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On Solutions of First-Order Partial Differential-Functional Equations in an Unbounded Domain

Z. KAMONT and K. PRZADKA

Unter der Voraussetzung der Stetigkeit und einmaligen partiellen Differenzierbarkeit der Lösungen wird bewiesen, daß das Cauchy-Problem

$$
z_x^{(i)}(x, y) = f^{(i)}(x, y, z(x, y), z, z_y^{(i)}(x, y))
$$

\n
$$
z^{(i)}(x, y) = \varphi_i(x, y) \quad \text{für} \quad (x, y) \in [-\tau_0, 0] \times \mathbb{R}^n \qquad (i = 1, ..., m)
$$

nicht mehr als eine Lösung besitzt, falls die Funktion $j = (f^{(1)}, ..., f^{(m)})$ der Variablen (x, y, p, z, q) die Lipschitzsche Bedingung bezüglich (p, z, q) oder die Lipschitzsche Bedingung bezüglich (p, z) und die Höldersche Bedingung bezüglich q erfüllt.

Под условием непрерывности и существования частных производных первого порядка решений доказывается, что задача Коши

$$
z_x^{(i)}(x, y) = f^{(i)}(x, y, z(x, y), z, z_y^{(i)}(x, y))
$$

$$
z^{(i)}(x, y) = \varphi_i(x, y) \quad \text{and} \quad (x, y) \in [-\tau_0, 0] \times \mathbb{R}^n \quad (i = 1, ..., m)
$$

может иметь только единственное решение, если функция $j = (f^{(1)}, ..., f^{(m)})$ переменных (x, y, p, z, q) удовлетворяет условию. Липшица по переменным (p, z, q) или условию Липшица по переменным (p, z) и условию Гёльдера по переменной q.

Under the assumptions of continuity and the existence of first-order partial derivatives of the solutions it is proved that the Cauchy problem

$$
z_x^{(i)}(x, y) = f^{(i)}(x, y, z(x, y), z, z_y^{(i)}(x, y))
$$

\n
$$
z^{(i)}(x, y) = \varphi_i(x, y) \text{ for } (x, y) \in [-\tau_0, 0] \times \mathbb{R}^n \qquad (i = 1, ..., m)
$$

admits at most one solution if the function $f = (f^{(1)}, \ldots, f^{(m)})$ of the variables (x, y, p, z, q) satisfies a Lipschitz condition with respect to (p, z, q) , or a Lipschitz condition with respect to (p, z) and a Hölder condition with respect to q.

1. Introduction

First-order partial differential inequalities were first treated by A. HAAR [9] and by M. NA-GUMO [20]. The classical theory of partial inequalities is described in detail in [17]. The investigation of properties of partial differential functional equations of first order is strongly connected with the theory of differential and differential functional inequalities. In [11] differential-functional inequalities are applied to the estimation of the difference between solutions of two systems of partial differential-functional equations and to the formulation of a criterion of uniqueness of solutions of such systems. Difference inequalities corresponding to the differential inequalities with a retarded argument are considered in [13]. The paper [14] contains sufficient conditions for the stability and asymptotic stability of solutions of
non-linear partial differential-functional equations. The basic tool in these investigations are differential-functional inequalities and Lapunow functions. Generalized solutions of partial equations and inequalities are considered in $[1-3, 7, 15, 16, 18]$.

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The problem of existence of solutions for partial differential-functional equations is also strongly connected "With differential inequalities. A global existence of solutions together with an estimation of the existence domain is considered in [12, 24]. At the present moment there exist numerous papers on this subject. For more detailed information and references see [2, 12, 24].

This note deals with solutions of differential-functional equations and inequalities defined in an unbounded zone. The solutions are supposed to be continuous and to have first partial derivatives. The theorems discussed here are known, under the st'onger assumptions that the solutions possess the total differentials 111, 24].

Denote by \mathbf{R}^n the *n*-dimensional Euclidean space and by $y = (y_1, \ldots, y_n)$ its ele-The Protectricity **R**^{*n*} the *n*-dimensional Euclidean space and by $y = (y_1, ..., y_n)$ its elements. Let $I_0 = \{0\} \times \mathbb{R}^n$, $D_0 = [-\tau_0, 0] \times \mathbb{R}^n$ and $D = [0, a) \times \mathbb{R}^n$, where $0 \leq \tau_0$ have first partial derivatives. The theorems discussed here are known under the stronger assumptions that the solutions possess the total differentials [11, 24].

Theoret by \mathbf{R}^n the *n*-dimensional Euclidean' space The problem of existence of solutions for partial differential-functional equations is also
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there exist numerous papers on this s on *Q;* then we denote *z* = *(z1 (1 , ..., zm)).* $(z^{(1)}, ..., z^{(m)})$ is a function c

If z possesses first-order part
 $(z_{y_1}^{(i)}, ..., z_{y_n}^{(i)}), z_y = [z_{y_j}^{(i)}]_{i=1...}$ ments. Let $I_0 = \{0\} \times \mathbb{R}^n$, $D_0 = [-\tau_0, 0, \tau_0]$
and $0 < a \leq +\infty$. Suppose that $z = (z$
 (x, y) defined in a domain $\Omega \subset \mathbb{R}^{1+n}$. If
on Ω , then we denote
 $z_x = (z_x^{(1)}, ..., z_x^{(m)}), \quad z_y^{(i)} = (z$
Further, denote by $C(X, Y$

$$
z_x = (z_x^{(1)}, \ldots, z_x^{(m)}); \quad z_y^{(i)} = (z_{y_1}^{(i)}, \ldots, z_{y_n}^{(i)}), \quad z_y = [z_{y_j}^{(i)}]_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}.
$$

Further, denote by $C(X, \, Y)$ the set of all continuous functions defined in X taking $z_x = (z_x^{(1)}, \ldots, z_x^{(m)}); \quad z_y^{(i)} = (z_{y_1}^{(i)}, \ldots, z_{y_n}^{(i)}), \quad z_y = [z_{y_j}^{(i)}]_{i=1,\ldots,m}$.

Further, denote by $C(X, Y)$ the set of all continuous functions defined in X taking values in Y; X, Y being arbitrary metric spaces. Supp values in Y; X, Y being arbitrary metric spaces. Suppose that $\varphi = (\varphi_1, ..., \varphi_m) \in C(D_0, \mathbb{R}^m)$ and

$$
f = (f^{(1)}, \ldots, f^{(m)}) : D \times \mathbf{R}^m \times C(D_0 \cup D, \mathbf{R}^m) \times \mathbf{R}^n \to \mathbf{R}^m.
$$

If $z \in C(D_0 \cup D, \mathbb{R}^m)$ and possesses first-order partial derivatives on $D \setminus I_0$, then

we define for
$$
(x, y) \in D \setminus I_0
$$

\n
$$
f(x, y, z(x, y), z, z_y(x, y))
$$
\n
$$
= (f^{(1)}(x, y, z(x, y), z, z_y^{(1)}(x, y)), ..., f^{(m)}(x, y, z(x, y), z, z_y^{(m)}(x, y))).
$$
\nIn this paper we shall deal with the Cauchy problem for partial differential-
\nfunctional systems of first order

functional systems of first order

$$
= (f^{(1)}(x, y, z(x, y), z, z_y^{(1)}(x, y)), \dots, f^{(m)}(x, y, z(x, y), z, z_y^{(m)}(x, y))).
$$

\ns paper we shall deal with the Cauchy problem for partial differential-
\nal systems of first order
\n
$$
z_x(x, y) = f(x, y, z(x, y), z, z_y(x, y)), \qquad (x, y) \in D \setminus I_0,
$$

\n
$$
z(x, y) = \varphi(x, y), \qquad (x, y) \in D_0.
$$

\n(1)

The solutions are supposed to be continuous in $D_0 \cup D$ and to have first-order partial derivatives in $D \setminus I_0$. As a particular case we obtain differential-integral equations and systems with a retarded argument.

First-order partial differential-functional equations have applications in different branches of knowledge. Hyperbolic differential and differential-integral systems of first order have recently been proposed [3] as simple mathematical models for the non-linear phenomenon of harmonic generation of laser radiation through piezoelectric crystals for non-dispersive materials and of Maxwell-Hopkinson type. There are various problems in non-linear optics which lead to non-linear hyperbolic differential-integral problems. For more detailed information and references see [4]. Non-linear equations may be used to describe the growth' of a population of cells which constantly differentiate (change their properties) in time [8]. First-order partial differential-integral telegraphic equations are' examined in [19]. Our results in .this paper are also motivated by applications of partial differential-integral equations tial derivatives in $D \setminus I_0$. As a partitions and systems with a retarded ar

fitions and systems with a retarded ar

First-order partial differential-function

of knowledge. Hyperbolic differential a

recently been prop

For $(x, y) \in D$ we define

$$
T(x, y) = \{(t, s) \in D_0 \cup D: -\tau_0 \leqq t \leqq x \text{ and } ||s||_* \leqq ||y||_*\},\
$$

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where $\|\cdot\|_*$ is the Euclidean norm in \mathbb{R}^n . We assume that the function *f* satisfies the

following *Volterra condition*: if $z, \bar{z} \in C(D_0 \cup D, \mathbb{R}^m)$ a where $\lVert \cdot \rVert_{*}$ is the Euclidean norm in \mathbb{R}^{n} . We assume that the function f satisfies the $f(x, y, p, z, q) = f(x, y, p, \overline{z}, q)$ for all $p \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$.

In this note we prove that if the function f of the variables (x, y, p, z, g) satisfies a Lipschitz condition with respect to (p, z, q) then the problem (1) admits at most one solution in *D.* We obtain this uniqueness theorem as a particular- case of some general comparison theorem for: partial differential-functional inequalities. At the end we prove the uniqueness of solutions in the case when / satisfies a Lipschitz condition with respect to p , z and a Hölder condition with respect to q . In this note we prove that if the function f of the
a Lipschitz condition with respect to (p, z, q) then t
one solution in D. We obtain this uniqueness theorer
egeneral comparison theorem for partial differential-
end we p *z* in the numeral interaction of the value of p , z , y , y , z , y , y , z , y) inter the problem (1) admits in the problem (1) admits or
comparison theorem for: partial

The system (1)-is of special hyperbolic type since each equation contains first-order derivatives of only one unknown function. This is a weakly coupled system. The existence and uniqueness of solutions of initial problems for strongly coupled systems is examined by using slightly different methods and under more restrictive assumptions than for (1). We illustrate this in more detail by the example of the Cauchy problem for the system without a functional argument *z*(*c*) *z*(*x*) *z*(*j* and solution contains motion to the pelod system. The existence and systems is examined by ustions than for (1). We illustry the system without a function $i = 1, ..., m$, $i = 1, ..., m$,

$$
z_x^{(i)}(x, y) = F_i(x, y, z(x, y), z_{y_i}(x, y), z_{y_i}(i)(x, y)), \qquad i = 1, ..., m,
$$

..'

 $z_x^{(i)}(x, y) = F_i(x, y, z(x, y), z_y(x, y), z_y^{(i)}(x, y)), \quad i = 1, ..., m,$
 $z(0, y) = \gamma(y),$

where $F = (F_A, ..., F_m): D \times \mathbb{R}^{m+m-n} \to \mathbb{R}^m$. The derivatives $(z_{y_1}^{(1)}, ..., z_{y_1}^{(m)}) = z_{y_i}$,

sible for the system to be strongly coupled. Solutions of th are responsible for the system to be strongly coupled. Solutions of the above problem are supposed .to belong to a special class of analytic functions with respect to the variable y_1 . This class was
first taken advantage of by K. NICKEL [22] in the theory of strongly coupled parabolic systems
of non-linear second order first taken advantage of by K. NICKEL [22] in the theory of strongly coupled parabolic systems of non-linear second order differential equations. The analyticity of z with respect to y_i is essential in questions treated in this paper. This is shown by a counter-example constructed by A. PLIS [23] in which for a strongly coupled system of two linear equations there is no uniqueness for the Cauchy problem in the class C^{∞} . The local uniqueness (and also existence) of a solution which is analytic with respect to y and belongs to the class C^1 with respect to x was proved by M. NAGUMO [21] under assumptions on analyticity of the right-hand sides of strongly coupled systems with respect to all arguments except the variable x . The problem of uniqueness of solutions for strongly coupled differential-functional systems in a class of where $F = (F_A, \ldots, F_m) : D \times \mathbb{R}^{m+n} \to \mathbb{R}^m$. The derivatives $(z_{11}^{(1)}, \ldots, z_{1n}^{(m)})$ =
sible for the system to be strongly coupled. Solutions of the above problem a
belong to a special class of analytic functions with where $F = (F_n, \ldots, F_m) : D \times \mathbb{R}^{m+n-n} \to \mathbb{R}^m$. The derivatis
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For $p, \bar{p} \in \mathbb{R}^m$ we write $p \leq \bar{p}$ if the components satisfy $p_j \leq \bar{p}_j$ for all indices uniqueness for the Cauchy problem in the class C^{∞} . The local uniqueness (and also existence)
of a solution which is analytic with respect to y and belongs to the class C^1 with respect to x
was proved by M. Natouv For $p, \bar{p} \in \mathbb{R}^m$ we write $p \leq \bar{p}$ if the components satisfy $p_j \leq \bar{p}_j$ for all indices $j = 1, ..., m$, and in case of a fixed index $p \leq \bar{p}$ if $p \leq \bar{p}$ and $p_i = \bar{p}_i$. For p we define $||p||_m = |p_1| + \cdots + |$ $w = (w_1, ..., w_m) \in C((\alpha_0, \beta_0), \mathbf{R}^m)$ then
 $||w||_{[\alpha, \beta]} = ||w_1||_{[\alpha, \beta]} + \cdots + ||w_m||_{[\alpha, \beta]}$ 2. A comparison lemma

for $p, \bar{p} \in \mathbb{R}^m$ we write $p \leq \bar{p}$ if the components satis
 $j = 1, ..., m$, and in case of a fixed index $p \leq \bar{p}$ if $p \leq \bar{p}$ a
 $||p||_m = |p_1| + \cdots + |p_m|$, $[lp] = (|p_1|, ..., |p_m|)$ and if C :

t

where $\|\cdot\|_{[\alpha,\beta]}$ is the usual max-norm of $C([\alpha,\beta], R)$. For $(t, y) \in D$ we define $S(t, y)$ $=\{(t, s): ||s||_*\leq ||y||_*\}$. We shall denote a function w of the variable $t \in [\alpha, \beta]$ by $w(\cdot)$ or $(w(t))_{[a,\beta]}$. If $z \in C(D_0 \cup D, \mathbb{R}^m)$ and $x \in [0, a)$ then we define a vector-valued function $||w||_{[\alpha,\beta]} = ||w_1||_{[\alpha,\beta]} + \cdots + ||w_m||_{[\alpha,\beta]}$
 $|a,\beta|$ is the usual max-norm of $C([\alpha, \beta], \mathbf{R})$. For $||s||_* \le ||y||_*$. We shall denote a function w
 $v(t)|_{[\alpha,\beta]}$. If $z \in C(D_0 \cup D, \mathbf{R}^m)$ and $x \in [0, a)$ the
 $(\max_{s \in S(t,y)} [z(t,s)]_{[-\tau_0,z]}$ $w\|_{[\alpha,\beta]} = \|w_1\|_{[\alpha,\beta]} + \cdots + \|w_m\|_{[\alpha,\beta]}$
 $\|s\|_* \le \|y\|_*$. We shall denote a function w of the variable $t \in [\alpha,$
 $(t)\|_{[\alpha,\beta]}$. If $z \in C(D_0 \cup D, \mathbb{R}^m)$ and $x \in [0, a)$ then we define a vector-v
 $\max_{\delta \in St(t,p)} [z(t,s)]_{[-t_0,x]} = \$

 $, ..., \left(\max_{s \in S(t,w)} |z^{(m)}(t,s)| \right)_{t=t,x} \bigg).$

In order to simplify the formulation of subsequent theorems we introduce the following Assumption H on a function Z. KAMONT and K. PRZADKA

er to simplify the formulation of subsequent t

Assumption H on a function
 $\sigma = (\sigma_1, ..., \sigma_m): [0, a) \times \mathbf{R}_+^m \times C([\mathbf{r}_0, a), \mathbf{R}_+^m])$
 $= [0, +\infty):$

$$
\sigma=(\sigma_1,\, \dots,\, \sigma_m)\colon [0,a) \times \mathbf{R_+}^m \times C([\ \div \tau_0,\, a),\, \mathbf{R_+}^m) \to \mathbf{R_+}^m.
$$

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In order to simplify the formulation of subsequent theorems we introduce the

following Assumption H on a function
 $\sigma = (\sigma_1, ..., \sigma_m) : [0, a) \times \mathbb{R}_+^m \times C([\mathbf{I} - \tau_0, a), \mathbb{R}_+^m) \to \mathbb{R}_+^m$

whe where $R_+ = [0, +\infty)$:
1. σ satisfies the following *Volterra condition*: if $w, \bar{w} \in C([-r_0, a), R_+^m]$ and $w = \overline{w}$ on $[-\tau_0, x]$, $0 \le x \le a$, then $\sigma(x, p, w) = \sigma(x, p, \overline{w})$ for all $p \in \mathbb{R}_+^m$. *2. p* $\leq \overline{p}$ implies $\sigma_i(x, p, w) \leq 0$, $\mathbb{R}_+^m \times C\left(\{-\tau_0, a\}, \mathbb{R}_+^m\right) \to \mathbb{R}_+^m$.
 2. $p \leq \overline{p}$ implies $\sigma_i(x, p, w) \leq \sigma_i(x, \overline{p}, w)$ for $x \in [0, a)$, \mathbb{R}_+^m , $\overline{w} \in C\left(\{-\tau_0, a\}, \mathbb{R}_+^m\right)$ and In order to simplify the formulation of subsequent theorems we introduce the

following Assumption H on a function
 $\sigma = (\sigma_1, ..., \sigma_m) : [0, a) \times \mathbb{R}_+^m \times C([-\tau_0, a), \mathbb{R}_+^m) \to \mathbb{R}_+^m$

where $\mathbb{R}_+ = [0, +\infty)$:

1. σ sati

4. σ satisfies the one-sided Lipschitz condition

$$
\sigma_i(x, p, w) - \sigma_i(x, \overline{p}, \overline{w}) \le K ||p - \overline{p}||_m + M ||w - \overline{w}||_{[-r_0, x]},
$$

for $i = 1,..., m$, where $x \in [0, a), p, \overline{p} \in \mathbb{R}_+^m$ with $p \geq \overline{p}$ and $w, \overline{w} \in C([-r_0, a), \mathbb{R}_+^m)$ with $w(t) \geq \overline{w}(t)$ for $t \in [-r_0, x]$.

• • i i following <i>Assumption H on a function $\sigma = (\sigma_1, ..., \sigma_m) : [0, a) \times \mathbb{R}_+^m \times C([\mathbf{i}-\tau_0, a), \mathbb{R}_+^m) \to \mathbb{R}_+$
where $\mathbb{R}_+ = [0, +\infty)$:
 • • $w = \overline{w}$ on $[-\tau_0, x]$, $0 \le x \le a$, then $\sigma(x, p, w) = \sigma(x, p$ In dealing with applications of ordinary differential inequalities to partial differential equations, we have to estimate the solutions of such equations, which are functions of several variables, by functions of one variable. In this section we shall prove the following comparison lemma. \overline{w} on $[-\tau_0, x], 0 \le x \le a$, then $\sigma(x, p, w)$
 $2. p \le \overline{p}$ implies $\sigma_i(x, p, w) \le \sigma_i(x, \overline{p}, w)$ for 3. If $w, \overline{w} \in C([- \tau_0, a), \mathbf{R}_{+}^{m})$ and $w(t) \le$
 $\sigma(x, p, \overline{w})$ for all $p \in \mathbf{R}_{+}^{m}$.

4. σ satisfies the one 2. $p \leq \overline{p}$ implies $\sigma_i(x, p, w) \leq \sigma_i(x, \overline{p}, w)$ for $x \in [0, a)$ and $w \in C([- \tau_0, a),$
3. If $w, \overline{w} \in C([- \tau_0, a), \mathbf{R}_{+}^{m})$ and $w(t) \leq \overline{w}(t)$ for $t \in [- \tau_0, x]$ then $\sigma(2)$
4. σ satisfies the one-sided Lipschitz co $v_i(x, p, w) = v_i(x)$

for $i = 1, ..., m$, where $x \in W$ with $w(t) \ge \overline{w}(t)$ for $t \in [-1, 1]$ dealing with applice

ential equations, we have

functions of several varia

prove the following competes

Lemma 1: Suppose tha

1. the fun fferential inequalities to positions of such equation

one variable. In this sec
 \therefore
 l possesses first-order

differential-functional ine
 $\Psi(t, s)$]]
 $|_{(-\tau_0, t)}$
 $(x, y) \in D \setminus I_0$,

requality

- 1. *the function a satisfies Assumption H,*

2. $\Psi = (\psi^{(1)}, \ldots, \psi^{(m)}) \in C(D_0 \cup D, \mathbb{R}^m)$ and possesses first-order partial derivatives on $D \setminus I_0$,

(2)

 μ ₂. there exists a constant $L \geq 0$ such that the differential-functional inequality

ential equations, we have to estimate the solutions of such equations, with functions of several variables, by functions of one variable. In this section we prove the following comparison lemma.

\nLemma 1: Suppose that

\n1. the function
$$
\sigma
$$
 satisfies Assumption H,

\n2. $\Psi = (\psi^{(1)}, \ldots, \psi^{(m)}) \in C(D_0 \cup D, \mathbb{R}^m)$ and possesses first-order partial values on $D \setminus I_0$.

\n3. there exists a constant $L \geq 0$ such that the differential-functional inequality

\n
$$
\llbracket \Psi_x(x, y) \rrbracket \leq \sigma \left(x, \llbracket \Psi(x, y) \rrbracket, \left(\max \llbracket \Psi(t, s) \rrbracket \right) \right)_{(-\tau_0, x)}
$$
\nwhere $L^{(m)} = (L_0, \ldots, L) \in \mathbb{R}^m$, and the initial inequality

\n
$$
\llbracket \Psi(x, y) \rrbracket \leq \eta(x), \qquad (x, y) \in D_0,
$$
\nwhere $\eta = (\eta_1, \ldots, \eta_m) \in C([-\tau_0, 0], \mathbb{R}_+^m)$, are satisfied,

\n4. the right-hand solution $\tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_m) \text{ of the problem}$

\n
$$
\omega'(x) = \sigma(x, \omega(x), \omega)
$$

\n
$$
\omega(x) = \eta(x) \quad \text{for} \quad x \in [-\tau_0, 0],
$$

\nexists on $[0, a)$.

\nexists on $[0, a)$.

where $L^{[m]} = (L, ..., L) \in \mathbb{R}^m$, and the initial inequality

$$
\llbracket \Psi(\dot{x}, y) \rrbracket \leq \eta(x), \qquad (x, y) \in D_0,
$$

$$
\omega'(x) = \sigma(x, \, \omega(x), \, \omega)
$$

 $\omega(x) = \eta(x)$ for $x \in [-\tau_0, 0],$

exists on [0, *a*).
Under these assumptions $[\![\Psi(x, y)]\!] \leq \tilde{\omega}(x)$ *for* $(x, y) \in D$. $\omega'(x) = \sigma(x, \omega(x), \omega)$
 $\omega(x) = \eta(x) \quad \text{for} \quad x \in [-\tau_0, 0]$
 Under these assumptions $[\![\Psi(x, y)]\!] \leq$
 Proof: It is natterned on that given

 $\begin{aligned}\n&\max_{f \in S(t,y)} \mathbb{I} \Psi(t,s) \mathbb{I} \Big|_{(-r_0,x)}\n\end{aligned}$, $(x, y) \in D \setminus I_0$,
 itial inequality
 $\begin{aligned}\n&\sum_{j=1}^{n} \delta_j(x,y) \text{ for } (x,y) \in D.\n\end{aligned}$
 $\begin{aligned}\n&\delta(x) &\text{ for } (x,y) \in D.\n\end{aligned}$
 $\begin{aligned}\n&\text{if } [\delta]. \text{ We will show that at an arbitra } w(\cdot) = \mathbb{I} \Psi(\cdot) \math$ **• Proof:** It is patterned on that given in [5]. We will show that at an arbitrary $\omega'(x) = \sigma(x, \omega(x), \omega)$
 $\omega(x) = \eta(x) \text{ for } x \in [-\tau_0, 0],$

exists on [0, a].

Under these assumptions $[\![\Psi(x, y)]\!] \leq \tilde{\omega}(x) \text{ for } (x, y) \in D.$

Proof: It is patterned on that given in [5]. We will show that at an arbitrary

point $(\$ Leve $L^{[m]} = (L_0, ..., L) \in \mathbb{R}^m$, and the ini
 $[\![\Psi](x, y)]\!] \leq \eta(x), \qquad (x, y) \in D_0$

Leve $\eta = (\eta_1, ..., \eta_m) \in C([\pm \tau_0, 0], \mathbb{R}_+^m)$

4. the right-hand solution $\tilde{\omega} = (\tilde{\omega}_1, ..., \tilde{\omega}')(x) = \sigma(x, \omega(x), \omega)$
 $\omega(x) = \eta(x) \quad \text{for} \quad x \in [-\tau_0$ (3)
 $\omega(x) = \eta(x)$ for $x \in [-\tau_0, 0],$
 $\omega(x) = \eta(x)$ for $x \in [-\tau_0, 0],$
 $\omega(x) = \eta(x)$ for $x \in [-\tau_0, 0],$
 $\omega(x) = \omega(x)$ or $(x, y) \in D$.
 $\omega(x) = \omega(x) + \omega(x)$
 $\omega(x) = \omega(x) + \omega(x) + \omega(x)$
 $\omega(x) = \omega(x) + \omega(x) + \omega(x)$ and
 $\phi_0(t) = \max_{\substack{1 \le i \le m \ x \$

point $(\bar{x}, \bar{y}) \in D$, $\bar{x} \neq 0$, the function $w(\cdot) = [\![\Psi](\cdot)]\!] - \tilde{w}(\cdot)$ is non-positive. Let $\bar{x} < b < a$ and
 $\phi_0(t) = \max_{\substack{1 \le i \le m \\ ||y||_s \le t}} \max_{\substack{x \in [0,b] \\ ||y||_s \le t}} w^{(t)}(x, y)$.

Let $\Phi \in C^1(\mathbb{R}_+, \mathbb{R}_+,)$, $\phi(t) \uparrow +$ $\Phi_0(t) = \max_{1 \le i \le m} \max_{\substack{x \in \{0, b\} \\ ||y||_s \le t}} w^{(i)}(x, y).$

Let $\Phi \in C^1(\mathbf{R}_+, \mathbf{R}_+), \Phi(t) \uparrow + \infty$ as $t \to +\infty$
 $t \ge 0$. It is easy to verify that the function $\frac{1}{x}$
 $\frac{1}{t}$

$$
H(x, y) = \exp \left[\phi(nLx + (||y||_{*}^{2} + 1)^{1/2}) + x(m(K + M) + 1)\right]
$$

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satisfies the inequality

$$
m(K+M) H(\cdot) + L \|H_{\nu}(\cdot)\|_{n} - H_{\nu}(\cdot) \leq -H(\cdot) < 0 \text{ on } D. \tag{5}
$$

It follows from the definition of Φ that for every $\varepsilon > 0$ there exists $r = r(\varepsilon)$ such that for $i = 1, ..., m$

$$
w^{(i)}(x, y) < \varepsilon H(x, y) \quad \text{for} \quad 0 \leq x \leq b \quad \text{and} \quad r \leq \|y\|_{\ast}.
$$

Let $\overline{\Psi} = \Psi/H$, $\overline{\omega} = \tilde{\omega}/H$ and $\overline{w}(\cdot) = [\Psi(\cdot)] - \overline{\omega}(\cdot)$. There exist an index i_0 and a doint (x_0, y_0) , $0 \le x_0 \le b$ and $||y_0||_* \le r$, such that

$$
\overline{w}^{(i_{\bullet})}(x_0, y_0) = \max_{1 \leq i \leq m} \max_{\substack{0 \leq x \leq b \\ ||y||_{\bullet} \leq r}} \overline{w}^{(i)}(x, y).
$$
\n(6)

It follows from the initial estimate that $\overline{w}^{(i_0)}(x_0, y_0) \leq 0$ if $x_0 = 0$. We prove this inequality now if $x_0 + 0$ and $||y_0||_* < r$. Suppose the contrary. Then we have $|\psi^{(i_0)}(x_0, y_0)| > 0$ and

$$
\begin{aligned} |w_x^{(i_0)}(x_0, y_0) + \sigma_{i_0}(x_0, \tilde{\omega}(x_0), \tilde{\omega})| &= |v_x^{(i_0)}(x_0, y_0)| \\ &\leq \sigma_{i_0}\big(x_0, \, \mathbb{I}\psi(x_0, y_0)\mathbb{I}, \, \Big(\max_{s \in S(t, y_0)} \mathbb{I}\Psi(t, s)\mathbb{I}\Big)_{[-r_0, x]}\Big) + L \, \|w_y^{(i_0)}(x_0, y_0)\|_n. \end{aligned}
$$

Because $\overline{w}^{(i_0)}(x_0, y_0) > 0$, $\overline{w}_x^{(i_0)}(x_0, y_0) \ge 0$ and $\overline{w}_y^{(i_0)}(x_0, y_0) = 0$, then we obtain

$$
0 \leq \overline{w}_x^{(i_0)}(x_0, y_0) H(x_0, y_0)
$$

\n
$$
\leq \overline{w}^{(i_0)}(x_0, y_0) [L || H_y(x_0, y_0) ||_n - H_x(x_0, y_0)]
$$

\n
$$
+ \sigma_{i_0} (x_0, \llbracket H(x_0, y_0) \ \overline{\Psi}(x_0, y_0) \rrbracket, \left(\max_{a \in S(t, y_0)} \llbracket \Psi(t, s) \rrbracket \right)_{[-\tau_0, x]}
$$

\n
$$
- \sigma_{i_0}(x_0, H(x_0, y_0) \ \overline{\omega}(x_0, y_0), \ \omega).
$$

Let $I = \{i : \overline{w}^{(i)}(x_0, y_0) > 0\}$ and $\Gamma = (\gamma^{(1)}, \ldots, \gamma^{(m)})$ where $\gamma^{(i)}(x, y) = \begin{cases} H(x, y) | \overline{\Psi}^{(i)}(x, y) | & \text{if } i \in I \\ H(x, y) | \overline{\omega}^{(i)}(x, y) & \text{if } i \notin I. \end{cases}$

Then we have

$$
\llbracket H(x_0, y_0) \overline{\Psi}(x_0, y_0) \rrbracket \leq \Gamma(x_0, y_0).
$$

Let $A = (\lambda_1, ..., \lambda_m)$ be a function defined by

$$
\lambda_i(x) = \begin{cases} \tilde{\omega}_i(x), & x \in [-\tau_0, 0] \\ \max \left[\tilde{\omega}_i(x), \max_{s \in S(x,y_0)} |\psi^{(i)}(x, s)| \right], & x \in (0, a) \end{cases}
$$

Then we have $\Lambda \in C([-\tau_0, a), \mathbb{R}_+^m)$ and

$$
\max_{s \in S(x,y_0)} \llbracket \Psi(x,s) \rrbracket \leq \Lambda(x) \quad \text{for } x \in [-\tau_0, a).
$$

From the monotonicity conditions for σ and from (8), (9) we get

$$
\sigma_{i_0}\left(x_0,\llbracket H(x_0,y_0)\right)\overline{\Psi}(x_0,y_0)\rrbracket,\left(\max_{s\in S(t,y_0)}\llbracket \Psi(t,s)\rrbracket\right)_{t-t_0,x_0}\right)
$$

$$
\leq \sigma_{i_0}\left(x_0,\Gamma(x_0,y_0),\Lambda\right).
$$

(8)

 (9)

 (10)

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Applying the Lipschitz condition for σ_{i_0} and (8) we obtain
 $\sigma_{i_0}(x_0, T(x_0, y_0), \vec{A}) - \sigma_{i_0}(x_0, H(x_0, y_0), \vec{\omega})$ Applying the Lipschitz condition for σ_{i_0} and (8) we obtain.

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\nApplying the Lipschitz condition for
$$
\sigma_{i_0}
$$
 and (8) we obtain
\n
$$
\sigma_{i_0}(x_0, \Gamma(x_0, y_0), \Lambda) - \sigma_{i_0}(x_0, \Pi(x_0, y_0) \overline{\omega}(x_0, y_0), \hat{\omega})
$$
\n
$$
\leq K \sum_{j \in I} [\gamma^{(j)}(x_0, y_0) - H(x_0, y_0) \overline{\omega}^{(j)}(x_0, y_0)] + M ||A - \hat{\omega}||_{[-\tau_{0}, \tau_{0}]}
$$
\n
$$
\leq mKH(x_0, y_0) \overline{\omega}^{(i_0)}(x_0, y_0) + M ||A - \hat{\omega}||_{[-\tau_{0}, \tau_{0}]}
$$
\n11)
\nIt follows from the definition of A that there exist points $(x_j, y^{(j)})$, $j = 1, ..., m$,
\nsuch that $0 \leq x_j \leq x_0$, $||y^{(j)}||_{*} \leq ||y_0||_{*}$ and
\n $||\lambda_j - \hat{\omega}_j||_{[-\tau_{0}, \tau_{0}]} = \max_{g \in SL(\tau_{j}, \mu_{0})} |\psi^{(j)}(x_j, s)| - \hat{\omega}_j(x_j) = |\psi^{(j)}(x_j, y^{(j)})| - \hat{\omega}_j(x_j)$.
\nThen we have
\n $||\lambda_j - \hat{\omega}_j||_{[-\tau_{0}, \tau_{0}]} = \overline{w}^{(j)}(x_j, y^{(j)}) H(x_j, y^{(j)}) \leq H(x_0, y_0) \overline{w}^{(i_0)}(x_0, y_0)$. (12)
\nEstimates (7), (10)–(12) imply, in contradiction to (5),
\n $0 \leq \overline{w}_x^{(i_0)}(x_0, y_0) H(x_0, y_0)$
\n $< \overline{w}^{(i_0)}(x_0, y_0) \left[\overline{w}(K + M) H(x_0, y_0) + L ||H(x_0, y_0)||_{*} - H(x_0, y_0) \right]$.

It follows from the definition of *A* that there exist points $(x_i, y^{(i)})$, $j = 1, ..., m$, lly_{oll∗,} and App
 •
 1t fo

such
 •

Then $j=1, ...$
 $\tilde{\omega}_j(x_j)$.

$$
U \subseteq x_j \leq x_0, ||y^{(j)}||_* \leq ||y_0||_* \text{ and}
$$

\n
$$
||\lambda_j - \tilde{\omega}_j||_{[-t_0, x_0]} = \max_{s \in S(x_j, y_0)} |\psi^{(j)}(x_j, s)| - \tilde{\omega}_j(x_j) = |\psi^{(j)}(x_j, y^{(j)})| - \tilde{\omega}_j(x_j).
$$

\nhave
\n
$$
||\lambda_j - \tilde{\omega}_j||_{[-t_0, x_0]} = \overline{w}^{(j)}(x_j, y^{(j)}) H_j(x_j, y^{(j)}) \leq H(x_0, y_0) \overline{w}^{(i_0)}(x_0, y_0).
$$

Then we have

have
\n
$$
\mathfrak{g}(\mathcal{S}(x_j, y_i))
$$
\nhave
\n
$$
||\lambda_j - \tilde{\omega}_j||_{[-\tau_i, x_i]} = \overline{w}^{(j)}(x_j, y^{(j)}) H(x_j, y^{(j)}) \leq H(x_0, y_0) \overline{w}^{(i_0)}(x_0, y_0).
$$
\n(12)

such that
$$
0 \leq x_j \leq x_0
$$
, $||y^{(j)}||_* \geq ||y_0||_*$, and
\n
$$
||\lambda_j - \tilde{\omega}_j||_{[-\tau_0, x_0]} = \max_{s \in S(x_j, y_0)} |\varphi(x_j, s)| - \tilde{\omega}_j(x_j) = |\varphi^{(j)}(x_j, y^{(j)})| - \tilde{\omega}_j(x_j).
$$
\nThen we have\n
$$
||\lambda_j - \tilde{\omega}_j||_{[-\tau_0, x_1]} = \overline{w}^{(j)}(x_j, y^{(j)}) H(x_j, y^{(j)}) \leq H(x_0, y_0) \overline{w}^{(i_0)}(x_0, y_0).
$$
\n(12)\nEstimates (7), (10)–(12) imply, in contradiction to (5),\n
$$
0 \leq \overline{w}_x^{(i_0)}(x_0, y_0) H(x_0, y_0)
$$
\n
$$
\leq \overline{w}^{(i_0)}(x_0, y_0) [m(K + M) H(x_0, y_0) + L ||H_y(x_0, y_0)||_n - H_x(x_0, y_0)].
$$
\nFinally, if $||y_0||_* = r$, then, from the definition of \overline{w} , we obtain $\overline{w}^{(i_0)}(x_0, y_0) < \varepsilon$,
\n ε being arbitrary.
\nIf follows from the above considerations that $\overline{w}^{(i)}(\overline{x}, \overline{y}) \leq 0$, $i = 1, ..., m$. Since
\n $(\overline{x}, \overline{y})$ is an arbitrary point in *D*, we obtain the desired inequality \blacksquare
\nExample: Suppose that for $(x, p, w) \in [0, a) \times \mathbf{R}_+^m \times C[(-\tau_0, a), \mathbf{R}^m]$ and $i = 1$,
\n \dots , m we have
\n $\sigma_i(x, p, w) = \sigma_i (x, p, \max_i w_i(t), ..., \max_i w_m(t)).$
\nIf $z \in C(D_0 \cup D, \mathbf{R}^m)$ and $(x, y) \in D$, then we denote

-

It follows from the above considerations that $\overline{w}^{(i)}(\overline{x}, \overline{y}) < \varepsilon$ for arbitrary $\varepsilon > 0$. Then we have $\overline{w}^{(i)}(\overline{x}, \overline{y}) \leq 0$ and consequently $w^{(i)}(\overline{x}, \overline{y}) \leq 0, i = 1, ..., m$. Since (\bar{x}, \bar{y}) is an arbitrary point in D, we obtain the desired inequality \blacksquare Estimates (7), $(10) - (12)$ imply, in contradiction
 $0 \le \overline{w}_x$ ⁽ⁱv)(x_0, y_0) $H(x_0, y_0)$
 $\le \overline{w}_y$ ⁽ⁱv)(x_0, y_0) $H(x_0, y_0)$
 $\le \overline{w}$ ⁽ⁱv)(x_0, y_0) $[m(K + M) H(x_0, y_0) +$

Finally, if $||y_0||_* = r$, then, from th *e* being

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Then w
 $(\overline{x}, \overline{y})$ is
 E xar

..., *m* w

.

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If $z \in C$ Figure 1. The definition of
 i considerations that \overline{w} and consequently w of
 D , we obtain the desir
 $\text{or } (x, p, w) \in [0, a) \times \mathbf{R}$
 $\max_{t \in [-\tau_a, x]} w_1(t), \dots, \max_{t \in [-\tau_b, x]}$
 $\in D$, then we denote $\leq w^{(1)}(x_0, y_0) |m(K + M) H(x_0, y_0) + L \|H_y(x_0, y_0)\|_n -$

Finally, if $||y_0||_* = r$, then, from the definition of \overline{w} , we obtain
 ϵ being arbitrary.

It follows from the above considerations that $\overline{w}^{(t)}(\overline{x}, \overline{y}) < \epsilon$

Example: Suppose that for $(x, p, w) \in [0, a) \times \mathbb{R}^m \times C([-t_0, a), \mathbb{R}^m)$ and $i = 1$, ..., *m* we have **•** *Example: Suppose that for* $(x, p, w) \in [0, a) \times \mathbb{R}_+^m$ **
** \ldots, m **we have
** $\sigma_i(x, p, w) = \sigma_i \left(x, p, \max_{t \in [-\tau_0, x]} w_1(t), \ldots, \max_{t \in [-\tau_0, x]} w_m(t) \right)$ **

if** $z \in C(D_0 \cup D, \mathbb{R}^m)$ and $(x, y) \in D$, then we denote
 $\max_{(t, s) \in T(x, y)}$

$$
\sigma_i(x, p, w) = \sigma_i\left(x, p, \max_{t \in [-\tau_0, x]} w_1(t), \ldots, \max_{t \in [-\tau_0, x]} w_m(t)\right).
$$

If $z \in C(D_0 \cup D, R^m)$ and $(x, y) \in D$, then we denote

$$
U_{\epsilon}(-\tau_{\epsilon},x)
$$
\n
$$
U_{0} \cup D, \mathbb{R}^{m}
$$
 and $(x, y) \in D$, then we denote\n
$$
\max_{(t,s)\in T(x,y)} \mathbb{E}[z(t,s)] = \left(\max_{(t,s)\in T(x,y)} |z^{(1)}(t,s)|, \dots, \max_{(t,s)\in T(x,y)} |z^{(m)}(t,s)|\right).
$$
\n
$$
\max_{t\in [-\tau_{\epsilon},x]} w(t) = \left(\max_{t\in [-\tau_{\epsilon},x]} w_{1}(t), \dots, \max_{t\in [-\tau_{\epsilon},x]} w_{m}(t)\right).
$$
\n
$$
\text{umption (2) of Lemma 1 has the form}
$$

Let

S

$$
\max_{t\in[-\tau_0,x]}w(t)=\Big(\max_{t\in[-\tau_0,x]}w_1(t),\ldots,\max_{t\in[-\tau_0,x]}w_m(t)\Big).
$$

Then assumption (2) of Lemma 1 has the form

Then we have
$$
w^{(s)}(x, y) \le 0
$$
 and consequently $w^{(s)}(x, y) \le 0$, $i = 1, ..., m$. Since
\n $(\overline{x}, \overline{y})$ is an arbitrary point in *D*, we obtain the desired inequality \blacksquare
\n
$$
\blacksquare
$$

\nExample: Suppose that for $(x, p, w) \in [0, a) \times \mathbf{R}_+^m \times C \big([-\tau_0, a), \mathbf{R}^m \big)$ and $i =$
\n \blacksquare
\n $\sigma_i(x, p, w) = \sigma_i \left(x, p, \max_i w_1(t), ..., \max_i w_m(t) \right)$.
\nIf $z \in C(D_0 \cup D, \mathbf{R}^m)$ and $(x, y) \in D$, then we denote
\n
$$
\max_{(t,s) \in T(x,y)} [\![z(t,s)]\!] = \left(\max_{(t,s) \in T(x,y)} |z^{(1)}(t,s)|, ..., \max_{(t,s) \in T(x,y)} |z^{(m)}(t,s)| \right)
$$
\nLet
\n
$$
\max_{t \in [-\tau_0, x]} w(t) = \left(\max_{t \in [-\tau_0, x]} w_1(t), ..., \max_i w_m(t) \right)
$$
.
\nThen assumption (2) of Lemma 1 has the form
\n
$$
\llbracket \Psi_x(x, y) \rrbracket \le \sigma\Big(x, \llbracket \Psi(x, y) \rrbracket, \max_i \llbracket \Psi(t, s) \rrbracket \Big) + L^{\lfloor m \rfloor} \llbracket \Psi_y(x, y) \rrbracket^T, \quad (x, y) \in D \setminus I_0,
$$
\nand the comparison problem (3) is

$$
[\![\Psi_x(x, y)]\!] \leq \sigma[x, [\![\Psi(x, y)]\!], \max_{(t, s) \in T(x, y)} [\![\Psi(t, y)]\!],
$$

and the comparison problem (3) is

$$
\omega'(x) = \sigma[x, \omega(x), \max_{t \in [-\tau_0, z]} \omega(t)],
$$

$$
\omega(x) = \eta(x) \text{ for } x \in [-\tau_0, 0].
$$

and the comparison problem (3) is
 $\omega'(x) = \sigma(x, \omega(x), \max_{t \in [-\tau, x]} \omega(t)),$
 $\omega(x) = \eta(x) \text{ for } x \in [-\tau_0, 0].$

If η is non-decreasing then the above problem is equivalent with $\omega'(x) = \sigma(x, \omega(x))$, $\omega(0) = \eta(0)$.

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3. Eiimation of the difference between two solutions, uniqueness criteria

Let us consider the initial problem (1) and the problem

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\n127
\n2. Estimation of the difference between two solutions, uniqueness criteria
\nLet us consider the initial problem (1) and the problem
\n
$$
z_x(x, y) = g(x, y, z(x, y), z, z_y(x, y)),
$$
 $(x, y) \in D \setminus I_0,$
\n $z(x, y) = \tilde{\varphi}(x, y),$ $(x, y) \in D_0.$
\nThe following theorem allows us to estimate the difference between solutions of (1)
\nand (13).
\nTheorem 1: Suppose that
\n1. the functions $f = (f^{(1)}, ..., f^{(m)})$ and $g = (g^{(1)}, ..., g^{(m)})$ are defined on $D \times \mathbb{R}^m$
\n2. the function σ satisfies Assumption H,
\n3. the estimates
\n $\mathbb{I}/(x, y, p, z, q) - g(x, y, \bar{p}, \bar{z}, q)$]
\n $\leq \sigma(x, [\![p - \bar{p}]\!], (\![\text{max} [[\![z(t, s) - \bar{z}(t, s)]\!])\!]$

The following theorem allows us to estimate the difference between solutions of (1) and (13) .

Theorem 1' : *Suppose that*

1. the functions $f = (f^{(1)}, ..., f^{(m)})$ *and* $g = (g^{(1)}, ..., g^{(m)})$ *are defined on* $D \times \mathbb{R}^m$

$$
z_x(x, y) = g(x, y, z(x, y), z, z_y(x, y)), \quad (x, y) \in D
$$

\n
$$
z(x, y) = \tilde{\varphi}(x, y), \quad (x, y) \in D_0.
$$

\nThe following theorem allows us to estimate the difference
\nand (13).
\nTheorem 1: Suppose that
\n1. the functions $f = (f^{(1)}, \ldots, f^{(m)})$ and $g = (g^{(1)}, \ldots, g^{(m)})$
\n $\times C(D_0 \cup D, \mathbb{R}^m) \times \mathbb{R}^n$ and satisfy the Volterra condition,
\n2. the function σ satisfies Assumption H,
\n3. the estimates
\n
$$
\mathbb{I}\{(x, y, p, z, q) - g(x, y, \overline{p}, \overline{z}, q)\}\
$$

\n $\leq \sigma \left(x, \mathbb{I}p - \overline{p}\right], \left(\max_{a \in S(t, y)} \left[\overline{z}(t, s) - \overline{z}(t, s)\right]\right)_{[-t_0, x]})$
\nand
\n
$$
||f(x, y, p, z, q) - g(x, y, p, z, \overline{q})||_m \leq L||q - \overline{q}||_n
$$

\nare satisfied on $D \times \mathbb{R}^m \times C(D, \dots, D, \mathbb{R}^m) \times \mathbb{R}^n$.

and

,

$$
||f(x, y, p, z, q) - g(x, y, p, z, \overline{q})||_m \leq L||q - \overline{q}||_n
$$

are satisfied on $D \times \mathbb{R}^m \times C(D_0 \cup D, \mathbb{R}^m) \times \mathbb{R}^n$,

4. $u, v \in C(D_0 \cup D, \mathbb{R}^m)$ are solutions of (1) and (13), respectively, having first-order *partial derivatives on* $D \setminus I_0$ *, and*
 $\left\|\left\|(x, y, p, z, q) - g(x, y, p, z, \overline{q})\right\|_{n} \le L\|q - \overline{q}\|_{n}$
 are satisfied on $D \times \mathbb{R}^m \times C(D_0 \cup D, \mathbb{R}^m) \times \mathbb{R}^n$,

4. u, $v \in C(D_0 \cup D, \mathbb{R}^m)$ are solutions of (1) and (13), respectively, hai

partial de

5. *there exists a function* $\eta \in C([-r_0, 0], \mathbb{R}, \mathbb{R})$ *such that*

$$
\llbracket u(x, y) - v(x, y) \rrbracket \leq \eta(x) \quad \text{for} \quad (x, y) \in D_0
$$

right-hand solution $\tilde{\omega}$ of the problem (3) exists

ir these assumptions $[\![u(x, y) - v(x, y)]\!] \leq \tilde{\omega}(x)$

of: The function $\Psi = u - v$ satisfies all the

the desired inequality holds true $[\![\!]$
 $\|\cdot\| \leq C(D_0 \cup D, \mathbb{R}^m)$

Proof: The function $\Psi = u - v$ satisfies all the conditions of Lemma 1 and, ience, the desired inequality holds true \blacksquare

For $z \in C(D_0 \cup D, \mathbb{R}^m)$ and $(x, y) \in D$ denote

$$
||z||_{xy} = \sum_{i=1}^m ||\dot{z}^{(i)}||_{zy} \text{ where } ||z^{(i)}||_{xy} = \max_{(t,s)\in T(x,y)} |z^{(i)}(t,s)|.
$$

The next theorem is an immediate consequence of Theorem 1.

Theorem 2: *Suppose that*

1. the function $f: D \times \mathbb{R}^m \times C(D_0 \cup D, \mathbb{R}^m) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies the Volterra condition and

The next theorem is an immediate consequence of Theorem 1.
 Theorem 2: Suppose that
 1. the function $f: D \times \mathbb{R}^m \times C(D_0 \cup D, \mathbb{R}^m) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies the Volterra co
 tion and
 2. $||f(x,y, p; z, q) - f(x, y$

 $||z||_{xy} = \sum_{i=1}^{m} ||\dot{z}^{(i)}||_{xy}$ where $||z^{(i)}||_{xy} = \max_{(t,s)\in T(x,y)} |z^{(i)}(t, \cdot)|$.
The next theorem is an immediate consequence of Theorem
Theorem 2: Suppose that
1. the function $f: D \times \mathbf{R}^m \times C(D_0 \cup D, \mathbf{R}^m) \times \mathbf{R}^n \to \$ Under these assumptions, the solution u of (1), which is continuous in D_0 \cup D and *has first-order partial derivatives on* $D \setminus I_0$, *is unique and depends continuously on* φ *and f.* For $z \in C(D_0 \cup D, \mathbb{R}^m)$ and
 $||z||_{xy} = \sum_{i=1}^m ||z^{(i)}||_{xy}$
 And $||z||_{xy} = \sum_{i=1}^m ||z^{(i)}||_{xy}$ *

<i>and* $x = 1$ *. the function* $f: D \times \mathbb{R}^m >$
 and $2. ||f(x, y, p, z, q) - f(x, y, z)$
 and $x = 2, \overline{z} \in C(D_0 \cup D, \mathbb{R}^m)$, *p*, *D*

The following theorem concerns weak inequalities between vector-valued func tions satisfying first-order partial differential-functional inequalities in an unbound-The following theorem concerns weak inequalities between vector-valued functions satisfying first-order partial differential-functional inequalities in an unbound-
ed zone.

 $\mathbf{e}^{\frac{1}{2} \left(\mathbf{e}^{\frac{1}{2} \left$

• Theorem **3:** *Suppose that*

 $\label{eq:2.1} \frac{1}{\sqrt{2\pi}}\sum_{i=1}^N\frac{1}{\sqrt{2\pi}}\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int_{0}^{1}d\mu\int$

1. *the function* $f: D \times \mathbf{R}^m \times C(D_0 \cup D, \mathbf{R}^m) \times \mathbf{R}^n \to \mathbf{R}^m$ satisfies the Volterra condi*tion and the following monotonicity conditions:*

(i) if $z, \bar{z} \in C(D_0 \cup D, \mathbb{R}^m)$ and $z(t, s) \leq \bar{z}(t, s)$, for $(t, s) \in T(x, y)$, $(x, y) \in D$, then $f(x, y, p, z, q) \leq f(x, y, p, \overline{z}, q)$ for all $p \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$,

1. the function $f: D \times \mathbf{R}^m \times C(D_0 \cup D_n)$
 n and the following monotonicity cond

(i) if $z, \overline{z} \in C(D_0 \cup D_1, \mathbf{R}^m)$ and $z(t, s_0, y, p, z, q) \leq f(x, y, p, \overline{z}, q)$ for all $p \in$

(ii) $p \leq \overline{p}$ implies $f^{(i)}(x, y, p, z, q$ (ii) $p \leq \overline{p}$ implies $f^{(i)}(x, y, p, z, q) \leq f^{(i)}(x, y, \overline{p}, z, q)$ for all $(x, y) \in D$, $q \in \mathbb{R}^n$ and $z\in C(D_0\cup D,\, \mathbf{R}^m),$ (ii) $p \leq \overline{p}$ implies $f^{(i)}(x, y, p, z, q) \leq f^{(i)}(x, y, \overline{p}, z, q)$ for all $(x, y) \in D, q \in \mathbb{R}^n$ and
 $z \in C(D_0 \cup D, \mathbb{R}^m)$,

2. there exist constants $K, M, L \geq 0$ such that for $i = 1, ..., m$
 $f^{(i)}(x, y, p, z, q) - f^{(i)}(x, y, \over$

2. there exist constants $K, M, L \geq 0$ such that for $i = 1, ..., m$

$$
h, z, q) \leq f(x, y, p, z, q) \text{ for all } p \in \mathbb{R}^n \text{ and } q \in \mathbb{R}^n,
$$

\n
$$
\leq \overline{p} \text{ implies } f^{(i)}(x, y, p, z, q) \leq f^{(i)}(x, y, \overline{p}, z, q)
$$

\n
$$
h \circ \cup D, \mathbb{R}^m),
$$

\n
$$
h \circ \cup D, \mathbb{R}^m),
$$

\n
$$
h \circ \cup B, \mathbb{R}^m, h \circ \mathbb{R}^m \text{ is a constant.}
$$

\n
$$
K, M, L \geq 0 \text{ such that for } i = f^{(i)}(x, y, p, z, q) - f^{(i)}(x, y, \overline{p}, \overline{z}, \overline{q})
$$

\n
$$
\leq K ||p - \overline{p}||_m + M_1 ||z - \overline{z}||_{xy} + L ||q - \overline{q}_n||,
$$

\n
$$
x, y) \in D, p, \overline{p} \in \mathbb{R}^m \text{ with } p \geq \overline{p}, q, \overline{q} \in \mathbb{R}^n, z, \overline{z}
$$

\n
$$
h \circ \mathbb{R}^m \text{ with } p \geq \overline{p}, q, \overline{q} \in \mathbb{R}^n, z, \overline{z}
$$

\n
$$
h \circ \mathbb{R}^m \text{ with } p \geq \overline{p}, q, \overline{q} \in \mathbb{R}^n, z, \overline{z}
$$

\n
$$
h \circ \mathbb{R}^m \text{ with } p \geq \overline{p}, q, \overline{q} \in \mathbb{R}^n, z, \overline{z}
$$

\n
$$
h \circ \mathbb{R}^m \text{ with } p \geq \overline{p}, q, \overline{q} \in \mathbb{R}^n, z, \overline{z}
$$

\n
$$
h \circ \mathbb{R}^m \text{ with } p \geq \overline{p}, q, \overline{q} \in \mathbb{R}^n, z, \overline{z}
$$

\n $$

Theorem 3; Suppose that

1. the function $f: D \times \mathbb{R}^m \times C(D_0 \cup D, \mathbb{R}$

ion and the following monotonicity conditio

(i) if $z, \overline{z} \in C(D_0 \cup D, \mathbb{R}^m)$ and $z(t, s) \le$
 $f(x, y, p, z, q) \leq f(x, y, p, \overline{z}, q)$ for all $p \in \mathbb{R}$ $\geq \overline{z}(t, s)$ for $(t, s) \in T(x, y)$,
3. the functions u, v are continuous on $D_0 \cup D$, possess, first-order partial derivatives
on $D \setminus I_0$ and satisfy the initial inequality $u(x, y) \leq v(x, y)$ for $(x, y) \in D_0$, *onhere* $(x, y) \in D$, $p, \bar{p} \in \mathbb{R}^m$ *with* $p \geq \bar{p}$, $q, \bar{q} \in \mathbb{R}^n$, $z, \bar{z} \in C(D_0 \cup D, \mathbb{R}^m)$ if $\geq \bar{z}(t, s)$ for $(t, s) \in T(x, y)$,
3. the functions u, v are continuous on $D_0 \cup D$, possess first-order par $\leq K ||p - \overline{p}||_m + M ||z - \overline{z}||_{xy} + L ||q - \overline{q}_n||$

where $(x, y) \in D$, $p, \overline{p} \in \mathbb{R}^m$ with $p \geq \overline{p}$, $q, \overline{q} \in \mathbb{R}^n$, $z \geq \overline{z}(t, s)$ for $(t, s) \in T(x, y)$,

3. the functions u, v are continuous on $D_0 \cup D$, posses *I*⁽ⁱ⁾(*x, y, p, z, q*) $-f^{(i)}(x, y, \overline{p}, \overline{z}, \overline{q})$
 $\leq K ||p - \overline{p}||_m + M ||z - \overline{z}||_{xy} + L ||q - \overline{q}_n||$,
 ere $(x, y) \in D$, $p, \overline{p} \in \mathbb{R}^m$ *with* $p \geq \overline{p}$, $q, \overline{q} \in \mathbb{R}^n$, $z, \overline{z} \in C(D_0 \cup D, \mathbb{R}^m)$ \over

4. the differential-functional inequalities

$$
u_x(x, y) \leq f(x, y, u(x, y), u, u_y(x, y)),
$$

$$
v_x(x, y) \geq f(x, y, v(x, y), v, v_y(x, y))
$$

Figure satisfied on $D \setminus I_0$.
Under these assumptions $u(x, y) \leq v(x, y)$ for $(x, y) \in D$.
Proof: We have to show that the function $w = u - v$ is non-positive at an arbitrary point $(\bar{x}, \bar{y}) \in D$. Let $0 < \bar{x} < b < a$ and $v_x(x, y) \leq f(x, y, v).$

are satisfied on $D \setminus I_0$.
 Under these assumptions a

Proof: We have to sho

arbitrary point $(\bar{x}, \bar{y}) \in D$.
 $\Phi_0(t) = \max_{1 \leq i \leq m} \max_{\substack{x \in [0, b] \\ ||y||_s \leq t}}$

Let $\Phi \in C^1(\mathbf{R}_+, \mathbf{R}_+), \ \Phi(t) \uparrow$

$$
\Phi_0(t) = \max_{1 \leq i \leq m} \max_{\substack{x \in [0,b] \\ ||y||_0 \leq t}} w^{(i)}(x, y).
$$

arbitrary point $(\bar{x}, \bar{y}) \in D$. Let 0
 $\Phi_0(t) = \max_{1 \le i \le m} \max_{\substack{x \in [0, b] \\ ||y||_* \le t}} w^{(i)}$

Let $\Phi \in C^1(\mathbf{R}_+, \mathbf{R}_+), \ \Phi(t) \uparrow +\infty$
 $t \ge 0$. Let H be a function defi $\Phi_0(t) = \max_{1 \le i \le m} \max_{\substack{x \in [0,b] \\ \|y\|_{*} \le t}} w^{(i)}(x, y).$

Let $\Phi \in C^1(\mathbb{R}_+, \mathbb{R}_+,)$, $\Phi(t) \uparrow +\infty$ as $t \to +\infty$, satisfies the condition $\Phi(t) \ge \Phi_0(t)$ for $t \ge 0$. Let *H* be a function defined by (4). This *H* satisfi $t \geq 0$. Let H be a function defined by (4). This H satisfies the inequality (5). For every $\varepsilon > 0$ there exists $r = r(\varepsilon)$ such that $w^{(i)}(x, y) < \varepsilon H(x, y)$, $i = 1, ..., m$, for *t* ≥ 0 . Let *H* be a function defined by (4). This *H* satisfies the inequality (5). For every $\varepsilon > 0$ there exists $r = r(\varepsilon)$ such that $w^{(i)}(x, y) < \varepsilon H(x, y)$, $i = 1, ..., m$, for $0 \leq x \leq b$ and $r \leq ||y||_*$. Introduce n $0 \le x \le b$ and $r \le ||y||_*$. Introduce now the transformation $w = \overline{w}H$, $u = \overline{u}H$ and $v = \overline{v}H$ on $D_0 \cup D$. Let the index i_0 and the point $(x_0, y_0) \in D$ be defined by (6) with \overline{w} defined above. If $x_0 = 0$ th with \bar{w} defined above. If $x_0 = 0$ then by the initial inequality we have $\bar{w}^{(i_0)}(x_0, y_0)$ with \bar{w} defined above. If $x_0 = 0$ then by the initial inequality we have $w^{(1)}(x_0, y_0)$
 ≤ 0 . We will show that the last inequality holds if $x_0 \neq 0$ and $||y_0||_{\ast} < r$. Suppose Proof: We have to show that the function

arbitrary point $(\bar{x}, \bar{y}) \in D$. Let $0 < \bar{x} < b < a$ a
 $\Phi_0(t) = \max_{1 \le i \le m} \max_{\substack{x \in [0,b] \\ \|y\|_{\ast} \le t}} w^{(i)}(x, y)$.

Let $\Phi \in C^1(\mathbf{R}_+, \mathbf{R}_+)$, $\Phi(t) \uparrow + \infty$ as $t \to +\infty$, sa
 $t \ge 0$ \mathbf{R}_{+} , $\phi(t) \uparrow +\infty$ as $t \rightarrow +\infty$, satisfies the condition $\Phi(t) \geq \Phi_0$

e a function defined by (4). This *H* satisfies the inequality (5

ere exists $r = r(\varepsilon)$ such that $w^{(t)}(x, y) < \varepsilon H(x, y), i = 1, ..., n$,
 $r \leq \|y\|_{\$

with *w* defined above. If
$$
x_0 = 0
$$
 then by the initial inequality we have $w \le 0$.
\n ≤ 0 . We will show that the last inequality holds if $x_0 \ne 0$ and $||y_0||_{*} < r$.
\nthe contrary. Then we have $\overline{w}^{(i_0)}(x_0, y_0) > 0$, $\overline{w}_x^{(i_0)}(x_0, y_0) \ge 0$ and $\overline{w}_y^{(i_0)}(x_0, y_0)$
\nIt follows from (14) that
\n
$$
w_x^{(i_0)}(x_0, y_0) = \overline{w}_x^{(i_0)}(x_0, y_0) H(x_0, y_0) + \overline{w}^{(i_0)}(x_0, y_0) H_x(x_0, y_0)
$$
\n
$$
\le f^{(i_0)}(x_0, y_0, \overline{u}(x_0, y_0) H(x_0, y_0), u, \overline{u}_y^{(i_0)}(x_0, y_0) H(x_0, y_0)
$$
\n
$$
+ \overline{u}^{(i_0)}(x_0, y_0) H_y(x_0, y_0) - f^{(i_0)}(x_0, y_0, \overline{v}(x_0, y_0)
$$
\n
$$
\times H(x_0, y_0), v, \overline{v}_y^{(i_0)}(x_0, y_0) H(x_0, y_0) + \overline{v}^{(i_0)}(x_0, y_0) H_y(x_0, y_0)
$$
\nLet $I = \{i : \overline{w}^{(i)}(x_0, y_0) > 0\}$ and $\Gamma = (y^{(1)}, ..., y^{(m)})$ where
\n
$$
y^{(i)}(x, y) = \begin{cases} H(x, y) \overline{u}^{(i)}(x, y) & \text{if } i \in I \\ H(x, y) \overline{v}^{(i)}(x, y) & \text{if } i \notin I. \end{cases}
$$

Let $I = \{i : \overline{w}^{(i)}(x_0, y_0) > 0\}$ and $\Gamma = (\gamma^{(1)}, \dots, \gamma^{(m)})$ where

$$
\times H(x_0, y_0), v, \overline{v}_y \xrightarrow{\alpha_0} (x_0, y_0) H(x_0, y_0)
$$

\n
$$
\{i : \overline{w}^{(i)}(x_0, y_0) > 0\} \text{ and } \Gamma = (\gamma^{(1)})^{\gamma_0}, \dots, \gamma^{(n)}
$$

\n
$$
\gamma^{(i)}(x, y) = \begin{cases} H(x, y) \overline{v}^{(i)}(x, y) & \text{if } i \in I \\ H(x, y) \overline{v}^{(i)}(x, y) & \text{if } i \notin I. \end{cases}
$$

(14)

Li

On Solutions of First-Order Part. Diff.-Funct. Equ. 129

-

 $\mathcal{F}_{\mathcal{A}}$

On Solutions of First-Order Part. Diff. Funct. Equ. 129
\nThen we have
\n
$$
\overline{u}(x_0, y_0) H(x_0, y_0) \leq \Gamma(x_0, y_0).
$$
\nLet $A = (\lambda_1, ..., \lambda_m)$ be a function defined by
\n
$$
\lambda_i(x, y) = \max [u^{(i)}(x, y), v^{(i)}(x, y)] \text{ for } (x, y) \in D_0 \cup D.
$$
\nThen we have $A \in C(D_0 \cup D, \mathbb{R}^m)$ and
\n
$$
u(x, y) \leq A(x, y) \text{ for } (x, y) \in D_0 \cup D.
$$
\nThe monotonicity conditions of f and (15), (16) imply
\n
$$
t^{(i_0)}(x, y) = \overline{u}(x, y) H(x, y) + t^{(i_0)}(x, y) H(x, y).
$$
\n(16)

$$
\lambda_i(x, y) = \max [u^{(i)}(x, y), v^{(i)}(x, y)]
$$
 for $(x, y) \in D_0 \cup D$.

$$
u(x, y) \leq \Lambda(x, y) \quad \text{for} \ \ (x, y) \in D_0 \cup D.
$$

The monotonicity conditions of f and (15), (16) imply

Let have
\n
$$
\overline{u}(x_0, y_0) H(x_0, y_0) \leq \Gamma(x_0, y_0).
$$
\n(15)
\n
$$
= (\lambda_1, ..., \lambda_m) \text{ be a function defined by}
$$
\n
$$
\lambda_i(x, y) = \max [u^{(i)}(x, y), v^{(i)}(x, y)] \text{ for } (x, y) \in D_0 \cup D.
$$
\nwe have $A \in C(D_0 \cup D, \mathbb{R}^m)$ and
\n
$$
u(x, y) \leq A(x, y) \text{ for } (x, y) \in D_0 \cup D.
$$
\n(16)
\ncontonicity conditions of *f* and (15), (16) imply
\n
$$
f^{(i_0)}(x_0, y_0, \overline{u}(x_0, y_0)) H(x_0, y_0), u, \overline{u}_y^{(i_0)}(x_0, y_0) H(x_0, y_0) + \overline{u}^{(i_0)}(x_0, y_0) H_y(x_0, y_0))
$$
\n
$$
+ \overline{u}^{(i_0)}(x_0, y_0, \Gamma(x_0, y_0), A, \overline{u}_y^{(i_0)}(x_0, y_0) H(x_0, y_0) + \overline{u}^{(i_0)}(x_0, y_0) H_y(x_0, y_0))
$$
\n
$$
+ \overline{u}^{(i_0)}(x_0, y_0) H_y(x_0, y_0).
$$
\nng the Lipschitz condition for *f*^(i_0) and (15), (16) we obtain
\n
$$
\overline{u}_x^{(i_0)}(x_0, y_0) H(x_0, y_0) + \overline{u}^{(i_0)}(x_0, y_0) H_x(x_0, y_0)
$$
\n
$$
\leq K \sum_{j \in I} w^{(j)}(x_0, y_0) H(x_0, y_0) + M ||A - v||_{x,y},
$$
\n
$$
+ L\overline{w}^{(i_0)}(x_0, y_0) ||H_y(x_0, y_0)||_1.
$$
\nthus from the definition of *A* that there exist points $(x_j, y^{(j)})$, $j = 1, ..., m$, that $0 \$

- Applying the Lipschitz condition for *f(bo)* and (15), (16) we obtain

$$
+ u^{(1)}(x_0, y_0) H_y(x_0, y_0).
$$
\nApplying the Lipschitz condition for $f^{(i_0)}$ and (15), (16)
\n
$$
\overline{w}_x^{(i_0)}(x_0, y_0) H(x_0, y_0) + \overline{w}^{(i_0)}(x_0, y_0) H_x(x_0, y_0)
$$
\n
$$
\leq K \sum_{j \in I} w^{(j)}(x_0, y_0) H(x_0, y_0) + M ||A - v||_{x_0 y_0}
$$
\n
$$
+ L \overline{w}^{(i_0)}(x_0, y_0) ||H_y(x_0, y_0)||_n.
$$
\nIt follows from the definition of A that there exist such that $0 \leq x_j \leq x_0$, $||y^{(j)}||_* \leq ||y_0||_*$ and $||\lambda_j - v^{(j)}||_n$
\nhave, in contradiction to (5),
\n $0 \leq \overline{w}^{(i_0)}(x_0, y_0) H(x_0, y_0)$
\n $\leq \overline{w}^{(i_0)}(x_0, y_0) [m(K + M) H(x_0, y_0) + L_f ||H_y]$
\nFinally, if $||y_0||_* = r$, then we obtain from the definite

It follows from the definition of A that there exist points $(x_j, y^{(i)})$, $j = 1, ..., m$, $\leq K \sum_{j \in I} w^{(j)}(x_0, y_0) H(x_0, y_0) + M ||A - i$
+ $L \overline{w}^{(i_0)}(x_0, y_0) ||H_y(x_0, y_0)||_n$.
It follows from the definition of A that there ϵ
such that $0 \leq x_j \leq x_0$, $||y^{(j)}||_* \leq ||y_0||_*$ and $||\lambda_j$
have, in contradiction to $(v_j, y^{(i)})$, $j = 1, ..., m$,
 $-v^{(j)}\|_{x_{0}y_{0}} = w^{(j)}(x_j, y^{(j)})$. Then we have, in contradiction to (5) , +

follows free, in contract to θ :
 $\theta \leq$
 $\theta \leq$

$$
0 \leq \overline{w}^{(i_0)}(x_0, y_0) H(x_0, y_0)
$$

$$
\leq \overline{w}^{(i_0)}(x_0, y_0) [m(K + M) H(x_0, y_0) + L_f ||H_y(x_0, y_0)||_n - H_x(x_0, y_0)].
$$

Finally, if $||y_0||_* = r$, then we obtain from the definition of \overline{w} that \overline{w} ⁽ⁱ)(x₀, y₀) $\lt \epsilon$, ε being arbitrary.

It follows from the above considerations that $\overline{w}^{(i)}(\overline{x}, \overline{y}) < \varepsilon$, $i = 1, ..., m$, for an arbitrary $\varepsilon > 0$. Then we have $\overline{w}(\overline{x}, \overline{y}) \leq 0$ and $w(\overline{x}, \overline{y}) \leq 0$. Since $(\overline{x}, \overline{y})$ is an arbitrary point in arbitrary $\varepsilon > 0$. Then we have $\overline{w}(\overline{x}, \overline{y}) \leq 0$ and $w(\overline{x}, \overline{y}) \leq 0$. Since $(\overline{x}, \overline{y})$ is an arbitrary point in *D*, we obtain the desired inequality **I**

4. Uniqueness criterion with a Hölder condition

Now *we* prove the uniqueness of solutions of the initial problem (1) under weaker assumptions concerning the function f : the Lipschitz condition with respect to q is replaced by the Hölder condition. This will be a generalization of the results **4.** Uniqueness criterion with a Hölder condition
Now we prove the uniqueness of solutions of the initial problem (1) assumptions concerning the function f : the Lipschitz condition with
is replaced by the Hölder conditi

Theorem *4: ,Suppose that*

1. *the function f is defined on* $D \times \mathbf{R}^m \times C(D_0 \cup D, \mathbf{R}^m) \times \mathbf{R}^n$ and satisfies the Theorem 4: Suppose that
1. the function f is defined on $D \times \mathbb{R}^m \times C(D_0 \cup D, \mathbb{R}^m) \times \mathbb{R}^n$ and satisfies the
Volterra condition,

Analysis lid. *6,* heft 2 (1057)

2. the estimate

IM

Z. KAMONT and K. Przapka
\n*estimate*
\n
$$
||f(x, y, p, z, q) - f(x, y, \overline{p}, \overline{z}, \overline{q})||_{m}
$$
\n
$$
\leq K ||p - \overline{p}||_{m} + M ||z - \overline{z}||_{xy} + L \sum_{k=1}^{n} \max' (|q_{k} - \overline{q}_{k}|, |q_{k} - \overline{q}_{k}|^{a}) \qquad (17)
$$
\n
$$
|e d \text{ on } D \times \mathbf{R}^{m} \times C(D_{a} \cup D, \mathbf{R}^{m}) \times \mathbf{R}^{n}, \text{ where } \alpha \in (0, 1) \text{ and } K, L, M \geq 0.
$$

is satisfied on $D \times \mathbb{R}^m \times C(D_0 \cup D, \mathbb{R}^m) \times \mathbb{R}^n$, where $\alpha \in (0, 1)$ and $K, L, M \geq 0$.

Under these assumptions, the solution u of (1), which is continuous in $D_0 \cup D$ *and has first-order partial derivatives on* $D \setminus I_0$ *, is unique.*

Proof: Let $u, v \in C(D_0 \cup D, \mathbf{R}^m)$ be two solutions of (1). Denote $w = u - v$. By **Proof:** Let $u, v \in C(D_0 \cup D, \mathbb{R}^m)$ be two solutions of (1). Denote $w = u - v$. By (1) we have $w = 0$ in D_0 . We shall show that $w = 0$ in D . Let $0 < b < a$ and Φ_0 be the function defined in the proof of Theorem 3 with the above *w*. Let $\Phi \in C^1(\mathbb{R}, \mathbb{R}_+),$ **Proof:** Let $u, v \in C(D_0 \cup D, \mathbb{R}^m)$ be two solutions of (1). Denote $w = u - v$. By) we have $w = 0$ in D_0 . We shall show that $w = 0$ in D . Let $0 < b < a$ and Φ_0 the function defined in the proof of Theorem 3 with the r these assumptions, the solution u of (1), which is continuous in $D_0 \cup D$ and
t-order partial derivatives on $D \setminus I_0$, is unique.
f: Let $u, v \in C(D_0 \cup D, \mathbb{R}^m)$ be two solutions of (1). Denote $w = u - v$. By
have $w = 0$ $\Phi'(t) \geq 1$ as $t \geq 0$, satisfies the condition $\Phi(t) \geq \Phi_0(t)$ for $t \geq 0$. Then we have fo
 $i = 1, ..., m$
 $w^{(i)}(x, y) \leq \Phi(||y||_*)$ in $D' = [0, b] \times \mathbb{R}^n$. (18

Let $\varepsilon \in (0, 1)$ and $\beta = (1 - \alpha) \alpha^{-1}$. It is easy to verify *+ LE max (I HYk(x, y;* **i)!,** *!Hy^k (x, y; e)J).* (19)

$$
w^{(i)}(x, y) \leq \Phi(||y||_*) \quad \text{in} \quad D' = [0, b] \times \mathbf{R}^n. \tag{18}
$$

$$
H(x, y; \varepsilon) = \varepsilon \exp \left[\varPhi(\varepsilon \theta(\|y\|_{*}^{2} + 1)^{1/2} + nLx) + (m(K + M) + 1)x\right]
$$

satisfies in *D'* the inequality

$$
w^{(i)}(x, y) \leq \Phi(||y||_*) \text{ in } D' = [0, b] \times \mathbb{R}^n. \tag{18}
$$

Let $\varepsilon \in (0, 1)$ and $\beta = (1 - \alpha) \alpha^{-1}$. It is easy to verify that the auxiliary function

$$
H(x, y; \varepsilon) = \varepsilon \exp [\Phi(\varepsilon^{\beta}(||y||_*^2 + 1)^{1/2} + nLx) + (m(K + M) + 1)x]
$$

satisfies in D' the inequality

$$
H_x(x, y; \varepsilon) \geq [m(K + M) + 1] H(x, y; \varepsilon)
$$

$$
+ L \sum_{k=1}^n \max (|H_{y_k}(x, y; \varepsilon)|, |H_{y_k}(x, y; \varepsilon)|^{\alpha}). \tag{19}
$$

Moreover, since $\Phi(t) \uparrow + \infty$ as $t \to +\infty$ we have for every fixed $\varepsilon \in (0, 1)$:

$$
\frac{\Phi(||y||_*)}{H(x, y; \varepsilon)} \to 0 \text{ as } ||y||_* \to +\infty. \tag{20}
$$

Moreover, since $\Phi(t) \uparrow +\infty$ as $t \to +\infty$ we have for every fixed $\varepsilon \in (0, 1)$:

$$
\frac{\varPhi(||y||_{*})}{H(x, y; \varepsilon)} \to 0 \quad \text{as} \quad ||y||_{*} \to +\infty. \tag{20}
$$

Let $z(x, y) = w(x, y) - H(x, y; \varepsilon)$. By (18) and (19), for every $\varepsilon > 0$ there exists Let $z(x, y) = w(x, y) - H(x, y; \varepsilon)$. By (18) and (19), for every $\varepsilon > 0$ there exists $r = r(\varepsilon)$ such that $z(x, y) < 0$ for $0 \le x \le b$ and $r \le ||y||_*$. If $x = 0$ then $H(x, y; \varepsilon)$ > 0 and it follows from the initial condition that $z(x, y) < 0$ for $||y||_* < r$. We prove that $z(x, y) < 0$ for $x \neq 0$ and $||y||_{*} < r$. Suppose the contrary. Then there Let $z(x, y) = w(x, y) - H(x, y; \varepsilon)$. By (18) and (19), for every $\varepsilon > 0$ there exists $r = r(\varepsilon)$ such that $z(x, y) < 0$ for $0 \le x \le b$ and $r \le ||y||_*$. If $x = 0$ then $H(x, y; \varepsilon) > 0$ and it follows from the initial condition that z $Z_1 = 0$, $z^{(i)}(x_0, y_0) \leq 0$ and $z^{(i)}(x, y) < 0$ for $0 \leq x \leq x_0$ and $||y||_* < r$. Then we have $\begin{array}{l} \text{if } z(x, y) < 0 \text{ for } x \neq 0 \text{ and } ||y||_{*} < 0 \ \text{for } i_{0} \text{ and a point } (x_{0}, y_{0}), 0 < x_{0} \leqq 0 \ \text{for } y_{0}) \leqq 0 \text{ and } z^{(i)}(x, y) < 0 \text{ for } 0 \leqq 0 \ \text{and } z^{(i_{0})}(x_{0}, y_{0}) \geqq 0 \quad \text{and } z_{y}(i_{0})(x_{0}, y_{0}) = 0 \ \text{if } (x_{0}, y_{0}) \text{ we have} \ \begin{array}{l} \frac{\partial}{\partial y_{k$

$$
z_x^{(i_0)}(x_0, y_0) \ge 0 \quad \text{and} \quad z_y^{(i_0)}(x_0, y_0) = 0. \tag{20}
$$

Further, at (x_0, y_0) we have

$$
\left|\frac{\partial}{\partial y_k}|w^{(i_0)}(x, y)|\right| = \left|w^{(i_0)}_{y_k}(x, y)\right|, \qquad \frac{\partial}{\partial x}|w^{(i_0)}(x, y)| \leq |w_x^{(i_0)}(x, y)|,
$$

and by (20) we obtain

 $w_{\boldsymbol{\mathit{v}}}$ $\left| \psi^{(i_{0})}(x, y) \right| = \left| \psi^{(i_{0})}_{y_{k}}(x, y) \right|, \qquad \frac{\partial}{\partial x} \left| \psi^{(i_{0})}(x, y) \right| \leq \left| \psi_{x}^{(i_{0})}(x, y) \right|,$
we obtain
 $\psi^{(i_{0})}(x_{0}, y_{0}) \mathbb{I} = \mathbb{I}H_{y}(x_{0}, y_{0}; \varepsilon) \mathbb{I}, \qquad \left| \psi_{x}^{(i_{0})}(x_{0}, y_{0}) \right| - H_{x}(x_{0}, y_{0}; \varepsilon) \geq 0$ $0. (21)$ But on the other hand, by (1) and (17) we get

$$
|w_x^{(i_0)}(x_0,y_0)|\leq K\,||w(x_0,y_0)||_m+M\,||w||_{x_0y_0}
$$

+
$$
L \sum_{k=1}^{n} \max (|w_{y_k}^{(i_0)}(x_0, y_0)|, |w_{y_k}^{(i_0)}(x_0, y_0)|^{\alpha}),
$$

(22)

and by (19), (20) we obtain

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\nand by (19), (20) we obtain
\n
$$
|w_x^{(i_1)}(x_0, y_0)| - H_x(x_0, y_0; \varepsilon)
$$
\n
$$
\leq K ||w(x_0, y_0)||_m + M ||w||_{x,y_0}
$$
\n+ $L \sum_{k=1}^n \max (|w_{y_k}^{(i_0)}(x_0, y_0)|, |w_{y_k}^{(i_0)}(x_0, y_0)|^a)$
\n $- [m(K + M) + 1] \dot{H}(x_0, y_0; \varepsilon)$ (22)
\n $- L \sum_{k=1}^n \max (|H_{y_k}(x_0, y_0; \varepsilon)|, |H_{y_k}(x_0, y_0; \varepsilon)|^a)$
\n $\leq K \sum_{j=1}^m [|w^{(j)}(x_0, y_0)| - H(x_0, y_0; \varepsilon)] - H(x_0, y_0; \varepsilon)$
\n $+ M \sum_{j=1}^n [||w^{(j)}||_{x,y_0} - H(x_0, y_0; \varepsilon)].$
\nIt follows from the definition of w that there exist points $(x_i, y^{(j)}), j = 1, ..., m$,
\nsuch that $0 \leq x_j \leq x_0$; $||y^{(j)}||_* \leq ||y_0||_*$ and $||w^{(j)}||_{x,y_0} = w^{(j)}(x_j, y^{(j)})$. Then we get
\nfrom (22)
\n $|w_x^{(i_0)}(x_0, y_0)| - H_x(x_0, y_0; \varepsilon) \leq -H(x_0, y_0; \varepsilon) < 0$
\nwhich contradicts (21). So inequality $\gamma(x_0, y_0; \varepsilon) < 0$

such that $0 \leq x_j \leq x_0$, $||y^{(j)}||_* \leq ||y_0||_*$ and $||w^{(j)}||_{x,y} = w^{(j)}(x_j, y^{(j)})$. Then we get $\begin{aligned} \n\mathcal{L}_1 + M \sum_{j=1}^m \left[||w^{(j)}||_{x_i y_0} - H(x_0, y_0; \varepsilon) \right]. \n\end{aligned}$ ws from the definition of w that there exist point

hat $0 \le x_j \le x_0$, $||y^{(j)}||_* \le ||y_0||_*$ and $||w^{(j)}||_{x_0 y_0} = u$

(2)
 $|w_x^{(i_0)}(x_0, y_0)| - H_x(x_0, y_0; \v$

$$
w_x^{(\mathbf{i}_{\bullet})}(x_0, y_0)| - H_x(x_0, y_0; \varepsilon) \le -H(x_0, y_0; \varepsilon) < 0
$$

which contradicts (21). So inequality $z(x, y) < 0$ is satisfied for $(x, y) \in D$, $0 \le x \le b$. Now, letting $\varepsilon \to 0$ we obtain $w(x, y) = 0$ in *D'*. Since *b* may be chosen arbitrarily close to \hat{a} , the proof is complete \blacksquare

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