0$ this operator reduces to the $\tilde{\Pi}$-operator (see VEKUA [27: p. 210]).

Lemma 12: *If $\varrho \in C^{1+\alpha}(\Gamma)$, then P,*

$$P(z) = \frac{1}{2\pi i} \int_\Gamma \varrho(\zeta) \frac{\Phi(\zeta) + \Phi(z)}{\Phi(\zeta) - \Phi(z)} \frac{d\Phi(\zeta)}{\Phi(\zeta)} \qquad (z \in D),$$

*belongs to $C^{1+\alpha}(\hat{D})$ and satisfies the inequality $\|P\|_{1+\alpha} \leqq M \|\varrho\|_{1+\alpha,\Gamma}$ for some constant M depending only on $\alpha$ and D. Here $\Phi$ denotes the conformal mapping of D onto the unit disc* **D**.

Proof: Let $z_0 \in D$ be the zero of $\Phi$. Then from Privalov's theorem (see COURANT and HILBERT [18: p. 380]), we have $|P(z) - P(z_0)| \leqq kH_{\alpha,\Gamma}(\varrho) |z - z_0|^\alpha$ so that $\|P\|_0 \leqq k \|\varrho\|_{\alpha,\Gamma} \leqq \tilde{k} \|d\varrho/ds\|_{0,\Gamma} \leqq \tilde{k} \|\varrho\|_{1,\Gamma}$. Now $P' = (dP/d\Phi) \Phi'$, and an integration by part yields

$$\frac{dP(z)}{d\Phi(z)} = \frac{1}{2\pi i} \int_\Gamma \frac{d\varrho(\zeta)}{d\Phi(\zeta)} \frac{\Phi(\zeta) + \Phi(z)}{\Phi(\zeta) - \Phi(z)} \frac{d\Phi(\zeta)}{\Phi(\zeta)} + \frac{1}{2\pi i} \int_\Gamma \frac{d\varrho(\zeta)}{d\Phi(\zeta)} \frac{d\Phi(\zeta)}{\Phi(\zeta)}.$$

Again using Privalov's theorem, we get $\|P'\|_\alpha \leqq \hat{k} \|\varrho\|_{1+\alpha,\Gamma}$, where $\hat{k}$ is a constant independent of $\varrho$ ∎

## 5. Main results

This section contains various a priori estimates. These estimates are particularly useful, from the constructive point of view, for studying non-linear Riemann-Hilbert boundary value problems consisting of non-linear boundary and side conditions.

Theorem 1: *Let D be a $C^{1+\alpha}$ domain and let $a, b \in C^\alpha(\hat{D})$ with $\|a\|_\alpha + \|b\|_\alpha \leqq K$. Then there exist constants $\tilde{\gamma}_\nu$, $1 \leqq \nu \leqq 4$, depending only on $\alpha$, D, K, $\tau$, $\sigma$, and $z_k$ $(1 \leqq k \leqq n)$ but not on a and b such that for each $w \in C^{1+\alpha}(\hat{D})$ satisfying (24) with $e^{i\tau}$, $\psi \in C^{1+\alpha}(\Gamma)$, the following estimate holds:*

$$\|w\|_{1+\alpha} \leqq \tilde{\gamma}_1 \|\psi\|_{1+\alpha,\Gamma} + \tilde{\gamma}_2 |\varkappa| + \tilde{\gamma}_3 \sum_{k=1}^n |a_k| + \tilde{\gamma}_4 \|w_{\bar{z}} - aw - b\bar{w}\|_\alpha. \tag{26}$$

The estimate (26) is stated in WENDLAND [30: p. 20] based on the closed graph theorem where the constants $\tilde{\gamma}_\nu$, $1 \leqq \nu \leqq 4$, depend on $a$ and $b$ as well.

Proof of the theorem: Let us begin with the homogeneous data, $\psi = 0$, $\varkappa = 0$, $a_k = 0$ $(1 \leqq k \leqq n)$ and derive (26) for $a = b = 0$. From the representation formula for $w$ in Lemma 9, it is easy to see that $\|w\|_0$, $\|w\|_\alpha \leqq M \|w_{\bar{z}}\|_0$ (similarly as in BEGEHR and HSIAO [12]). Now from the representation formula for $w_z$, we shall establish the inequality

$$\|w_z\|_0 \leqq M \|w_{\bar{z}}\|_\alpha.$$

Here the proof is more involved, and some clarifications are needed. Indeed, in view of the representation formula for $w_z$, let us first make the following observations.

a) For the conformal mapping $\Phi$ in Section 4, there exists a positive constant $c_0$ depending on $D$ such that

$$c_0^{-1} \leqq \left| \frac{\Phi(\zeta) - \Phi(z)}{\zeta - z} \right| \leqq c_0 \qquad (z, \zeta \in \hat{D}).$$

In addition we have the estimate $\left|\dfrac{\Phi(\zeta)-\Phi(z)}{1-\overline{\Phi(\zeta)}\,\Phi(z)}\right| \leqq 1$ for all $z, \zeta \in \hat{D}$.

b) The function $\tilde{\Phi}$ defined by

$$\tilde{\Phi}(z) = i \int_{\Gamma} \tilde{\tau}[d_n G^{\mathrm{I}} - i\, dG^{\mathrm{II}}](\zeta, z), \qquad \tilde{\tau} = \tau + \sum_{k=1}^{n} \arg(z - z_k)$$

in the representation formula belongs to $C^{1+\alpha}(\hat{D})$ and is analytic in $D$. The Hölder coefficient of $\tilde{\Phi}'$ depends on that of $d\tau/ds$.

c) For given $\psi$ and $\varkappa$, not necessarily equal to zero, there holds the estimate

$$\left\| \frac{w(\cdot)}{\cdot - z_k} \right\|_0 \leqq M[\|\psi\|_{\alpha,\Gamma} + |\varkappa| + \|w_{\bar{z}}\|_\alpha].$$

This follows from the representation formula for $w$ by making use of Privalov's theorem, the estimates

$$\left| \frac{1}{\zeta - z_k} \left[ w_{\bar{\zeta}}(\zeta) - \sum_{l=1}^{n} w_{\bar{z}}(z_l)\, \overline{P_l'(\zeta)} \right] \right|$$

$$\leqq \frac{H_\alpha(w_{\bar{z}})}{|\zeta - z_k|^{1-\alpha}} + |w_{\bar{z}}(z_k)| \left| \frac{\overline{P_k'(\zeta)} - 1}{\zeta - z_k} \right| + \sum_{l \neq k}^{n} |w_{\bar{z}}(z_l)| \left| \frac{P_l'(\zeta)}{\zeta - z_k} \right|,$$

as well as typical inequalities of the form (see VEKUA [27: p. 39])

$$\int_D \frac{d\xi\, d\eta}{|\zeta - z_k|^{1-\alpha}\, |\zeta - z|} \leqq M(\alpha, D),$$

$$\int_D \frac{d\xi\, d\eta}{|\zeta - z_l|\, |\zeta - z|\, |\zeta - z_k|^{1-\alpha}} \leqq \frac{M(\alpha, D, z_l, z_k)}{|z - z_l|^{1-\alpha}} \qquad (z_l \neq z_k),$$

$$\int_D \prod_{l \neq k} \left| \frac{z - z_l}{\zeta - z_l} \right| \frac{d\xi\, d\eta}{|\zeta - z_k|^{1-\alpha}\, |\zeta - z|} \leqq M\big(\alpha, D, z_k \quad (1 \leqq k \leqq n)\big).$$

The last inequality follows from the second one by applying decomposition into partial fractions to the product.

It remains now to estimate the supremum norm of the area integral. This can be achieved by transforming it into an integral over the unit disc $\mathbf{D}$ to which we may apply Lemma 11 in view of a). The result here will be needed for the general case for nonvanishing $a$ and $b$.

We next consider the case $\psi \neq 0$, $\varkappa \neq 0$. Again we assume that $a$ vanishes and proceed from the representation formulas for $w$. To facilitate the presentation, let

$$J_1(z) := \theta_n(z)\, e^{\tilde{\Phi}(z)} \prod_{k=1}^{n} (z - z_k)$$

and denote by $J_2$ the area integral in the representation formula. We would like to show that the Hölder coefficients of $J_1$ and $J_2$ satisfy the estimates

$$H_\alpha(J_1) \leqq M[H_\alpha(\psi) + \|\psi\|_0 + |\varkappa| + \|w_{\bar{z}}\|_0] \quad \text{and} \quad H_\alpha(J_2) \leqq M\, \|w_{\bar{z}}\|_0.$$

The first estimate follows from Privalov's theorem. To establish the second one, it suffices to examine only the integral containing $G_\zeta^{\mathrm{I}} + \tilde{G}_\zeta^{\mathrm{II}}$, since the other integral can be treated similarly. We note that $G_\zeta^{\mathrm{I}} + \tilde{G}_\zeta^{\mathrm{II}}$ has a singularity of the form $(\zeta - z)^{-1}$ and that $(\cdot - z)/(\Phi(\cdot) - \Phi(z))$ is Hölder-continuous in $\hat{D}$. Let us first consider the product

$$S(\zeta, z) = \left(\prod_{k=1}^{n} \frac{z - z_k}{\zeta - z_k}\right) \frac{1}{\zeta - z}$$

in $J_2$ for $z = \zeta_1$ and $\zeta_2$. We observe that

$$S(\zeta, \zeta_1) - S(\zeta, \zeta_2) = \left\{\prod_{k=1}^{n} \frac{\zeta_1 - z_k}{\zeta - z_k} \frac{1}{(\zeta - \zeta_1)(\zeta - \zeta_2)} + \sum_{l=1}^{n} \prod_{k=1}^{l-1} \frac{\zeta_1 - z_k}{\zeta - z_k} \prod_{k=l+1}^{n} \frac{\zeta_2 - z_k}{\zeta - z_k} \frac{1}{(\zeta - z_l)(\zeta - \zeta_2)}\right\} (\zeta_1 - \zeta_2);$$

moreover, since $(\zeta - z_l)^{-1} = (\zeta - \zeta_1)^{-1} - (\zeta_1 - z_l)/[(\zeta - z_l)(\zeta - \zeta_1)]$ this difference $S(\zeta, \zeta_1) - S(\zeta, \zeta_2)$ can be rewritten in the form

$$\left\{\prod_{k=1}^{n} \frac{\zeta_1 - z_k}{\zeta - z_k} + \sum_{l=1}^{n} \prod_{k=1}^{l-1} \frac{\zeta_1 - z_k}{\zeta - z_k} \prod_{k=l+1}^{n} \frac{\zeta_2 - z_k}{\zeta - z_k} - \sum_{l=1}^{n} \prod_{k=1}^{l} \frac{\zeta_1 - z_k}{\zeta - z_k} \prod_{k=l+1}^{n} \frac{\zeta_2 - z_k}{\zeta - z_k}\right\} \frac{(\zeta_1 - \zeta_2)}{(\zeta - \zeta_1)(\zeta - \zeta_2)},$$

or

$$S(\zeta, \zeta_1) - S(\zeta, \zeta_2) = \left\{\sum_{l=1}^{n} \prod_{k=1}^{l-1} \frac{\zeta_1 - z_k}{\zeta - z_k} \prod_{k=l+1}^{n} \frac{\zeta_2 - z_k}{\zeta - z_k} - \sum_{l=1}^{n-1} \prod_{k=1}^{l} \frac{\zeta_1 - z_k}{\zeta - z_k} \prod_{k=l+1}^{n} \frac{\zeta_2 - z_k}{\zeta - z_k}\right\} \frac{\zeta_1 - \zeta_2}{(\zeta - \zeta_1)(\zeta - \zeta_2)}.$$

Now from the identity

$$\prod_{k=1}^{m} \frac{\zeta_i - z_k}{\zeta - z_k} \frac{1}{\zeta - \zeta_i} = \frac{1}{\zeta - \zeta_i} - \sum_{k=1}^{m} \left(\prod_{\nu \neq k} \frac{\zeta_i - z_\nu}{z_k - z_\nu}\right) \frac{1}{\zeta - z_k}$$

$(i = 1, 2)$ each term in the above sum for the difference $S(\zeta, \zeta_1) - S(\zeta, \zeta_2)$ may be reformulated, e.g.

$$\left(\prod_{k=1}^{l} \frac{\zeta_1 - z_k}{\zeta - z_k} \frac{1}{\zeta - \zeta_1}\right) \left(\prod_{k=l+1}^{n} \frac{\zeta_2 - z_k}{\zeta - z_k} \frac{1}{\zeta - \zeta_2}\right)$$
$$= \frac{1}{(\zeta - \zeta_1)(\zeta - \zeta_2)} - \sum_{k=1}^{l} \left(\prod_{\nu \neq k} \frac{\zeta_1 - z_\nu}{z_k - z_\nu}\right) \frac{1}{(\zeta - z_k)(\zeta - \zeta_2)}$$
$$- \sum_{k=l+1}^{n} \left(\prod_{\nu \neq k} \frac{\zeta_2 - z_\nu}{z_k - z_\nu}\right) \frac{1}{(\zeta - z_k)(\zeta - \zeta_1)}$$
$$+ \sum_{k=1}^{l} \sum_{\varkappa=l+1}^{n} \left(\prod_{\nu \neq k} \frac{\zeta_1 - z_\nu}{z_k - z_\nu}\right) \left(\prod_{\mu \neq \varkappa} \frac{\zeta_2 - z_\mu}{z_\varkappa - z_\mu}\right) \frac{1}{(\zeta - z_k)(\zeta - z_\varkappa)}.$$

Hence it is not difficult to see that the difference $J_2(\zeta_1) - J_2(\zeta_2)$ will contain now typical integrals of singular terms such as

$$\frac{1}{(\zeta - \zeta_1)(\zeta - \zeta_2)}, \quad \frac{1}{(\zeta - z_k)(\zeta - \zeta_1)}, \quad \frac{1}{(\zeta - z_k)(\zeta - \zeta_2)}, \quad \text{and}$$

$$\frac{1}{(\zeta - z_k)(\zeta - z_\nu)}.$$

All these integrals can be handled. In particular we see that (VEKUA [27: p. 39])

$$\int_D \frac{d\xi\, d\eta}{|\zeta - \zeta_1|\, |\zeta - \zeta_2|} \leqq M(1 + |\log |\zeta_1 - \zeta_2||).$$

This leads to the desired result.

In the same manner, we find

$$H_\alpha\left(\frac{w}{z - z_k}\right) \leqq M\big(H_\alpha(\psi) + \|\psi\|_0 + |\varkappa| + \|w_{\bar z}\|_0\big)$$

and

$$H_\alpha(w) \leqq M\big(H_\alpha(\psi) + \|\psi\|_0 + |\varkappa| + \|w_{\bar z}\|_0\big).$$

For the estimates of the Hölder coefficients of the $z$-derivatives of $J_1$ and $J_2$, we use Lemmas 11 and 12, and find (see VEKUA [27: p. 63])

$$H_\alpha\left(\frac{\partial}{\partial z} J_1\right) \leqq M(\|\psi\|_{\alpha,\Gamma} + \|\psi'\|_\alpha + \|w_{\bar z}\|_0) \quad \text{and} \quad H_\alpha\left(\frac{\partial}{\partial z} J_2\right) \leqq M\, \|w_{\bar z}\|_\alpha.$$

Thus, we obtain $H_\alpha(w_z) \leqq M(\|\psi\|_{1+\alpha,\Gamma} + |\varkappa| + \|w_{\bar z}\|_\alpha)$ and hence (26) follows in the special case $a_k = 0$ $(1 \leqq k \leqq n)$ with $a$ and $b$ equal to zero.

If now $w$ does not vanish at the $z_k$, then we may use the transformation

$$\Omega = w - f, \qquad f(z) = \sum_{k=1}^{n} a_k \prod_{l \neq k} \frac{z - z_l}{z_k - z_l}$$

and reduce it to the previous case for $\Omega$ satisfying

$$\left.\begin{aligned} &\operatorname{Re}\{e^{i\tau}\Omega\} = \psi - \operatorname{Re}\{e^{i\tau} f\} \quad \text{on } \Gamma \\ &\frac{1}{\Sigma}\int_\Gamma \operatorname{Im}\{e^{i\tau}\Omega\}\,\sigma\, ds = \varkappa - \frac{1}{\Sigma}\int_\Gamma \operatorname{Im}\{e^{i\tau} f\}\,\sigma\, ds \\ &\Omega(z_k) = 0 \qquad (1 \leqq k \leqq n). \end{aligned}\right\}$$

In this way (26) is proved for the general case but with vanishing $a$ and $b$.

We are now in a position to establish (26) for the most general case. Again we will begin with homogeneous boundary and side conditions. For arbitrary $a, b \in C^\alpha(\hat D)$, we consider the function $\omega$ uniquely given by

$$\omega_{\bar z} = \begin{cases} a + b\,\dfrac{\overline{w}}{w}, & w \neq 0, \\ a, & w = 0, \end{cases} \quad \operatorname{Im} \omega|_\Gamma = 0, \text{ and } \int_\Gamma \operatorname{Re} \omega(\zeta)\,\sigma\, ds = 0.$$

We note that $\tilde\omega = i\omega$ satisfies the homogeneous conditions (24) in the special case $n = 0$, $\tau = 0$. Although $\tilde\omega \notin C^{1+\alpha}(\hat D)$ in general, from the representation formula, as in the case for the homogeneous data, it can be shown that the inequality $\|\tilde\omega\|_\alpha \leqq M\, \|\tilde\omega_{\bar z}\|_0$ remains valid. Thus we have $\|\omega\|_\alpha \leqq M(\|a\|_0 + \|b\|_0) \leqq MK$, and hence $\|e^\omega\|_\alpha \leqq e^{MK}(1 + MK)$.

Next, let $f_0$ be analytic in $D$ and uniquely defined by

$$\operatorname{Re}\{e^{i\tau} f_0\}|_\Gamma = 0, \quad \int_\Gamma \operatorname{Im}\{e^{i\tau} f_0\}\,\sigma\, ds = \Sigma, \quad f_0(z_k) = 0 \quad (1 \leqq k \leqq n).$$

Then if we define $f$ by

$$f = w\, e^{-\omega} - A f_0 \quad \text{with} \quad A = \int_\Gamma \operatorname{Im}\{e^{i\tau} w\, e^{-\omega}\}\,\sigma\, ds,$$

then $f$ satisfies the equation $f_{\bar z} = (w_{\bar z} - aw - b\bar w)\,e^{-\omega}$ in $D$ together with the homogeneous boundary and side conditions

$$\operatorname{Re}\{e^{i\tau} f\}|_\Gamma = 0, \quad \int_\Gamma \operatorname{Im}\{e^{i\tau} f\}\,\sigma\,ds = 0, \quad f(z_k) = 0 \quad (1 \leqq k \leqq n),$$

if $w$ has vanishing data $\psi = 0$, $\varkappa = 0$, $a_k = 0$ $(1 \leqq k \leqq n)$. Because $f_{\bar z} \in C^\alpha(\hat D)$, we may apply the previously established estimates to $f$ and obtain $\|f\|_\alpha \leqq M\,\|f_{\bar z}\|_0$. Further, from the definition of $f$, we have $w = (f + Af_0)\,e^\omega$ and hence

$$A \int_\Gamma \operatorname{Im}\{e^{i\tau} f_0\,e^\omega\}\,\sigma\,ds + \int_\Gamma \operatorname{Im}\{e^{i\tau} f\,e^\omega\}\,\sigma\,ds = 0$$

for vanishing $\varkappa$. Observing the conditions satisfied by $f_0$ on $\Gamma$ and especially the consequence $0 < \operatorname{Im}\{e^{i\tau} f_0\}$ on $\Gamma$ leads to

$$\int_\Gamma \operatorname{Im}\{e^{i\tau} f_0\,e^\omega\}\,\sigma\,ds = \int_\Gamma \operatorname{Im}\{e^{i\tau} f_0\}\operatorname{Re} e^\omega \sigma\,ds \geqq \Sigma\,e^{-MK}.$$

Consequently we have

$$|A| \leqq e^{2MK}\,\|f\|_0, \text{ and } \|w\|_\alpha \leqq [\|f\|_\alpha + e^{2MK}\,\|f\|_0\,\|f_0\|_\alpha]\,\|e^\omega\|_\alpha \leqq \tilde M\,\|f\|_\alpha$$

for some constant $\tilde M$. Moreover, in view of the analyticity of $f_0$, we see that $w_{\bar z} = f_{\bar z}\,e^\omega + w\omega_{\bar z} = f_{\bar z}\,e^\omega + aw + b\bar w$. Hence,

$$\|w_z\|_\alpha \leqq M\,\|w_{\bar z}\|_\alpha \leqq \hat M[\|f\|_\alpha + \|f_{\bar z}\|_\alpha] \leqq \tilde{\tilde M}\,\|f_{\bar z}\|_\alpha.$$

Thus, we arrive at (26) in the special case for the homogeneous data:

$$\|w\|_{1+\alpha} \leqq \gamma_4\,\|w_{\bar z} - aw - b\bar w\|_\alpha. \tag{27}$$

Clearly here the constant $\gamma_4$ depends on $\|a\|_\alpha + \|b\|_\alpha$ but not on $a$ and $b$.

Suppose now $a_k = 0$ $(1 \leqq k \leqq n)$ but $\psi \neq 0$ and $\varkappa \neq 0$. Then let $\theta$ denote the analytic function from the representation formula (Lemma 9). Set

$$\omega = w + \tilde\theta \quad \text{with} \quad \tilde\theta(z) = \theta(z)\,e^{\tilde\phi(z)} \prod_{k=1}^n (z - z_k).$$

Then it can be verified that

$$\omega_{\bar z} = w_{\bar z}, \quad \omega(z_k) = 0 \ (1 \leqq k \leqq n), \quad \operatorname{Re}\{e^{i\tau}\omega\}|_\Gamma = 0,$$
$$\int_\Gamma \operatorname{Im}\{e^{i\tau}\omega\}\,\sigma\,ds = 0,$$

and hence $\omega$ satisfies (27). Thus

$$\|w\|_{1+\alpha} \leqq \|\omega\|_{1+\alpha} + \|\tilde\theta\|_{1+\alpha}, \quad \|\omega\|_{1+\alpha} \leqq \gamma_4\,\|w_{\bar z} - aw - b\bar w\|_\alpha + \gamma_4\,\|a\tilde\theta + b\bar{\tilde\theta}\|_\alpha.$$

This together with the already established estimates of $\|\tilde\theta\|_{1+\alpha}$ implies (26) with $a_k = 0$.

Finally if the similar transformation

$$\Omega = w - f \quad \text{with} \quad f(z) = \sum_{k=1}^n a_k \prod_{l \neq k} \frac{z - z_l}{z_k - z_l}$$

is utilized, the general inequality for the non-homogeneous data then follows immediately ∎

We emphasize that here (26) is found in a constructive way, and it is more suitable for numerical procedures such as the Newton embedding type (see WENDLAND [28], BEGEHR and HSIAO [14]).

The considerations in Section 3 enable us now to extend Theorem 1 to the following result for the general Beltrami equation.

Theorem 2: *Let $w \in C^{1+\alpha}(\hat{D})$ satisfy* (24) *with* $e^{i\tau}$, $\psi \in C^{\alpha}(\Gamma)$. *Let* $\mu_1, \mu_2 \in C^{1+\alpha}(\mathbf{C})$ *and* $\nu_1, \nu_2 \in C^{\alpha}(\hat{D})$ *be given fulfilling the assumptions*

a) $$\mu_1 = \mu_2 = 0 \quad in \quad \mathbf{C} \setminus \hat{D},$$

$$\|\mu_1\|_0 + \|\mu_2\|_0 \leqq q < 1,$$

$$\|\mu_{1z}\|_\alpha + \|\mu_{1\bar{z}}\|_\alpha + \|\mu_{2z}\|_\alpha + \|\mu_{2\bar{z}}\|_\alpha \leqq M_1,$$

*and*

b) $$\|\nu_1\|_\alpha + \|\nu_2\|_\alpha \leqq K,$$

*respectively. Then there holds the estimate*

$$\|w\|_{1+\alpha} \leqq \gamma_1 \|\psi\|_{1+\alpha,\Gamma} + \gamma_2 |\varkappa| + \gamma_3 \sum_{k=1}^{n} |a_k| + \gamma_4 \|w_{\bar{z}} - \mu_1 w_z - \mu_2 w_{\bar{z}} - \nu_1 w - \nu_2 \overline{w}\|_\alpha, \tag{28}$$

*where $\gamma_k$ $(1 \leqq k \leqq 4)$ are constants depending on $D$, $z_k$ $(1 \leqq k \leqq n)$, $\sigma$, $\tau$, $\alpha$, $q$, $M_1$, $K$ but not on $w$, $\mu_1$, $\mu_2$, $\nu_1$, $\nu_2$, $\psi$, $\varkappa$, $a_k$ $(1 \leqq k \leqq n)$.*

Proof: The transformation (20), (21) reduces the differential equation

$$w_{\bar{z}} = \mu_1 w_z + \mu_2 \overline{w}_z + \nu_1 w + \nu_2 \overline{w} + \lambda, \qquad \lambda \in C^{\alpha}(\hat{D}),$$

together with the boundary and side conditions (24) to the canonical problem defined by the differential equation

$$\omega_{\bar{\zeta}} = A\omega + B\overline{\omega} + C \quad \text{in} \quad \tilde{D}$$

and the boundary and side conditions (25). Here the coefficients $A$, $B$, and $C$ are given explicitly by

$$A = \frac{(1 - \bar{\mu}_1\mu)(\nu_1 + \bar{a}\nu_2) + \mu_2\mu(\bar{a}\bar{\nu}_1 + \bar{\nu}_2)}{(1 - |a|^2)(|1 - \bar{\mu}_1\mu|^2 - |\mu_2\mu|^2)\,\overline{\zeta_z}} - \frac{a_{\bar{\zeta}}\bar{a}}{1 - |a|^2},$$

$$B = \frac{(1 - \bar{\mu}_1\mu)(a\nu_1 + \nu_2) + \mu_2\mu(\bar{\nu}_1 + a\bar{\nu}_2)}{(1 - |a|^2)(|1 - \bar{\mu}_1\mu|^2 - |\mu_2\mu|^2)\,\overline{\zeta_z}} - \frac{a_{\bar{\zeta}}}{1 - |a|^2},$$

$$C = \frac{(1 - \bar{\mu}_1\mu)\lambda + \mu_2\mu\bar{\lambda}}{(|1 - \bar{\mu}_1\mu|^2 - |\mu_2\mu|^2)\,\overline{\zeta_z}}.$$

Obviously, these coefficients belong to $C^{\alpha}(\hat{\tilde{D}})$, $\hat{\tilde{D}} := \tilde{D} \cup \partial\tilde{D}$, and are bounded according to

$$\|A\|_{\alpha,\tilde{D}} + \|B\|_{\alpha,\tilde{D}} \leqq \tilde{K} \quad \text{and} \quad \|C\|_{\alpha,\tilde{D}} \leqq M\,\|\lambda\|_{\alpha,D}$$

with constants $\tilde{K}$ and $M$ depending only on $q$, $M_1$, $K$, and $D$. Hence the estimate (26) is valid for $\omega$ in $\tilde{D}$. Now from the transformation (21), we see that

$$w = \frac{\omega + a\overline{\omega}}{1 - |a|^2} \quad \text{and hence} \quad \|w\|_{1+\alpha,D} \leqq M_0\,\|\omega\|_{1+\alpha,D}.$$

However, in order to apply (26) to $\omega$, we first show that the right-hand side is dominated by $\|\omega\|_{1+\alpha,\tilde{D}}$. To this end, let us consider the relations

$$|\zeta_1 - \zeta_2| = \int_{\zeta[\overrightarrow{z_1 z_2}]} d\tilde{s} = \int_{\overrightarrow{z_1 z_2}} \frac{d\tilde{s}}{ds}\, ds = \int_{\overrightarrow{z_1 z_2}} |\zeta_z|\, |z'(s) + \overline{\mu z'(s)}|\, ds .$$

Here $\overrightarrow{z_1 z_2}$ is the line segment between $z_1$ and $z_2$, and lies in $D$ for $|z_1 - z_2|$ sufficiently small while $\zeta[\overrightarrow{z_1 z_2}]$ is the image of $\overrightarrow{z_1 z_2}$ under the mapping $\zeta$. Hence by using the estimate $\|\zeta_z\|_0 \leqq e^{MM_1}$ from (10), we obtain the inequality $|\zeta_1 - \zeta_2| \leqq e^{MM_1}(1 + q) \times |z_1 - z_2|$. This together with the identities $\omega_z = \omega_\zeta \zeta_z + \omega_{\bar\zeta} \overline{\mu \zeta_z}$ and $\omega_{\bar z} = \omega_\zeta \mu \zeta_z + \omega_{\bar\zeta} \bar\zeta_z$ implies that

$$\|\omega_z\|_{\alpha,D} + \|\omega_{\bar z}\|_{\alpha,D} \leqq M\, e^{\alpha M M_1}(1 + q)^\alpha\, [\|\omega_\zeta\|_{\alpha,\tilde{D}} + \|\omega_{\bar\zeta}\|_{\alpha,\tilde{D}}].$$

Consequently we have

$$\|\omega\|_{1+\alpha,D} \leqq M\, e^{\alpha M M_1}(1 + q)^\alpha\, \|\omega\|_{1+\alpha;\tilde{D}},$$

and from (26), $\|\omega\|_{1+\alpha,\tilde{D}}$ is bounded by the appropriate terms with respect to norms on $\tilde{D}$ and also on $\tilde{\Gamma}$. In particular, to recover those bounds on $\tilde{\Gamma}$ with respect to the norm on $\Gamma$, we see that for $f \in C^\alpha(\tilde{\Gamma})$

$$\|f\|_{\alpha,\tilde{\Gamma}} \leqq \frac{e^{\alpha M M_1}}{(1 - q)^\alpha} \|f\|_{\alpha,\Gamma},$$

where we have used the similar estimate as before

$$|\zeta_1 - \zeta_2| = \int_{\overrightarrow{\zeta_1 \zeta_2}} d\tilde{s} = \int_{z[\overrightarrow{\zeta_1 \zeta_2}]} |\zeta_z|\, |z'(s) + \overline{\mu z'(s)}|\, ds \geqq e^{-MM_1}(1 - q) \int_{z[\zeta_1 \zeta_2]} ds$$
$$\geqq e^{-MM_1}(1 - q)\, |z(\zeta_1) - z(\zeta_2)| .$$

Similarly from (25) we have, in view of Lemma (6),

$$\left\| \frac{1 - |a|^2}{\varrho} \psi \right\|_{1+\alpha,\tilde{\Gamma}} \leqq \left\| \frac{ds}{d\tilde{s}} \right\|_{\alpha,\tilde{\Gamma}} \frac{e^{\alpha M M_1}}{(1 - q)^\alpha} \left\| \frac{1 - |a|^2}{\varrho} \psi \right\|_{1+\alpha,\Gamma} ;$$

Moreover, it is not difficult to see that

$$|\varkappa_1| \leqq (1 + q) \left[ |\varkappa| + \frac{2q}{(1 - q)^2} \|\psi\|_{0,\Gamma} \right] \quad \text{and} \quad |a_k + a\bar{a}_k| \leqq (1 + q)\, |a_k| .$$

These estimates give

$$\|w\|_{1+\alpha,D} \leqq \gamma_1 \|\psi\|_{1+\alpha,\Gamma} + \gamma_2 |\varkappa| + \gamma_3 \sum_{k=1}^{n} |a_k| + \gamma_4 \|\lambda\|_{\alpha,D},$$

which proves the desired a priori estimate ∎

If $\mu_1$, $\mu_2$ are only Hölder continuous and $e^{i\tau}$, $\psi \in C^\alpha(\Gamma)$, one can still derive an a priori estimate using a subnorm of $w$:

$$\|w\|_\alpha \leqq \hat{\gamma}_1 \|\psi\|_\alpha + \hat{\gamma}_2 |\varkappa| + \hat{\gamma}_3 \sum_{k=1}^{n} |a_k| + \hat{\gamma}_4 \|w_{\bar z} - \mu_1 w_z - \mu_2 \bar{w}_z - \nu_1 w - \nu_2 \bar{w}\|_0 .$$

We emphasize again that a priori estimates such as (28) are most desirable for establishing existence and uniqueness results for the nonlinear Hilbert problem consisting of nonlinear boundary and side conditions. In this regard, we refer to BEGEHR and HSIAO [14] where (28) is utilized to treat such problems.

To conclude the paper, we now state a similar result concerning an a priori estimate for functions with generalized derivatives.

Theorem 3: *Let $\mu_1$, $\mu_2$ be two measurable functions in $\hat{D}$ fulfilling* (2), *and let $\nu_1, \nu_2 \in L_p(\hat{D})$ for $p > 2$ but sufficiently close to 2. Then there exist constants $\gamma_k$ $(1 \leqq k \leqq 4)$ such that for $w \in W_p^1(\hat{D})$ satisfying* (24), *the inequality*

$$\|w\|_0 + \|w_z\|_p + \|w_{\bar{z}}\|_p$$
$$\leqq \gamma_1 \|\psi\|_{\alpha,\Gamma} + \gamma_2 |\varkappa| + \gamma_3 \sum_{k=1}^{n} |a_k| + \gamma_4 \|w_{\bar{z}} - \mu_1 w_z - \mu_2 \overline{w_z} - \nu_1 w - \nu_2 \overline{w}\|_p$$

*holds.*

The proof of this theorem is given in BEGEHR and HSIAO [15][3]). Again this a priori estimate can be employed to establish existence and uniqueness theorems for the Hilbert boundary value problem on nonlinear equations of the form (3) with nonlinear boundary and side conditions. For details, we refer to BEGEHR and HSIAO [15].

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VERFASSER:

Prof. Dr. HEINRICH BEGEHR
I. Mathematisches Institut der Freien Universität
D-1000 Berlin-West 33, Arnimallee 3

Prof. Dr. GEORGE C. HSIAO
Department of Mathematical Science
University of Delaware
Newark, Delaware 19716, USA