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On Solutions of First-Order Partial Differential-Functional Equations in an Unbounded Domain

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Unter der Voraussetzung der Stetigkeit und einmaligen partiellen Differenzierbarkeit der Lösungen wird bewiesen, daß das Cauchy-Problem

$$\begin{aligned} z_x^{(i)}(x, y) &= f^{(i)}(x, y, z(x, y), z, z_y^{(i)}(x, y)) \\ z^{(i)}(x, y) &= \varphi_i(x, y) \quad \text{für} \quad (x, y) \in [-\tau_0, 0] \times \mathbf{R}^n \end{aligned} \qquad (i = 1, \ldots, m)$$

nicht mehr als eine Lösung besitzt, falls die Funktion $f = (f^{(1)}, \ldots, f^{(m)})$ der Variablen (x, y, p, z, q) die Lipschitzsche Bedingung bezüglich (p, z, q) oder die Lipschitzsche Bedingung bezüglich (p, z) und die Höldersche Bedingung bezüglich q erfüllt.

Под условием непрерывности и существования частных производных первого порядка решений доказывается, что задача Коши

$$\begin{aligned} z_x^{(i)}(x, y) &= f^{(i)}(x, y, z(x, y), z, z_y^{(i)}(x, y)) \\ z^{(i)}(x, y) &= \varphi_i(x, y) \quad \text{для} \quad (x, y) \in [-\tau_0, 0] \times \mathbf{R}^n \end{aligned} \qquad (i = 1, \ldots, m)$$

может иметь только единственное решение, если функция $f = (f^{(1)}, \ldots, f^{(m)})$ переменных (x, y, p, z, q) удовлетворяет условию Липшица по переменным (p, z, q) или условию Липшица по переменным (p, z) и условию Гёльдера по переменной q.

Under the assumptions of continuity and the existence of first-order partial derivatives of the solutions it is proved that the Cauchy problem

$$\begin{aligned} z_x^{(i)}(x, y) &= f^{(i)}(x, y, z(x, y), z, z_y^{(i)}(x, y)) \\ z^{(i)}(x, y) &= \varphi_i(x, y) \quad \text{for} \quad (x, y) \in [-\tau_0, 0] \times \mathbf{R}^n \end{aligned} \qquad (i = 1, \ldots, m)$$

admits at most one solution if the function $f = (f^{(1)}, \ldots, f^{(m)})$ of the variables (x, y, p, z, q) satisfies a Lipschitz condition with respect to (p, z, q), or a Lipschitz condition with respect to (p, z) and a Hölder condition with respect to q.

1. Introduction

First-order partial differential inequalities were first treated by A. Haar [9] and by M. Nagumo [20]. The classical theory of partial inequalities is described in detail in [17]. The investigation of properties of partial differential-functional equations of first order is strongly connected with the theory of differential and differential-functional inequalities. In [11] differential-functional inequalities are applied to the estimation of the difference between solutions of two systems of partial differential-functional equations and to the formulation of a criterion of uniqueness of solutions of such systems. Difference inequalities corresponding to the differential inequalities with a retarded argument are considered in [13]. The paper [14] contains sufficient conditions for the stability and asymptotic stability of solutions of non-linear partial differential-functional equations. The basic tool in these investigations are differential-functional inequalities and Lapunow functions. Generalized solutions of partial equations and inequalities are considered in [1–3, 7, 15, 16, 18].

The problem of existence of solutions for partial differential-functional equations is also strongly connected with differential inequalities. A global existence of solutions together with an estimation of the existence domain is considered in [12, 24]. At the present moment there exist numerous papers on this subject. For more detailed information and references see [2, 12, 24].

This note deals with solutions of differential-functional equations and inequalities defined in an unbounded zone. The solutions are supposed to be continuous and to have first partial derivatives. The theorems discussed here are known under the stronger assumptions that the solutions possess the total differentials [11, 24].

Denote by $\mathbf{R}^n$ the n-dimensional Euclidean space and by $y = (y_1, \ldots, y_n)$ its elements. Let $I_0 = \{0\} \times \mathbf{R}^n$, $D_0 = [-\tau_0, 0] \times \mathbf{R}^n$ and $D = [0, a) \times \mathbf{R}^n$, where $0 \leqq \tau_0$ and $0 < a \leqq +\infty$. Suppose that $z = (z^{(1)}, \ldots, z^{(m)})$ is a function of the variables (x, y) defined in a domain $\Omega \subset \mathbf{R}^{1+n}$. If z possesses first-order partial derivatives on Ω, then we denote

$$z_x = (z_x^{(1)}, \ldots, z_x^{(m)}), \quad z_y^{(i)} = (z_{y_1}^{(i)}, \ldots, z_{y_n}^{(i)}), \quad z_y = [z_{y_j}^{(i)}]_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}}.$$

Further, denote by $C(X, Y)$ the set of all continuous functions defined in X taking values in Y; X, Y being arbitrary metric spaces. Suppose that $\varphi = (\varphi_1, \ldots, \varphi_m) \in C(D_0, \mathbf{R}^m)$ and

$$f = (f^{(1)}, \ldots, f^{(m)}) \colon D \times \mathbf{R}^m \times C(D_0 \cup D, \mathbf{R}^m) \times \mathbf{R}^n \to \mathbf{R}^m.$$

If $z \in C(D_0 \cup D, \mathbf{R}^m)$ and possesses first-order partial derivatives on $D \setminus I_0$, then we define for $(x, y) \in D \setminus I_0$

$$\begin{aligned} &f\big(x, y, z(x, y), z, z_y(x, y)\big) \\ &= \big(f^{(1)}(x, y, z(x, y), z, z_y^{(1)}(x, y)), \ldots, f^{(m)}(x, y, z(x, y), z, z_y^{(m)}(x, y))\big). \end{aligned}$$

In this paper we shall deal with the Cauchy problem for partial differential-functional systems of first order

$$\begin{aligned} &z_x(x, y) = f\big(x, y, z(x, y), z, z_y(x, y)\big), \qquad (x, y) \in D \setminus I_0, \\ &z(x, y) = \varphi(x, y), \qquad (x, y) \in D_0. \end{aligned} \tag{1}$$

The solutions are supposed to be continuous in $D_0 \cup D$ and to have first-order partial derivatives in $D \setminus I_0$. As a particular case we obtain differential-integral equations and systems with a retarded argument.

First-order partial differential-functional equations have applications in different branches of knowledge. Hyperbolic differential and differential-integral systems of first order have recently been proposed [3] as simple mathematical models for the non-linear phenomenon of harmonic generation of laser radiation through piezoelectric crystals for non-dispersive materials and of Maxwell-Hopkinson type. There are various problems in non-linear optics which lead to non-linear hyperbolic differential-integral problems. For more detailed information and references see [4]. Non-linear equations may be used to describe the growth of a population of cells which constantly differentiate (change their properties) in time [8]. First-order partial differential-integral telegraphic equations are examined in [19]. Our results in this paper are also motivated by applications of partial differential-integral equations considered in [10].

For $(x, y) \in D$ we define

$$T(x, y) = \{(t, s) \in D_0 \cup D \colon -\tau_0 \leqq t \leqq x \text{ and } \|s\|_* \leqq \|y\|_*\},$$

where $\|\cdot\|_*$ is the Euclidean norm in $\mathbf{R}^n$. We assume that the function f satisfies the following *Volterra condition*: if $z, \bar{z} \in C(D_0 \cup D, \mathbf{R}^m)$ and $z = \bar{z}$ on $T(x, y)$ then $f(x, y, p, z, q) = f(x, y, p, \bar{z}, q)$ for all $p \in \mathbf{R}^m$ and $q \in \mathbf{R}^n$.

In this note we prove that if the function f of the variables (x, y, p, z, g) satisfies a Lipschitz condition with respect to (p, z, q) then the problem (1) admits at most one solution in D. We obtain this uniqueness theorem as a particular case of some general comparison theorem for partial differential-functional inequalities. At the end we prove the uniqueness of solutions in the case when f satisfies a Lipschitz condition with respect to p, z and a Hölder condition with respect to q.

The system (1) is of special hyperbolic type since each equation contains first-order derivatives of only one unknown function. This is a weakly coupled system. The existence and uniqueness of solutions of initial problems for strongly coupled systems is examined by using slightly different methods and under more restrictive assumptions than for (1). We illustrate this in more detail by the example of the Cauchy problem for the system without a functional argument

$$z_x^{(i)}(x, y) = F_i(x, y, z(x, y), z_{y_1}(x, y), z_y^{(i)}(x, y)), \quad i = 1, \ldots, m,$$

$$z(0, y) = \gamma(y),$$

where $F = (F_1, \ldots, F_m): D \times \mathbf{R}^{m+m-n} \to \mathbf{R}^m$. The derivatives $(z_{y_1}^{(1)}, \ldots, z_{y_1}^{(m)}) = z_{y_1}$ are responsible for the system to be strongly coupled. Solutions of the above problem are supposed to belong to a special class of analytic functions with respect to the variable y_1. This class was first taken advantage of by K. Nickel [22] in the theory of strongly coupled parabolic systems of non-linear second order differential equations. The analyticity of z with respect to y_1 is essential in questions treated in this paper. This is shown by a counter-example constructed by A. Pliś [23] in which for a strongly coupled system of two linear equations there is no uniqueness for the Cauchy problem in the class C^∞. The local uniqueness (and also existence) of a solution which is analytic with respect to y and belongs to the class C^1 with respect to x was proved by M. Nagumo [21] under assumptions on analyticity of the right-hand sides of strongly coupled systems with respect to all arguments except the variable x. The problem of uniqueness of solutions for strongly coupled differential-functional systems in a class of analytic functions was considered in [25].

2. A comparison lemma

For $p, \bar{p} \in \mathbf{R}^m$ we write $p \leqq \bar{p}$ if the components satisfy $p_j \leqq \bar{p}_j$ for all indices $j = 1, \ldots, m$, and in case of a fixed index $p \underset{i}{\leqq} \bar{p}$ if $p \leqq \bar{p}$ and $p_i = \bar{p}_i$. For p we define $\|p\|_m = |p_1| + \cdots + |p_m|$, $[\![p]\!] = (|p_1|, \ldots, |p_m|)$ and if $C = [c_{ij}]$ is an $m \times n$-matrix then $[\![C]\!] = [|c_{ij}|]$. C^T is the transposed matrix of C. Let $[\alpha, \beta] \subset (\alpha_0, \beta_0) \subset \mathbf{R}$. If $w = (w_1, \ldots, w_m) \in C\big((\alpha_0, \beta_0), \mathbf{R}^m\big)$ then

$$\|w\|_{[\alpha,\beta]} = \|w_1\|_{[\alpha,\beta]} + \cdots + \|w_m\|_{[\alpha,\beta]}$$

where $\|\cdot\|_{[\alpha,\beta]}$ is the usual max-norm of $C([\alpha, \beta], \mathbf{R})$. For $(t, y) \in D$ we define $S(t, y) = \{(t, s): \|s\|_* \leqq \|y\|_*\}$. We shall denote a function w of the variable $t \in [\alpha, \beta]$ by $w(\cdot)$ or $\big(w(t)\big)_{[\alpha,\beta]}$. If $z \in C(D_0 \cup D, \mathbf{R}^m)$ and $x \in [0, a)$ then we define a vector-valued function

$$\Big(\max_{s\in S(t,y)} [\![z(t, s)]\!]_{[-\tau_0,x]}\Big) = \Big(\Big(\max_{s\in S(t,y)} |z^{(1)}(t, s)|\Big)_{[-\tau_0,x]}, \ldots, \Big(\max_{s\in S(t,y)} |z^{(m)}(t, s)|\Big)_{[-\tau_0,x]}\Big).$$

In order to simplify the formulation of subsequent theorems we introduce the following **Assumption** H on a function

$$\sigma = (\sigma_1, \ldots, \sigma_m): [0, a) \times \mathbf{R}_+{}^m \times C([-\tau_0, a), \mathbf{R}_+{}^m) \to \mathbf{R}_+{}^m$$

where $\mathbf{R}_+ = [0, +\infty)$:

1. σ satisfies the following *Volterra condition*: if $w, \overline{w} \in C([-\tau_0, a), \mathbf{R}_+{}^m)$ and $w = \overline{w}$ on $[-\tau_0, x]$, $0 \leqq x \leqq a$, then $\sigma(x, p, w) = \sigma(x, p, \overline{w})$ for all $p \in \mathbf{R}_+{}^m$.

2. $p \leqq \overline{p}$ implies $\sigma_i(x, p, w) \leqq \sigma_i(x, \overline{p}, w)$ for $x \in [0, a)$ and $w \in C([-\tau_0, a), \mathbf{R}_+{}^m)$.

3. If $w, \overline{w} \in C([-\tau_0, a), \mathbf{R}_+{}^m)$ and $w(t) \leqq \overline{w}(t)$ for $t \in [-\tau_0, x]$ then $\sigma(x, p, w) \leqq \sigma(x, p, \overline{w})$ for all $p \in \mathbf{R}_+{}^m$.

4. σ satisfies the one-sided Lipschitz condition

$$\sigma_i(x, p, w) - \sigma_i(x, \overline{p}, \overline{w}) \leqq K \|p - \overline{p}\|_m + M \|w - \overline{w}\|_{[-\tau_0, x]},$$

for $i = 1, \ldots, m$, where $x \in [0, a)$, $p, \overline{p} \in \mathbf{R}_+{}^m$ with $p \geqq \overline{p}$ and $w, \overline{w} \in C([-\tau_0, a), \mathbf{R}_+{}^m)$ with $w(t) \geqq \overline{w}(t)$ for $t \in [-\tau_0, x]$.

In dealing with applications of ordinary differential inequalities to partial differential equations, we have to estimate the solutions of such equations, which are functions of several variables, by functions of one variable. In this section we shall prove the following comparison lemma.

Lemma 1: *Suppose that*

1. *the function σ satisfies Assumption H,*

2. $\Psi = (\psi^{(1)}, \ldots, \psi^{(m)}) \in C(D_0 \cup D, \mathbf{R}^m)$ *and possesses first-order partial derivatives on $D \setminus I_0$,*

3. *there exists a constant $L \geqq 0$ such that the differential-functional inequality*

$$[\![\Psi_x(x, y)]\!] \leqq \sigma\Big(x, [\![\Psi(x, y)]\!], \big(\max_{s \in S(t,y)} [\![\Psi(t, s)]\!]\big)_{[-\tau_0, x]}\Big) + L^{[m]}[\![\Psi_y(x, y)]\!]^T, \qquad (x, y) \in D \setminus I_0, \tag{2}$$

where $L^{[m]} = (L, \ldots, L) \in \mathbf{R}^m$, and the initial inequality

$$[\![\Psi(x, y)]\!] \leqq \eta(x), \qquad (x, y) \in D_0,$$

where $\eta = (\eta_1, \ldots, \eta_m) \in C([-\tau_0, 0], \mathbf{R}_+{}^m)$, are satisfied,

4. *the right-hand solution $\tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_m)$ of the problem*

$$\begin{aligned} &\omega'(x) = \sigma(x, \omega(x), \omega) \\ &\omega(x) = \eta(x) \quad \text{for} \quad x \in [-\tau_0, 0], \end{aligned} \tag{3}$$

exists on $[0, a)$.

Under these assumptions $[\![\Psi(x, y)]\!] \leqq \tilde{\omega}(x)$ for $(x, y) \in D$.

Proof: It is patterned on that given in [5]. We will show that at an arbitrary point $(\overline{x}, \overline{y}) \in D$, $\overline{x} \neq 0$, the function $w(\cdot) = [\![\Psi(\cdot)]\!] - \tilde{\omega}(\cdot)$ is non-positive. Let $\overline{x} < b < a$ and

$$\Phi_0(t) = \max_{1 \leqq i \leqq m} \max_{\substack{x \in [0,b] \\ \|y\|_* \leqq t}} w^{(i)}(x, y).$$

Let $\Phi \in C^1(\mathbf{R}_+, \mathbf{R}_+)$, $\Phi(t) \uparrow +\infty$ as $t \to +\infty$, satisfies the condition $\Phi(t) \geqq \Phi_0(t)$ for $t \geqq 0$. It is easy to verify that the function

$$H(x, y) = \exp\left[\Phi\big(nLx + (\|y\|_*{}^2 + 1)^{1/2}\big) + x\big(m(K + M) + 1\big)\right] \tag{4}$$

satisfies the inequality

$$m(K + M)\, H(\cdot) + L\, \|H_y(\cdot)\|_n - H_x(\cdot) \leqq -H(\cdot) < 0 \text{ on } D. \tag{5}$$

It follows from the definition of Φ that for every $\varepsilon > 0$ there exists $r = r(\varepsilon)$ such that for $i = 1, \ldots, m$

$$w^{(i)}(x, y) < \varepsilon H(x, y) \quad \text{for} \quad 0 \leqq x \leqq b \quad \text{and} \quad r \leqq \|y\|_*.$$

Let $\overline{\Psi} = \Psi/H$, $\overline{\omega} = \tilde{\omega}/H$ and $\overline{w}(\cdot) = [\![\Psi(\cdot)]\!] - \overline{\omega}(\cdot)$. There exist an index i_0 and a doint (x_0, y_0), $0 \leqq x_0 \leqq b$ and $\|y_0\|_* \leqq r$, such that

$$\overline{w}^{(i_0)}(x_0, y_0) = \max_{1 \leqq i \leqq m} \max_{\substack{0 \leqq x \leqq b \\ \|y\|_* \leqq r}} \overline{w}^{(i)}(x, y). \tag{6}$$

It follows from the initial estimate that $\overline{w}^{(i_0)}(x_0, y_0) \leqq 0$ if $x_0 = 0$. We prove this inequality now if $x_0 \neq 0$ and $\|y_0\|_* < r$. Suppose the contrary. Then we have $|\psi^{(i_0)}(x_0, y_0)| > 0$ and

$$\begin{aligned} &\left|w_x^{(i_0)}(x_0, y_0) + \sigma_{i_0}\big(x_0, \tilde{\omega}(x_0), \tilde{\omega}\big)\right| = |\psi_x^{(i_0)}(x_0, y_0)| \\ &\leqq \sigma_{i_0}\Big(x_0, [\![\psi(x_0, y_0)]\!], \Big(\max_{s \in S(t, y_0)} [\![\Psi(t, s)]\!]\Big)_{[-\tau_0, x]}\Big) + L\, \|w_y^{(i_0)}(x_0, y_0)\|_n. \end{aligned}$$

Because $\overline{w}^{(i_0)}(x_0, y_0) > 0$, $\overline{w}_x^{(i_0)}(x_0, y_0) \geqq 0$ and $\overline{w}_y^{(i_0)}(x_0, y_0) = 0$, then we obtain

$$\begin{aligned} 0 &\leqq \overline{w}_x^{(i_0)}(x_0, y_0)\, H(x_0, y_0) \\ &\leqq \overline{w}^{(i_0)}(x_0, y_0)\, [L\, \|H_y(x_0, y_0)\|_n - H_x(x_0, y_0)] \\ &\quad + \sigma_{i_0}\Big(x_0, [\![H(x_0, y_0)\, \overline{\Psi}(x_0, y_0)]\!], \Big(\max_{s \in S(t, y_0)} [\![\Psi(t, s)]\!]\Big)_{[-\tau_0, x]}\Big) \\ &\quad - \sigma_{i_0}\big(x_0, H(x_0, y_0)\, \overline{\omega}(x_0, y_0), \tilde{\omega}\big). \end{aligned} \tag{7}$$

Let $I = \{i : \overline{w}^{(i)}(x_0, y_0) > 0\}$ and $\Gamma = (\gamma^{(1)}, \ldots, \gamma^{(m)})$ where

$$\gamma^{(i)}(x, y) = \begin{cases} H(x, y)\, |\overline{\Psi}^{(i)}(x, y)| & \text{if } i \in I \\ H(x, y)\, \overline{\omega}^{(i)}(x, y) & \text{if } i \notin I. \end{cases}$$

Then we have

$$[\![H(x_0, y_0)\, \overline{\Psi}(x_0, y_0)]\!] \leqq \Gamma(x_0, y_0). \tag{8}$$

Let $\Lambda = (\lambda_1, \ldots, \lambda_m)$ be a function defined by

$$\lambda_i(x) = \begin{cases} \tilde{\omega}_i(x), & x \in [-\tau_0, 0] \\ \max\Big[\tilde{\omega}_i(x), \max_{s \in S(x, y_0)} |\psi^{(i)}(x, s)|\Big], & x \in (0, a). \end{cases}$$

Then we have $\Lambda \in C\big([-\tau_0, a), \mathbf{R}_+^m\big)$ and

$$\max_{s \in S(x, y_0)} [\![\Psi(x, s)]\!] \leqq \Lambda(x) \quad \text{for } x \in [-\tau_0, a). \tag{9}$$

From the monotonicity conditions for σ and from (8), (9) we get

$$\begin{aligned} &\sigma_{i_0}\Big(x_0, [\![H(x_0, y_0)\, \overline{\Psi}(x_0, y_0)]\!], \Big(\max_{s \in S(t, y_0)} [\![\Psi(t, s)]\!]\Big)_{[-\tau_0, x_0]}\Big) \\ &\leqq \sigma_{i_0}\big(x_0, \Gamma(x_0, y_0), \Lambda\big). \end{aligned} \tag{10}$$

Applying the Lipschitz condition for σ_{i_0} and (8) we obtain

$$\sigma_{i_0}\big(x_0, \Gamma(x_0, y_0), \Lambda\big) - \sigma_{i_0}\big(x_0, H(x_0, y_0)\,\overline{w}(x_0, y_0), \tilde{\omega}\big)$$
$$\leqq K \sum_{j\in I} [\gamma^{(j)}(x_0, y_0) - H(x_0, y_0)\,\overline{w}^{(j)}(x_0, y_0)] + M\,\|\Lambda - \tilde{\omega}\|_{[-\tau_0, x_0]}$$
$$\leqq mKH(x_0, y_0)\,\overline{w}^{(i_0)}(x_0, y_0) + M\,\|\Lambda - \tilde{\omega}\|_{[-\tau_0, x_0]}. \tag{11}$$

It follows from the definition of Λ that there exist points $(x_j, y^{(j)})$, $j = 1, \dots, m$, such that $0 \leqq x_j \leqq x_0$, $\|y^{(j)}\|_* \leqq \|y_0\|_*$ and

$$\|\lambda_j - \tilde{\omega}_j\|_{[-\tau_0, x_0]} = \max_{s\in S(x_j, y_0)} |\psi^{(j)}(x_j, s)| - \tilde{\omega}_j(x_j) = |\psi^{(j)}(x_j, y^{(j)})| - \tilde{\omega}_j(x_j).$$

Then we have

$$\|\lambda_j - \tilde{\omega}_j\|_{[-\tau_0, x_0]} = \overline{w}^{(j)}(x_j, y^{(j)})\,H(x_j, y^{(j)}) \leqq H(x_0, y_0)\,\overline{w}^{(i_0)}(x_0, y_0). \tag{12}$$

Estimates (7), (10)—(12) imply, in contradiction to (5),

$$0 \leqq \overline{w}_x^{(i_0)}(x_0, y_0)\,H(x_0, y_0)$$
$$\leqq \overline{w}^{(i_0)}(x_0, y_0)\,[m(K+M)\,H(x_0, y_0) + L\,\|H_y(x_0, y_0)\|_n - H_x(x_0, y_0)].$$

Finally, if $\|y_0\|_* = r$, then, from the definition of $\overline{w}$, we obtain $\overline{w}^{(i_0)}(x_0, y_0) < \varepsilon$, ε being arbitrary.

It follows from the above considerations that $\overline{w}^{(i)}(\bar{x}, \bar{y}) < \varepsilon$ for arbitrary $\varepsilon > 0$. Then we have $\overline{w}^{(i)}(\bar{x}, \bar{y}) \leqq 0$ and consequently $w^{(i)}(\bar{x}, \bar{y}) \leqq 0$, $i = 1, \dots, m$. Since $(\bar{x}, \bar{y})$ is an arbitrary point in D, we obtain the desired inequality ∎

Example: Suppose that for $(x, p, w) \in [0, a) \times \mathbf{R}_+^m \times C\big([-\tau_0, a), \mathbf{R}^m\big)$ and $i = 1, \dots, m$ we have

$$\sigma_i(x, p, w) = \sigma_i\Big(x, p, \max_{t\in[-\tau_0, x]} w_1(t), \dots, \max_{t\in[-\tau_0, x]} w_m(t)\Big).$$

If $z \in C(D_0 \cup D, \mathbf{R}^m)$ and $(x, y) \in D$, then we denote

$$\max_{(t,s)\in T(x,y)} [\![z(t, s)]\!] = \Big(\max_{(t,s)\in T(x,y)} |z^{(1)}(t, s)|, \dots, \max_{(t,s)\in T(x,y)} |z^{(m)}(t, s)|\Big).$$

Let

$$\max_{t\in[-\tau_0, x]} w(t) = \Big(\max_{t\in[-\tau_0, x]} w_1(t), \dots, \max_{t\in[-\tau_0, x]} w_m(t)\Big).$$

Then assumption (2) of Lemma 1 has the form

$$[\![\Psi_x(x, y)]\!] \leqq \sigma\Big(x, [\![\Psi(x, y)]\!], \max_{(t,s)\in T(x,y)} [\![\Psi(t, s)]\!]\Big) + L^{[m]}[\![\Psi_y(x, y)]\!]^T, \quad (x, y) \in D \setminus I_0,$$

and the comparison problem (3) is

$$\omega'(x) = \sigma\Big(x, \omega(x), \max_{t\in[-\tau_0, x]} \omega(t)\Big),$$
$$\omega(x) = \eta(x) \quad \text{for} \quad x \in [-\tau_0, 0].$$

If η is non-decreasing then the above problem is equivalent with $\omega'(x) = \sigma\big(x, \omega(x), \omega(x)\big)$, $\omega(0) = \eta(0)$.

3. Estimation of the difference between two solutions, uniqueness criteria

Let us consider the initial problem (1) and the problem

$$\begin{aligned} z_x(x, y) &= g(x, y, z(x, y), z, z_y(x, y)), \qquad (x, y) \in D \setminus I_0, \\ z(x, y) &= \tilde{\varphi}(x, y), \qquad (x, y) \in D_0. \end{aligned} \tag{13}$$

The following theorem allows us to estimate the difference between solutions of (1) and (13).

Theorem 1: *Suppose that*

1. *the functions* $f = (f^{(1)}, \ldots, f^{(m)})$ *and* $g = (g^{(1)}, \ldots, g^{(m)})$ *are defined on* $D \times \mathbf{R}^m \times C(D_0 \cup D, \mathbf{R}^m) \times \mathbf{R}^n$ *and satisfy the Volterra condition,*
2. *the function* σ *satisfies Assumption H,*
3. *the estimates*

$$\begin{aligned} &[\![f(x, y, p, z, q) - g(x, y, \bar{p}, \bar{z}, q)]\!] \\ &\leqq \sigma\left(x, [\![p - \bar{p}]\!], \left(\max_{s \in S(t,y)} [\![z(t, s) - \bar{z}(t, s)]\!]\right)_{[-\tau_0, x]}\right); \end{aligned}$$

and

$$\|f(x, y, p, z, q) - g(x, y, p, z, \bar{q})\|_m \leqq L\|q - \bar{q}\|_n$$

are satisfied on $D \times \mathbf{R}^m \times C(D_0 \cup D, \mathbf{R}^m) \times \mathbf{R}^n$,

4. $u, v \in C(D_0 \cup D, \mathbf{R}^m)$ *are solutions of* (1) *and* (13), *respectively, having first-order partial derivatives on* $D \setminus I_0$,
5. *there exists a function* $\eta \in C([-\tau_0, 0], \mathbf{R}_+{}^m)$ *such that*

$$[\![u(x, y) - v(x, y)]\!] \leqq \eta(x) \quad \text{for} \quad (x, y) \in D_0$$

and the right-hand solution $\bar{\omega}$ *of the problem* (3) *exists on* $[0, a)$.

Under these assumptions $[\![u(x, y) - v(x, y)]\!] \leqq \bar{\omega}(x)$ *for* $(x, y) \in D$.

Proof: The function $\Psi = u - v$ satisfies all the conditions of Lemma 1 and, hence, the desired inequality holds true ∎

For $z \in C(D_0 \cup D, \mathbf{R}^m)$ and $(x, y) \in D$ denote

$$\|z\|_{xy} = \sum_{i=1}^{m} \|z^{(i)}\|_{xy} \quad \text{where} \quad \|z^{(i)}\|_{xy} = \max_{(t,s) \in T(x,y)} |z^{(i)}(t, s)|.$$

The next theorem is an immediate consequence of Theorem 1.

Theorem 2: *Suppose that*

1. *the function* $f: D \times \mathbf{R}^m \times C(D_0 \cup D, \mathbf{R}^m) \times \mathbf{R}^n \to \mathbf{R}^m$ *satisfies the Volterra condition and*
2. $\|f(x, y, p, z, q) - f(x, y, \bar{p}, \bar{z}, \bar{q})\|_m \leqq K \|p - \bar{p}\|_m + M \|z - \bar{z}\|_{xy} + L \|q - \bar{q}\|_n$, *where* $z, \bar{z} \in C(D_0 \cup D, \mathbf{R}^m)$, $p, \bar{p} \in \mathbf{R}^m$ *and* $q, \bar{q} \in \mathbf{R}^n$.

Under these assumptions, the solution u *of* (1), *which is continuous in* $D_0 \cup D$ *and has first-order partial derivatives on* $D \setminus I_0$, *is unique and depends continuously on* φ *and* f.

The following theorem concerns weak inequalities between vector-valued functions satisfying first-order partial differential-functional inequalities in an unbounded zone.

Theorem 3: *Suppose that*

1. *the function* $f: D \times \mathbf{R}^m \times C(D_0 \cup D, \mathbf{R}^m) \times \mathbf{R}^n \to \mathbf{R}^m$ *satisfies the Volterra condition and the following monotonicity conditions:*

(i) *if* $z, \bar{z} \in C(D_0 \cup D, \mathbf{R}^m)$ *and* $z(t, s) \leqq \bar{z}(t, s)$ *for* $(t, s) \in T(x, y)$, $(x, y) \in D$, *then* $f(x, y, p, z, q) \leqq f(x, y, p, \bar{z}, q)$ *for all* $p \in \mathbf{R}^m$ *and* $q \in \mathbf{R}^n$,

(ii) $p \leqq \bar{p}$ *implies* $f^{(i)}(x, y, p, z, q) \leqq f^{(i)}(x, y, \bar{p}, z, q)$ *for all* $(x, y) \in D$, $q \in \mathbf{R}^n$ *and* $z \in C(D_0 \cup D, \mathbf{R}^m)$,

2. *there exist constants* $K, M, L \geqq 0$ *such that for* $i = 1, \ldots, m$

$$f^{(i)}(x, y, p, z, q) - f^{(i)}(x, y, \bar{p}, \bar{z}, \bar{q})$$
$$\leqq K\|p - \bar{p}\|_m + M\|z - \bar{z}\|_{xy} + L\|q - \bar{q}_n\|,$$

where $(x, y) \in D$, $p, \bar{p} \in \mathbf{R}^m$ *with* $p \geqq \bar{p}$, $q, \bar{q} \in \mathbf{R}^n$, $z, \bar{z} \in C(D_0 \cup D, \mathbf{R}^m)$ *with* $z(t, s) \geqq \bar{z}(t, s)$ *for* $(t, s) \in T(x, y)$,

3. *the functions* u, v *are continuous on* $D_0 \cup D$, *possess first-order partial derivatives on* $D \setminus I_0$ *and satisfy the initial inequality* $u(x, y) \leqq v(x, y)$ *for* $(x, y) \in D_0$,

4. *the differential-functional inequalities*

$$\begin{aligned} u_x(x, y) &\leqq f\big(x, y, u(x, y), u, u_y(x, y)\big), \\ v_x(x, y) &\geqq f\big(x, y, v(x, y), v, v_y(x, y)\big) \end{aligned} \tag{14}$$

are satisfied on $D \setminus I_0$.

Under these assumptions $u(x, y) \leqq v(x, y)$ *for* $(x, y) \in D$.

Proof: We have to show that the function $w = u - v$ is non-positive at an arbitrary point $(\bar{x}, \bar{y}) \in D$. Let $0 < \bar{x} < b < a$ and

$$\Phi_0(t) = \max_{1 \leqq i \leqq m} \max_{\substack{x \in [0, b] \\ \|y\|_* \leqq t}} w^{(i)}(x, y).$$

Let $\Phi \in C^1(\mathbf{R}_+, \mathbf{R}_+)$, $\Phi(t) \uparrow +\infty$ as $t \to +\infty$, satisfies the condition $\Phi(t) \geqq \Phi_0(t)$ for $t \geqq 0$. Let H be a function defined by (4). This H satisfies the inequality (5). For every $\varepsilon > 0$ there exists $r = r(\varepsilon)$ such that $w^{(i)}(x, y) < \varepsilon H(x, y)$, $i = 1, \ldots, m$, for $0 \leqq x \leqq b$ and $r \leqq \|y\|_*$. Introduce now the transformation $w = \bar{w}H$, $u = \bar{u}H$ and $v = \bar{v}H$ on $D_0 \cup D$. Let the index i_0 and the point $(x_0, y_0) \in D$ be defined by (6) with $\bar{w}$ defined above. If $x_0 = 0$ then by the initial inequality we have $\bar{w}^{(i_0)}(x_0, y_0) \leqq 0$. We will show that the last inequality holds if $x_0 \neq 0$ and $\|y_0\|_* < r$. Suppose the contrary. Then we have $\bar{w}^{(i_0)}(x_0, y_0) > 0$, $\bar{w}_x^{(i_0)}(x_0, y_0) \geqq 0$ and $\bar{w}_y^{(i_0)}(x_0, y_0) = 0$. It follows from (14) that

$$\begin{aligned} w_x^{(i_0)}(x_0, y_0) &= \bar{w}_x^{(i_0)}(x_0, y_0) H(x_0, y_0) + \bar{w}^{(i_0)}(x_0, y_0) H_x(x_0, y_0) \\ &\leqq f^{(i_0)}\big(x_0, y_0, \bar{u}(x_0, y_0) H(x_0, y_0), u, \bar{u}_y^{(i_0)}(x_0, y_0) H(x_0, y_0) \\ &\quad + \bar{u}^{(i_0)}(x_0, y_0) H_y(x_0, y_0)\big) - f^{(i_0)}\big(x_0, y_0, \bar{v}(x_0, y_0) \\ &\quad \times H(x_0, y_0), v, \bar{v}_y^{(i_0)}(x_0, y_0) H(x_0, y_0) + \bar{v}^{(i_0)}(x_0, y_0) H_y(x_0, y_0)\big). \end{aligned}$$

Let $I = \{i: \bar{w}^{(i)}(x_0, y_0) > 0\}$ and $\Gamma = (\gamma^{(1)}, \ldots, \gamma^{(m)})$ where

$$\gamma^{(i)}(x, y) = \begin{cases} H(x, y)\,\bar{u}^{(i)}(x, y) & \text{if } i \in I \\ H(x, y)\,\bar{v}^{(i)}(x, y) & \text{if } i \notin I. \end{cases}$$

Then we have

$$\overline{u}(x_0, y_0)\, H(x_0, y_0) \leqq \Gamma(x_0, y_0). \tag{15}$$

Let $\Lambda = (\lambda_1, \ldots, \lambda_m)$ be a function defined by

$$\lambda_i(x, y) = \max\,[u^{(i)}(x, y), v^{(i)}(x, y)] \quad \text{for } (x, y) \in D_0 \cup D.$$

Then we have $\Lambda \in C(D_0 \cup D, \mathbf{R}^m)$ and

$$u(x, y) \leqq \Lambda(x, y) \quad \text{for } (x, y) \in D_0 \cup D. \tag{16}$$

The monotonicity conditions of f and (15), (16) imply

$$\begin{aligned} &f^{(i_0)}\big(x_0, y_0, \overline{u}(x_0, y_0)\, H(x_0, y_0), u, \overline{u}_y^{(i_0)}(x_0, y_0)\, H(x_0, y_0) \\ &\quad + \overline{u}^{(i_0)}(x_0, y_0)\, H_y(x_0, y_0)\big) \\ &\leqq f^{(i_0)}\big(x_0, y_0, \Gamma(x_0, y_0), \Lambda, \overline{u}_y^{(i_0)}(x_0, y_0)\, H(x_0, y_0) \\ &\quad + \overline{u}^{(i_0)}(x_0, y_0)\, H_y(x_0, y_0)\big). \end{aligned}$$

Applying the Lipschitz condition for $f^{(i_0)}$ and (15), (16) we obtain

$$\begin{aligned} &\overline{w}_x^{(i_0)}(x_0, y_0)\, H(x_0, y_0) + \overline{w}^{(i_0)}(x_0, y_0)\, H_x(x_0, y_0) \\ &\leqq K \sum_{j \in I} w^{(j)}(x_0, y_0)\, H(x_0, y_0) + M\, \|\Lambda - v\|_{x_0 y_0} \\ &\quad + L\overline{w}^{(i_0)}(x_0, y_0)\, \|H_y(x_0, y_0)\|_n. \end{aligned}$$

It follows from the definition of Λ that there exist points $(x_j, y^{(j)})$, $j = 1, \ldots, m$, such that $0 \leqq x_j \leqq x_0$, $\|y^{(j)}\|_* \leqq \|y_0\|_*$ and $\|\lambda_j - v^{(j)}\|_{x_0 y_0} = w^{(j)}(x_j, y^{(j)})$. Then we have, in contradiction to (5),

$$\begin{aligned} 0 &\leqq \overline{w}^{(i_0)}(x_0, y_0)\, H(x_0, y_0) \\ &\leqq \overline{w}^{(i_0)}(x_0, y_0)\, [m(K + M)\, H(x_0, y_0) + L_r\|H_y(x_0, y_0)\|_n - H_x(x_0, y_0)]. \end{aligned}$$

Finally, if $\|y_0\|_* = r$, then we obtain from the definition of $\overline{w}$ that $\overline{w}^{(i_0)}(x_0, y_0) < \varepsilon$, ε being arbitrary.

It follows from the above considerations that $\overline{w}^{(i)}(\overline{x}, \overline{y}) < \varepsilon$, $i = 1, \ldots, m$, for an arbitrary $\varepsilon > 0$. Then we have $\overline{w}(\overline{x}, \overline{y}) \leqq 0$ and $w(\overline{x}, \overline{y}) \leqq 0$. Since $(\overline{x}, \overline{y})$ is an arbitrary point in D, we obtain the desired inequality ∎

4. Uniqueness criterion with a Hölder condition

Now we prove the uniqueness of solutions of the initial problem (1) under weaker assumptions concerning the function f: the Lipschitz condition with respect to q is replaced by the Hölder condition. This will be a generalization of the results published in [6].

Theorem 4: *Suppose that*

1. *the function f is defined on $D \times \mathbf{R}^m \times C(D_0 \cup D, \mathbf{R}^m) \times \mathbf{R}^n$ and satisfies the Volterra condition,*

2. *the estimate*

$$\|f(x, y, p, z, q) - f(x, y, \bar{p}, \bar{z}, \bar{q})\|_m$$
$$\leqq K\,\|p - \bar{p}\|_m + M\,\|z - \bar{z}\|_{xy} + L\sum_{k=1}^{n}\max\left(|q_k - \bar{q}_k|, |q_k - \bar{q}_k|^{\alpha}\right) \tag{17}$$

is satisfied on $D \times \mathbf{R}^m \times C(D_0 \cup D, \mathbf{R}^m) \times \mathbf{R}^n$, *where* $\alpha \in (0, 1)$ *and* $K, L, M \geqq 0$.

Under these assumptions, the solution u *of* (1), *which is continuous in* $D_0 \cup D$ *and has first-order partial derivatives on* $D \setminus I_0$, *is unique.*

Proof: Let $u, v \in C(D_0 \cup D, \mathbf{R}^m)$ be two solutions of (1). Denote $w = u - v$. By (1) we have $w = 0$ in D_0. We shall show that $w = 0$ in D. Let $0 < b < a$ and Φ_0 be the function defined in the proof of Theorem 3 with the above w. Let $\Phi \in C^1(\mathbf{R}, \mathbf{R}_+)$, $\Phi'(t) \geqq 1$ as $t \geqq 0$, satisfies the condition $\Phi(t) \geqq \Phi_0(t)$ for $t \geqq 0$. Then we have for $i = 1, \ldots, m$

$$w^{(i)}(x, y) \leqq \Phi(\|y\|_*) \quad \text{in} \quad D' = [0, b] \times \mathbf{R}^n. \tag{18}$$

Let $\varepsilon \in (0, 1)$ and $\beta = (1 - \alpha)\alpha^{-1}$. It is easy to verify that the auxiliary function

$$H(x, y; \varepsilon) = \varepsilon \exp\left[\Phi\left(\varepsilon^{\beta}(\|y\|_*^{2} + 1)^{1/2} + nLx\right) + \left(m(K + M) + 1\right)x\right]$$

satisfies in D' the inequality

$$H_x(x, y; \varepsilon) \geqq [m(K + M) + 1]\,H(x, y; \varepsilon)$$
$$+ L\sum_{k=1}^{n}\max\left(|H_{y_k}(x, y; \varepsilon)|, |H_{y_k}(x, y; \varepsilon)|^{\alpha}\right). \tag{19}$$

Moreover, since $\Phi(t) \uparrow +\infty$ as $t \to +\infty$ we have for every fixed $\varepsilon \in (0, 1)$:

$$\frac{\Phi(\|y\|_*)}{H(x, y; \varepsilon)} \to 0 \quad \text{as} \quad \|y\|_* \to +\infty. \tag{20}$$

Let $z(x, y) = w(x, y) - H(x, y; \varepsilon)$. By (18) and (19), for every $\varepsilon > 0$ there exists $r = r(\varepsilon)$ such that $z(x, y) < 0$ for $0 \leqq x \leqq b$ and $r \leqq \|y\|_*$. If $x = 0$ then $H(x, y; \varepsilon) > 0$ and it follows from the initial condition that $z(x, y) < 0$ for $\|y\|_* < r$. We prove that $z(x, y) < 0$ for $x \neq 0$ and $\|y\|_* < r$. Suppose the contrary. Then there exist an index i_0 and a point (x_0, y_0), $0 < x_0 \leqq b$ and $\|y_0\|_* < r$, such that $z^{(i_0)}(x_0, y_0) = 0$, $z^{(i)}(x_0, y_0) \leqq 0$ and $z^{(i)}(x, y) < 0$ for $0 \leqq x \leqq x_0$ and $\|y\|_* < r$. Then we have

$$z_x^{(i_0)}(x_0, y_0) \geqq 0 \quad \text{and} \quad z_y^{(i_0)}(x_0, y_0) = 0. \tag{20}$$

Further, at (x_0, y_0) we have

$$\left|\frac{\partial}{\partial y_k}\,|w^{(i_0)}(x, y)|\right| = \left|w_{y_k}^{(i_0)}(x, y)\right|, \qquad \frac{\partial}{\partial x}\,|w^{(i_0)}(x, y)| \leqq |w_x^{(i_0)}(x, y)|,$$

and by (20) we obtain

$$[\![w_y^{(i_0)}(x_0, y_0)]\!] = [\![H_y(x_0, y_0; \varepsilon)]\!], \quad |w_x^{(i_0)}(x_0, y_0)| - H_x(x_0, y_0; \varepsilon) \geqq 0. \tag{21}$$

But on the other hand, by (1) and (17) we get

$$|w_x^{(i_0)}(x_0, y_0)| \leqq K\,\|w(x_0, y_0)\|_m + M\,\|w\|_{x_0 y_0}$$
$$+ L\sum_{k=1}^{n}\max\left(\left|w_{y_k}^{(i_0)}(x_0, y_0)\right|, \left|w_{y_k}^{(i_0)}(x_0, y_0)\right|^{\alpha}\right),$$

and by (19), (20) we obtain

$$\begin{aligned}
&|w_x^{(i_0)}(x_0, y_0)| - H_x(x_0, y_0; \varepsilon) \\
&\leqq K \, \|w(x_0, y_0)\|_m + M \, \|w\|_{x_0 y_0} \\
&\quad + L \sum_{k=1}^{n} \max \left(|w_{y_k}^{(i_0)}(x_0, y_0)|, |w_{y_k}^{(i_0)}(x_0, y_0)|^\alpha\right) \\
&\quad - [m(K + M) + 1] \, H(x_0, y_0; \varepsilon) \\
&\quad - L \sum_{k=1}^{n} \max \left(|H_{y_k}(x_0, y_0; \varepsilon)|, |H_{y_k}(x_0, y_0; \varepsilon)|^\alpha\right) \\
&\leqq K \sum_{j=1}^{m} [|w^{(j)}(x_0, y_0)| - H(x_0, y_0; \varepsilon)] - H(x_0, y_0; \varepsilon) \\
&\quad + M \sum_{j=1}^{m} [\|w^{(j)}\|_{x_0 y_0} - H(x_0, y_0; \varepsilon)].
\end{aligned} \tag{22}$$

It follows from the definition of w that there exist points $(x_j, y^{(j)})$, $j = 1, \ldots, m$, such that $0 \leqq x_j \leqq x_0$, $\|y^{(j)}\|_* \leqq \|y_0\|_*$ and $\|w^{(j)}\|_{x_0 y_0} = w^{(j)}(x_j, y^{(j)})$. Then we get from (22)

$$|w_x^{(i_0)}(x_0, y_0)| - H_x(x_0, y_0; \varepsilon) \leqq -H(x_0, y_0; \varepsilon) < 0$$

which contradicts (21). So inequality $z(x, y) < 0$ is satisfied for $(x, y) \in D$, $0 \leqq x \leqq b$. Now, letting $\varepsilon \to 0$ we obtain $w(x, y) = 0$ in D'. Since b may be chosen arbitrarily close to a, the proof is complete ∎

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