Zeitschrift für Analysis
und ihre Anwendungen
Bd. 6 (2) 1987, S. 183–187

On the Convergence of Some Random Series

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Es seien $\{X_n\}$ eine Folge unabhängiger Zufallsgrößen mit Werten in einem separablen Banachraum, $S_n = X_1 + \cdots + X_n$ und $\{a_n\}$ eine Folge nichtnegativer Zahlen. Es wird ein Kriterium für die fast sichere Konvergenz der Reihe $\sum a_n S_n$ bewiesen. Insbesondere wird der Spezialfall $a_n = n^{-\alpha}$, $\alpha > 0$, untersucht.

Пусть $\{X_n\}$ последовательность независимых случайных элементов со значениями в сепарабельном банаховом пространстве, $S_n = X_1 + \cdots + X_n$ и $\{a_n\}$ последовательность неотрицательных чисел. Доказывается критерий для сходимости почти наверное ряда $\sum a_n S_n$. В частности, рассматривается случай $a_n = n^{-\alpha}$, $\alpha > 0$.

Let $\{X_n\}$ be a sequence of independent random elements with values in a separable Banach space, $S_n = X_1 + \cdots + X_n$ and $\{a_n\}$ a sequence of nonnegative numbers. There is proved a criterion for the almost sure convergence of the series $\sum a_n S_n$. The special case $a_n = n^{-\alpha}$, $\alpha > 0$, is also studied.

1. Introduction

Throughout the paper $\{X_n, n \geqq 1\}$ is a sequence of independent random elements with values in a separable Banach space $(B, \|\cdot\|)$ and $\{a_n, n \geqq 1\}$ is a sequence of nonnegative numbers. Assume $a_n > 0$ infinitely often. Put

$$b_j = \sum_{i=j}^{\infty} a_i, \quad b_{n,j} = \sum_{i=j}^{n} a_i, \quad S_n = \sum_{i=1}^{n} X_i \quad \text{for } j \geqq 1, \ n \geqq j.$$

In this note we study the a.s. (almost sure) behaviour of the series $\sum a_n S_n$. Such a series has been considered in connection with problems of learning theory (cf. [2]), but its a.s. convergence is also a measure of the rate of convergence in the strong law of large numbers $\lim a_n S_n = 0$ a.s. Koopmans et al. [2] considered the series $\sum n^{-\alpha-1} S_n$ for sequences of real-valued X_n and positive α. They showed that this series diverges a.s. for each sequence of independent identically distributed X_n if $\alpha < 1/2$. Gaposhkin [6] obtained the following result: Assume $\sum a_n < \infty$. Then for each sequence of independent symmetric X_n the series $\sum a_n S_n$ converges a.s. iff the series $\sum b_n X_n$ converges a.s. He also showed that $\sum n^{-\alpha-1} S_n$ diverges a.s. for each sequence of independent and identically distributed real-valued X_n if $\alpha \leqq 1/2$. In this note we prove a criterion for the a.s. convergence of the series $\sum a_n S_n$. Furthermore, the results of Koopmans et al. and Gaposhkin concerning the a.s. divergence of $\sum n^{-\alpha-1} S_n$ are generalized to the Banach space setting.

2. Auxiliary results

The lemmas of this section are fundamental for the proofs of the results in Section 3.

Lemma 1 (Martikainen [7: Lemma 1]): *Let $\{Y_n, n \geqq 1\}$, $\{Z_n, n \geqq 1\}$ be two sequences of B-valued random elements such that the distributions of $(Z_1, \ldots, Z_n, Y_n)$ and $(Z_1, \ldots, Z_n, -Y_n)$ are identical for $n \geqq 1$. If $\lim (Y_n + Z_n) = 0$ a.s. then $\lim Y_n = 0$ a.s.*

Lemma 2: *Suppose that one of the X_i is nondegenerate. If $\sum a_n S_n$ converges a.s. then $\sum a_n < \infty$.*

Proof: If $\sum a_n S_n$ converges a.s. then $\sum a_n S_n^s$ converges a.s., where $S_n^s = X_1^s + \cdots + X_n^s$, $n \geqq 1$, and $\{X_n^s, n \geqq 1\}$ is a symmetrized version of $\{X_n, n \geqq 1\}$. Let X_{i_0} be nondegenerate. Put

$$Y_n = b_{n,i_0} X_{i_0}, \qquad Z_n = \sum_{k=1}^{n} a_k S_k^s - Y_n, \quad \text{for } n \geqq i_0.$$

Then

$$\lim \sum_{k=1}^{n} a_k S_k^s = \lim (Y_n + Z_n) = Y \quad \text{a.s.}$$

for some B-valued random element Y. Fix $\varepsilon > 0$. By Lévy's inequality,

$$2P(\|Y_n + Z_n\| > \varepsilon) \geqq P(\|Y_n\| > \varepsilon) \geqq P(\|Y_n\| > \varepsilon, X_{i_0}^s \neq 0).$$

Since X_{i_0} is nondegenerate we have $P(X_{i_0}^s \neq 0) > 0$. Assume $\sum a_n = \infty$. Then $\lim \|Y_n\| = \infty$ on $\{X_{i_0}^s \neq 0\}$. Hence $4P(\|Y_n + Z_n\| > \varepsilon) \geqq P(X_{i_0}^s \neq 0)$ for sufficiently large n. Thus $4P(\|Y\| > \varepsilon) \geqq P(X_{i_0}^s \neq 0) > 0$. Since $\varepsilon > 0$ is arbitrary we have $P(\|Y\| = \infty) > 0$ in contradiction to the a.s. convergence of $\sum a_n S_n$. Hence $\sum a_n < \infty$ ∎

Lemma 3: *Suppose that $\sum a_n < \infty$ and $\sum a_n S_n$ converges a.s. Then there exists a sequence $\{c_n, n \geqq 1\}$ of constants such that*

$$\lim b_{n+1}(S_n - c_n) = 0 \quad a.s. \tag{1}$$

If the X_n, $n \geqq 1$, are symmetric or $b_{n+1} S_n \xrightarrow{P} 0$ as $n \to \infty$, then (1) holds with $c_n = 0$.

Proof: Assume that the X_n are symmetric. We have

$$\sum_{i=1}^{n} a_i S_i = \sum_{i=1}^{n} b_{n,i} X_i = \sum_{i=1}^{n} b_i X_i - b_{n+1} S_n. \tag{2}$$

Thus

$$\sum_{i=1}^{n} a_i S_i - \sum_{i=1}^{m} a_i S_i = b_{n,m+1} S_m + \sum_{i=m+1}^{n} b_{n,i} X_i$$

for $m \leqq n$. Hence

$$\lim_{m\to\infty} \left(b_{m+1} S_m + \lim_{n\to\infty} \sum_{i=m+1}^{n} b_{n,i} X_i \right) = 0 \quad \text{a.s.}$$

The a.s. existence of $Y_m = \lim_n (b_{n,m+1} X_{m+1} + \cdots + b_{n,n} X_n)$ is a consequence of (2) and the a.s. convergence of $\sum a_n S_n$. Let $Z_m = b_{m+1} S_m$. Then $\lim (Y_m + Z_m) = 0$ a.s. Apply Lemma 1 to obtain (1) with $c_n = 0$.

Now, let the X_n, $n \geqq 1$, be not necessarily symmetric. Applying the first part of the proof to a symmetrized version $\{X_n^s, n \geqq 1\}$, we obtain (1) for this version and for $\{X_n, n \geqq 1\}$ it follows by standard desymmetrization arguments. Furthermore, if (1) holds and $b_{n+1}S_n \xrightarrow{P} 0$ as $n \to \infty$, then $b_{n+1}c_n \to 0$ as $n \to \infty$ ∎

Lemma 4: *Assume that $\sum b_n X_n$ converges a.s. Then $\sum a_n S_n$ converges a.s. and $\lim b_n S_n = 0$ a.s.*

This lemma follows from (2), the a.s. convergence of $\sum b_n X_n$ and Kronecker's lemma. It is not difficult to see that Lemma 4 remains true for not necessarily independent random elements.

3. Main results

In this section we study the a.s. behaviour of $\sum a_n S_n$. In general, there is no $0-1$ law for the convergence of this series. Indeed, let X_1 be nondegenerate, $X_2 = X_3 = \cdots = 0$ and $\sum a_n = \infty$. Then $\sum a_n S_n$ converges on $\{X_1 = 0\}$ and diverges on $\{X_1 \neq 0\}$. But if $\{X_n, n \geqq 1\}$ is independent and identically distributed then the Hewitt-Savage $0-1$ law is applicable to $\sum a_n S_n$. The following theorem characterizes the a.s. convergence of this series.

Theorem 1: *Suppose that one of the X_i is nondegenerate. The following statements are equivalent:*

(A) *The series $\sum a_n S_n$ converges a.s.*

(B) *$\sum a_n < \infty$ and there are constants d_n, $n \geqq 1$, such that*

$$\lim_{n\to\infty} \left(b_{n+1}S_n - \sum_{i=1}^{n} b_i d_i\right) = 0 \quad a.s. \tag{3}$$

and the series $\sum b_i(X_i - d_i)$ converges a.s.

(C) *$\sum a_n < \infty$, $\lim a_n S_n = 0$ a.s. and there are constants d_n, $n \geqq 1$, such that*

$$\lim_{n\to\infty} \sum_{i=1}^{n} (b_i - b_n)\, d_i = 0 \tag{4}$$

and $\sum b_i(X_i - d_i)$ converges a.s.

Proof: (A) ⇒ (B), (C): By Lemma 2 we have $\sum a_n < \infty$. By Lemma 3 there is a sequence $\{c_n, n \geqq 1\}$ such that $\lim b_{n+1}(S_n - c_n) = 0$ a.s. Define d_n by $b_{n+1}c_n = b_1 d_1 + \cdots + b_n d_n$, $n \geqq 1$. Then $\lim b_{n+1}(S_n - c_n) = 0$ a.s. implies (3). By (2) we have

$$V_n = \sum_{i=1}^{n} a_i S_i = \left\{\sum_{i=1}^{n} b_i X_i - b_{n+1}c_n\right\} - b_{n+1}(S_n - c_n)$$

$$= \sum_{i=1}^{n} b_i(X_i - d_i) - \left\{b_{n+1}S_n - \sum_{i=1}^{n} b_i d_i\right\}.$$

Let $T_n = b_1(X_1 - d_1) + \cdots + b_n(X_n - d_n)$. Using (3), we have $\lim (V_n - T_n) = 0$ a.s. Thus, by (A), the sequence $\{T_n, n \geqq 1\}$ converges a.s. The relation $\lim a_n S_n = 0$ a.s. is an immediate consequence of (A). By Kronecker's lemma and the a.s. convergence of $\{T_n, n \geqq 1\}$

$$\lim b_n \left(S_n - \sum_{i=1}^{n} d_i\right) = 0 \text{ a.s.} \tag{5}$$

By (5), (3) and $\lim a_nS_n = 0$ a.s.,

$$\lim_{n\to\infty} \sum_{i=1}^{n} (b_i - b_n)\, d_i = \lim_{n\to\infty} \left\{ b_n \left(S_n - \sum_{i=1}^{n} d_i\right) + (-b_nS_n + a_nS_n) + \sum_{i=1}^{n} b_id_i \right\}$$

$$= \lim_{n\to\infty} b_n \left(S_n - \sum_{i=1}^{n} d_i\right) - \lim_{n\to\infty} \left(b_{n+1}S_n - \sum_{i=1}^{n} b_id_i\right) = 0 \quad \text{a.s.,}$$

which implies (4). Thus (B) and (C) follow from (A).

(C) $\Rightarrow$ (B): The relations (5), (4), $\lim a_nS_n = 0$ a.s. and

$$b_{n+1}S_n - \sum_{i=1}^{n} b_id_i = -a_nS_n + b_n \left(S_n - \sum_{i=1}^{n} d_i\right) + \sum_{i=1}^{n} (b_n - b_i)\, d_i$$

imply (B). (B) $\Rightarrow$ (A): This follows from $\lim (V_n - T_n) = 0$ a.s. ∎

The following theorem is more practicable than Theorem 1. In case $B = \mathbf{R}$, Gaposhkin's criterion on the a.s. convergence of $\sum a_nS_n$ (see Section 1 and [5]) is an immediate corollary of

Theorem 2: *Assume $\sum a_n < \infty$. Suppose that $b_{n+1}S_n \xrightarrow{P} 0$ as $n \to \infty$, or the X_n, $n \geqq 1$, are symmetric. Then $\sum b_nX_n$ converges a.s. if and only if $\sum a_nS_n$ converges a.s. If $\sum b_nX_n$ converges a.s. then $\sum a_nS_n = \sum b_nX_n$ a.s.*

Proof: By Lemma 4 and (2) it is sufficient to prove that the a.s. convergence of $\sum a_nS_n$ implies that $\lim b_{n+1}S_n = 0$ a.s. But this follows from Lemma 3 and the assumptions of the theorem ∎

By the same methods we obtain the following: Assume $\sum a_n < \infty$. Then $\sum b_nX_n$ converges a.s. iff $\sum a_nS_n$ converges a.s. and $b_{n+1}S_n \xrightarrow{P} 0$ as $n \to \infty$. In this way, the problem of a.s. convergence of $\sum a_nS_n$ can be reduced to the problem of convergence of $\sum b_nX_n$. Conditions for the convergence of this series are well known. In Banach spaces they depend on the geometry of B (cf. [3, 4]).

The results of Koopmans et al. and Gaposhkin concerning the convergence of $\sum n^{-\alpha-1}S_n$ (see Section 1) follow from

Theorem 3: *Let B be a Hilbert space. Assume that $\{X_n, n \geqq 1\}$ is independent and identically distributed. For fixed $\alpha > 1/2$ the following statements are equivalent:*

(D) *The series $\sum n^{-\alpha-1}S_n$ converges a.s.*

(E) *The series $\sum n^{-\alpha}X_n$ converges a.s.*

(F) *$E\,\|X_1\|^{1/\alpha} < \infty$. Furthermore, $EX_1 = 0$ for $\alpha \in (1/2, 1]$ and $\sum n^{-1}EX_1I(\|X_1\| \leqq n)$ converges for $\alpha = 1$.*

If $\alpha \leqq 1/2$ then $\sum n^{-\alpha-1}S_n$ diverges a.s.

Proof: Let $a_n = n^{-\alpha-1}$. Obviously, $\alpha b_nn^\alpha \sim 1$ as $n \to \infty$. Assume $\alpha \leqq 1/2$. Thus, by Lemma 3 the condition $\lim n^{-\alpha}(S_n - c_n) = 0$ a.s. (for appropriate c_n) is necessary for the a.s. convergence of $\sum n^{-\alpha-1}S_n$. But this is impossible by the central limit theorem in Hilbert spaces (see [1]). Hence $\sum n^{-\alpha-1}S_n$ is not a.s. convergent. By the Hewitt-Savage 0–1 law this series diverges a.s.

Assume $\alpha > 1/2$. By the 3-series theorem in Hilbert spaces (see [5]) and by standard arguments (E) and (F) are equivalent. Using $\alpha b_nn^\alpha \sim 1$, it is not difficult to show that (E) and the a.s. convergence of $\sum b_nX_n$ are equivalent.

Assume (E). Then $\sum b_nX_n$ converges a.s. and (D) is an immediate consequence of Lemma 4.

Assume (D). Lemma 3 and $\alpha b_n n^\alpha \sim 1$ imply $\lim n^{-\alpha}(S_n - c_n) = 0$ a.s. Thus $E\,\|X_1\|^{1/\alpha} < \infty$. Let $\alpha \in (1/2, 1]$ and assume $EX_1 \neq 0$. By the strong law of large numbers, $\sum n^{-\alpha-1} S_n$ converges a.s. iff $\sum n^{-\alpha} < \infty$. This is impossible. Hence $EX_1 = 0$. Now, $E\,\|X_1\|^{1/\alpha} < \infty$ and $EX_1 = 0$ for $\alpha \in (1/2, 1]$ imply that $b_{n+1} S_n \xrightarrow{P} 0$, as $n \to \infty$. Apply Theorem 2 to obtain that $\sum b_n X_n$ converges a.s. Hence $\sum n^{-\alpha} X_n$ converges a.s. ∎

Acknowledgement: The author is grateful to Alexander Martikainen for valuable discussions. In particular, this work was inspired by Martikainen's Lemma 1.

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Manuskripteingang: 10. 12. 1984; in revidierter Fassung 11. 09. 1985

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