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An Application of B. N. Sadovskij's Fixed Point Principle to Nonlinear Singular Equations

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Diese Arbeit befaßt sich mit der Anwendbarkeit des Sadovskijschen Fixpunktprinzips auf die Lösbarkeit nichtlinearer singulärer Integralgleichungen der Form $x = \lambda SFx$, wo F ein nichtlinearer Superpositionsoperator und S ein linearer singulärer Integraloperator ist. Bildet der Operator F den „kleinen" Hölderraum H_α^0 oder einen Raum $J_{\alpha,\beta}$ von Funktionen, die einer Hölderbedingung in Integralform genügen, in sich ab, so ist er unter recht allgemeinen Bedingungen k-verdichtend, so daß die obige Gleichung wenigstens für kleines λ eine Lösung besitzt. Andererseits kann man unter diesen allgemeinen Voraussetzungen den klassischen Fixpunktsatz von Schauder nur bedingt und den von Banach-Caccioppoli gar nicht anwenden; es wird in der Tat gezeigt, daß sich das Banach-Caccioppoli-Prinzip nur dann anwenden läßt, wenn die obige Gleichung linear ist. Darüber hinaus wird gezeigt, daß man zur Untersuchung der obigen Gleichung auch die topologische Abbildungsgradtheorie für Vektorfelder mit verdichtenden Operatoren heranziehen kann.

В статье обсуждается возможность использования для доказательства разрешимости нелинейных сингулярных интегральных уравнений вида $x = \lambda SFx$ (здесь F — нелинейный оператор суперпозиции, S — линейный сингулярный интегральный оператор) принципа неподвижной точки Б. Н. Садовского. Если оператор F действует в „малом" пространстве Гёльдера H_α^0 или в пространстве функций $J_{\alpha,\beta}$, удовлетворяющих интегральному условию Гёльдера, то оказывается, что он, вообще говоря, является k-уплотняющим, что и позволяет установить разрешимость рассматриваемого уравнения по крайней мере для малых λ. С другой стороны, в условиях этих теорем классический принцип неподвижной точки Шаудера трудно, а принцип Банаха—Каччиопполи вообще нельзя применять; в самом деле, показано, что возможность применения принципа Банаха—Каччиопполи в рассматриваемом случае означает, что рассматриваемое уравнение оказывается линейным. Кроме того, в статье показана также возможность для исследования уравнений рассматриваемого типа использовать теорию вращения векторных полей с уплотняющими операторами.

This paper is concerned with the applicability of Sadovskij's fixed point principle to the solvability of nonlinear singular integral equations of the form $x = \lambda SFx$, with F being a nonlinear superposition operator, and S a singular linear integral operator. If the operator F acts in the "little" Hölder space H_α^0, or in some space $J_{\alpha,\beta}$ of functions which satisfy an integral-type Hölder condition, F turns out to be k-condensing under fairly general hypotheses such that the above equation has a solution at least for small λ. On the other hand, under these general hypotheses the classical fixed point principles of Schauder does not apply immediately, and that of Banach-Caccioppoli not at all; in fact, it is shown that the Banach-Caccioppoli principle applies only if the above equation is linear. Moreover, in this paper it is shown that the above equation may be studied as well by means of the topological degree theory for vector fields involving condensing operators.

0. Introduction

Let X be a real Banach space and A a nonlinear operator in X. Recall that A is called *k-condensing* if the estimate

$$\psi(AM \mid X) \leqq k\psi(M \mid X) \qquad \big(M \in \mathscr{B}(X)\big) \tag{1}$$

holds, where $\mathscr{B}(X)$ denotes the family of all bounded subsets of X, and $\psi(\cdot \mid X)$ is some measure of noncompactness in X. Sadovskij's fixed point principle states that a continuous condensing operator A has fixed points in each closed ball $B_r(x_0) = \{x \in X : \|x - x_0 \mid X\| \leqq r\}$ which is mapped by A into itself, provided that $k < 1$ ([34, 36], see also [17]). Since both compact operators and contractions (in the norm of X) are condensing (for example, with respect to the Hausdorff measure of noncompactness, see below), this result generalizes Schauder's classical fixed point principle as well as the contraction mapping principle of Banach-Caccioppoli.

The purpose of the present paper is to apply Sadovskij's fixed point principle to the nonlinear singular integral equation

$$x(s) = \lambda \int_0^1 \frac{k(s, t)}{s - t} f\big(t, x(t)\big)\, dt. \tag{2}$$

This equation can be written as operator equation

$$x = \lambda SFx, \tag{3}$$

where

$$Fx(t) = f\big(t, x(t)\big) \tag{4}$$

is a nonlinear superposition operator, and

$$Sy(s) = \int_0^1 \frac{k(s, t)}{s - t} y(t)\, dt \tag{5}$$

is a linear singular operator. The problem of verifying condition (1) for the nonlinear operator $A = SF$, as well as that of finding an invariant ball for A, obviously leads then to the corresponding problems for the superposition operator (4), since the contribution of the linear part (5) is completely described by its "essential norm" $\chi(S)$ and its usual norm $\|S\|$, respectively. Consequently, the natural problem arises to find conditions for the superposition operator to be condensing, as well as upper estimates for its growth on a given ball.

Let us make some observations on the first point. The most useful and appropriate measure of noncompactness in applications is the so-called *Hausdorff measure of noncompactness*

$$\chi(M \mid X) = \inf E(M), \tag{6}$$

where $E(M)$ denotes the set of all positive reals ε for which M admits a finite ε-net. As already mentioned, a sufficient condition for some nonlinear operator A to satisfy (1) is a Lipschitz condition with the same constant k. "Unfortunately", it may happen that the converse is also true: a Lipschitz condition with constant k may turn out to be necessary for (1). This means, as a matter of fact, that the application of Sadovskij's fixed point principle does not give new results in comparison with the classical Banach-Caccioppoli theorem. This situation occurs, for instance, if one considers the superposition operator (4) in the space C of continuous functions on $[0, 1]$, with the usual maximum norm. In this case one can show [1] that condition (1) (for the measure of noncompactness (6)) is equivalent to both

$$\|Fx - Fy \mid C\| \leqq k\,\|x - y \mid C\| \quad (x, y \in C)$$

and

$$|f(s, u) - f(s, v)| \leqq k\,|u - v| \quad (u, v \in \mathbf{R}).$$

A similar statement holds in the space L_p of all measurable functions which are p-integrable on [0, 1] (see again [1]).

It turns out, however, that the situation is quite different in other function spaces. For example, if one considers the operator (4) in the Hölder space H_α, equipped with the norm

$$\|x \mid H_\alpha\| = |x(0)| + \sup_{s \neq t} \frac{|x(s) - x(t)|}{|s - t|^\alpha}, \tag{7}$$

then condition (1) is satisfied for a reasonable class of functions, while F is Lipschitz in this norm only if the function f has the form $f(\cdot, u) = g(\cdot)\, u + h(\cdot)$, i.e. is linear in u!

It is interesting to notice that the Hölder spaces H_α are fundamental in the study of singular integral operators. (Singular integral operators do not act in the space C; they do act in L_p for $1 < p < \infty$, but there are "too few" nonlinear operators acting in L_p: in fact, the superposition operator F maps L_p into itself only if the generating function f is sublinear in u [22: p. 349]). To summarize, Hölder-type spaces like H_α are the right "candidates" to provide "essential" applications of Sadovskij's fixed point principle to nonlinear singular integral equations.

It is worth-while mentioning, by the way, that the advantage of using Sadovskij's principle rather than the Banach-Caccioppoli theorem is already evident by the fact that, in order to verify condition (1) for the operator $A = SF$, the contribution of S is given by the essential norm $\chi(S)$, while for a Lipschitz condition one must take into account the operator norm $\|S\|$. In most applications the difference between these two norms is considerably large. This fact allows one to strengthen existence theorems for nonlinear singular equations like (2) not only in Hölder spaces H_α, but even in Lebesgue spaces L_p (acting conditions for classical singular linear operators in H_α can be found e.g. in [32], in L_p e.g. in [33]).

The plan of the paper is as follows: In the first section we introduce and study a class of spaces $J_{\alpha,\beta}$ $(0 < \alpha \leqq 1,\ 0 < \beta \leqq \infty)$ consisting of continuous functions x on [0, 1] whose modulus of continuity $\omega(x, \sigma)$ is integrable with some weight in σ $(\beta < \infty)$ or is controlled by some power of σ $(\beta = \infty)$; in particular, the space $J_{\alpha,\infty}$ coincides with the Hölder space H_α. Special attention will be paid to the study of the Hausdorff measure of noncompactness (6). Section 2 is devoted to properties of the superposition operator (4) in these spaces; in particular, we shall be interested in condition (1). Finally, some applications to nonlinear singular integral equations will be indicated in Section 3, and some possible extensions will be discussed at the end of the paper.

1. The spaces $J_{\alpha,\beta}$

Throughout this section, let $0 < \alpha \leqq 1$ and $0 < \beta \leqq \infty$. Given a continuous function x on [0, 1], let

$$\omega(x, \sigma) = \sup \{|x(s) - x(t)| : 0 \leqq s, t \leqq 1,\ |s - t| \leqq \sigma\}$$

denote its modulus of continuity. We write $x \in J_{\alpha,\beta}$ $(\beta < \infty)$ if

$$j_{\alpha,\beta}(x) = \int_0^1 \sigma^{-(\beta+1)} \omega(x, \sigma)^{\beta/\alpha}\, d\sigma < \infty.$$

For $\beta \geqq \alpha$ the set $J_{\alpha,\beta}$ with the norm $\|\cdot \mid J_{\alpha,\beta}\| = \max\{\|\cdot \mid C\|, j_{\alpha,\beta}(\cdot)^{\alpha/\beta}\}$ is a Banach space. Similarly, we write $x \in J_{\alpha,\infty}$, if $\omega(x, \sigma) = O(\sigma^\alpha)$, i.e. $j_{\alpha,\infty}(x) = \sup\{\sigma^{-\alpha}\omega(x, \sigma) : 0 < \sigma \leqq 1\} < \infty$. The set $J_{\alpha,\infty}$ with the norm $\|\cdot \mid J_{\alpha,\infty}\| = \max\{\|\cdot \mid C\|, j_{\alpha,\infty}(\cdot)\}$ is a

Banach space. Observe that $J_{\alpha,\infty}$ is nothing else than the Hölder space H_α; the equivalence of the norms $\|\cdot \mid J_{\alpha,\infty}\|$ and $\|\cdot \mid H_\alpha\|$ is a simple consequence of the estimate $\|x \mid C\| \leqq |x(0)| + j_{\alpha,\infty}(x)$. Moreover, we remark that $J_{\alpha,\beta}$ coincides with Muhtarov's space $I_{\varphi,p}$ (see [21: p. 79]) for $\varphi(\sigma) = \sigma^{1-\alpha}$ and $p = \beta/\alpha$.

For each subinterval $(a, b) \subseteq [0, 1]$, define a linear operator $P_{a,b}$ by

$$P_{a,b}x(s) = \begin{cases} x(a) & \text{if } 0 \leqq s \leqq a, \\ x(s) & \text{if } a \leqq s \leqq b, \\ x(b) & \text{if } b \leqq s \leqq 1. \end{cases} \tag{8}$$

Obviously, $P_{a,b}$ acts in each of the spaces $J_{\alpha,\beta}$, and has norm 1. For $0 < \beta \leqq \infty$, let $J^0_{\alpha,\beta}$ be the subspace of all functions $x \in J_{\alpha,\beta}$ for which

$$\lim_{b-a\to 0} j_{\alpha,\beta}(P_{a,b}x) = 0. \tag{9}$$

It is not hard to see that, for $\beta < \infty$, $J^0_{\alpha,\beta}$ coincides with the whole space $J_{\alpha,\beta}$. On the other hand, the subspace $J^0_{\alpha,\infty}$ $(0 < \alpha < 1)$ consists precisely of those functions $x \in J_{\alpha,\infty}$ for which $\omega(x, \sigma) = o(\sigma^\alpha)$ as $\sigma \to 0$, and hence coincides with the "little" Hölder space $H_\alpha{}^0$; finally, for $\alpha = 1$ the subspace $J^0_{1,\infty}$ contains only constant functions. (We remark that the "little" Hölder space $H_\alpha{}^0$ $(0 < \alpha < 1)$ can be characterized equivalently as closure of the space C^1 of continuously differentiable functions on [0, 1] with respect to the norm (8), see [38], and has important applications in both partial differential equations, see [25], and singular integral equations, see [21] and below.)

Let us point out an analogy between the spaces $J_{\alpha,\beta}$ and the Lorentz spaces $L_{p,q}$ which arise naturally in interpolation theory for linear operators (see e.g. [5, 23, 31]). In fact, if for some measurable function x on [0, 1] we denote by $\omega(x, \sigma)$ the measure of the Lebesgue set $\{s : |x(s)| < 1/\sigma\}$, the condition $\int_0^1 \sigma^{-(q+1)}\omega(x, \sigma)^{q/p}\, d\sigma < \infty$ holds if and only if x belongs to the Lorentz space $L_{p,q}$ (in particular, the Lebesgue space L_p for $p = q$); similarly, the condition $\sup\{\sigma^{-p}\omega(x, \sigma): 0 < \sigma \leqq 1\} < \infty$ holds if and only if x belongs to the Marcinkiewicz space $M_p = L_{p,\infty}$ (see, e.g. [16, 41]). Moreover, if the operator (8) is replaced by the multiplication operator

$$P_D x(s) = \chi_D(s)\, x(s), \tag{10}$$

where χ_D denotes as usual the characteristic function of $D \subseteq [0, 1]$, and if (9) is replaced by the analoguous condition $\lim_{\text{mes}\, D\to 0} \|P_D x \mid L_{p,q}\| = 0$, then instead of $J^0_{\alpha,\beta}$ one obtains the space $L^0_{p,q}$ of all functions $x \in L_{p,q}$ with absolutely continuous norm.

Let us still remark that other choices of the function $\omega(x, \cdot)$ (in terms of measurable or continuous x) lead to other well-known function spaces. For example, letting

$$\omega(x, \sigma) = \sup\left\{\int_\alpha^\beta |x(t)|\, dt : 0 \leqq \alpha < \beta \leqq 1, \beta - \alpha \leqq \sigma\right\},$$

one gets a class of spaces which contains in particular the Morrey-Campanato spaces (which are fundamental in the theory of elliptic systems, see e.g. [24, 29]). Similarly, letting

$$\omega(x, \sigma) = \sup\left\{\left|\int_\alpha^\beta x(t)\, dt\right| : 0 \leqq \alpha < \beta \leqq 1, \beta - \alpha \leqq \sigma\right\},$$

one obtains a useful generalization of the Bogoljubov spaces (which occur in averaging methods for ordinary differential equations, see e.g. [40]).

As for the Lorentz spaces $L_{p,q}$, the spaces $J_{\alpha,\beta}$ are "decreasing" in α and "increasing" in β; more precisely, the following holds.

Proposition 1: *For $0 < \varepsilon < \alpha \leqq 1$, $0 < \beta < \infty$, and $0 < \gamma \leqq \infty$ the imbeddings $J_{\alpha,\beta} \subseteq J^0_{\alpha,\infty} \subseteq J_{\alpha,\infty} \subseteq J_{\alpha-\varepsilon,\gamma}$ hold.*

Proof: First, for $x \in J_{\alpha,\beta}$ we have

$$2^{-(\beta+1)}\sigma^{-\beta}\omega(x, \sigma)^{\beta/\alpha} \leqq \int_\sigma^{2\sigma} \tau^{-(\beta+1)}\omega(x, \tau)^{\beta/\alpha}\, d\tau = o(1) \quad (\sigma \to 0)$$

and hence $\omega(x, \sigma) = o(\sigma^\alpha)$ as $\sigma \to 0$. The second inclusion is trivial. Finally, for $x \in J_{\alpha,\infty}$ we have

$$j_{\alpha-\varepsilon,\gamma}(x) \leqq j_{\alpha,\infty}(x)^{\gamma/(\alpha-\varepsilon)} \int_0^1 \sigma^{-(\gamma+1)}\sigma^{\alpha\gamma/(\alpha-\varepsilon)}\, d\sigma$$

$$= j(_{\alpha,\infty}(x)^{\gamma/(\alpha-\varepsilon)} \int_0^1 \sigma^{(\gamma\varepsilon-\alpha+\varepsilon)/(\alpha-\varepsilon)}\, d\sigma < \infty;$$

and hence the third inclusion holds ∎

In what follows, the notion of a measure of noncompactness will be of fundamental importance. Following the axiomatic setting of B. N. Sadovskij [36], we call a real nonnegative function $\psi(\cdot \mid X)$ which is defined on the system $\mathscr{B}(X)$ of all bounded subsets of a Banach space X, a *measure of noncompactness*, if it has the following properties ($\overline{M}$ denotes the closure and co the convex hull of M):

(i) $\psi(M \mid X) = 0$ iff $\overline{M}$ is compact,

(ii) $\psi(\overline{\text{co}}\, M \mid X) = \psi(M \mid X)$,

(iii) $\psi(M \cup N \mid X) = \max\{\psi(M \mid X), \psi(N \mid X)\}$,

(iv) $\psi(M + N \mid X) \leqq \psi(M \mid X) + \psi(N \mid X)$,

(v) $\psi(\lambda M \mid X) = |\lambda|\, \psi(M \mid X)$,

(vi) $\psi(M \mid X) \leqq \psi(N \mid X)$ if $M \subseteqq N$.

For example, the Hausdorff measure of noncompactness (6) satisfies all these properties, but it is also often convenient to invent new measures of noncompactness which are in some sense "natural" for the problem under consideration (see, for example, the monograph [4]).

The next theorem provides explicit formulas for the Hausdorff measure of noncompactness in the spaces $J_{\alpha,\beta}$ ($\alpha \leqq \beta \leqq \infty$) and $J^0_{\alpha,\infty}$.

Proposition 2: *The equality*

$$\chi(M \mid J_{\alpha,\beta}) = \psi_{\alpha,\beta}(M) \qquad \big(M \in \mathscr{B}(J_{\alpha,\beta})\big)$$

holds, where

$$\psi_{\alpha,\beta}(M) = \overline{\lim_{\sigma\to 0}} \sup_{x\in M} \left\{\int_0^\sigma \tau^{-(\beta+1)}\omega(x, \tau)^{\beta/\alpha}\, d\tau\right\}^{\alpha/\beta};$$

similarly, the equality

$$\chi(M \mid J^0_{\alpha,\infty}) = \psi^0_{\alpha,\infty}(M) \qquad \big(M \in \mathscr{B}(J^0_{\alpha,\infty})\big)$$

holds, where

$$\psi^0_{\alpha,\infty}(M) = \overline{\lim_{\sigma\to 0}} \sup_{x\in M} \sigma^{-\alpha}\omega(x, \sigma).$$

Proof: Let us prove the second equality; the analogous first equality is proved in the same way with only a few minor changes. So let, $\eta > \chi(M \mid J^0_{\alpha,\infty})$, and let $\{x_1, \ldots, x_m\}$ be a finite η-net for M in $J^0_{\alpha,\infty}$. Fix $x \in M$ and choose x_j such that

$j_{\alpha,\infty}(x - x_j) = \sup_{\sigma>0} \sigma^{-\alpha}\omega(x - x_j, \sigma) \leqq \eta$. Since $x_i \in J^0_{\alpha,\infty}$, we have $\sigma^{-\alpha}\omega(x_j, \sigma) \leqq \varepsilon$ for σ sufficiently small, and hence $\psi^0_{\alpha,\infty}(M) \leqq \chi(M \mid J^0_{\alpha,\infty}) + \varepsilon$. Since ε is arbitrary, we have $\psi^0_{\alpha,\infty}(M) \leqq \chi(M \mid J^0_{\alpha,\infty})$.

On the other hand, given $\varepsilon > 0$, we can find $\delta \in (0, 1)$ such that $\sigma^{-\alpha}\omega(x, \sigma) \leqq \psi^0_{\alpha,\infty}(M) + \varepsilon$ for $\sigma \leqq \delta$ (uniformly in $x \in M$). Since M is bounded in $J^0_{\alpha,\infty}$, it is compact in C; thus we can find a finite $\varepsilon\delta^\alpha$-net $\{x_1, \ldots, x_m\}$ for M in C which consists (without loss of generality) of functions $x_j \in M$, i.e. fulfills $\sigma^{-\alpha}\omega(x_j, \sigma) \leqq \varepsilon$ for $\sigma \leqq \delta$ and $j = 1, \ldots, m$. Given $x \in M$, choose x_j such that $\|x - x_j \mid C\| \leqq \varepsilon\delta^\alpha$. We have then

$$\sigma^{-\alpha}\omega(x - x_j, \sigma) \leqq \begin{cases} \delta^{-\alpha} 2\, \|x - x_j \mid C\| \leqq 2\varepsilon & \text{for} \quad \sigma > \delta \\ \sigma^{-\alpha}\omega(x, \sigma) + \sigma^{-\alpha}\omega(x_j, \sigma) \leqq (\eta + \varepsilon) + \varepsilon & \text{for} \quad \sigma \leqq \delta. \end{cases}$$

This shows that $\|x - x_j \mid J^0_{\alpha,\infty}\| \leqq \max\{\varepsilon\delta^\alpha, \eta + 2\varepsilon\} = \eta + 2\varepsilon$, and hence $\chi(M \mid J^0_{\alpha,\infty}) \leqq \psi^0_{\alpha,\infty}(M)$ ∎

Let us remark that, by the Arzelà-Ascoli theorem, the function

$$\psi_\infty(M) = \overline{\lim_{\sigma\to 0}} \sup_{x\in M} \omega(x, \sigma) \tag{11}$$

is a measure of noncompactness in C; it is already mentioned in [36] in the form

$$\psi_\infty(M) = \overline{\lim_{\sigma\to 0}} \sup_{x\in M} \max_{\tau\leq\sigma} \|x - x_\tau \mid C\|, \tag{12}$$

where x_τ is the shift

$$x_\tau(s) = \begin{cases} x(s + \tau) & \text{for} \quad 0 \leqq s \leqq 1 - \tau, \\ x(1) & \text{for} \quad 1 - \tau \leqq s \leqq 1. \end{cases}$$

(The equality of (11) and (12) follows from the fact that the maximum in (12) equals $\omega(x, \sigma)$.) By standard techniques one can show that the two-sided estimate $2^{-1}\psi_\infty(M) \leqq \chi(M \mid C) \leqq 2\psi_\infty(M)$ ($M \in \mathscr{B}(C)$) holds.

One could expect that similar estimates can be obtained for the Hausdorff measure of noncompactness in the Hölder space $J_{\alpha,\infty} = H_\alpha$ (which is not covered by Prop. 2). It turns out, however, that this is not possible. Moreover, the compactness criteria in the space H_α (or even more general spaces) with can be found in the literature (as e.g. [19, 21, 28]) are false. In fact, in the book [21], for instance, the authors claim that a necessary and sufficient condition for a set M to be compact in the space H_α is that M be closed and bounded, and that the family of functions of two variables

$$x^\alpha(s, t) = \frac{|x(s) - x(t)|}{|s - t|^\alpha} \quad (s \neq t) \tag{13}$$

(for x running over M) be equicontinuous either on each closed subset which is entirely contained in the upper triangle $\triangle_+ = \{(s, t) : 0 \leqq s < t \leqq 1\}$ or lower triangle $\triangle_- = \{(s, t) : 0 \leqq t < s \leqq 1\}$, or on the union $\triangle_+ \cup \triangle_-$. (As a matter of fact, this equicontinuity condition is formulated not very precisely in the above mentioned papers, and the error is just due to the confusion between these two conditions.) It is not hard to see that the equicontinuity condition on each closed subset of $\triangle_+$ or $\triangle_-$ is necessary for the compactness of M in H_α. Nevertheless, it is not sufficient, since it is satisfied already by any set M which is compact merely in the space C. On the other hand, the equicontinuity condition on the whole set $\triangle_+ \cup \triangle_-$, together with closedness and boundedness, is certainly sufficient for a set M to be compact in H_α. Nevertheless, it is not necessary, since it fails to hold even for the singleton $M = \{x_0\}$, where $x_0(t) = t^{2\alpha} |\sin t^{-1}|^\alpha \in H_\alpha$. In fact, considering the sequences

$$(s_n, t_n) = ([2n\pi + \pi/2]^{-1}, [2n\pi + \pi]^{-1}) \quad \text{and} \quad (s_n', t_n') = ([2n\pi]^{-1}, [2n\pi + \pi/2]^{-1}),$$

one obviously has $(s_n, t_n) - (s_n', t_n') \to 0$, but

$$|x_0^\alpha(s_n, t_n) - x_0^\alpha(s_n', t_n')| = \frac{2\alpha}{\pi^\alpha} \frac{(2n+1)^\alpha + (2n)^\alpha}{\left(2n + \frac{1}{2}\right)^\alpha} \to \frac{2^{1+\alpha}}{\pi^\alpha} \neq 0,$$

and hence the function (13) can not be uniformly continuous on $\Delta_+ \cup \Delta_-$.

Let us observe that a similar situation occurs in the Marcinkiewicz space $M_p = L_{p,\infty}$: One does not know any simple compactness criterion in this space, nor any appropriate measure of noncompactness.

Let us return to the spaces $J_{\alpha,\beta}$, and compare Prop. 2 with what one can say in the spaces $L_{p,q}$. Recall that a subset $M \subset L_{p,q}$ is called *absolutely bounded* [41] if

$$\overline{\lim_{\operatorname{mes} D \to 0}} \sup_{x \in M} \|P_D x \mid L_{p,q}\| = 0,$$

where P_D is defined in (10). Absolute boundedness is essentially weaker than compactness: Indeed, the imbedding of $L_{p,q}$ into $L_{p-\varepsilon,r}$ (in particular, of L_p into $L_{p-\varepsilon}$) is absolutely bounded, but not compact. On the other hand, Prop. 2 shows that the relation

$$\overline{\lim_{b-a \to 0}} \sup_{x \in M} j_{\alpha,\beta}(P_{a,b} x) = 0$$

(where $P_{a,b}$ is now given by (8)) describes precisely the compact subsets M of $J^0_{\alpha,\beta}$: in particular, one verifies easily that the imbedding of $J_{\alpha,\beta}$ into $J_{\alpha-\varepsilon,\gamma}$ is compact. It is due to these facts that the behaviour of the superposition operator in Hölder spaces is quite different from that in Lebesgue spaces. A detailed analysis of the superposition operator in the spaces $J_{\alpha,\beta}$ and $J_{\alpha,\infty}$ will be carried out in the following section.

2. The superposition operator

Let f be a real function on $[0, 1] \times \mathbf{R}$, and let $Fx(s) = f(s, x(s))$ be the corresponding superposition operator. If one considers the operator F between two normed spaces X and Y, many of its analytical and topological properties can be described by means of the *growth function*

$$\mu_F(r) = \sup \{\|Fx \mid Y\| : \|x \mid X\| \leqq r\}.$$

For example, the relation $\mu_F(r) = O(r)$ gives information on invariant balls for F, while the relation $\mu_F(r) = o(r)$ $(r \to 0$ or $r \to \infty)$ means, roughly speaking, that F is differentiable, respectively, at zero or infinity.

It is easy to see that the operator F maps the space C into itself if and only if f is continuous on the product $[0, 1] \times \mathbf{R}$ (see, e.g., [37]); moreover, in this case F is always continuous and bounded, and $\mu_F(r) = \max \{|f(s, u)| : 0 \leqq s \leqq 1, |u| \leqq r\}$. This shows that the operator F exhibits in the space C a similar behaviour as in Lebesgue spaces, where also the acting condition $F(L_p) \subseteq L_q$ $(q < \infty)$ implies already the boundedness and continuity of F (see e.g. [22: §17]). In Hölder spaces H_α, however, the situation is completely different: In fact, the acting condition $F(H_\alpha) \subseteq H_\beta$ does not imply the continuity or boundedness of F (see [6, 10]). Moreover, if F maps H_α into H_β, the generating function f need not be continuous, and hence F does not act in the space C [10]. If the function f does not depend on s, however, the acting condition $F(H_\alpha) \subseteq H_\beta$ does imply the boundedness of F, but F may still be discontinuous [3, 6]. Let us remark that the superposition operator has been studied in Hölder spaces by several authors [3, 6–14, 18, 27, 30, 39] but the analysis is far from being complete. For instance, one does not know necessary and sufficient conditions for the continuity of F on the whole space H_α.

Let us recall the following basic result [13, 14] which we shall need in the sequel.

Proposition 3: *The superposition operator F which is generated by the function f maps H_α (or H_α^0) into H_β and is bounded if and only if the relation*

$$\psi(r) := \sup_{0<\tau\leqq 1} \tau^{-\beta} W(r, \tau, r\tau^\alpha) < \infty \tag{14}$$

holds for any $r > 0$, where

$$W(r, \tau, \xi) = \sup \{|f(s, u) - f(t, v)| : \\ 0 \leqq s, t \leqq 1 \ \text{ with } \ |s - t| \leqq \tau;\ 0 \leqq |u|, |v| \leqq r \ \text{ with } \ |u - v| \leqq \xi\}. \tag{15}$$

In this case the growth function μ_F is given by

$$\mu_F(r) = \max \{\varphi(r), \psi(r)\}, \tag{16}$$

where

$$\varphi(r) = \sup \{|f(s, u)| : 0 \leqq s \leqq 1, |u| \leqq r\}. \tag{17}$$

We point out again that the boundedness of the operator F does not follow from the fact that F maps H_α into H_β; consequently, Prop. 3 does not provide just a necessary and sufficient acting condition $F(H_\alpha) \subseteqq H_\beta$ (or $F(H_\alpha^0) \subseteqq H_\beta$). Moreover, under the hypotheses of Prop. 3 the operator F need not be continuous on H_α; as mentioned above, global continuity criteria for F in terms of the function f are not known.

Similarly, the following holds.

Proposition 4: *The superposition operator F which is generated by the function f maps H_α^0 (or H_α) into H_β^0 and is bounded if and only if, in addition to* (14), *the relation*

$$\lim_{\tau,\varepsilon\to 0} \tau^{-\beta} W(r, \tau, \varepsilon\tau^\alpha) = 0 \left(\textit{resp.} \lim_{\tau\to 0} \tau^{-\beta} W(r, \tau, r\tau^\alpha) = 0\right)$$

holds for any $r > 0$. In this case, the growth function μ_F is again given by (16), *and F is continuous (resp. completely continuous).*

We now pass to the problem of characterizing the functions f which generate a Lipschitz continuous superposition operator; we confine ourselves to the case $\alpha = \beta$.

Proposition 5: *The superposition operator F which is generated by the function f satisfies a Lipschitz condition in H_α (or H_α^0) if and only if f has the form $f(\cdot, u) = g(\cdot)\, u + h(\cdot)$ with $g, h \in H_\alpha$ (resp. H_α^0).*

Proof: The fact that such a function generates a Lipschitz continuous superposition operator in H_α and H_α^0 is obvious. Conversely, suppose that F satisfies a Lipschitz condition in the norm (7), i.e.

$$|Fx(0) - Fy(0)| + j_{\alpha,\infty}(Fx - Fy) \leqq L[|x(0) - y(0)| + j_{\alpha,\infty}(x - y)] \tag{18}$$

for any $x, y \in H_\alpha$. Fix $\sigma, \tau \in [0, 1]$, $\sigma < \tau$, and $u_1, u_2, v_1, v_2 \in \mathbf{R}$, and define two functions x_1 and x_2 by

$$x_i(s) = \begin{cases} u_i & \text{if } 0 \leqq s \leqq \sigma, \\ \dfrac{u_i - v_i}{|\sigma - \tau|^\alpha}(s - \tau)^\alpha + v_i & \text{if } \sigma \leqq s \leqq \tau, \\ v_i & \text{if } \tau \leqq s \leqq 1 \end{cases}$$

$(i = 1, 2)$. Obviously,

$$|(x_1 - x_2)(s) - (x_1 - x_2)(t)|$$
$$= \frac{1}{|\sigma - \tau|^\alpha} [(u_1 - v_1)(s - \tau)^\alpha - (u_2 - v_2)(s - \tau)^\alpha$$
$$- (u_1 - v_1)(t - \tau)^\alpha + (u_2 - v_2)(t - \tau)^\alpha]$$
$$\leqq \frac{1}{|\sigma - \tau|^\alpha} |u_1 - v_1 - u_2 + v_2| \, |(s - \tau)^\alpha - (t - \tau)^\alpha|$$

for $\sigma \leqq s \leqq t \leqq \tau$; since $|(s - \tau)^\alpha - (t - \tau)^\alpha| \leqq |s - t|^\alpha$, it follows that $j_{\alpha,\infty}(x_1 - x_2) \leqq |u_1 - v_1 - u_2 + v_2|/|\sigma - \tau|^\alpha$. Substituting $x = x_1$ and $y = x_2$ in (18) gives

$$|f(0, u_1) - f(0, u_2)| + \frac{|f(\sigma, u_1) - f(\sigma, u_2) - f(\tau, v_1) + f(\tau, v_2)|}{|\sigma - \tau|^\alpha}$$
$$= L\left[|u_1 - u_2| + \frac{|u_1 - v_1 - u_2 + v_2|}{|\sigma - \tau|^\alpha}\right].$$

Multiplying both sides of this inequality by $|\sigma - \tau|^\alpha$ and letting $|\sigma - \tau|$ tend to zero yields (by the continuity of $f(\cdot, u)$)

$$|f(\sigma, u_1) - f(\sigma, u_2) - f(\sigma, v_1) + f(\sigma, v_2)| \leqq L \, |u_1 - u_2 - v_1 + v_2| . \qquad (19)$$

Now define the function φ_σ on $\mathbf{R}$ by $\varphi_\sigma(w) = f(\sigma, w) - f(\sigma, 0)$; setting $u_1 = a + b$, $u_2 = a$, $v_1 = b$, $v_2 = 0$ in (19), we get $\varphi_\sigma(a + b) = \varphi_\sigma(a) + \varphi_\sigma(b)$, i.e. φ_σ is additive. Further, setting $v_1 = v_2 = 0$ in (19) gives $|\varphi_\sigma(u_1) - \varphi_\sigma(u_2)| \leqq L \, |u_1 - u_2|$, i.e. φ_σ is continuous. Thus φ_σ is linear which means that $\varphi_\sigma(w) = g(\sigma) \, w$, hence, as claimed, $f(s, u) = \varphi_s(u) + f(s, 0) = g(s) \, u + h(s)$, where both functions $h = F\theta$ and $g = F1 - F\theta$ belong to H_α (with θ being the zero function).

In case F maps $H_\alpha{}^0$ into itself the proof remains almost unchanged; the only difference is that instead of the functions x_1 and x_2 one must consider functions $x_1{}^\varepsilon$ and $x_2{}^\varepsilon$, defined by

$$x_i{}^\varepsilon(s) = \begin{cases} u_i & \text{if } 0 \leqq s \leqq \sigma, \\ \dfrac{u_i - v_i}{(\sigma - \tau)^{\alpha+\varepsilon}} (s - \tau)^{\alpha+\varepsilon} + v_i & \text{if } \sigma \leqq s \leqq \tau, \\ v_i & \text{if } \tau \leqq s \leqq 1, \end{cases}$$

with ε positive ∎

We remark that the basic idea of this proof is taken from [26], where an analogous result is proved in the space H_1 of Lipschitzian functions.

We are now interested in conditions on f which ensure that F is condensing with respect to the measure of noncompactness (6) in $H_\alpha{}^0$; we confine ourselves again to the case $\alpha = \beta$.

Proposition 6: *Suppose that the superposition operator F which is generated by the function f acts in $H_\alpha{}^0$ and is bounded. Then F is $k(r)$-condensing on the ball $B_r(\theta)$, where*

$$k(r) = \sup_{0<\varrho\leqq r} \overline{\lim_{\tau\to 0}} \, \varrho^{-1}\tau^{-\alpha} W(r, \tau, \varrho\tau^\alpha) \qquad (20)$$

with W given by (15).

Proof: Let M be a subset of $B_r(\theta)$ and $a = \chi(M \mid H_\alpha^0)$ its Hausdorff measure of noncompactness. Further, given $\eta > 0$, choose $\tau_\eta > 0$ such that $\tau^{-\alpha} W(r, \tau, \varrho\tau^\alpha) \leqq (k(r) + \eta)\, \varrho$ $(0 < \tau < \tau_\eta)$ and $\omega(x, \tau) \leqq (a + \eta)\, \tau^\alpha$ $(x \in M,\ 0 < \tau < \tau_\eta)$. Then for any $x \in M$ the inequality

$$\tau^{-\alpha}\omega(Fx, \tau) \leqq \tau^{-\alpha} W\big(r, \tau, \omega(x, \tau)\big) \leqq \tau^{-\alpha} W\big(r, \tau, (a + \eta)\, \tau^\alpha\big) \leqq \big(k(r) + \eta\big)(a + \eta)$$

holds $(0 < \tau < \tau_\eta)$, and hence, by Prop. 2, $\chi(FM \mid H_\alpha^0)$ does not exceed $\big(k(r) + \eta\big)$ $(a + \eta)$. Since $\eta > 0$ is arbitrary, we have $\chi(FM \mid H_\alpha^0) \leqq k(r)\, \chi(M \mid H_\alpha^0)$ as claimed ∎

As already pointed out above, the operator F is k-condensing in the space C iff it satisfies a Lipschitz condition (with Lipschitz constant k). The preceding two propositions show that in the space H_α^0 the operator F may very well be k-condensing without satisfying a Lipschitz condition. To give a very simple example, already such a "harmless" nonlinearity like $f(u) = |u|$ generates a superposition operator F which is 1-condensing, but does not satisfy a Lipschitz condition, by Prop. 2, in any of the spaces H_α or H_α^0 (the latter fact may be proved also directly by considering the functions $x_n(s) = s - 1/n$ and $y_n(s) = s$).

Let us now consider the superposition operator in the spaces $J_{\alpha,\beta}$ $(\alpha \leqq \beta < \infty)$.

Proposition 7: *Let the function f be continuous on $[0, 1] \times \mathbf{R}$ and let $\alpha\delta = \beta\gamma$. Suppose that for any $r \in (0, \infty)$ the estimate*

$$\tau^{-(\delta+1)} W(r, \tau, \tau^\alpha \varrho)^{\delta/\gamma} \leqq c(r, \tau, \lambda) + \lambda \tau^{-1} \varrho^{\beta/\alpha} \qquad \big(\lambda \in \Lambda(r),\ \varrho \leqq r\big) \tag{21}$$

holds for some set $\Lambda(r) \subseteqq (0, \infty)$, where W is given by (15), *and $c(r, \cdot, \lambda)$ is integrable on* $[0, 1]$ *for $r \in (0, \infty)$ and $\lambda \in \Lambda(r)$. Then the superposition operator F which is generated by f maps $J_{\alpha,\beta}$ into $J_{\gamma,\delta}$, is bounded, and satisfies*

$$\mu_F(r) \leqq \max\,\{\varphi(r), \psi(r)\}, \tag{22}$$

where φ is again given by (17), *and*

$$\psi(r) = \inf_{\lambda \in \Lambda(r)} \left[\int_0^1 c(r, \tau, \lambda)\, d\tau + \lambda r^{\beta/\alpha}\right]^{\gamma/\delta}.$$

Moreover, F is continuous and $k(r)$-condensing on each ball $B_r(\theta)$, where

$$k(r) \leqq \inf \Lambda(r). \tag{23}$$

Proof: Choose $x \in J_{\alpha,\beta}$ with $\|x \mid J_{\alpha,\beta}\| \leqq r$. For $\lambda \in \Lambda(r)$ we have

$$\int_0^1 \tau^{-(\delta+1)} \omega(Fx, \tau)^{\delta/\gamma}\, d\tau \leqq \int_0^1 \tau^{-(\delta+1)} W\big(r, \tau, \omega(x, \tau)\big)^{\delta/\gamma}\, d\tau$$

$$\leqq \int_0^1 c(r, \tau, \lambda)\, d\tau + \lambda \int_0^1 \tau^{-(\beta+1)} \omega(x, \tau)^{\beta/\alpha}\, d\tau = \int_0^1 c(r, \tau, \lambda)\, d\tau + \lambda j_{\alpha,\beta}(x),$$

hence

$$Fx \in J_{\gamma,\delta} \quad \text{and} \quad j_{\gamma,\delta}(Fx) \leqq \int_0^1 c(r, \tau, \lambda)\, d\tau + \lambda j_{\alpha,\beta}(x).$$

Since f is continuous, we have $\varphi(r) < \infty$, and thus (22) holds.

We show now that F is $k(r)$-condensing on the ball $B_r(\theta)$, with $k(r)$ as in (23). To this end, choose $M \subseteqq B_r(\theta)$, and let $a > \chi(M \mid J_{\alpha,\beta})$. Given $x \in M$ and $\lambda \in \Lambda(r)$, we have

$$\int_0^\sigma \tau^{-(\delta+1)}\omega(Fx, \tau)^{\delta/\gamma}\,d\tau \leqq \int_0^\sigma c(r, \tau, \lambda)\,d\tau + \lambda \int_0^\sigma \tau^{-(\beta+1)}\omega(x, \tau)^{\beta/\alpha}\,d\tau\,.$$

For σ sufficiently small, the right-hand side of this inequality does not exceed $\int_0^\sigma c(r, \tau, \lambda)\,d\tau + \lambda a^{\beta/\alpha}$, by Prop. 2; but this means that

$$\overline{\lim_{\sigma\to 0}} \sup_{x\in M} \int_0^\sigma \tau^{-(\delta+1)}\omega(Fx, \tau)^{\delta/\gamma}\,d\tau \leqq \lambda a^{\delta/\gamma},$$

and hence, again by Prop. 2, $\chi(FM \mid J_{\gamma,\delta}) \leqq \lambda a$. Since $a > \chi(M \mid J_{\alpha,\beta})$ is arbitrary, we have $\chi(FM \mid J_{\gamma,\delta}) \leqq k(r)\,\chi(M \mid J_{\alpha,\beta})$ as claimed.

It remains to prove the continuity of F. Since F is condensing, it suffices to show that F maps any sequence which converges in $J_{\alpha,\beta}$ into a uniformly convergent sequence. But this follows from the continuity of f ∎

We do not know if equality holds in the estimates (22) and (23), and whether or not condition (21) is necessary for F to map $J_{\alpha,\beta}$ into $J_{\gamma,\delta}$.

To illustrate Prop. 7 we consider the case when the function (15) satisfies an estimate

$$W(r, \tau, \xi) \leqq a(r, \tau) + b(r)\,\xi^{\varkappa} \tag{24}$$

for some $\varkappa \in (0, \infty)$. Then condition (21) leads to the two relations

$$\int_0^1 \tau^{-(\delta+1)}a(r, \tau)^{\delta/\gamma}\,d\tau < \infty \tag{25}$$

and

$$b(r)\,\tau^{-(\delta+1)+\varkappa\beta}\varrho^{\varkappa\delta/\gamma} \leqq c(r, \tau, \lambda) + \lambda\tau^{-1}\varrho^{\beta/\alpha} \qquad (0 \leqq \varrho \leqq r), \tag{26}$$

where

$$\int_0^1 c(r, \tau, \lambda)\,d\tau < \infty \qquad (r \in (0, \infty), \lambda \in \Lambda(r)). \tag{27}$$

The first relation is simply analyzed: In fact, it holds for $\gamma \leqq \gamma_0$ and $\delta \geqq \delta_0$, with two positive numbers γ_0 and δ_0, possibly except for the extremal values γ_0 and δ_0 themselves. The second relation (26) is harder to analyze; for sake of definiteness, let us introduce the auxiliary function

$$\Phi(\mu, \nu, u, \lambda, r) = \max_{0\leqq\varrho\leqq r} \{u\varrho^{\mu} - \lambda\tau^{-1}\varrho^{\nu}\}\,.$$

If $c(r, \cdot, \lambda)$ is integrable, as claimed in (27), relation (26) may be written as

$$\int_0^1 \Phi\left(\frac{\varkappa\delta}{\gamma}, \frac{\beta}{\alpha}, b(r)\,\tau^{-(\delta+1)+\varkappa\beta}, \lambda, r\right) d\tau < \infty\,. \tag{28}$$

Hence, this integrability condition has to be verified in order to prove (26). By an elementary calculation one obtains the explicit formula

$$\Phi(\mu, \nu, u, \lambda, r) = \begin{cases} ur^{\mu} - \lambda\tau^{-1}r^{\nu} & \text{if } \mu \leqq \nu \text{ and } \nu\lambda r^{\nu-\mu} \leqq \mu u\tau \\ (\nu - \mu)\,(\nu^{-1}u)^{\nu/(\nu-\mu)}\,(\mu^{-1}\tau^{-1}\lambda)^{\mu/(\mu-\nu)} & \text{if } \mu \leqq \nu \text{ and } \nu\lambda r^{\nu-\mu} > \mu u\tau \\ \max\,\{0, ur^{\mu} - \lambda\tau^{-1}r^{\nu}\} & \text{if } \mu > \nu\,. \end{cases} \tag{29}$$

By means of this formula it is possible to analyze condition (28) more explicitly:

The case $\varkappa\alpha > \gamma$: If $\varkappa \leqq 1$, the condition $\nu\lambda r^{\nu-\mu} > \mu u\tau$ appearing in (29) reads $\tau^{\varkappa\beta-\delta} < \lambda r^{(1-\varkappa)\beta/\alpha}/\varkappa b(r)$ and hence implies that, for any $\lambda > 0$ and sufficiently small $\tau > 0$,

$$\Phi(\varkappa\delta/\gamma, \beta/\alpha, b(r)\, \tau^{-(\delta+1)+\varkappa\beta}, \lambda, r) = c\tau^{-(\delta+1-\varkappa\beta)/(1-\varkappa)+\varkappa/(1-\varkappa)}$$

with some constant $c > 0$. Since $-(\delta + 1 - \varkappa\beta)/(1 - \varkappa) + \varkappa/(1 - \varkappa) > -1$, condition (28) holds, and $\Lambda(r) = (0, \infty)$. On the other hand, if $\varkappa > 1$, the expression $ur^\mu - \lambda\tau^{-1}r^\nu$ under the maximum sign in (29) becomes negative if and only if $\tau < \lambda u^{-1}r^{\nu-\mu}$; it follows that, for any $\lambda > 0$ and sufficiently small $\tau > 0$,

$$\Phi(\varkappa\delta/\gamma, \beta/\alpha, b(r)\, \tau^{-(\delta+1)+\varkappa\beta}, \lambda, r) = 0.$$

Consequently, condition (28) holds again, and $\Lambda(r) = (0, \infty)$. To summarize, under the hypotheses $\varkappa\alpha > \gamma$, (24) and (25), the operator F maps $J_{\alpha,\beta}$ into $J_{\gamma,\delta}$ and has the properties mentioned in Prop. 7.

The case $\varkappa\alpha = \gamma$: The formula for Φ shows that then $\Phi(\varkappa\delta/\gamma, \beta/\alpha, b(r)\, \tau^{-(\delta+1)+\varkappa\beta}, \lambda, r) = c\tau^{-1}$ for sufficiently small $\tau > 0$ (with some constant c), and hence integrability fails except for $c = 0$. The latter condition holds only if $\mu \geqq \nu$ or, equivalently, if $\delta \geqq \beta$ and $ur^\mu \leqq \lambda\tau^{-1}r^\nu$ hence $\lambda \geqq \beta\delta^{-1}b(r)\, r^{(\delta-\beta)/\alpha}$. Consequently, the operator F maps $J_{\alpha,\beta}$ into $J_{\gamma,\delta}$ in this case only for $\delta \geqq \beta$, and the set $\Lambda(r)$ reduces to the interval $[\beta\delta^{-1}b(r)\, r^{(\delta-\beta)/\alpha}, \infty)$.

The case $\varkappa\alpha < \gamma$: The formula for Φ shows immediately that the function $\Phi(\varkappa\delta/\gamma, \beta/\alpha, b(r)\, \tau^{-(\delta+1)+\varkappa\beta}, \lambda, r)$ is never integrable on $[0, 1]$.

We may summarize our discussion as follows: *Condition* (24) *guarantees that the superposition operator F maps $J_{\alpha,\beta}$ into $J_{\gamma,\delta}$ if either $\varkappa\alpha > \gamma$, or $\varkappa\alpha = \gamma$ and $\delta \geqq \beta$. In the first case F is completely continuous, in the second case only condensing.*

3. Singular integral equations

Combining the results of the previous section on the superposition operator with boundedness results on linear singular integral operators (more precisely, upper estimates for the norm and the radius of the essential spectrum of such operators in spaces of type $J_{\alpha,\beta}$), one obtains various existence theorems for the singular equation (2). In this section we shall give just one simple result which, however, cannot be obtained by the classical theorems of Schauder and Banach-Caccioppoli. Consider the nonlinear integral equation (2) or, equivalently, the nonlinear operator equation (3). Suppose that the kernel k of the linear part (5) satisfies the following assumptions:

(S1) k is continuous on $[0, 1] \times [0, 1]$ with $k(0, 0) = k(1, 1) = 0$;

(S2) $|k(s', 0) - k(s'', 0)|\, |\ln |s' - s''|| = o(|s' - s''|^\alpha)$ as $|s' - s''| \to 0$;

(S3) $|k(0, t') - k(0, t'')| = o(|t' - t''|^\alpha)$ as $|t' - t''| \to 0$.

These conditions ensure the boundedness of the operator (5) in the "little" Hölder space $H_\alpha{}^0$ $(0 < \alpha < 1)$ [21: p. 188]). In order to apply Sadovskij's fixed point principle to equation (2), it is necessary to find estimates for both the norm and essential norm of the operator (5) which are, in general, different. Without going into details, we just mention that, for example, the classical Hilbert operator

$$Sy(s) = \int_0^1 \cot \pi(s - t)\, y(t)\, dt$$

has essential norm $\chi(S) = 1$, while the operator norm $\|S\|$ heavily depends on the space involved (see [32] for Hölder spaces and [33] for Lebesgue spaces) and may be, generally speaking, much larger than $\chi(S)$.

Suppose now that the condition

$$\lim_{\tau,\varepsilon\to 0} \tau^{-\alpha} W(r, \tau, \varepsilon\tau^\alpha) = 0$$

holds, with W given by (15). By Prop. 4, this implies that the superposition operator F acts in $H_\alpha{}^0$ and is continuous. Let φ be defined as in (17), and let

$$\psi(r) = \sup_{0<\tau\leq 1} \tau^{-\alpha} W(r, \tau, r\tau^\alpha) \quad \text{and} \quad k(r) = \sup_{0<\varrho\leq r} \overline{\lim_{\tau\to 0}}\, \varrho^{-1}\tau^{-\alpha} W(r, \tau, \varrho\tau^\alpha);$$

by Prop. 4 and 6, F is bounded in $H_\alpha{}^0$ if $\psi(r) < \infty$, and condensing in $H_\alpha{}^0$ if $k(r) < \infty$. Applying Sadovskij's fixed point principle to the operator equation (3), we arrive at the following existence result.

Proposition 8: *Suppose that, for some $r > 0$, the inequalities*

$$\lambda \|S\|\, \varphi(r) \leq r, \qquad \lambda \|S\|\, \psi(r) \leq r, \qquad \lambda\chi(S)\, k(r) < 1$$

hold. Then equation (2) *has a solution in the ball* $\|x \mid H_\alpha{}^0\| \leqq r$.

Similarly, using Prop. 7 instead of Prop. 4—6 one obtains existence theorems for singular equations in the spaces $J_{\alpha,\beta}$; we do not discuss this in detail.

Let us still make an additional remark. In this paper we discussed only the applicability of Sadovskij's fixed point principle to equation (2). However, one could arrive at similar conclusions by means of the topological degree theory for condensing vector fields developed by Sadovskij in [34—36]. Moreover, for studying equation (2), one could use analogues of the Leray—Schauder alternative principle, of Krasnosel'skij's fixed point principle for asymptotically linear operators, or of various bifurcation theorems, and similar methods which build on differentiability conditions. To this end, necessary and sufficient differentiability criteria for the superposition operator (4) in Hölder spaces, as given in [2], are useful.

In this connection, the following fact is worth mentioning: In the case of "classical" singular integral operators S which act in L_2, the composite operator $A = SF$ is completely continuous (as operator in L_2) on each ball in H_α, by classical compactness criteria in Lebesgue spaces. Consequently, for such equations one gets solvability results already by means of the Schauder principle. On the other hand, not every singular integral operator acts in L_2: just consider operators of the form $S = S_0 + S_1$, where S_0 is "classical", and S_1 is a compact operator in H_α which is not defined on L_2. Furthermore, we point out that, in order to apply topological degree techniques, one must deal with domains which are representable as closure of open sets; this excludes the possibility of considering compact sets of the above mentioned type (i.e. balls in H_α) as subsets of Lebesgue spaces.

4. Concluding remarks

In this final section we want to indicate a possible extension of the preceding results to spaces of smooth functions which are suitable for (ordinary or partial) differential equations. More precisely, let us consider the space $H_{n,\alpha}$ ($= C^{n+\alpha}$) of all functions $x \in C^n$ for which the norm

$$\|x \mid H_{n,\alpha}\| = \max \{\|x \mid C\|, \|x^{(n)} \mid H_\alpha\|\}$$

makes sense and is finite, or the space $J_{n,\alpha,\beta}$ of all functions $x \in C^n$ for which the norm

$$\|x \mid J_{n,\alpha,\beta}\| = \max \{\|x \mid C\|, \|x^{(n)} \mid J_{\alpha,\beta}\|\}$$

makes sense and is finite. We do not describe here in detail how the results of the first section carry over to such spaces; it just suffices to replace x by $x^{(n)}$.

It is natural to search for acting conditions (sufficient or necessary) for the superposition operator (4) in such spaces. This is a rather delicate problem; in particular, the results of the second section do not carry over immediately. Sufficient conditions, of course, are easily found. For example, from the formula for higher derivatives of composite functions (see e.g. [20, 22])

$$\frac{d^{(m)}}{ds^{(m)}} f(s, x(s)) = \sum_{\alpha_0+\alpha_1+2\alpha_2+\cdots+m\alpha_m=m} A_m^{-1} f_{s^{\alpha_0} u^{\alpha_1+\cdots\alpha_m}}^{(\alpha_0+\alpha_1+\cdots\alpha_m)} [s, x(s)]\, x'(s)^{\alpha_1} \cdots x^{(m)}(s)^{\alpha_m}$$

(with $A_m = [\alpha_0!(1!)^{\alpha_1}\alpha_1! \ldots (m!)^{\alpha_m}\alpha_m!]/m!$) it follows that the operator F acts between $H_{n,\alpha}$ and $H_{m,\beta}$ if either $n > m$ or $n = m$ and $\alpha \geqq \beta$, provided all partial derivatives $f_{s^\alpha u^\beta}^{(m)}$ $(\alpha + \beta = m)$ of the function f are continuous. In this case F will be automatically bounded, continuous, and condensing; an exact computation of its growth function μ_F, however, has not been carried out yet.

We still point out that the existence of the partial derivatives $f_{s^\alpha u^\beta}^{(m)}$ $(\alpha + \beta = m)$ of f is, in general, not necessary for the acting condition $F(H_{n,\alpha}) \subseteqq H_{m,\beta}$, but only under the additional assumption that F be bounded. In this connection, we refer to the necessary and sufficient conditions for F to be bounded in the space C^1 given in [11]. Moreover, we recall a recent result of J. BRÜNING [15] which states that, if the function f generates a continuous superposition operator F from C^n into C^m $(m \leqq n)$, it must be m-times continuously differentiable on the product $[0, 1] \times \mathbf{R}$.

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