A New Computer-Assisted Analytic Method for the Dirichlet and Neumann Problems[1])

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Es wird eine neue Methode zur Behandlung des Dirichlet- und Neumann-Problems für die Laplacegleichung im $\mathbf{R}^2$ und $\mathbf{R}^3$ eingeführt. Mit Hilfe der Computersprache FORMAC wird gezeigt, wie eine vollständige Folge harmonischer Polynome in einem Gebiet G der Lösung orthonormiert werden kann. Dann ist die Lösung als Entwicklung in diesen orthonormierten Polynomen darstellbar. Wenn die Randwerte gewissen Bedingungen genügen, können die Koeffizienten automatisch berechnet werden.

Вводится новый метод решения задач Дирихле и Неймана для уравнения Лапласа в $\mathbf{R}^2$ и $\mathbf{R}^3$. С помощью языка программирования FORMAC показывается как орто-нормировать полную последовательность гармонических многочленов в области G решения. Тогда решение представимо в виде разложения по этим ортонормированным многочленам. Если краевые значения удовлетворяют некоторым условиям то коэффи-циенты разложения могут быть вычислены автоматически.

The authors introduce a new method for the Dirichlet and Neumann problems (in $\mathbf{R}^2$ and $\mathbf{R}^3$) based on certain theorems of Zaremba and Bergmann. With the help of a symbolic computing language (FORMAC with PL/1), the authors show how it is possible to orthonormalize a com-plete sequence of harmonic polynomials on the domain G of the solution. The solution is then given by an expansion in these orthonormalized polynomials. Under certain conditions on the boundary values, the expansion coefficients themselves may also be automatically calculated. j

Let G be a simply-connected domain in $\mathbf{R}^2$ or $\mathbf{R}^3$ whose closure is compact, and whose boundary ∂G is at least piecewise smooth. (In $\mathbf{R}^3$, we suppose that G is 3-connected.) We consider for G *Dirichlet's problem*: to find a function u such that

$$\nabla^2 u = 0 \quad \text{on } G,$$
$$u = F \quad \text{on } \partial G. \tag{1}$$

(where F is a given function, and $\nabla^2 u = $ Laplacian of u), as well as *Neumann's problem*:

$$\nabla^2 u = 0 \quad \text{on } G,$$
$$\frac{\partial u}{\partial N} = F \quad \text{on } \partial G, \tag{2}$$

(where, in (2), the boundary value function F must also satisfy the compatibility condition $\int_{\partial G} F \, d\theta = 0$, where the integral is the line (surface) integral around ∂G in $\mathbf{R}^2$ ($\mathbf{R}^3$); $\partial u / \partial N$ is taken in the sense of the outer-normal to G).

It is well known that there exist few methods for these problems which are both *analytic* and *computationally effective*. For example, among all orthogonal coordinate systems, Laplace's equation separates in only a few; these may be found in (for example) MOON and SPENCER [11]. In such cases, the method is well known, leading

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[1]) Vortrag auf der Konferenz „Complex Analysis" (s. Z. Anal. Anw. H. 2 (1986)).

to solutions in the form of multiple infinite series or integrals. In $\mathbf{R}^2$, on the other hand, there is the possibility of obtaining a conformal mapping of G onto the disc and/or the upper half-plane. However, if G is (for example) the interior of a real polygon (i.e., with no vertex at infinity), then the mapping function will be in general at least hyperelliptic. Even with the aid of approximation techniques, it seems fair to say that such methods are really effective only in the hands of a few specialists. (See, e.g., the collection of papers in BECKENBACH [2].)

Our method is based upon certain theorems of ZAREMBA [17, 18] and BERGMANN [3]. Although obtained many years ago, their theorems have only recently become computationally effective with the development of high-speed symbolic computing methods. With Zaremba, begin with the Green's identity

$$\int_G f \, \nabla^2 g \, d\tau + \int_G \nabla f \cdot \nabla g \, d\tau = \int_{\partial G} f \frac{\partial g}{\partial N} \, d\sigma \tag{3}$$

for sufficiently smooth functions f, g. Define the inner product

$$(f, g) = \int_G \nabla f \cdot \nabla g \, d\tau \tag{4}$$

and the associated *gradient norm* $\|f\|^2 = (f, f)$. Let now $\{P_m\} = P_0, P_1, P_2, \ldots$ be a sequence of harmonic functions, complete in the space of harmonic functions on G, and orthonormal on G in the gradient norm. We seek a solution u of Dirichlet's problem (1) in the form

$$u = \sum_{m=0}^{\infty} a_m P_m \tag{5}$$

and, in (3), set $f = u$ and $g = P_n$ for an arbitrary fixed n. Then (3) becomes

$$\int_G u \, \nabla^2 P_n \, d\tau + \sum_{m=1}^{\infty} a_m (P_m, P_n) = \int_{\partial G} F \frac{\partial P_n}{\partial N} \, d\sigma \tag{6}$$

since $f = u = F$ on ∂G. But since $\nabla^2 P_n = 0$, (6) reduces to

$$a_n = \int_{\partial G} F \frac{\partial P_n}{\partial N} \, d\sigma. \tag{7}$$

For Neumann's problem (2), proceed similarly but with $f = P_n$ and $g = u$, so that

$$\int_G P_n \, \nabla^2 u \, d\tau + \sum_{m=1}^{\infty} a_m (P_n, P_m) = \int_{\partial G} P_n F \, d\sigma.$$

Here we have $\nabla^2 u = 0$ on G, so that

$$a_n = \int_{\partial G} P_n F \, d\sigma. \tag{8}$$

In both cases, we see that the expansion coefficients a_n are all determined except for the coefficient a_0 of $P_0 = \text{const}$, which is lost upon taking the gradient of (5). But we have $u = a_0 + \sum a_m P_m = a_0 + v$ (say) and, if Q is a conveniently chosen point of the boundary ∂G, then $a_0 = u(Q) - v(Q)$. Let us notice, finally, that the calculations (3)−(8) are entirely independent of the dimension. The theorems of Bergmann (op. cit.) show that the resulting expansions converge absolutely and almost

uniformly on G. (These results of Zaremba were discussed by NEHARI [12] who, however, does not contemplate the kind of direct implementation of them given below.)

For Zaremba's method to be computationally effective, one must be able to produce the necessary complete orthonormal sequence $\{P'_m\}$ of harmonic functions on G. In $\mathbf{R}^2$, the real and imaginary parts of $(x + iy)^m$, viz.,

$$1, x, y, x^2 - y^2, 2xy, \dots \tag{9}$$

are certainly harmonic and complete on every such domain G, but obviously not orthonormal. In principle, one may orthonormalize the sequence (9) (up to some finite degree p) by the Gram-Schmidt algorithm, which entails repeatedly computing double integrals of polynomials over G. Clearly, such computations are not manually feasible for moderate-to-large p, even for geometrically simple domains G.

At the time when we began these researches, we had access to large-scale digital computing facilities equipped with the symbolic computing capacity known as FORMAC (a symbolic pre-processing auxiliary language associated with PL/1; *vide* BAHR [1]). FORMAC possesses, in particular, the capability of automatic integration of polynomials.

It is beyond the scope of this paper to go into details of a FORMAC program necessary to carry out the kind of calculation we have described. (We plan to give such an exposition in [15]; *vide* also WILKERSON [16].)

However, we do want to emphasize the *analytic* (as opposed to numerical) character of the method by exhibiting certain portions of the program's output. In the first part of any such program, one obtains the polynomials (9), up to some specified degree p. (Any portion of the calculation — including the specified subset of the polynomials (9) — may be printed out for the analyst's inspection.) In the second part of the program, one encodes the necessary instructions to orthonormalize the polynomials (9) over the domain G in question.

We have already remarked that our method is essentially independent of the simension of the space. One must, of course, be able to produce *ab initio* the necesdary complete set of harmonic polynomials. In $\mathbf{R}^3$, it is well known that there are exactly $2m + 1$ linearly independent homogeneous harmonic polynomials of degree $m = 1, 2, 3, \dots$ To generate them, introduce the *hypercomplex variable* $Z = x + iy + jz$, where $j^2 = -1 - i^2$. Then Z^m has one real and $2m$ "imaginary parts", all linearly independent homogeneous harmonic polynomials of degree m, given as coefficients in the expansion of Z^m over the first $2m + 1$ of the elements $1, i, j, ij, i^2, i^2j, i^3, i^3j, \dots$ (the totality of which forms the basis of an infinite-dimensional commutative algebra over $\mathbf{R}$; *vide* SNYDER [14] and ROSCULET [13]). Any finite subset of polynomials generated in this way is in principle machine-computable. We do see some possibilities of our method for the Robin's problem: $\partial u/\partial N = \alpha u + F$ on ∂G (with α piecewise-polynomial). We shall discuss these and other extensions elsewhere.

It seems clear that our method should be applicable to Dirichlet and/or Neumann problems for a large class of homogeneous partial differential equations of elliptic type. In particular, for linear elliptic equations $Lu = 0$ with $L = P(\partial/\partial x_1, \dots, \partial/\partial x_n)$, P a homogeneous polynomial in the partials, can one characterize a large class of such equations which possess a complete set of polynomial solutions? Some important special results along these lines have been obtained by MILES and WILLIAMS [6—10] and by HORVATH [5].

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