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On a Coupled System of Partial Differential Equations

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Es werden verschiedene Randwertaufgaben für ein gekoppeltes System stationärer partieller Differentialgleichungen, das Modellen des Eindringens von Verunreinigungen in Halbleiter entlehnt und den Gleichungen der inneren Elektronik ähnlich ist, mittels energetischer Methoden auf Existenz von Lösungen untersucht. Daneben werden Abschätzungen von oben für den Durchmesser der Lösungsmenge gewonnen.

Исследуются разные краевые задачи для системы стационарных уравнений в частных производных, которая проистекает от моделей инжектирования зарядов в полупроводники и подобна уравнениям распределения плотностей зарядов. Доказывается существование решений при помощи энергетических методов. При этом получаются и оценки сверху для диаметра множества решений.

Boundary value problems are considered for a system of stationary partial differential equations that stems from the models describing the implantation of impurities into semiconductor devices and is similar to the equations of carrier transport. The existence of solutions by functional analytic means (energetic method) is proved. Besides, an upper estimate for the set of solutions is obtained.

We investigate the following coupled system of stationary partial differential equations:

$$\left.\begin{aligned} &\nabla\cdot(D_i\,\nabla u_i) + \nabla\cdot\left(k_i z_i u_i (a + b u_0^2)^{-1/2}\,\nabla\Psi\right) = f_i \\ &-\nabla\cdot(D_0\,\nabla\Psi) = k(N + u_0), \qquad i = 1, \dots, M \end{aligned}\right\} \tag{1}$$

where u_i, f_i, Ψ and N are real functions on a bounded domain G of the n-dimensional Euclidean space $\mathbf{R}_n$ allowing partial integration and the Sobolev imbedding theorems. $\nabla\cdot(D\,\nabla u)$ means div (D grad u). Let further $u_0 = z_1u_1 + \dots + z_Mu_M$, $z_i = \pm 1$, $U_0 = 1/(a + bu_0^2)^{1/2}$, M a natural number, k, k_i, a, b positive constants. The diffusion coefficients may be of the form $D_i = D_i(|\nabla u_i|)$ and $D_0 = D_0(|\nabla\Psi|)$. For all functions u_i and Ψ we suppose the same type of boundary conditions:

$$\left.\begin{aligned} &u_i|_{\partial G_1} = \underline{y}_i, \qquad i = 1, \dots, M, \qquad \Psi|_{\partial G_1} = \underline{y}_0 \\ &J_i\cdot\vec{n}|_{\partial G_2} = 0, \qquad i = 0, 1, \dots, M \end{aligned}\right\} \tag{2}$$

where $J_0 = -D_0\,\nabla\Psi$ and $J_i = -D_i\,\nabla u_i - k_i z_i u_i (a + bu_0^2)^{-1/2}\,\nabla\Psi$ are the flow densities, ∂G_1 and ∂G_2 are portions of the boundary of G with the properties $\partial G_1 \cap \partial G_2 = \emptyset$, $\partial G = \partial G_1 \cup \partial G_2$, mes $\partial G_1 > 0$ with respect to the boundary measure, and $\vec{n}$ is the outer normal to the boundary of G.

The system (1) is similar to the stationary equations of carrier transport in semiconductors (cf. [3, 7]). Instead of two equations, here an arbitrary number of carriers is supposed. The mobilities are replaced by the terms $z_ik_iu_i(a + bu_0^2)^{-1/2}$ and the recombination rates are lacking. The coupling of the quantities u_i by the last equation of (1) here takes place in an analoguous way. The system (1) has its origin in the models describing the implantation of impurities into semiconductor devices as they were used by Dutton (cf. [5]) and others for numerical simulations.

It is the purpose of this paper to prove the existence of a solution of the problem (1), (2) by functional analytical means. We proceed in a similar way as in [3, 7].

In Section 1 we consider first the problem with homogeneous boundary conditions

$$\left.\begin{aligned} &u_i|_{\partial G_1} = 0, \quad i = 1, \ldots, M, \qquad \Psi|_{\partial G_1} = 0 \\ &J_i \cdot n|_{\partial G_2} = 0, \qquad i = 0, 1, \ldots, M. \end{aligned}\right\} \tag{3}$$

The closure of $\{v \in C^2(\bar{G}) : v|_{\partial G_1} = 0\}$ with respect to the norm

$$\|v\|_{1,q} = \Big(\int_G [|v|^2 + |\nabla v|_n^2]^{q/2}\, dG\Big)^{1/q}, \quad q > 1,$$

of the Sobolev space $W_q^1(G) = \{v \in L_q(G) : D^\alpha v \in L_q(G)\}$ ($|\cdot|_n$ is the Euclidean norm of $\mathbf{R}_n$) is denoted by $W_q(G)$. Let us assume that G has a Lipschitzian boundary (cf. [9]). Then

$$\|v\|_{0,q} = \Big(\int_G |\nabla v|_n^q\, dG\Big)^{1/q}$$

and $\|\cdot\|_{1,q}$ are equivalent norms on $W_q(G)$. We ask for solutions $[u_1, \ldots, u_M, \Psi]$ of the problem (1), (3) in the sense of the notion of the solution in variation (weak solution) in the space $W^M \times W_r(G)$, where $W^M = W \times \cdots \times W$ is the M-fold product of $W = W_p(G)$, $r, p \geqq 2$. For the sake of simplicity we suppose $p > n$ in order to be sure of the compact imbedding of $W_p^1(G)$ into the space $C(\bar{G})$ of continuous functions on $\bar{G}$. We assume further that $N \in L_{r'}(G)$, $1/r' + 1/r = 1$, $f_i \in L_{p'}(G)$, $1/p' + 1/p = 1$. Concerning the diffusivities we assume

$$D_i(|\nabla u_i|) = d^i(|\nabla u_i|)\, |\nabla u_i|^{p-2} \quad \text{and} \quad D_0(|\nabla \Psi|) = |\nabla \Psi|^{r-2}, \tag{4}$$

where the d^i are defined on $\mathbf{R}_+ = [0, +\infty)$, continuous, non-decreasing, and $0 < d_{i0} \leqq d^i(s) \leqq d_{i1}$, $s \in \mathbf{R}_+$. Under these assumptions the diffusivity terms generate uniformly monotone operators in $W_p(G)$ and $W_r(G)$, respectively. An operator T of the whole of a real, reflexive Banach space B_0 into its adjoint space $(B_0)^*$ is called *uniformly monotonous* if there is a function $\delta \in (\mathbf{R}_+ \to \mathbf{R}_+)$ with $\delta(1) > 0$ such that $(Tu - Tv, u - v) \geqq \delta(\|u - v\|)$ holds for $u, v \in B_0$. In our cases either $\delta(s) = c_1 s^p$ or $\delta(s) = c_0 s^r$ hold. We will show that the operator equation $Tu = 0$ corresponding to the homogeneous problem has a bounded, coercive, pseudomonotone operator T, and therefore a solution exists by a well-known theorem of Brézis [1].

In Section 2 the inhomogeneous problem (1), (2) will be investigated, reducing it to the homogeneous case. In this way a "disturbed" operator equation will arise that will then be solved by the same means as the undisturbed one considered in Section 1.

1. The homogeneous boundary value problem

Here we consider the homogeneous boundary value problem (1), (3). Let us begin with the following useful abstract lemma, which is partially contained in many papers (cf. e.g. [2, 4, 6]).

Lemma: *Let $(H, (\cdot, \cdot))$ be a Hilbert space, $|\cdot| = (\cdot, \cdot)^{1/2}$ the norm and α a real number. For $u, v \in H$ we have*

$$(|u|^\alpha u - |v|^\alpha v, u - v) \geqq \begin{cases} K\,|u - v|^{\alpha+2} & \text{for } \alpha \geqq 0 \\ K(|u|^{-\alpha} + |v|^{-\alpha})^{-1}\,|u - v|^2 & \text{for } -1 < \alpha \leqq 0 \end{cases} \tag{1.1}$$

and

$$||u|^\alpha u - |v|^\alpha v| \leqq \begin{cases} G(|u|^\alpha + |v|^\alpha)\,|u - v| & \text{for } \alpha \geqq 0 \\ G\,|u - v|^{\alpha+1} & \text{for } -1 < \alpha \leqq 0 \end{cases} \tag{1.2}$$

where K and G are positive constants depending on α.

Proof: We are going to prove the Lemma in an elementary algebraic manner. We begin with (1.1). Division of (1.1) by $|u|^{\alpha+2}$ (we assume $|u| \geqq |v| > 0$) yields

$$\begin{aligned} &\big(u/|u| - (|v|/|u|)^\alpha v/|u|,\ u/|u| - v/|u|\big) \\ &\geqq \begin{cases} K|u/|u| - v/|u||^{\alpha+2} & \text{for } \alpha \geqq 0 \\ K\big(1 + (|v|/|u|)^{-\alpha}\big)^{-1}\,|u/|u| - v/|u||^2 & \text{for } -1 < \alpha \leqq 0. \end{cases} \end{aligned}$$

Consequently, it suffices to show for $|x| = 1$ and $|y| \leqq 1$ that

$$(x - |y|^\alpha y, x - y) \geqq \begin{cases} K\,|x - y|^{\alpha+2} & \text{for } \alpha \geqq 0 \\ K(1 + |y|^{-\alpha})^{-1}\,|x - y|^2 & \text{for } -1 < \alpha \leqq 0. \end{cases}$$

Setting $t = |y|$ and $s = (x, y)$ with $|x| = 1$ we get

$$\begin{aligned} &(1 - t)(1 - t^{\alpha+1}) + (t^\alpha + 1)(t - s) \\ &\geqq \begin{cases} K\big((1 - t)^2 + 2(t - s)\big)^{\alpha/2+1} & \text{for } \alpha \geqq 0 \\ K(1 + t^{-\alpha})^{-1}\,[(1 - t)^2 + 2(t - s)] & \text{for } -1 < \alpha \leqq 0 \end{cases} \end{aligned} \tag{1.3}$$

for $-t \leqq s \leqq t \leqq 1$ with $t \geqq 0$. This is what we have to show.

First let us consider the special cases $t = 1$ and $t = s$. The case $t = 1$ is easy and clear. In the case $t = s$ and $\alpha \geqq 0$ we argue that, because of $1 \geqq (1 - t)^{\alpha+1} + t^{\alpha+1}$, $(1 - t)^{\alpha+1} \leqq 1 - t^{\alpha-1}$ holds, i.e. $(1 - t)(1 - t^{\alpha+1}) \geqq (1 - t)^{\alpha+2}$. For $0 \leqq \varepsilon < 1$ we have $(1 - t^\varepsilon)/(1 - t) \geqq \varepsilon$, as the difference quotient equals $f'(t_0)$ for $f(t) = t^\varepsilon$ and some $t_0 \leqq 1$. Because of the monotonicity of f' then $f'(t_0) \geqq f'(1) = \varepsilon$. Then, in the case $t = s$ and $-1 < \alpha \leqq 0$,

$$(1 + t^{-\alpha})(1 - t^{\alpha+1})/(1 - t) \geqq (1 - t^{\alpha+1})/(1 - t) \geqq \alpha + 1 = K > 0.$$

Now we are going to show (1.3) in case $t \neq s$ and $t \neq 1$. Because of $(1 - t)^{\alpha+1} \leqq 1 - t^{\alpha+1}$ we have for $\alpha \geqq 0$

$$\begin{aligned} &\big((1 - t)^2 + 2(t - s)\big)^{\alpha/2+1}/[(1 - t)(1 - t^{\alpha+1}) + (t^\alpha + 1)(t - s)] \\ &\leqq 2^{\alpha/2}[(1 - t)^{\alpha+2} + 2^{\alpha/2+1}(t - s)^{\alpha/2+1}]/[(1 - t)(1 - t^{\alpha+1}) + (t^\alpha + 1)(t - s)] \\ &\leqq 2^{\alpha/2}[(1 - t)^{\alpha+2}/(1 - t)(1 - t^{\alpha+1}) + 2^{\alpha/2+1}(t - s)^{\alpha/2+1}/(t^\alpha + 1)(t - s)] \\ &\leqq 2^{\alpha/2}[1 + 2^{\alpha/2+1}\,2^{\alpha/2}] = 1/K. \end{aligned}$$

At last, for $-1 < \alpha \leqq 0$, using $(1 - t^\varepsilon)/(1 - t) \geqq \varepsilon$ for $0 \leqq \varepsilon < 1$, we have

$$\begin{aligned}
&[(1-t)^2 + 2(t-s)]/(1-t^{-\alpha})\,[(1-t)(1-t^{\alpha+1}) + (t^\alpha+1)(t-s)] \\
&\leqq (1-t)^2/(1+t^{-\alpha})\,(1-t)\,(1-t^{\alpha+1}) \\
&\quad + 2(t-s)/(1+t^{-\alpha})\,(t^\alpha+1)(t-s) \\
&= (1-t)/(1+t^{-\alpha})\,(1-t^{\alpha+1}) + 2/(1+t^{-\alpha})\,(t^\alpha+1) \\
&\leqq 1/(\alpha+1) + 2 = 1/K.
\end{aligned}$$

The proof of (1.1) is complete.

As to (1.2), we first observe that the inverse of the operator $T_\alpha u = |u|^\alpha u$ is just the operator $T_\alpha^{-1}u = T_{-\alpha/(\alpha+1)}u = |u|^{-\alpha/(\alpha+1)}u = T_\beta u$, and $\alpha \geqq 0$ iff $-1 < \beta \leqq 0$, $-1 < \alpha \leqq 0$ iff $\beta \geqq 0$ hold. This way, to prove (1.2) for T_α we use (1.1) for $T_\alpha^{-1} = T_\beta$. Let $u_1 = T_\alpha u$ and $v_1 = T_\alpha v$, i.e. $u = T_\beta u_1$ and $v = T_\beta v_1$. Then we have

$$|T_\beta u_1 - T_\beta v_1|\,|u_1 - v_1| \geqq (T_\beta u_1 - T_\beta v_1, u_1 - v_1)$$

$$\geqq \begin{cases} K(\beta)\,(|u_1|^{-\beta} + |v_1|^{-\beta})^{-1}\,|u_1 - v_1|^2 & \text{for } \alpha \geqq 0 \\ K(\beta)\,|u_1 - v_1|^{\beta+2} = K(\beta)\,|u_1 - v_1|^{1+1/(\alpha+1)} & \text{for } -1 < \alpha \leqq 0, \end{cases}$$

$$|T_\alpha u - T_\alpha v| = |u_1 - v_1| \leqq \begin{cases} |u - v|\,K(\beta)^{-1}\,(|u_1|^{-\beta} + |v_1|^{-\beta}) \\ (|u - v|\,K(\beta)^{-1})^{\alpha+1}. \end{cases}$$

This is (1.2), taking into account that $|u_1|^{-\beta} = |T_\alpha u|^{-\beta} = \big||u|^\alpha u\big|^{\alpha/(\alpha+1)} = |u|^\alpha$ holds ∎

Using the boundary conditions (3) and partial integration we come to the weak formulation of the problem (1). The function $[u_1, \ldots, u_M, \Psi] \in W^M \times W_r(G)$ is called *solution* of our problem if

$$\int_G D_i \nabla u_i \nabla w_i\,dG + \int_G k_i z_i u_i U_0 \nabla\Psi \nabla w_i\,dG + \int_G f_i w_i\,dG = M, \quad i = 1, \ldots, 0, \tag{1.4}$$

and

$$\int_G D_0 \nabla\Psi \nabla w_0\,dG = \int_G k(N + u_0)\,w_0\,dG \tag{1.5}$$

hold for all $[w_1, \ldots, w_M, w_0] \in W^M \times W_r(G)$. If $D_0 = |\nabla\Psi|^{r-2}$, the left-hand side of (1.5) generates a uniformly monotonous, coercive operator on $W_r(G)$. Hence (1.5) is uniquely solvable for fixed N and u_0. Since this solution (for generally fixed N) only depends on $u = [u_1, \ldots, u_M]$ we denote it by Su. Substituting this into (1.4), we obtain the problem: Find $u = [u_1, \ldots, u_M) \in W^M$ such that

$$\int_G D_i \nabla u_i \nabla w_i\,dG + \int_G k_i z_i u_i U_0 \nabla Su \nabla w_i\,dG + \int_G f_i w_i\,dG = 0$$

hold for $i = 1, \ldots, M$. In the sequel we will transform this problem into the operator equation $Au + Bu = f$ in $(W^M)^*$ and investigate the properties of the operators A and B.

Now, let $\langle\cdot,\cdot\rangle$ be the dual pairing between W^* and $W = W_p(G)$. We generate operators A_i and B_i mapping W^M into W^*:

$$\langle A_i v, w\rangle = \int_G D_i \nabla v_i \nabla w\,dG,$$

$$\langle B_i v, w\rangle = z_i k_i \int_G v_i (a + b v_0^2)^{-1/2} \nabla Sv \nabla w\,dG.$$

Under suitable conditions on D_i, the operator A_i will be uniformly monotonous (cf. (8) and [3]):

$$\langle A_i u - A_i v, u_i - v_i\rangle \geqq c_i \|u_i - v_i\|^p, \qquad c_i > 0. \tag{1.6}$$

Now, after proving some estimates for $\|B_i u - B_i v\|_{W^*}$ and $\|B_i u\|_{W^*}$ we will show that B_i is increasing continuous. First we estimate $b_i = |\langle B_i u - B_i v, w\rangle|$. We have

$$b_i \leqq |k_i| \int_G |u_i U_0 \nabla Su - v_i V_0 \nabla Sv| \, |\nabla w| \, dG,$$

where $U_0 = (a + bu_0^2)^{-1/2}$ and $V_0 = (a + bv_0^2)^{-1/2}$. As the derivative of the function $(a + bx^2)^{-1/2}$ is bounded we have $|U_0 - V_0| \leqq k(a, b)\,|u_0 - v_0|$. We calculate

$$\begin{aligned}&|u_i U_0 \nabla Su - v_i V_0 \nabla Sv|\\ &\leqq |u_i U_0(\nabla Su - \nabla Sv) + u_i(U_0 - V_0)\nabla Sv + (u_i - v_i) V_0 \nabla Sv|\\ &\leqq a^{-1/2}(|u_i|\,|\nabla Su - \nabla Sv| + |u_i - v_i|\,|\nabla Sv|) + |u_i|\,|\nabla Sv|\,|u_0 - v_0|\,k(a, b).\end{aligned}$$

Then,

$$\begin{aligned}b_i \leqq k_i^1 \Big(&\int_G |u_i|\,|\nabla Su - \nabla Sv|\,|\nabla w|\,dG\\ &+ \int_G |u_i|\,|\nabla Sv|\,|u_0 - v_0|\,|\nabla w|\,dG + \int_G |u_i - v_i|\,|\nabla Sv|\,|\nabla w|\,dG\Big),\end{aligned} \tag{1.7}$$

where k_i^1 is a positive constant not depending on u and v. For $p > n$, as W is imbedded into C, we get by the Hölder inequality, putting $q = p/(p - 1)$ and denoting the L_q-norm by $\|\cdot\|_q$, the C-norm by $|\cdot|_C$,

$$\begin{aligned}b_i \leqq k_i^1(&|u_i|_C \|\nabla Su - \nabla Sv\|_q + |u_i|_C |u_0 - v_0|_C \|\nabla Sv\|_q\\ &+ |u_i - v_i|_C \|\nabla Sv\|_q) \|\nabla w\|_p.\end{aligned}$$

Next, we estimate $\|\nabla Su - \nabla Sv\|_q$ and $\|\nabla Sv\|_q$. We have

$$\nabla\cdot(|\nabla Su|^{r-2}\nabla Su) = k(N + u_0) \quad \text{and} \quad \nabla\cdot(|\nabla Sv|^{r-2}\nabla Sv) = k(N + v_0).$$

From this we get by subtraction and partial integration for $w \in W_r(G)$

$$\int_G (|\nabla Su|^{r-2}\nabla Su - |\nabla Sv|^{r-2}\nabla Sv)\,\nabla w\,dG = -k\int_G (u_0 - v_0)\,w\,dG.$$

For $s, t \in \mathbf{R}_n$ and $r \geqq 2$ the Lemma gives $(s\,|s|^{r-2} - t\,|t|^{r-2})(s - t) \geqq \text{const}\,|s - t|^r$. Thus, if we take $w = Su - Sv \in W_r(G)$ then we get

$$\begin{aligned}&\text{const}\int_G |\nabla Su - \nabla Sv|^r\,dG\\ &\leqq \int_G (|\nabla Su|^{r-2}\nabla Su - |\nabla Sv|^{r-2}\nabla Sv)(\nabla Su - \nabla Sv)\,dG\\ &\leqq \int_G |u_0 - v_0|\,|Su - Sv|\,dG \leqq \text{const}\,|u_0 - v_0|_C \|\nabla Su - \nabla Sv\|_r\end{aligned}$$

as $W_r \subset W_r^1 \subset L_r \subset L_1$ with continuous imbeddings. We get

$$\|\nabla Su - \nabla Sv\|_r^r \leqq \text{const}\,|u_0 - v_0|_C \|\nabla Su - \nabla Sv\|_r,$$

i.e. $\|\nabla Su - \nabla Sv\|_r \leqq \text{const}\,|u_0 - v_0|_C^{1/(r-1)}$. As $q \leqq r$ we also have

$$\|\nabla Su - \nabla Sv\|_q \leqq \text{const}\,|u_0 - v_0|_C^{1/(r-1)}. \tag{1.8}$$

By multiplication with Sv and partial integration the equation $\nabla \cdot (|\nabla Sv|^{r-2} \nabla Sv) = k(N + v_0)$ gives

$$\int_G |\nabla Sv|^{r-2} \nabla Sv \nabla Sv \, dG = -k \int_G (N + v_0) \, Sv \, dG,$$

i.e. $\|\nabla Sv\|_r^r \leqq k \|N + v_0\|_{r'} \|Sv\|_r$. From $\|Sv\|_r \leqq \text{const} \|\nabla Sv\|_r$ we get as before

$$\|\nabla Sv\|_q \leqq \text{const} \|N + v_0\|_{r'}^{1/(r-1)}. \tag{1.9}$$

Now, taking into account that $\|\nabla w\|_p = \|w\|_{0,p}$, (1.7) and (1.8) yield

$$\|B_i u - B_i v\|_{W^*} \leqq \text{const} \left(|u_i|_C \, |u_0 - v_0|_C{}^{1/(r-1)} + |u_i|_C \, |u_0 - v_0|_C \, \|\nabla Sv\|_q \right.$$
$$\left. + |u_i - v_i|_C \, \|\nabla Sv\|_q \right). \tag{1.10}$$

In the same way as in [3] we are now able to show that B_i is increasing continuous. Indeed, let $u^j \rightharpoonup u$ in W^M, i.e. $u_i^j \rightharpoonup u_i$ and $u_0^j \rightharpoonup u_0$ in W. Then, because of the compact imbedding $W \subset C$, $u_i^j \to u_i$ and $u_0^j \to u_0$ in C, and (1.10) gives us $\|B_i u^j - B_i u\|_{W^*} \to 0$ since $|u_i^j|_C$ is bounded.

Let us now define operators $A, B \colon W^M \to (W^M)^*$ as

$$\langle Au, w \rangle_M = \sum_1^M \langle A_i u, w_i \rangle \quad \text{and} \quad \langle Bu, w \rangle_M = \sum_1^M \langle B_i u, w_i \rangle,$$

where $\langle \cdot, \cdot \rangle_M$ is the dual pairing between $(W^M)^*$ and W^M. Then, if we define $\|u\|_{W^M}^2 = \|u_1\|_{0,p}^2 + \cdots + \|u_M\|_{0,p}^2$, from (1.6) we get

$$\langle Au - Av, u - v \rangle_M \geqq c \sum_1^M \|u_i - v_i\|_{0,p}^p \geqq \bar{c} \, \|u - v\|_{W^M}^p. \tag{1.11}$$

Furthermore, the operator B is increasing continuous because the B_i are. Indeed, we have for all $w \in W^M$

$$|\langle Bu - Bv, w \rangle_M| \leqq \sum_1^M |\langle B_i u - B_i v, w_i \rangle| \leqq \sum_1^M \|B_i u - B_i v\|_{W^*} \|w_i\|_W$$
$$\leqq \left(\sum_1^M \|B_i u - B_i v\|_{W^*}^2 \right)^{1/2} \|w\|_{W^M},$$

i.e.

$$\|Bu - Bv\|_{(W^M)^*} \leqq \left(\sum_1^M \|B_i u - B_i v\|_{W^*}^2 \right)^{1/2}.$$

Besides, we need an estimate for $|\langle Bv, v \rangle_M|$. The Hölder inequality gives

$$|\langle B_i v, v_i \rangle| = |k_i| \left| \int_G v_i (a + b v_0^2)^{-1/2} \nabla Sv \, \nabla v_i \, dG \right| \leqq \text{const} \, |v_i|_C \, \|\nabla Sv\|_q \, \|v_i\|_W.$$

From (1.9) and $|v_i|_C \leqq \text{const} \|v_i\|_{0,p}$ we then obtain

$$|\langle B_i v, v_i \rangle| \leqq \text{const} \|N + v_0\|_{r'}^{1/(r-1)} \|v_i\|_{0,p}^2.$$

This gives

$$|\langle Bv, v \rangle|_M \leqq \sum_1^M |\langle B_i v, v_i \rangle| \leqq \text{const} \|N + v_0\|_{r'}^{1/(r-1)} \|v\|_{W^M}^2. \tag{1.12}$$

Furthermore, we can estimate

$$\|N + v_0\|_{r'}^{1/(r-1)} \leqq \text{const}\,(\|N\|_{r'}^{1/(r-1)} + \|v\|_{W^M}^{1/(r-1)}) \tag{1.13}$$

since

$$\|v_0\|_{r'} \leqq \text{const}\,\|v_0\|_{0,p} \leqq \text{const}\,\sum_1^M \|v_i\|_{0,p} \leqq \text{const}\,\|v\|_{W^M}.$$

Now we are ready to prove the coerciveness of $T = A + B$. Because of (1.11), (1.12) and $A0 = 0$ we have

$$\langle Tv, v\rangle_M = \langle Av, v\rangle_M + \langle Bv, v\rangle_M \geqq \text{const}\,\|v\|_{W^M}^p - \text{const}\,\|N + v_0\|_{r'}^{1/(r-1)}\,\|v\|_{W^M}^2,$$

and with (1.13)

$$\langle Tv, v\rangle_M/\|v\|_M = \text{const}\,\|v\|_M{}^{p-1} - \text{const}\,\|v\|_M\,(\text{const} + \|v\|_M{}^{1/(r-1)}),$$

converging to ∞ if $\|v\|_M \to \infty$, provided $1 + 1/(r-1) < p - 1$, i.e. $r > (p-1)/(p-2)$.

This way, if $p > n$ and $r > (p-1)/(p-2)$, T is a coercive and pseudomonotone operator (T is the sum of a uniformly monotone, continuous operator and an increasing continuous one). Consequently, by a standard theorem of monotonicity theory the problem $Tu = 0$ has a solution $u \in W^M$, taking into account that T is also bounded. We have obtained

Theorem 1: *Let be $p, r \geqq 2$, $p > n$, $r > (p-1)/(p-2)$ and let our assumptions for D_i, $i = 1, \ldots, M$, and D_0 be fulfilled. Then the homogeneous boundary value problem* (1), (3) *has at least one solution.*

2. The inhomogeneous problem

In this section we seek solutions $[v_1, \ldots, v_M, \Phi]$ of the inhomogeneous boundary value problem (1), (2). We suppose that there exist elements $y_i \in W_p{}^1(G)$, $i = 1, \ldots, M$, and $y_0 \in W_r{}^1(G)$ such that

$$y_i|_{\partial G_1} = \underline{y}_i, \qquad i = 1, \ldots, M, \quad \text{and} \quad y_0|_{\partial G_1} = \underline{y}_0. \tag{2.1}$$

Then, for $u_i = v_i - y_i$ and $\Psi = \Phi - y_0$ we have the homogeneous conditions (3). Now, we can reduce the inhomogeneous problem to the homogeneous one considered in Section 1. To derive the corresponding operator equation we substitute $v_i = u_i + y_i$, $i = 1, \ldots, M$, and $\Phi = \Psi + y_0$ into the original differential equations (1), and we consider these equations in the variables u_i and Ψ. Then we again generate operator equations in $W_p(G)$ and $W_r(G)$. Besides the homogeneous boundary conditions after the substitution we obtain the differential equations

$$\left.\begin{aligned} &\nabla\cdot\big(D_i(|\nabla(u_i + y_i)|)\,\nabla(u_i + y_i)\big) + \nabla\cdot\big(k_i z_i (u_i + y_i)\,V_0\,\nabla(\Psi + y_0)\big) = f_i \\ &-\nabla\cdot\big(D_0(|\nabla(\Psi + y_0)|)\,\nabla(\Psi + y_0)\big) = k(N + v_0), \qquad i = 1, \ldots, M \end{aligned}\right\} \tag{2.2}$$

where $V_0 = (a + bv_0{}^2)^{-1/2}$ with $v_0 = z_1(u_1 + y_1) + \cdots + z_M(u_M + y_M)$.

Now, let us generate corresponding operators E_i and F_i. We put

$$(F_i u_i, w) = I_i{}^F(u_i, w), \qquad u_i \in W, \quad w \in W^M$$

$$\big(E_i(u_i, \Psi), w\big) = I_i{}^E(u_i, \Psi, w), \qquad \Psi \in W_r(G)$$

$$(F_0\Psi, w_0) = I^F(\Psi, w_0), \qquad \Psi, w_0 \in W_r(G)$$

where

$$I_i^F(u_i, w) = \int_G D_i\big(|\nabla(u_i + y_i)|\big)\, \nabla(u_i + y_i)\, \nabla w\, dG,$$

$$I_i^E(u_i, \Psi, w) = \int_G k_i z_i(u_i + y_i)\, V_0\, \nabla(\Psi + y_0)\, \nabla w\, dG,$$

$$I^F(\Psi, w_0) = \int_G D_0\big(|\nabla(\Psi + y_0)|\big)\, \nabla(\Psi + y_0)\, \nabla w_0\, dG.$$

We have $|I_i^F(u_i, w)| \leqq d_{i1}\|\nabla(u_i + y_i)\|_p^{p-1}\, \|w\|_{0,p}$. Consequently, there exists an $F_i \in \big(W \to (W^M)^*\big)$ with

$$(F_i u_i, w) = I_i^F(u_i, w) \quad \text{and} \quad \|F_i u_i\| \leqq d_{i1}\, 2^{p-2}(\|u_i\|_{0,p}^{p-1} + \|y_i\|_{1,p}^{p-1}).$$

Further, there exists an $E_0 \in \big(W_r(G) \to (W_r(G))^*\big)$ with

$$(F_0 \Psi, w_0) = I^F(\Psi, w_0) \quad \text{and} \quad \|F_0 \Psi\| \leqq 2^{p-2}(\|\Psi\|_{0,p}^{p-1} + \|y_0\|_{1,p}^{p-1}).$$

At last, we justify $\big(E_i(u_i, \Psi_0), w\big) = I_i^E(u_0, \Psi, w)$. We obtain, with $1/p + 1/p' = 1$,

$$\begin{aligned} |I_i^E(u_i, \Psi, w)| &\leqq a^{-1/2}\, |u_i + y_i|_C\, \|\nabla(\Psi + y_0)\|_{p'}\, \|w\|_{0,p} \\ &\leqq a^{-1/2}(|u_i|_C + |y_i|_C)\, (\|\Psi\|_{0,p'} + \|y_0\|_{1,p'})\, \|w\|_{0,p}. \end{aligned}$$

Next, we investigate the properties of the operators F_i, F_0 and E_i. Under the assumptions (4) and (2.1) F_i is uniformly monotonous with $\delta_i(t) = g_i t^p$. This follows from the fact that $F_i(u_i)$ can be understood as $A_i(u_i + y_i)$, because A_i can be generated also on elements $u_i + y_i$ as an operator in u_i on $W_p(G)$. The uniform monotonicity of F_i then results from the uniform monotonicity of A_i (cf. (1.6)). Regarding F_0 we can proceed in an analogous way. Then for $u \in W^M$ there exists a unique solution (again denoted by) Su of the problem (2.2) (now with v_0 instead of u_0) under the boundary conditions (3) at Ψ. Obviously, S is an increasing continuous mapping. We are going to show that, after substituting Su, the operator $E_i(u) = E_i(u_i, \nabla Su)$ is increasing continuous, too. The inequality

$$\begin{aligned} &|\langle E_i u^1 - E_i u^2, w\rangle| \\ &\leqq k_i \left(\int_G |(u_i^1 + y_i)\, V_0^1\, \nabla Su^1 - (u_i^2 + y_i)\, V_0^2\, \nabla Su^2|\, |\nabla w|\, dG\right) \\ &\leqq k_i^1 \left(\int_G |u_i^1 + y_0|\, |\nabla(Su^1 - Su^2)|\, |\nabla w|\, dG\right. \\ &\qquad \left. + \int_G |u_i^1 + y_i|\, |\nabla Su^2|\, |v_0^1 - v_0^2|\, dG + \int_G |u_i^1 - u_i^2|\, |\nabla Su^2|\, |\nabla w|\, dG\right) \end{aligned}$$

holds true, where k_i^1 is a constant and $v_0^j = v_0(u^j)$, $V_0^j = \big(a + b(v_0^j)^2\big)^{-1/2}$, $j = 1, 2$. The increasing continuity then follows in the same way as in Section 1. The solution of the inhomogeneous boundary value problem is equivalent to the solution of the operator equation $Fu + Eu = f$, where $F = [F_1, \ldots, F_M]$ is a uniformly monotonous operator and $E = [E_1, \ldots, E_M]$ is an increasing continuous one.

Theorem 2: *Under the formulated assumptions, especially* (4) *on* D_i, (2.1) *on* y_i *and* $r > 2$, $p > (2r - 1)/(r - 1)$, *the boundary value problem* (1), (2) *has at least one solution. The solution set is strongly compact and weakly closed and its diameter can be estimated from above by a concrete finite number.*

Proof: The operator $F + E$ is bounded, coercive and pseudomonotone. The coerciveness follows from the uniform monotonicity of F with $\delta(t) = gt^p$, $p > (2r - 1)/(r - 1)$, and the fact that (Eu, u) must increase to infinity at least as fast as $\|u\|_{0,p}^{(2r-1)/(r-1)}$. The solution set is weakly compact and weakly closed (cf. e.g. [6]). $F + E$ satisfies the S_+-condition, i.e. especially that every weakly (to a solution) converging sequence of solutions also converges strongly. From this, the strong compactness of the solution set follows.

Now to the last assertion. Because of the coerciveness of the mapping $F + G$ and the resulting a priori boundedness we have for $v^1, v^2 \in W^M$ and $r > 2$:

$$(Fv^1 - Fv^2, v^1 - v^2) + (Ev^1 - Ev^2, v^1 - v^2)$$
$$\geqq g\,\|v^1 - v^2\|_{0,p}^p - c_1\,\|v^1 - v^2\|_{0,p}^2 - c_2\,\|v^1 - v^2\|_{0,p}^{r/(r-1)}$$
$$= \|v^1 - v^2\|_{0,p}^{r/(r-1)} \left(\|v^1 - v^2\|_{0,p}^{(r-2)/(r-1)} (g\|v^1 - v^2\|_{0,p}^{p-2} - c_1) - c_2\right)$$
$$\geqq \varepsilon_2\|v^1 - v^2\|_{0,p}^{r/(r-1)}$$

if

$$\|v^1 - v^2\|_{0,p} \geqq \max\{[(c_1 + \varepsilon_1)/g]^{1/(p-2)}, [(c_2 + \varepsilon_2)/\varepsilon_1]^{(r-1)/(r-2)}\} = K(\varepsilon_1, \varepsilon_2)$$

is assumed with arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Hence for solutions v^1, v^2 we can conclude $\|v^1 - v^2\|_{0,p} \leqq K(\varepsilon_1, \varepsilon_2)$ ∎

Remark: The constants c_1 and c_2 in the proof of Theorem 2 enclose e.g. the imbedding constant of $W_p^1(G)$ into $C(\bar{G})$.

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