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Pseudo-Differential Operators in $F^s_{p,q}$-Spaces

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Für $a(x, \xi) \in S^0_{1,\delta}$ mit $0 \leqq \delta < 1$ ist der Pseudodifferentialoperator $a(x, D)$ stetig in $F^s_{p,q}$, wobei $-\infty < s < \infty$ und entweder $0 < p < \infty$, $0 < q \leqq \infty$ oder $p = q = \infty$ gilt.

Для $a(x, \xi) \in S^0_{1,\delta}$ с $0 \leqq \delta < 1$ псевдодифференциальный оператор $a(x, D)$ непрерывен в $F^s_{p,q}$, где $-\infty < s < \infty$ и либо $0 < p < \infty$, $0 < q \leqq \infty$ или $p = q = \infty$.

If $a(x, \xi) \in S^0_{1,\delta}$ with $0 \leqq \delta < 1$, then the pseudo-differential operator $a(x, D)$ is continuous in $F^s_{p,q}$, where $-\infty < s < \infty$ and either $0 < p < \infty$, $0 < q \leqq \infty$ or $p = q = \infty$.

1. Introduction

Let $\mathbf{R}_n$ be the Euclidean n-space. The symbol class $S^0_{1,\delta}$ with $0 \leqq \delta \leqq 1$ consists of all $a(x, \xi) \in C^\infty(\mathbf{R}_n \times \mathbf{R}_n)$ with the property that for any multi-indices α, β there exists a constant $c_{\alpha,\beta}$ such that

$$|D_x{}^\beta D_\xi{}^\alpha a(x, \xi)| \leqq c_{\alpha,\beta}(1 + |\xi|)^{-|\alpha|+\delta|\beta|}, \qquad x, \xi \in \mathbf{R}_n. \tag{1}$$

The corresponding pseudo-differential operator $a(x, D)$ is defined by

$$a(x, D) f(x) = \int_{\mathbf{R}^n} e^{ix\xi} a(x, \xi) \hat{f}(\xi) \, d\xi, \qquad x \in \mathbf{R}_n, \tag{2}$$

where $\hat{f}$ stands for the Fourier transform of f and $x\xi$ is the scalar product in $\mathbf{R}_n$. The operator $a(x, D)$ maps S into S, where S denotes the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}_n$. By duality, $a(x, D)$ maps S' into S', where S' is the usual space of tempered distributions. The main aim of the paper is to give a new proof of the following result, which is essentially due to L. Päivärinta, cf. [16].

Theorem: *Let $-\infty < s < \infty$ and let either $0 < p < \infty$, $0 < q \leqq \infty$ or $p = q = \infty$. Let $0 \leqq \delta < 1$ and $a(x, \xi) \in S^0_{1,\delta}$. Then $a(x, D)$ maps $F^s_{p,q}$ continuously into itself, in particular there exists a constant c such that*

$$\|a(x, D) f \mid F^s_{p,q}\| \leqq c \, \|f \mid F^s_{p,q}\|, \qquad f \in F^s_{p,q}. \tag{3}$$

Via real interpolation this result can be extended to the spaces $B^s_{p,q}$, where $-\infty < s < \infty$, $0 < p \leqq \infty$ and $0 < q \leqq \infty$. These two scales $B^s_{p,q}$ and $F^s_{p,q}$ of (isotropic non-homogeneous) spaces on $\mathbf{R}_n$ cover many well-known spaces of functions and distributions on $\mathbf{R}_n$:

(i) the classical Besov-Lipschitz spaces $\Lambda^s_{p,q} = B^s_{p,q}$ if $s > 0$, $1 < p < \infty$ and $1 \leqq q \leqq \infty$;

(ii) the Bessel-potential spaces $H_p^s = F^s_{p,2}$ if $-\infty < s < \infty$ and $1 < p < \infty$, with the Sobolev spaces $W_p^m = H_p^m$ if $1 < p < \infty$ and m non-negative integer, as special cases;

(iii) the Hölder-Zygmund spaces $\mathscr{C}^s = F^s_{\infty,\infty}$ if $s > 0$;

(iv) the (non-homogeneous) Hardy spaces $H_p = F^0_{p,2}$ if $0 < p < \infty$.

In other words, the above theorem and its counterpart with $B^s_{p,q}$ instead of $F^s_{p,q}$ cover known results of the above type for L_2, L_p; with $1 < p < \infty$, and $\mathscr{C}^s$ with $s > 0$, cf. also Remark 9, where we give some references.

The plan of the paper is as follows. In Section 2 we define the spaces $F^s_{p,q}$ and collect those results which will be needed in the sequel. Section 3 contains the proof of the above theorem. In Section 4 we give some references and add few remarks, mostly about further possibilities. In particular we wish to convince the reader that the method presented in this paper is especially well-adapted to problems of the above type and that it may be useful in connection with other classes of pseudo-differential operators and more complicated problems.

2. The spaces $F^s_{p,q}$

2.1. Definition

We follow essentially [20, in particular 2.3.1]. Recall that $\mathbf{R}_n$, S and S' have been introduced above. Because all spaces in this paper are defined on $\mathbf{R}_n$ we omit "$\mathbf{R}_n$" in the respective definitions. Let

$$\|f \mid L_p\| = \left(\int_{\mathbf{R}_n} |f(x)|^p\, dx\right)^{1/p}$$

(usual modification if $p = \infty$). Let Φ be the collection of all systems $\{\varphi_j\}_{j=0}^{\infty} \subset S$ with the following properties:

(i) $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ if $j = 1, 2, \ldots$ and $\xi \in \mathbf{R}_n$;

(ii) $\operatorname{supp} \varphi_0 \subset \{\xi \mid |\xi| \leqq 2\}$ and $\operatorname{supp} \varphi \subset \{\xi \mid 1/2 \leqq |\xi| \leqq 2\}$;

(iii) $\sum_{j=0}^{\infty} \varphi_j(\xi) = 1$ for every $\xi \in \mathbf{R}_n$.

Let $\varphi_j(D)\, f(x)$ be given by (2) with $a(x, \xi) = \varphi_j(\xi)$ and $f \in S'$ (in the sense of the above interpretation). By the Paley-Wiener-Schwartz theorem, $\varphi_j(D)\, f(x)$ is an analytic function with respect to $x \in \mathbf{R}_n$, where $f \in S'$.

Definition: Let $\{\varphi_j\} \in \Phi$. Let $-\infty < s < \infty$ and $0 < q \leqq \infty$.

(i) Let $0 < p < \infty$. Then

$$F^s_{p,q} = \left\{f \in S' \mid \|f \mid F^s_{p,q}\|^{\{\varphi_j\}} = \left\|\left(\sum_{j=0}^{\infty} 2^{jsq}\, |\varphi_j(D)\, f(\cdot)|^q\right)^{1/q} \middle| L_p\right\| < \infty\right\}$$

(usual modification if $q = \infty$).

(ii) Let $0 < p \leqq \infty$. Then

$$B^s_{p,q} = \left\{f \in S' \mid \|f \mid B^s_{p,q}\|^{\{\varphi_j\}} = \left(\sum_{j=0}^{\infty} 2^{jsq}\, \|\varphi_j(D)\, f \mid L_p\|^q\right)^{1/q} < \infty\right\}$$

(usual modification if $q = \infty$).

Remark 1: The theory of these spaces has been developed in [20]. All spaces are quasi-Banach spaces (Banach spaces if $p \geqq 1$ and $q \geqq 1$) and they cover the classical spaces mentioned in the introduction. Different choices of $\{\varphi_j\} \in \Phi$ yield equivalent quasi-norms in the respective spaces. We shall not distinguish between equivalent quasi-norms in a given space. In this sense we write in the sequel $\|f \mid F^s_{p,q}\|$ instead of $\|f \mid F^s_{p,q}\|^{\{\varphi_j\}}$ with $\{\varphi_j\} \in \Phi$ or instead of any of the equivalent quasi-norms decribed in the next subsection.

Remark 2: We put $F^s_{\infty,\infty} = B^s_{\infty,\infty}$ in the sense of the above theorem, $-\infty < s < \infty$. Let $(\cdot, \cdot)_{\theta,q}$ with $0 < \theta < 1$ and $0 < q \leq \infty$ be the real interpolation method for quasi-Banach spaces. Let $-\infty < s_0 < s_1 < \infty$, $0 < p \leq \infty$ and $s = (1 - \theta) s_0 + \theta s_1$. Then

$$B^s_{p,q} = (F^{s_0}_{p,p}, F^{s_1}_{p,p})_{\theta,q}, \tag{4}$$

cf. [20: 2.4.2]. In other words, if the above theorem is proved, then (4) yields a corresponding result with $B^s_{p,q}$ instead of $F^s_{p,q}$, where $-\infty < s < \infty$, $0 < p \leq \infty$ and $0 < q \leq \infty$, cf. Remark 6.

2.2. Equivalent quasi-norms

In order to prove the above theorem we need two ingredients: well-adapted equivalent quasi-norms and sufficiently strong Fourier multiplier assertions for the spaces $F^s_{p,q}$. We formulate the corresponding results in this subsection and in the following one. Let $k \in S$ with $\operatorname{supp} k \subset \{y \mid |y| < 1\}$ and $\hat{k}(0) \neq 0$. Let $k_N = \Delta^N k$, where Δ stands for the Laplacian and $N = 0, 1, \ldots$ (with $k_0 = k$). We introduce the means

$$K(k_N, t) f(x) = \int_{\mathbf{R}_n} k_N(y) f(x + ty) \, dy, \qquad x \in \mathbf{R}_n, \quad t > 0. \tag{5}$$

This makes sense for any $f \in S'$ (usual interpretation).

Proposition 1: *Let either $0 < p < \infty$, $0 < q \leq \infty$ or $p = q = \infty$. Let $-\infty < s < \infty$ and let $N \in \mathbf{N}$ with $2N > s$ and $2N > n(1/\min(p, 1) - 1)$. Let $0 < \varepsilon < \infty$ and $0 < r < \infty$. Then*

$$\|K(k, \varepsilon) f \mid L_p\| + \left\| \left(\int_0^r t^{-sq} |K(k_N, t) f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right| L_p \right\| \tag{6}$$

(modification if $q = \infty$) is an equivalent quasi-norm in $F^s_{p,q}$.

Remark 3: We developed the theory of these equivalent quasi-norms in [21, 22]. The above formulation coincides essentially with Theorem A in [23], as far as the spaces $F^s_{p,q}$ are concerned (there is a corresponding assertion for the spaces $B^s_{p,q}$). The advantage of (5), (6) in comparison with the quasi-norm $\|\cdot \mid F^s_{p,q}\|^{\{\varphi_j\}}$ is its strictly local nature: in order to calculate $K(k_N, t) f(x)$ in a given point $x \in \mathbf{R}_n$ one needs only a knowledge of f in a neighborhood of x (the positive numbers ε and r in (6) may be chosen as small as one wants).

Remark 4: We add two further remarks. As in the definition of $F^s_{p,q}$ one can replace the continuous version in (6) by its discrete counterpart

$$\|K(k, \varepsilon) f \mid L_p\| + \left\| \left(\sum_{j=L}^{\infty} 2^{sjq} |K(k_N, 2^{-j}) f(\cdot)|^q \right)^{1/q} \right| L_p \right\|, \tag{7}$$

where L is an arbitrary integer. This replecement is a technical matter and it is often more convenient to use (7) instead of (6). Secondly we remark that if the Tauberian condition $\hat{k}(0) \neq 0$ is not necessarily satisfied, then (6) and (7) can at least be estimated from above by $c \|f \mid F^s_{p,q}\|$, cf. Remark 3 in [22].

2.3. Fourier multipliers

Let $a(x, \xi) \in S^0_{1,\delta}$ with $0 \leqq \delta \leqq 1$. If $z \in \mathbf{R}_n$ is fixed, then we write

$$a(z, D) f(x) = \int_{\mathbf{R}_n} e^{ix\xi} a(z, \xi) \hat{f}(\xi) \, d\xi, \qquad x \in \mathbf{R}_n. \tag{8}$$

Furthermore, $\xi \to a(z, \xi)$ is a Fourier multiplier for $F^s_{p,q}$, where $-\infty < s < \infty$, and either $0 < p < \infty$, $0 < q \leqq \infty$ or $p = q = \infty$, i.e.,

$$\|a(z, D) f \mid F^s_{p,q}\| \leqq c \, \|f \mid F^s_{p,q}\|, \tag{9}$$

cf. [20: 2.3.7], where c is not only independent of f but also of $z \in \mathbf{R}_n$. One can strengthen (9) by

$$\left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \sup_{z\in\mathbf{R}_n} |\varphi_j(D) \circ a(z, D) f(\cdot)|^q \right)^{1/q} \middle| L_p \right\| \leqq c \, \|f \mid F^s_{p,q}\|, \tag{10}$$

where the φ_j have the same meaning as in the above Definition, cf. [20: Theorem 2.3.6]. We need a modification of (10) and of a known maximal inequality.

Proposition 2: *Let either* $0 < p < \infty$, $0 < q \leqq \infty$ *or* $p = q = \infty$. *Let* $a(x, \xi) \in S^0_{1,\delta}$ *with* $0 \leqq \delta \leqq 1$.

(i) *Let* $-\infty < s < \infty$, $0 < r < \infty$ *and* $0 < \varepsilon < \infty$. *Let* N *be a sufficiently large natural number. Then there exists a positive constant* c *such that*

$$\left\| \sup_{z\in\mathbf{R}_n} |a(z, D) K(k, \varepsilon) f| \mid L_p \right\| + \left\| \left(\int_0^r t^{-sq} \sup_{z\in\mathbf{R}_n} |a(z, D) K(k_N, t) f|^q \frac{dt}{t} \right)^{1/q} \middle| L_p \right\| \leqq c \, \|f \mid F^s_{p,q}\| \tag{11}$$

(*usual modification if* $q = \infty$) *for all* $f \in F^s_{p,q}$.

(ii) *Let* $s > n/p$ *and* $b > 0$. *Then there exists a positive constant* c *such that*

$$\left\| \sup_{|x-y|\leqq b} \sup_{z\in\mathbf{R}_n} \left|\left(a(z, D) f\right)(y)\right| \middle| L_p \right\| \leqq c \, \|f \mid F^s_{p,q}\| \tag{12}$$

for all $f \in F^s_{p,q}$.

Remark 5: $a(z, D) K(k, \varepsilon) f$ and $a(z, D) K(k_N, t) f$ must be understood in the sense of (8) as functions of x, and the L_p-quasi-norm in (11) is taken with respect to this variable x. Similarly, in (10) and (12) the integrands must be interpreted as functions of x (which makes sense at least via some limiting arguments). A proof of (11) (including the fact that we prefer continuous version in contrast to the discrete versions in (10)) is completely provided by the Proposition, the Theorem and the technique of estimates developed in [20; 2.3.6], cf. also [21] as far as the reformulation of $K(k_N, t) f$ in the language of Fourier multipliers is concerned. Here one needs that N is large. (12) is essentially a consequence of the estimates at the end of the proof of Corollary 1 in [20: 2.5.9, p. 100] and the technique from [20: 2.3.6].

3. Proof of the theorem

We prove the Theorem formulated in the Introduction in two steps:

Step 1: Let s be large, whereas all the other assumptions are the same as in the Theorem. In order to use (5) and (6), with $a(x, D) f(x)$ instead of f, we first calculate

$a(x + ty, D) f(x + ty)$, where $x \in \mathbf{R}_n$, $|y| \leqq 1$ and $0 < t < r$. We have

$$a(x + ty, D) f(x + ty) = \int_{\mathbf{R}_n} e^{i(x+ty)\xi} a(x + ty, \xi) \hat{f}(\xi) \, d\xi$$

$$= \sum_{|\gamma| \leqq L-1} \frac{t^{|\gamma|}}{\gamma!} y^\gamma \int_{\mathbf{R}_n} e^{i(x+ty)\xi} D_x^\gamma a(x, \xi) \hat{f}(\xi) \, d\xi$$

$$+ \sum_{|\gamma| = L} \frac{t^L}{\gamma!} y^\gamma \int_{\mathbf{R}_n} e^{i(x+ty)\xi} D_x^\gamma a(x + \vartheta t y, \xi) \hat{f}(\xi) \, d\xi, \tag{13}$$

where L is a natural number with $L > s$ and $0 \leqq \vartheta \leqq 1$. Let $k_\gamma(y) = y^\gamma k_N(y)$, where k_N has the above meaning. Furthermore, $K(k_\gamma, t) f(x)$ is defined by (5) with k_γ instead of k_N. Then we have

$$K(k_N, t) a(x, D) f(x)$$
$$= \sum_{|\gamma| \leqq L-1} c_\gamma t^{|\gamma|} \int_{\mathbf{R}_n} e^{ix\xi} D_x^\gamma a(x, \xi) \widehat{K(k_\gamma, t) f}(\xi) \, d\xi + R(x, t).$$

The integral version of the last term in (13) yields

$$|R(x, t)| \leqq c t^L \sum_{|\gamma| = L} \sup_{|x-y| \leqq r} \sup_{z \in \mathbf{R}_n} |D_z^\gamma a(z, D) f(y)|.$$

If s and L are fixed and if N is chosen sufficiently large, then the kernels k_γ are finite sums of kernels which satisfy the hypotheses of Proposition 1, maybe with exception of $\hat{k}(0) \neq 0$. However, because we are only interested in estimates from above, the last lines of Remark 4 can be applied. Recall that

$$I_b f(x) = \int_{\mathbf{R}_n} e^{ix\xi} (1 + |\xi|^2)^{b/2} \hat{f}(\xi) \, d\xi \tag{14}$$

is a lift in $F^s_{p,q}$: it maps $F^s_{p,q}$ isomorphically onto $F^{s-b}_{p,q}$, where $-\infty < b < \infty$, cf. [20: 2.3.8]. Let

$$a_\gamma(x, \xi) = D_x^\gamma a(x, \xi) (1 + |\xi|^2)^{-\frac{\delta}{2}|\gamma|}, \tag{15}$$

where δ has the above meaning, cf. (1). In particular $a_\gamma(x, \xi) \in S^0_{1,\delta}$. We have

$$|K(k_N, t) a(x, D) f(x)|$$
$$\leqq c \sum_{|\gamma| \leqq L-1} t^{|\gamma|} \sup_{z \in \mathbf{R}_n} |a_\gamma(z, D) K(k_\gamma, t) I_{\delta|\gamma|} f(x)| + |R(x, t)|.$$

By the above remarks about the kernels k_γ, Proposition 2(i) with k_γ and a_γ instead of k_N' and a, respectively, can be applied. We obtain

$$\left\| \left(\int_0^r t^{-sq} |K(k_N, t) a(x, D) f(x)|^q \frac{dt}{t} \right)^{1/q} \middle| L_p \right\|$$
$$\leqq c \sum_{|\gamma| \leqq L-1} \|I_{\delta|\gamma|} f \mid F^{s-|\gamma|}_{p,q}\| + c \sum_{|\gamma| = L} \left\| \sup_{|x-y| \leqq r} \sup_{z \in \mathbf{R}_n} |a_\gamma(z, D) I_{\delta|\gamma|} f(y)| \middle| L_p \right\|, \tag{16}$$

where we used $L > s$ and that N is chosen large enough. The first terms on the right-hand side of (16) can be estimated from above by $c\|f \mid F^{s-|\gamma|(1-\delta)}_{p,q}\|$ and hence by $c'\|f \mid F^s_{p,q}\|$ (this is true even if $\delta = 1$). We apply Proposition 2(ii) to the last term on the right-hand side of (16). Hence this term can be estimated from above by $c \, \|I_{\delta L} f \mid F^{\sigma}_{p,q}\|$ with $\sigma > n/p$, and consequently by $c \, \|f \mid F^{\sigma+\delta L}_{p,q}\|$. The term $\|K(k, \varepsilon) \times a(x, D) f \mid L_p\|$ can be treated in the same way. Now we fix our assumptions for s by

$$n/p + \delta L < s < L, \qquad L \text{ a suitable natural number.} \tag{17}$$

If p (and q) are fixed, then all large values of s are covered by (17) (here one needs for the first time that $\delta < 1$). Under these assumptions for s, the above considerations yield (3).

Step 2: Let s be an arbitrary real number. By the composition theorem for pseudo-differential operators, cf. [7: pp. 71/94], we have $a^d(x, \xi) \in S^0_{1,\delta}$, where the corresponding pseudo-differential operator $a^d(x, D)$ is defined by $a^d(x, D) = I_{-d} \circ a(x, D) \circ I_d$, $-\infty < d < \infty$, cf. (14) (here we used again $\delta < 1$). Hence we obtain $a(x, D) = I_d \circ a^d(x, D) \circ I_{-d}$. We choose d in such a way that Step 1 can be applied to $s + d$ instead of s. Then I_{-d} maps $F^s_{p,q}$ onto $F^{s+d}_{p,q}$, $a^d(x, D)$ maps $F^{s+d}_{p,q}$ into $F^{s+d}_{p,q}$, and I_d maps $F^{s+d}_{p,q}$ onto $F^s_{p,q}$. This completes the proof of the Theorem ∎

Remark 6: After the Theorem has been proved the real interpolation formula (4) yields a corresponding result for the $B^s_{p,q}$-spaces. In other words, let $-\infty < s < \infty$, $0 < p \leq \infty$ and $0 < q \leq \infty$, then there exists a constant c such that $\|a(x, D) f \mid B^s_{p,q}\| \leq c \, \|f \mid B^s_{p,q}\|$ for all $f \in B^s_{p,q}$.

4. Further remarks

On the basis of the two propositions our method is not very complicated. The Taylor expansion (13) is the crucial formula. Furthermore we used the composition theorem for pseudo-differential operators of the symbol class $S^0_{1,\delta}$ with $0 \leq \delta < 1$. The advantage of our method is the strictly local nature as far as the x-variable is concerned (for the ξ-variable nothing of this type can be expected). The question is to apply the above method to other (more complicated) symbol classes.

Remark 7 (*Weak-exotic symbol class*): The symbol class $S^0_{1,1}$ is sometimes called exotic. An extension of the above theorem to this class is not possible. There exist symbols $a(x, \xi) \in S^0_{1,1}$ such that $a(x, D)$ does not map L_2 into itself. On the other hand it has been shown by Y. MEYER [10] and T. RUNST [18] that $a(x, D)$ with $a(x, \xi) \in S^0_{1,1}$ preserves $F^s_{p,q}$ and $B^s_{p,q}$ if s is large. Unfortunately, the above method is not strong enough to cover Runst's result. But a weaker version can be obtained without additional efforts. We say that $a(x, \xi) \in C^\infty(\mathbf{R}_n \times \mathbf{R}_n)$ belongs to the weak-exotic class if

$$|D_x^\beta D_\xi^\alpha a(x, \xi)| \leq c_{\alpha,\beta}(1 + |\xi|)^{-|\alpha| + m_{|\beta|}}$$

for all $x \in \mathbf{R}_n$, $\xi \in \mathbf{R}_n$ with $j - m_j \uparrow \infty$ if $j \to \infty$. Maybe it is reasonable to assume that $j - m_j$ tends monotonically to infinity, but this is not really necessary. Of course, any $a(x, \xi) \in S^0_{1,\delta}$ with $0 \leq \delta < 1$ is weak-exotic. Step 1 of the above proof can be applied to this class of pseudo-differential operators, including the counterparts of (15)–(17). Hence, if p and q are given as above, then $a(x, D)$ maps $F^s_{p,q}$ into itself (and consequently also $B^s_{p,q}$ into itself) if s is large. We have not checked whether the composition theorem for this class of pseudo-differential operators holds. Hence it is at least doubtful whether Step 2 can be applied.

Remark 8 (*Elementary symbols*): A symbol $a(x, \xi)$ is called elementary if it can be represented as

$$a(x, \xi) = \sum_{j=0}^{\infty} A_j(x) \, B_j(\xi).$$

These symbols seem to be especially well-adapted to the above method because our approach is strictly local as far as the x-variable is concerned. The reduction of general symbols to elementary symbols is due to R. R. COIFMAN and Y. MEYER, cf. [5]. These ideas have been used in [6, 17] to investigate pseudo-differential operators in $B^s_{p,q}$- and $F^s_{p,q}$-spaces.

Remark 9: The first candidate in order to study mapping properties for various types of pseudo-differential operators is always L_2. However there has been done a lot in order to study different types of pseudo-differential operators in L_p with $1 < p < \infty$ and in Hölder spaces. We refer to [1–3, 8, 9, 11–15, 24] and [19: Chap. XI]. Some of these papers deal with the problem to weaken the smoothness asumptions for the symbol $a(x, \xi)$ in particular for the x-variable. Mapping properties in $B^s_{p,q}$- and $F^s_{p,q}$-spaces have been treated in [4, 6, 16, 17, 25, 26]. We repeat that the above-proven theorem is due to L. PÄIVÄRINTA, cf. [16]. The question is whether the above method can be used in order to attack other problems than those treated in this paper (maybe in connection with the quoted papers).

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