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On a Class of Quasi-Linear Riemann-Hilbert Problems

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Es wird die Existenz einer Lösung einer Klasse quasi-linearer Riemann-Hilbert-Probleme für eine holomorphe Funktion bewiesen.

Доказывается существование решения для некоторого класса квазилинейных краевых задач типа Римана-Гильберта для аналитических функций.

The existence of a solution of a class of quasi-linear Riemann-Hilbert problems for a holomorphic function is proved.

Introduction. In recent papers of one of the authors [5–7] a novel application of Schauder's fixed point theorem in the space of continuous functions is used for proving the existence of continuous solutions to some classes of nonlinear Riemann-Hilbert problems for holomorphic functions in the unit disk. Here in similar manner we prove the existence of solutions of the Hardy class H_2 to a class of quasi-linear Riemann-Hilbert problems with sufficiently smooth coefficients by means of Tikhonov's fixed point theorem in the space of quadratic summable functions. Also the proof of continuity for the fixed point mapping could be simplified in comparison to the corresponding proofs in [5–7]. The problem may be considered as a generalization of related quasilinear Riemann-Hilbert-Poincaré problems for holomorphic functions.

1. Statement of problem. Let $G: |z| < 1$ be the unit disk in the complex z plane with boundary $\Gamma: |t| = 1$, $t = e^{is} (-\pi \leq s \leq \pi)$. We deal with the following quasi-linear Riemann-Hilbert problem.

Problem Q: Find a holomorphic function $w(z) = u(z) + iv(z)$, $z = x + iy$, in G of the Hardy class H_2 which satisfies the boundary condition

$$u(t) + \Phi[u, v] (t) v(t) = \Psi[u, v] (t) \text{ a.e. on } \Gamma \tag{1}$$

and the additional condition

$$v(0) = 0 \quad \text{in} \quad z = 0. \tag{2}$$

The following basic *Assumption* A on the data is made:

(i) $\Phi[u, v]: L_2(\Gamma) \times L_2(\Gamma) \to C(\Gamma)$ is a (in general nonlinear) weakly-strongly continuous mapping.

(ii) $\Psi[u, v]: L_2(\Gamma) \times L_2(\Gamma) \to L_2(\Gamma)$ is a (in general nonlinear) weakly continuous mapping.

Assumption A is fulfilled, for instance, if

$$\Phi[u, v] (t) = F(t, (K_1 u) (t), (K_2 v) (t)), \tag{3}$$

where $F(t, U, V)$ is a real-valued continuous function on $\Gamma \times \mathbf{R} \times \mathbf{R}$ and $K_j : L_2(\Gamma) \to C(\Gamma)$, $j = 1, 2$, are linear compact operators (cf. [4: Chap. V, Th. 1.4]), and if

$$\Psi[u, v](t) = G(t, (M_1u)(t), (M_2v)(t)), \tag{4}$$

where the real-valued function $G(t, U, V)$ satisfies the Carathéodory condition (i.e., it is continuous with respect to $(U, V) \in \mathbf{R} \times \mathbf{R}$ for almost all $t \in \Gamma$ and measurable with respect to t on Γ for all $U \in \mathbf{R}$, $V \in \mathbf{R}$) and for all $r > 0$ an estimation of the form

$$|G(t, U, V)| \leqq g_r(t) \quad \text{as} \quad |U| + |V| \leqq r$$

with $g_r \in L_2(\Gamma)$ and $M_j : L_2(\Gamma) \to L_\infty(\Gamma)$, $j = 1, 2$, are linear compact operators (cf. [4: Chap. X, Th. 1.6 and Chap. V, § 1]). If $G(t, U, V) = a(t) + U + V$ with $a \in L_2(\Gamma)$, then it is only needed that M_j are linear continuous operators in $L_2(\Gamma)$.

2. Reduction to a fixed point equation. On account of (2) the boundary condition (1) is equivalent to the following nonlinear functional equation for $u(s) := u(e^{is})$

$$u(s) - \lambda(s)(Hu)(s) = g(s), \tag{5}$$

where

$$\lambda(s) \equiv \lambda(s, u) = \Phi[u, -Hu](t),$$

$$g(s) \equiv g(s, u) = \Psi[u, -Hu](t), \quad t = e^{is},$$

and H denotes the Hilbert transform

$$(Hu)(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\sigma) \cot \frac{\sigma - s}{2} \, d\sigma.$$

The corresponding linear equation (5) with $\lambda(s) = \lambda(s, \omega)$, $g(s) = g(s, \omega)$ for given function $\omega \in L_2(\Gamma)$ has the unique solution

$$u(s) = (N\omega)(s) \tag{6}$$

in $L_2(\Gamma)$ given by

$$(N\omega)(s) = \alpha(s)\, h(s) + \beta(s)\, e^{\nu(s)} H[e^{-\nu}h](s) + D\beta(s)\, e^{\nu(s)}, \tag{7}$$

where

$$\alpha = \frac{1}{\sqrt{1 + \lambda^2}}, \quad \beta = \frac{\lambda}{\sqrt{1 + \lambda^2}}, \quad \mu = \arctan \lambda, \quad \nu = H\mu, \quad h = \frac{g}{\sqrt{1 + \lambda^2}}$$

and the constant D has the value (cf. [1: § 31] and [7])

$$D = \tan \bar{\mu} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\nu(s)} h(s) \, ds, \quad \bar{\mu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu(s) \, ds.$$

Any solution $u \in L_2(\Gamma)$ of the fixed point equation $u = Nu$ yields with $v = -Hu$ by means of the Poisson integral formula (cf. [2: Chap. II, § 3, Cor. 3.2]) a solution $w(z) = u(z) + iv(z)$ of Problem **Q** with the boundary values $u(s)$ and $v(s)$ of its real and imaginary part, respectively.

3. Auxiliary lemmas. Preparing the proof of the existence of a fixed point to the equation (6) for $\omega = u$ in the next section, we state some auxiliary lemmas. At first we recall the well-known

Lemma 1: a) *Let μ be an arbitary bounded measurable function with oscillation* $2\gamma = \operatorname{ess\,sup} \mu(s) - \operatorname{ess\,inf} \mu(s)$, $s \in [-\pi, \pi]$. *Then*

$$A_\varkappa[\mu] := \int_{-\pi}^{\pi} e^{\varkappa(H\mu)(s)}\, ds \leqq \frac{2\pi}{\cos \varkappa\gamma} \tag{8}$$

for any real number $\varkappa$ with $|\varkappa| < \pi/2\gamma$.

b) *Let μ be a 2π-periodic continuous function on $[-\pi, \pi]$. Then* $\exp(H\mu) \in L_p(-\pi, \pi)$ *for any $p > 0$, i.e., $A_p[\mu]$ is finite for any $p > 0$.*

For proofs see [2, 3, 8], for instance. Further we prove

Lemma 2: *The operator $E: \mu \to \exp(H\mu)$ is a continuous mapping from $C(\Gamma)$ into $L_p(\Gamma)$ for any $p \in [1, \infty)$.*

Proof: Let $\mu_n \to \mu$ in $C(\Gamma)$. In a first step we show the convergence of the L_p-norms

$$\lim_{n\to\infty} \|\exp(H\mu_n)\|_p = \|\exp(H\mu)\|_p. \tag{9}$$

It follows from the estimation

$$\begin{aligned}
&|\|\exp(H\mu_n)\|_p{}^p - \|\exp(H\mu)\|_p{}^p| \\
&\leqq \int_{-\pi}^{\pi} |\exp[p(H\mu_n)(s)] - \exp[p(H\mu)(s)]|\, ds \\
&= \int_{-\pi}^{\pi} \exp(pH\mu)\, |\exp\big(p(H\mu_n - H\mu)\big) - 1|\, ds \\
&\leqq B_p[\mu] \left\{ \int_{-\pi}^{\pi} \left[\exp\big(2p(H\mu_n - H\mu)\big) - 2\exp\big(p(H\mu_n - H\mu)\big) + 1\right] ds \right\}^{1/2}
\end{aligned}$$

with the fixed constant

$$B_p[\mu] := \{A_{2p}[\mu]\}^{1/2} = \left\{ \int_{-\pi}^{\pi} \exp(2pH\mu)\, ds \right\}^{1/2} < \infty$$

(cp. Lemma 1b)) that it is sufficient to prove

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \exp\big(\varkappa(H\mu_n - H\mu)\big)\, ds = 2\pi$$

for $\varkappa = p, 2p$. We denote by γ_n the half of the oscillation of the functions $\mu_n - \mu$. Obviously, $\gamma_n \to 0$. Therefore, Lemma 1a) is applicable for sufficiently large n and

$$\int_{-\pi}^{\pi} \exp\big(\varkappa(H\mu_n - H\mu)\big)\, ds \leqq \frac{2\pi}{\cos \varkappa\gamma_n}. \tag{10}$$

On the other hand, by Cauchy-Schwarz' inequality we have

$$\int_{-\pi}^{\pi} \exp\big(\varkappa(H\mu_n - H\mu)\big)\, ds \int_{-\pi}^{\pi} \exp\big(-\varkappa(H\mu_n - H\mu)\big)\, ds \geqq (2\pi)^2.$$

Therefore, again by Lemma 1a)

$$\int_{-\pi}^{\pi} \exp\big(\varkappa(H\mu_n - H\mu)\big)\, ds \geqq 2\pi \cos \varkappa\gamma_n. \tag{11}$$

From (10) and (11) it follows (9) since $\gamma_n \to 0$.

In a second step we prove the weak convergence $\exp(H\mu_n) \rightharpoonup \exp(H\mu)$ in $L_p(\Gamma)$. Due to (9) the L_p-norms of $\exp(H\mu_n)$ are uniformly bounded. Therefore it suffices to prove $\exp(H\mu_n) \rightharpoonup \exp(H\mu)$ in $L_1(\Gamma)$. Let $\psi \in L_\infty(\Gamma)$ be arbitrary. Then

$$\left| \int_{-\pi}^{\pi} [\exp(H\mu_n)(s) - \exp(H\mu)(s)]\,\psi(s)\,ds \right|$$

$$\leqq \operatorname*{ess\,sup}_{-\pi\leqq s\leqq\pi} |\psi(s)| \int_{-\pi}^{\pi} |\exp(H\mu_n)(s) - \exp(H\mu)(s)|\,ds.$$

As shown in the proof above the right-hand integral expression tends to zero as n goes to infinity which implies the assertion.

Finally the strong convergence of these functions is a consequence of the weak convergence and the convergence of their norms (9) ∎

At last we have

Lemma 3: *Let μ be a 2π-periodic continuous function with oscillation $2\gamma < \pi/2$. Then for any function $h \in L_2(-\pi, \pi)$*

$$\|e^{H\mu} H[e^{-H\mu} h]\|_2 \leqq D_\gamma \|h\|_2 \quad \textit{with} \quad D_\gamma = \frac{1 + \cos 2\gamma}{1 - \sin 2\gamma}. \tag{12}$$

Proof: We introduce the function

$$\tilde{h}(s) = h(s) - \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-H\mu} h\,ds \cdot e^{(H\mu)(s)}$$

satisfying the orthogonality condition

$$\int_{-\pi}^{\pi} e^{-(H\mu)(s)}\,\tilde{h}(s)\,ds = 0.$$

The Lemma in [6] implies the inequality

$$\|e^{H\mu} H[e^{-H\mu}\tilde{h}]\|_2 \leqq C_\gamma \|\tilde{h}\|_2 \quad \text{with} \quad C_\gamma = \frac{\cos 2\gamma}{1 - \sin 2\gamma}.$$

But $H[e^{-H\mu}\tilde{h}] = H[e^{-H\mu}h]$ and

$$\|\tilde{h}\|_2 \leqq \|h\|_2 + \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-H\mu} h\,ds \right| \|e^{H\mu}\|_2$$

$$\leqq \|h\|_2 + \frac{1}{2\pi} \|e^{-H\mu}\|_2 \|h\|_2 \|e^{H\mu}\|_2 \leqq \|h\|_2 \left(1 + \frac{1}{\cos 2\gamma}\right),$$

where we used the Cauchy-Schwarz inequality and twice (8). Therefore the assertion follows ∎

Lemma 3 with the smaller constant $D_\gamma = 2/\sqrt{1 - \sin 2\gamma}$ also follows from the considerations in [2: Chap. IV, § 3].

4. Existence theorem. For proving the existence of a fixed point $u \in L_2(\Gamma)$ to the equation (6) for $\omega = u$ we make the additional *Assumption* **B**:

(i) It is

$$2\gamma = \mu_2 - \mu_1 < \pi/2 \quad \text{with} \quad \mu_j = \arctan \lambda_j,\ j = 1, 2, \tag{13}$$

where

$$\lambda_1 = \inf \Phi[u, -Hu](t) \quad \text{and} \quad \lambda_2 = \sup \Phi[u, -Hu](t),$$

the infimum and supremum are taken over $t \in \Gamma$, $u \in L_2(\Gamma)$.

(ii) There holds the inequality

$$\|\Psi[u, -Hu]\|_2 \leq g_0 \quad \text{for} \quad u \in L_2(\Gamma), \; g_0 \text{ a constant.} \tag{14}$$

Assumption **B** is fulfilled in case of (3) and (4) if (13) holds true with $\lambda_1 = \inf F(t, U, V)$ and $\lambda_2 = \sup F(t, U, V)$, where the infimum and supremum is taken over $t \in \Gamma$, $U \in \mathbf{R}$, $V \in \mathbf{R}$, and $|G(t, U, V)| \leq G_0(t)$ with a fixed function $G_0 \in L_2(\Gamma)$ uniformly in $U \in \mathbf{R}$, $V \in \mathbf{R}$.

Lemma 4: *Under assumptions* **A** *and* (13) *the operator* $N: L_2(\Gamma) \to L_2(\Gamma)$ *is weakly continuous.*

Proof: Let $\omega_n \rightharpoonup \omega$ in $L_2(\Gamma)$ and put $u = N\omega$, $u_n = N\omega_n$. We denote by $\alpha, \beta, \gamma, \lambda, \mu, \nu, g, h, D, \bar{\mu}$ and $\alpha_n, \beta_n, \gamma_n, \lambda_n, \mu_n, \nu_n, g_n, h_n, D_n, \bar{\mu}_n$ the functions (resp. constants) in formula (7) corresponding to ω and ω_n, respectively. Since $\omega_n \rightharpoonup \omega$, also $H\omega_n \rightharpoonup H\omega$ in $L_2(\Gamma)$. Due to Assumption **A** $\lambda_n \to \lambda$ in $C(\Gamma)$ and $g_n \rightharpoonup g$ in $L_2(\Gamma)$. Then $\alpha_n, \beta_n, \mu_n \in C(\Gamma)$ converge uniformly to $\alpha, \beta, \mu \in C(\Gamma)$, respectively, and $h_n \in L_2(\Gamma)$ converge weakly to $h \in L_2(\Gamma)$. On account of Lemma 2 $\exp(\pm\nu_n) \to \exp(\pm\nu)$ in $L_2(\Gamma)$ and moreover in any $L_p(\Gamma)$, $1 \leq p < \infty$. Since $\bar{\mu}_n \to \bar{\mu}$ with $|\bar{\mu}| < \pi/2$ and $h_n \rightharpoonup h$ in $L_2(\Gamma)$, the convergence $\exp(-\nu_n) \to \exp(-\nu)$ in $L_2(\Gamma)$ implies the convergence $D_n \to D$.

There remains to show that

$$\chi_n(s) := e^{\nu_n(s)} H[e^{-\nu_n} h_n](s) \rightharpoonup \chi(s) := \exp(\nu(s)) H[\exp(-\nu) h](s)$$

in $L_2(\Gamma)$. From the weak convergence of the h_n their uniform boundedness in $L_2(\Gamma)$ follows. Hence due to the assumption (13) Lemma 3 yields the uniform boundedness of the χ_n in $L_2(\Gamma)$. Therefore it suffices to prove $\chi_n \rightharpoonup \chi$ in $L_1(\Gamma)$ only. Now the convergences $h_n \rightharpoonup h$ in $L_2(\Gamma)$ and $\exp(-\nu_n) \to \exp(-\nu)$ in $L_4(\Gamma)$ imply $\exp(-\nu_n) h_n \rightharpoonup \exp(-\nu) h$ in $L_{4/3}(\Gamma)$. Then also $H[\exp(-\nu_n) h_n] \rightharpoonup H[\exp(-\nu) h]$ in $L_{4/3}(\Gamma)$. But from this and $\exp \nu_n \to \exp \nu$ in $L_4(\Gamma)$ again, $\chi_n \rightharpoonup \chi$ follows ∎

Further, under the additional Assumption **B** the operator N maps the whole space $L_2(\Gamma)$ into a fixed ball of $L_2(\Gamma)$.

Lemma 5: *Let* γ *be defined by* (13) $\lambda_0 = \max(\lambda_2, -\lambda_1)$, *and* D_γ *given by* (12). *Then*

$$\|N\omega\|_2 \leq R \quad \textit{for any} \quad \omega \in L_2(\Gamma)$$

with the uniform constant $R = \left(\sqrt{1 + D_\gamma^2} + 2\pi\lambda_0/\cos^2 2\gamma\right) g_0$.

Proof: From (7) and Lemma 3 we get

$$\|N\omega\|_2 \leq \sqrt{1 + D_\gamma^2}\, \|h\|_2 + |D|\, \|e^\nu\|_2 .$$

Due to the assumption (14) we have $\|h\|_2 \leq g_0$ and from (8) the estimation $\|e^\nu\|_2 \leq 2\pi/\cos 2\gamma$ follows. Further

$$|D| \leq \frac{\lambda_0}{2\pi} \|e^{-\nu}\|_2 \|h\|_2 \leq \frac{\lambda_0}{2\pi} \frac{2\pi}{\cos 2\gamma} g_0 = \frac{\lambda_0}{\cos 2\gamma} g_0 .$$

Therefore, we finally obtain the assertion ∎

We now apply Tikhonov's fixed point theorem to the equation (6) in the convex weakly compact subset $\mathfrak{K} = \{\omega \in L_2(\Gamma) : \|\omega\|_2 \leq R\}$ of $L_2(\Gamma)$ with the constant R

from Lemma 5. By Lemma 4 and 5 the operator N maps $\mathfrak{K}$ weakly continuously into itself. Therefore, we proved the

Theorem: *Under Assumption* **A** *and* **B** *the Problem* **Q** *possesses a solution w of the Hardy class H_2.*

The boundary values of the real and imaginary part of the solution $w = u + iv$ satisfy the inequalities $\|u\|_2, \|v\|_2 \leqq R$ with the constant R given by Lemma 5.

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