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Martinelli-Bochner Type Formulae in Complex Clifford Analysis

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Es werden verschiedene Lösungstypen des Systems $(D_x + iD_y) f = 0$ betrachtet, wobei D_x und D_y Operatoren von Diracschem Typ im $\mathbf{R}^m$ sind. In Verallgemeinerung der klassischen Martinelli-Bochner-Formel für holomorphe Funktionen wird solch eine Formel für C_1-Lösungen dieses Systems bewiesen. Es werden solche Formeln auch für andere überbestimmte Systeme aus der Clifford-Analysis erhalten.

Рассматриваются разного типа решения системы $(D_x + iD_y) f = 0$, где D_x и D_y операторы типа Дирака в $\mathbf{R}^m$. В обобщении классической формулы Мартинелли-Бохнера для голоморфных функций доказывается такая формула для C_1-решений этой системы. Такие формулы получаются также для других переопределенных систем, встречающихся в анализе Клиффорда.

Various types of solutions of the systems $(D_x + iD_y) f = 0$ are considered, where D_x and D_y are Dirac type operators in $\mathbf{R}^m$. Generalizing the classical Martinelli-Bochner formula for holomorphic functions, such a formula is proved for the C_1-solutions of this system. Martinelli-Bochner formulae are also obtained for other overdetermined systems occuring in Clifford analysis.

Introduction

Let $\mathcal{A}$ be the complex Clifford algebra constructed over $\mathbf{R}^m$. Then we consider $\mathcal{A}$-valued functions f, defined in open subsets $\Omega \subseteq \mathbf{C}^m = \mathbf{R}^m \times \mathbf{R}^m$, which satisfy the so-called *weak complex monogenic system*:

$$(D_x + iD_y) f = 0,$$

where

$$D_x = \sum_{j=1}^m e_j \frac{\partial}{\partial x_j} \quad \text{and} \quad D_y = \sum_{j=1}^m e_j \frac{\partial}{\partial y_j}$$

are Dirac type operators and where $\{e_1, \ldots, e_m\}$ is an orthonormal basis of $\mathbf{R}^m$ (see [1]). As special classes of solutions to this system we obtain the holomorphic functions of several complex variables and the solutions to the so-called *left biregular system* $D_x f = D_y f = 0$, which is an overdetermined system in $\mathbf{R}^m \times \mathbf{R}^m$.

In the first section, which is of an introductory nature, we describe the basic elementary properties of the weak complex monogenic system and we give several examples of weak complex monogenic functions occuring in mathematics and physics.

In the second section we start from the kernel

$$E(\vec{x} + i\vec{y}) = \frac{1}{\omega_{2m}} \frac{i\vec{y} - \vec{x}}{(|\vec{x}|^2 + |\vec{y}|^2)^m},$$

defined in $\mathbf{C}^m \setminus \{0\}$, where ω_{2m} is the area of the unit sphere in $\mathbf{C}^m$. Although this kernel is itself not weak complex monogenic, it gives rise to a singular integral kernel $(D_x + iD_y)\, E(\vec{x} + i\vec{y})$, which leads to a special higher Riesz transform A_C (see [8]). Next, using a generalized Cauchy formula, we obtain a Martinelli-Bochner formula for weak complex monogenic functions, in which the transform A_C occurs. Furthermore, using the fact that $(D_x + iD_y)\, E(\vec{x} + i\vec{y})$ takes values in the space of imaginary bivectors, we are able to split this formula in such a way that we obtain the classical Martinelli-Bochner formula for holomorphic functions as well as a Martinelli-Bochner formula for left biregular functions. Finally we show that the Martinelli-Bochner formula, obtained for two-sided biregular functions in [2], follows immediately from a more general formula in complex Clifford analysis.

1. Complex monogenic systems

In this paper $\mathcal{A}$ denotes the complex Clifford algebra over $\mathbf{R}^m$, while $\mathcal{A}_\mathbf{R}$ denotes the real part of $\mathcal{A}$. This means that elements of $\mathcal{A}$ and $\mathcal{A}_\mathbf{R}$ are of the form

$$a = \sum_{A \subseteq N} a_A e_A, \qquad a_A \in \mathbf{C} \quad \text{and} \quad a_A \in \mathbf{R} \text{ respectively},$$

where $N = \{1, \ldots, m\}$ and where for $A = \{\alpha_1, \ldots, \alpha_k\}$, $\alpha_1 < \cdots < \alpha_h$, $e_A = e_{\alpha_1} \ldots e_{\alpha_h}$. The product in $\mathcal{A}$ is determined by the relations $e_i e_j + e_j e_i = -2\delta_{ij}$, $i, j = 1, \ldots, m$, and the unity in $\mathcal{A}$ is denoted by $e_\emptyset = e_0 = 1$. Let $\vec{z} = \vec{x} + i\vec{y}$, $(\vec{x}, \vec{y}) \in \mathbf{R}^m \times \mathbf{R}^m$; then we shall identify $\vec{z}$ with the Clifford element $e_1 z_1 + \cdots + e_m z_m$.

Furthermore we introduce the Dirac operators

$$D_x = \sum_{j=1}^m e_j \frac{\partial}{\partial x_j} \quad \text{and} \quad D_y = \sum_{j=1}^m e_j \frac{\partial}{\partial y_j}$$

and we shall consider the systems of differential equations

$$(D_x + iD_y)\, f = \sum_{j=1}^m e_j \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}\right) f = 0, \tag{1}$$

$$D_x f = D_y f = 0, \tag{2}$$

$$\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}\right) f = 0, \qquad j = 1, \ldots, m, \tag{3}$$

$$\sum_{j=1}^m e_j \frac{\partial}{\partial z_j} f = 0, \qquad \frac{\partial}{\partial \bar{z}_j} f = 0, \qquad j = 1, \ldots, m. \tag{4}$$

The system (1) is the main subject of our study and will be called *weak complex monogenic*, while the system (4) is known as the *complex left monogenic* or *complex left regular system* (see [3, 5, 7]). The system (2) is called *left biregular* and forms, together with the *two-sided biregular system* $D_x f = f D_y = 0$ (see [2, 6]), a generalization to Clifford analysis of the *holomorphic Cauchy-Riemann system* (3) for $m = 2$.

Notice that all solutions to (2) and (3) are solutions to (1), whereas the solutions to (4) are the simultaneous solutions to (2) and (3). Hence the system (1) is, in some sense, the union of (2) and (3), while (4) is the intersection of (2) and (3).

Definition 1: Let $\Omega \subseteq \mathbf{C}^m$ be open and let E be a space of functions or distributions in Ω. Then by $M_{(r),E}(\Omega; \mathcal{A})$ we denote the right $\mathcal{A}$-module of all solutions in E to the differential system (1). If $E = C_\infty(\Omega; \mathcal{A}) = \mathcal{E}_{(r)}(\Omega; \mathcal{A})$, $E = \mathcal{D}'_{(l)}(\Omega; \mathcal{A})$, $E = \mathcal{S}'_{(e)}(\Omega; \mathcal{A})$, we use the notations $M_{(r),\mathcal{E}}(\Omega; \mathcal{A})$, $M_{(r),\mathcal{D}'}(\Omega; \mathcal{A})$ and $M_{(r),\mathcal{S}'}(\Omega; \mathcal{A})$.

We now give some examples of special solutions to (1) as a motivation for our study:

1. If a function f satisfies the system (1) and takes values in the space of scalars $e_0\mathbf{C}$, then it satisfies (3), i.e. it is a holomorphic function of several complex variables.

2. If a solution f to (1) takes values in the real Clifford algebra $\mathcal{A}_{\mathbf{R}}$, then it satisfies the left biregular system (2).

3. If $\frac{\partial}{\partial x_1} f = \frac{\partial}{\partial y_2} f = \frac{\partial}{\partial y_3} f = \frac{\partial}{\partial y_4} f = 0$, $m = 4$, f satisfies the Dirac equation for massless fields $\left(ie_1 \frac{\partial}{\partial y_1} + \sum_{j=2}^{4} e_j \frac{\partial}{\partial x_j}\right) f = 0$. Hence (1) generalizes the classical Dirac system. Sometimes we shall call $\vec{x}$ the *space variable* and $\vec{y}$ the *time variable*.

4. Very important is the link between the weak complex monogenic system and the classical operator $\bar{\partial} = \sum_{j=1}^{m} \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j$. If we identify the Clifford elements $e_A \in \mathcal{A}$ with the basic differential forms $d\bar{z}_A = d\bar{z}_{\alpha_1} \dots d\bar{z}_{\alpha_h}$, we obtain that $(D_x + iD_y) f = [\bar{\partial} - \bar{\partial}^*] \wedge f$, $*$ being the Hodge star operator. Hence, if f is a k-vector, the system $(D_x + iD_y) f$ splits into the system $\bar{\partial} \wedge f = 0$, $\bar{\partial}^* \wedge f = 0$. Furthermore, the inhomogeneous equation $(D_x + iD_y) f = g$ may be expressed purely in terms of the operators $\bar{\partial}$ and $\bar{\partial}^*$.

Remarks: **1.** We have that

$$(D_x \pm iD_y)^2 = -\sum_{j=1}^{m} \left(\frac{\partial}{\partial x_j} \pm i \frac{\partial}{\partial y_j}\right)^2 = P_{\pm}.$$

Hence the equation $(D_x \pm iD_y) f = g$, $g \in \mathcal{E}_{(r)}(\Omega, \mathcal{A})$ has a solution $f \in \mathcal{E}_{(r)}(\Omega, \mathcal{A})$ if and only if the equation $P_{\pm} f = g$ is solvable in C_∞-sense, which is equivalent to Ω being $P_{\pm}$-convex (see [9]). Furthermore the isotropic cone $\{\vec{z} : \vec{z}^2 = 0\}$ is the characteristic variety of $P_{\pm}$. Hence the entire solutions of $(D_x \pm iD_y) f = 0$ may be expressed as linear superpositions of plane wave solutions of the form $(\vec{t} \pm i\vec{u}) \exp (i\langle \vec{x}, \vec{t}\rangle + i\langle \vec{y}, \vec{u}\rangle)$, where $(\vec{t} + i\vec{u})^2 = 0$.

A fundamental solution K for $D_x + iD_y$ may be constructed as follows. Let $(D_x + iD_y) K = \delta$; then the Fourier transform $\hat{K}$ of K satisfies the equation $(\vec{x} + i\vec{y}) \hat{K} = 1$. A solution in $\mathcal{S}'_{(e)}(\mathbf{R}^{2m}; \mathcal{A})$ to this equation is given by

$$\langle \hat{K}, \varphi\rangle = -\lim_{\varepsilon > 0} \int\limits_{|\Sigma z_j^2| > \varepsilon} \varphi(\vec{x}, \vec{y}) \frac{\vec{x} + i\vec{y}}{\sum_{j=1}^{m} (x_j + iy_j)^2} d\vec{x}\, d\vec{y}, \quad \varphi \in \mathcal{S}_{(l)}(\mathbf{R}^{2m}; \mathcal{A}).$$

2. We have the identity

$$(D_x - iD_y)(D_x + iD_y) = -(\Delta_x + \Delta_y) + i[D_x, D_y],$$

where

$$[D_x, D_y] = \sum_{i<j} e_{ij} \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} - \frac{\partial}{\partial x_j} \frac{\partial}{\partial y_i}\right).$$

Hence our theory includes the elliptic second order system $(\Delta_x + \Delta_y) f = [D_x, D_y] f = 0$, which is still satisfied by both the left biregular and the holomorphic functions. It is the simplest elliptic system containing both classes of functions in a non-trivial way, but it has a much more complicated structure than system (1). It also leads to the study of the $\mathbf{C}$-valued system

$$(\Delta_x + \Delta_y) f = 0,$$

$$\left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} - \frac{\partial}{\partial x_j} \frac{\partial}{\partial y_i}\right) f = 0, \qquad i \neq j,$$

which is still satisfied by the holomorphic functions.

3. Let E be a right module over the ring $\mathcal{H}(\Omega; \mathcal{A})$ of $\mathcal{A}$-valued holomorphic functions in $\Omega \subseteq \mathbf{C}^n$. Then also $M_{(r),E}(\Omega; \mathcal{A})$ is a right module over $\mathcal{H}(\Omega; \mathcal{A})$. Indeed, if $f \in E$ satisfies $(D_x + iD_y) f = 0$ and $h \in \mathcal{H}(\Omega; \mathcal{A})$, then $fh \in E$ and

$$(D_x + iD_y)(fh) = ((D_x + iD_y) f) h + \sum_{j=1}^{m} e_j f \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) h = 0.$$

Furthermore, if $(D_x + iD_y) f = g$; then also $(D_x + iD_y)(f \cdot h) = g \cdot h$. Hence, in particular, the equation $(D_x + iD_y) f = h$, $h \in \mathcal{H}(\Omega; \mathcal{A})$, always admits a solution in C_∞-sense. Moreover, if $T \in E'$, $\varphi \in E$ and $h = \sum_A h_A e_A \in \mathcal{H}(\Omega; \mathcal{A})$, h_A being $\mathbf{C}$-valued holomorphic; then we put

$$\langle hT, \varphi \rangle = \sum_A \langle e_A T, \varphi h_A \rangle.$$

Hence E' is a left module over $\mathcal{H}(\Omega; \mathcal{A})$ and so, using the Hahn-Banach extension theorem, the dual module $M'_{(r),E}(\Omega; \mathcal{A})$ of $M_{(r),E}(\Omega; \mathcal{A})$ is also a left module over $\mathcal{H}(\Omega; \mathcal{A})$, where $M_{(r),E}(\Omega; \mathcal{A})$ is considered as a subspace of E itself, provided with the topology induced by E.

2. Martinelli-Bochner type theorems

As a first application of the theory of weak complex monogenic functions, we show that the classical Martinelli-Bochner theorem for holomorphic functions, as well as the corresponding theorem for left biregular functions may be derived from a more general Martinelli-Bochner formula, involving a singular integral. We start from the kernel

$$E(\vec{x} + i\vec{y}) = \frac{1}{\omega_{2m}} \frac{i\vec{y} - \vec{x}}{(|\vec{x}|^2 + |\vec{y}|^2)^m},$$

ω_{2m} being the area of S^{2m-1}. It is clear that $E(\vec{x} + i\vec{y}) \in L_1^{\mathrm{loc}}(\mathbf{R}^{2m})$. Using the notation $\vec{x} \wedge \vec{y} = \frac{1}{2}(\vec{x}\vec{y} - \vec{y}\vec{x})$, we have

Lemma 1: *For every* $(\vec{x}, \vec{y}) \in \mathbf{R}^m \times \mathbf{R}^m \setminus \{0\}$,

$$(D_x + iD_y) E(\vec{x} + i\vec{y}) = \frac{4mi}{\omega_{2m}} \frac{\vec{y} \wedge \vec{x}}{(|\vec{x}|^2 + |\vec{y}|^2)^{m+1}}.$$

Proof: The identity is obtained by straight-forward calculation, making use of the relation $(\vec{x} + i\vec{y})(i\vec{y} - \vec{x}) = (|\vec{x}|^2 + |\vec{y}|^2) - 2i\vec{y} \wedge \vec{x}$ ∎

Notice that $(D_x + iD_y) E(\vec{x} + i\vec{y})$ is no longer locally integrable. Hence it has to be considered as a singular integral kernel. Furthermore it takes values in the space of imaginary bivectors, a fact which will be of central importance in our argument.

In order to obtain our version of the Martinelli-Bochner formula, we shall use a generalized Cauchy-type formula for the operator $D_x + iD_y$. Let C be a compact set with C_1-boundary ∂C in $\mathbf{R}^{2m}$. Then by $\vec{e}_n = \vec{e}_{nx} + i\vec{e}_{ny}$ we denote the unit normal on ∂C for the usual inner product in $\mathbf{R}^{2m}$, where $\vec{e}_{nx}, \vec{e}_{ny} \in \mathbf{R}^m$. Furthermore an oriented surface measure on ∂C is given by

$$d\vec{\sigma} = d\vec{\sigma}_x + i d\vec{\sigma}_y, \qquad d\vec{\sigma}_x = \vec{e}_{nx}\, dS, \qquad d\vec{\sigma}_y = \vec{e}_{ny}\, dS,$$

dS being the Lebesgue measure on ∂C. In terms of differential forms, we have that

$$d\vec{\sigma}_x = \sum_{j=1}^{m} e_j(-1)^j \, dx_1 \wedge \cdots \wedge d\hat{x}_j \wedge \cdots \wedge dx_m \wedge dy_1 \wedge \cdots \wedge dy_m,$$

$$d\vec{\sigma}_y = \sum_{j=1}^{m} e_j(-1)^j \, dy_1 \wedge \cdots \wedge d\hat{y}_j \wedge \cdots \wedge dy_m \wedge dx_1 \wedge \cdots \wedge dx_m.$$

Hence, using Stokes' formula, we can easily prove

Theorem 1: *Let $\Omega \subseteqq \mathbf{R}^{2m}$ be open, $f, g \in C_1(\Omega; \mathcal{A})$ and let $C \subseteqq \Omega$ be compact with C_1-boundary ∂C. Then we have that*

(i) $$\int_C [(fD_x)\, g + f(D_x g)]\, dx\, dy = \int_{\partial C} f\, d\vec{\sigma}_x g,$$

$$\int_C [(fD_y)\, g + f(D_y g)]\, dx\, dy = \int_{\partial C} f\, d\vec{\sigma}_y g,$$

(ii) $$\int_C \left[\left(f(D_x + iD_y)\right) g + f\left((D_x + iD_y)\, g\right)\right] dx\, dy = \int_{\partial C} f\, d\vec{\sigma} g.$$

Next we introduce a singular integral operator A_C as follows. Let $f \in C_1(\Omega; \mathcal{A})$, let $C \subseteqq \Omega$ be compact with C_1-boundary and let $\vec{z}_0 = \vec{x}_0 + i\vec{y}_0 \in \mathring{C}$. Then we put

$$A_C(f)\,(\vec{z}_0) = \int_{C-\vec{z}_0} \frac{4mi}{\omega_{2m}} \frac{\vec{y} \wedge \vec{x}}{(|\vec{x}|^2 + |\vec{y}|^2)^{m+1}} f(\vec{z} + \vec{z}_0),$$

which may be regarded as the limit for $\varepsilon \to 0$ of the converging integrals

$$A_{C,\varepsilon}(f)\,(\vec{z}_0) = \int_{(C-\vec{z}_0)\setminus B(0,\varepsilon)} \frac{4mi}{\omega_{2m}} \frac{\vec{y} \wedge \vec{x}}{(|\vec{x}|^2 + |\vec{y}|^2)^{m+1}} f(\vec{z} + \vec{z}_0).$$

Notice that $\vec{y} \wedge \vec{x} = \sum_{j<k} (y_j x_k - y_k x_j)\, e_{jk}$ and that the functions $y_j x_k - y_k x_j$ are spherical harmonics of degree 2 in $\mathbf{R}^{2m}$. Hence the integral kernel $\frac{4mi}{\omega_{2m}} \times \frac{\vec{y} \wedge \vec{x}}{(|\vec{x}|^2 + |\vec{y}|^2)^{m+1}}$ is of the form $\frac{P_k(x)}{|x|^{k+n}}$, $n = 2m$, $k = 2$, P_k being spherical harmonic of degree k in $\mathbf{R}^n$. These singular kernels were studied by e.g. E. Stein in [8]. Hence, if $\mathcal{F}f$ denotes the Fourier transform, we have that

$$\mathcal{F}\left(\frac{4mi}{\omega_{2m}} \frac{\vec{y} \wedge \vec{x}}{(|\vec{x}|^2 + |\vec{y}|^2)^{m+1}}\right) = -\frac{4mi}{\omega_{2m}} \frac{\pi^m}{m!} \frac{\vec{y} \wedge \vec{x}}{|\vec{x}|^2 + |\vec{y}|^2},$$

and so the transform A_C is a higher Riesz transform of degree 2, which implies that it may act on C_1-functions. We now come to the Martinelli-Bochner formula for the operator $D_x + iD_y$.

Theorem 2: *Let $\Omega \subseteqq \mathbf{R}^{2m}$ be open, let $C \subseteqq \Omega$ be compact with C_1-boundary ∂C and let $z_0 \in \mathring{C}$. Then for every $f \in C_1(\Omega; \mathcal{A})$,*

$$f(\vec{z}_0) = \int_{\partial C} E(\vec{z} - \vec{z}_0)\, d\vec{\sigma}_z f(\vec{z}) - \int_C E(\vec{z} - \vec{z}_0) \left((D_x + iD_y)\, f(\vec{z})\right) + A_C(f)\,(\vec{z}_0).$$

Proof: By Theorem 1, we have that for every $\varepsilon > 0$,

$$\int_{\partial(C\setminus B(\vec z_0,\varepsilon))} E(\vec z - \vec z_0)\, d\vec\sigma_z f(\vec z)$$

$$= \int_{C\setminus B(\vec z_0,\varepsilon)} \left[E(\vec z - \vec z_0)\left((D_x + iD_y)\, f(\vec z)\right) - \frac{4mi}{\omega_{2m}} \frac{\vec y \wedge \vec x}{(|\vec x|^2 + |\vec y|^2)^{m+1}}\bigg|_{\vec z - \vec z_0} f(\vec z)\right].$$

Hence, we obtain that

$$\int_{\partial B(\vec z_0,\varepsilon)} E(\vec z - \vec z_0)\, d\vec\sigma_z f(\vec z)$$

$$= \int_{\partial C} E(\vec z - \vec z_0)\, d\vec\sigma_z f(\vec z) - \int_{C\setminus B(\vec z_0,\varepsilon)} E(\vec z - \vec z_0)\left((D_x + iD_y)\, f(\vec z)\right) + A_{C,\varepsilon}(f)\,(\vec z_0).$$

Furthermore, as we have that

$$\lim_{\varepsilon\to 0} \int_{\partial B(\vec z_0,\varepsilon)} E(\vec z - \vec z_0)\, d\vec\sigma_z f(\vec z)$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\omega_{2m}} \int_{\partial B(0,1)} (-\vec e_{nx} + i\vec e_{ny})\,(\vec e_{nx} + i\vec e_{ny})\, f(\vec z_0 + \varepsilon\vec\omega)\, dS_\omega$$

$$= f(\vec z_0)\left(1 - \frac{2i}{\omega_{2m}} \int_{\partial B(0,1)} \vec e_{nx} \wedge \vec e_{ny}\, dS_\omega\right) = f(\vec z_0)$$

and as $E(\vec z - \vec z_0) \in L_1^{\text{loc}}(\mathbf{R}^{2m})$ and $f \in C_1(\Omega; \mathcal{A})$, the theorem follows by taking the limit for $\varepsilon \to 0$ ∎

Corollary 1: *Let $f \in C_1(\Omega; \mathcal{A})$ such that $(D_x + iD_y)\, f = 0$ in Ω. Then for $C \subseteq \Omega$ compact with C_1-boundary and for $\vec z_0 \in C$,*

$$f(\vec z_0) = \int_{\partial C} E(\vec z - \vec z_0)\, d\vec\sigma_z f(\vec z) + A_C(f)\,(\vec z_0).$$

Notice that, still for weak complex monogenic functions, the singular integral term $A_C(f)$ occurs. This is due to the fact that the weak complex monogenic system is not elliptic. Hence, in order to get rid of the term $A_C(f)$, we have to restrict ourselves to special subclasses of weak complex monogenic functions, emerging from elliptic systems. Let us denote by $\langle \vec z, \vec w\rangle$ the complex inner product $z_1 w_1 + \cdots + z_m w_m$; then for the holomorphic system we obtain

Corollary 2: *Let f be holomorphic in Ω. Then for $\vec z_0 \in \mathring{C}$ we have that*

(i) $$f(\vec z_0) = \frac{1}{\omega_{2m}} \int_{\partial C} \frac{(\langle \vec x - \vec x_0, \vec e_{nx} + i\vec e_{ny}\rangle + \langle \vec y - \vec y_0, \vec e_{ny} - i\vec e_{nx}\rangle)}{(|\vec x - \vec x_0|^2 + |\vec y - \vec y_0|^2)^m}\, f(\vec z)\, dS,$$

(ii) $$A_C(f)\,(\vec z_0) = \frac{1}{\omega_{2m}} \int_{\partial C} \frac{(\vec x - \vec x_0) \wedge (\vec e_{nx} + i\vec e_{ny}) + (\vec y - \vec y_0) \wedge (\vec e_{ny} - i\vec e_{nx})}{(|\vec x - \vec x_0|^2 + |\vec y - \vec y_0|^2)^m}\, f(\vec z)\, dS.$$

Proof: As we may assume f to be $e_0\mathbf{C}$-valued, $A_C(f)\,(\vec z_0)$ is a bivector while $f(\vec z_0)$ is a scalar. Hence we obtain (i) and (ii) respectively as the scalar and the bivector part of the formula in Corollary 1 ∎

Notice that (i) corresponds to the classical Martinelli-Bochner formula (see [4]).

We also obtain Martinelli-Bochner formulae for f and $A_C(f)$ in the case where f is left biregular. As it is sufficient to consider f to take values in the real Clifford algebra $\mathcal{A}_{\mathbf{R}}$, we obtain these formulae respectively as the real and imaginary part of the formula in Corollary 1. We obtain

Corollary 3: *Let f be left biregular in Ω. Then for $\vec{z}_0 \in \mathring{C}$,*

$$\text{(i)}\quad f(\vec{z}_0) = -\frac{1}{\omega_{2m}} \int_{\partial C} \frac{(\vec{x} - \vec{x}_0)\,\vec{e}_{nx} + (\vec{y} - \vec{y}_0)\,\vec{e}_{ny}}{(|\vec{x} - \vec{x}_0|^2 + |\vec{y} - \vec{y}_0|^2)^m}\, f(\vec{z})\, dS,$$

$$\text{(ii)}\quad A_C(f)\,(\vec{z}_0) = \frac{i}{\omega_{2m}} \int_{\partial C} \frac{(\vec{x} - \vec{x}_0)\,\vec{e}_{ny} - (\vec{y} - \vec{y}_0)\,\vec{e}_{nx}}{(|\vec{x} - \vec{x}_0|^2 + |\vec{y} - \vec{y}_0|^2)^m}\, f(\vec{z})\, dS.$$

As to the Martinelli-Bochner formula for two-sided biregular functions (see [2]) we shall work in quite a similar way. Using Theorem 1, we obtain that, in the notations of the proof of Theorem 2,

$$\int_{\partial(C\setminus B(\vec{z}_0,\varepsilon))} [E(\vec{z} - \vec{z}_0)\, d\vec{\sigma}_x f(\vec{z}) + i f(\vec{z})\; d\vec{\sigma}_y E(\vec{z} - \vec{z}_0)]$$
$$= \int_{C\setminus B(\vec{z}_0,\varepsilon)} \big((E(\vec{z} - \vec{z}_0)\, D_x) f + i f(D_y E(\vec{z} - \vec{z}_0))\big),$$

f being two-sided biregular in a neighbourhood of C. As we may again consider f to be $\mathcal{A}_{\mathbf{R}}$-valued, we obtain by letting $\varepsilon \to 0$ and taking the real part, the formula obtained in [2]:

$$f(\vec{z}_0) = \frac{-1}{\omega_{2m}} \int_{\partial C} \frac{(\vec{x} - \vec{x}_0)\, d\vec{\sigma}_x f + f\, d\vec{\sigma}_y(\vec{y} - \vec{y}_0)}{(|\vec{x} - \vec{x}_0|^2 + |\vec{y} - \vec{y}_0|^2)^m}.$$

By taking the imaginary part, we obtain an identity of the form

$$\int_{\partial C} \frac{(\vec{y} - \vec{y}_0)\, d\vec{\sigma}_x f - f\, d\vec{\sigma}_y(\vec{x} - \vec{x}_0)}{(|\vec{x} - \vec{x}_0|^2 + |\vec{y} - \vec{y}_0|^2)^m}$$
$$= 2m \int_C \frac{[(\vec{x} - \vec{x}_0) \wedge (\vec{y} - \vec{y}_0), f]}{(|\vec{x} - \vec{x}_0|^2 + |\vec{y} - \vec{y}_0|^2)^{m+1}}\, dx\, dy,$$

where for $a, b \in \mathcal{A}$, $[a, b] = ab - ba$. The integral in the right-hand side of this formula may also be considered as a higher order Riesz transform (see [8]).

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