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The points and localisations of the topos of M-sets

Ilia Pirashvili

Abstract. In previous papers, we were able to prove that, much like in classical algebraic geometry, it is possible to recover our monoid scheme X from the topos $\mathsf{Qcoh}(X)$. This was achieved using topos points and localisations of $\mathsf{Qcoh}(X)$. With this philosophy in mind, the aim of this paper is to study the topos of M-sets for non-commutative monoids, especially their points and localisations. We will classify the points and localisations of M-sets for finite monoids in terms of the idempotent elements of M and idempotent ideals of M, respectively. Some of the results obtained in this paper can already be found in previous works, in direct or indirect forms. See the last part of the introduction.

1. Introduction

It is well known that the category of quasi-coherent sheaves over a quasi-separated scheme (of rings) is an abelian category, from which one can reconstruct the original scheme [2, 7, 20]. We were able to prove a similar result for the topos of quasi-coherent sheaves over a monoid scheme X, under some assumptions on X [17, 18].

One of our next overarching goals is to study "non-commutative monoid schemes". Much like in classical algebraic geometry, a sensible approach for this seems to be via studying quasi-coherent sheaves over them. The reason for this is that in the affine case, quasi-coherent sheaves are just M-sets, which is a topos irrespective of the commutativity of M. Throughout the paper, we shall restrict ourselves to the affine (non-commutative) case, though we will give a brief reminder of the non-affine commutative setting, in the introduction. It should also be pointed out that monoid schemes can be seen as a natural generalisation of toric varieties. As such, a better understanding of the non-commutative case could also be seen as progress towards understanding "non-commutative toric varieties".

We recall the following: let M be a commutative monoid and $\mathsf{Spec}(M)$ its set of prime ideals. Like for classical schemes, one can define the Zariski topology on $\mathsf{Spec}(M)$ as well and it too admits a structure sheaf $\mathcal{O}$, defined in much the same way as in classical algebraic geometry. This is called an *affine monoid scheme* and one obtains *monoid schemes* by gluing affine ones [6]. Monoid schemes play an important role in $\mathbb{F}_1$-mathematics and K-theory; see, for example, [3, 5, 6].

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As we noted in [17], the category $\mathfrak{Qc}(X)$ of quasi-coherent sheaves over a monoid scheme X, satisfying some finiteness condition, is a topos. In the case when $X = \mathsf{Spec}(M)$, for a commutative monoid M, the category $\mathfrak{Qc}(X)$ is the topos of M-sets. Our reconstruction theorem in the affine case says that if M is finitely generated and commutative, there are bijections

$$\mathsf{Spec}(M) \to \mathsf{F}(\mathscr{S}\mathrm{ets}_M) \quad \text{and} \quad \mathsf{Off}\left(\mathsf{Spec}(M)\right) \to \mathfrak{Loc}(\mathscr{S}\mathrm{ets}_M),$$

where $\mathscr{S}\mathrm{ets}_M$ is the topos of right M-sets and $\mathsf{Off}(X)$ is the set of all open subsets of a topological space X. For a topos $\mathscr{E}$, we denote with $\mathfrak{Loc}(\mathscr{E})$ the set of all localisations of $\mathscr{E}$ and with $\mathsf{F}(\mathscr{E})$ the set of isomorphism classes of the topos points of $\mathscr{E}$ [17, 18]. The structure sheaf $\mathcal{O}_X$, $X = \mathsf{Spec}(M)$ associates with an open set U of X, the centre of the localisation category corresponding to U.

Since points, localisations and the centre make sense for any topos $\mathscr{E}$, we can try to define a topology on $\mathsf{F}(\mathscr{E})$ by declaring a subset U of $\mathsf{F}(\mathscr{E})$ to be open if there exists a localisation $\mathscr{T}$ of $\mathscr{E}$ such that $\mathsf{p}_*(S) \in \mathscr{T}$, for any point $\mathsf{p} : \mathscr{S}\mathrm{ets} \to \mathscr{E}$ of U and any set S. Of course, we do not expect this to always work: $\mathsf{F}(\mathscr{E})$ or $\mathfrak{Loc}(\mathscr{E})$ could be a class and not a set, or on the other side, a topos might not have any points. But even under the assumption of enough points, there are other difficulties as well. For example, when $\mathscr{E} = \mathbf{Sh}(X)$ is the topos of sheaves over a sober topological space X, then $\mathsf{F}(\mathscr{E})$ is in a one-to-one correspondence with the points of X. In general, however, there are more localisations of $\mathbf{Sh}(X)$ than the open subsets of X.

The problem of equipping $\mathsf{F}(\mathscr{E})$ with a topology is, of course, not new and already appeared in [8, Section 2.6], where a topological space structure on the set $\mathsf{F}(\mathscr{E})$ is defined based on the subobjects of the terminal object of $\mathscr{E}$. In the case when $\mathscr{E}$ is the topos $\mathbf{Sh}(X)$ of sheaves over a sober topological space X, the space constructed in [8, Section 2.6] is homeomorphic to X; see [8, Remark 2.6.2]. However, in many other cases, the above topology can be quite trivial. Particularly, when $\mathscr{E} = \mathscr{S}\mathrm{ets}_M$ is the topos of right M-sets, the obtained topological space will only have two open sets, though $\mathsf{F}(\mathscr{S}\mathrm{ets}_M)$ can be arbitrarily big. Hence, it is natural to search for other topologies on $\mathsf{F}(\mathscr{E})$.

In recent years, there was a considerable attention to the problem of understanding the points and possible topological structures associated with the topos points of $\mathscr{S}\mathrm{ets}_M$. For example, Connes–Consani [4] and Le Bruyn [13] studied two interesting topologies in the case when $M = \mathbb{N}_+^\times$ is the multiplicative monoid of strictly positive natural numbers. See also a related paper by Hemelaer [9].

In this paper, we study the localisations and points of the topos $\mathscr{S}\mathrm{ets}_M$ of M-sets, for M being a noncommutative monoid. We first consider the case when M is a left zero monoid. That is, $M = \{1\} \cup S$, where S is any set and the multiplication is given by $st = s$ for all $s, t \in S$. We assume M is non-commutative; that is $|S| \geq 2$. Then, the topoi $\mathscr{S}\mathrm{ets}_M$ and ${}_M\mathscr{S}\mathrm{ets}$ of right and left M-sets are quite different.

For example, the cardinality of the subobject classifier of $\mathscr{S}\mathrm{ets}_M$ is equal to $1 + 2^{|S|}$ (see equality (4.4.1)), while the subobject classifier of ${}_M\mathscr{S}\mathrm{ets}$ only has three elements (see Proposition 4.4.3). As it turns out, however, $\mathsf{F}({}_M\mathscr{S}\text{ets}) = \mathsf{F}(\mathscr{S}\text{ets}_M)$ and $\mathsf{F}(\mathscr{S}\text{ets}_M)$ is independent from S. In more concrete terms, we prove that both topoi $\mathscr{S}\text{ets}_M$ and ${}_M\mathscr{S}\text{ets}$ have exactly two points (up to isomorphism) (see Corollary 4.1.3 and Lemma 4.4.1) and three localisations (see Propositions 4.4.2 and 4.4.3) and in both cases $\mathcal{O}$ is the constant sheaf corresponding to the trivial monoid. This shows that, in general, even if $\mathsf{F}(\mathscr{E})$ is a well-defined set, it carries limited information on $\mathscr{E}$.

There are some interesting questions that arise from these results. One of these questions is: Are localisations (resp., isomorphism classes of points) of the topoi $\mathscr{S}\text{ets}_M$ and ${}_M\mathscr{S}\text{ets}$ bijective sets?

The main results of this work give an affirmative answer to the above question, for finite M. We achieve this by proving that for finite M, the isomorphism classes of points of $\mathscr{S}\text{ets}_M$ is in a one-to-one correspondence with Green's $\mathscr{J}$-equivalence classes of idempotents (that is, $e\mathscr{J}f$ if and only if $MeM = MfM$). We also prove that for a finite monoid M, there is a bijection from the set of localisations $\mathfrak{Loc}(\mathscr{S}\text{ets}_M)$ to the set of all two-sided idempotent ideals of M. It follows that as topological spaces, $\mathsf{F}({}_M\mathscr{S}\text{ets})$ and $\mathsf{F}(\mathscr{S}\text{ets}_M)$ are both homeomorphic to the ordered topology of the poset of principal idempotent ideals.

The paper is organised as follows: Section 2 recollects some classical notions and facts related to localisations and points of a topos. In Section 3, we relate idempotents of a monoid M to the points of the topos $\mathscr{S}\text{ets}_M$ and recall classical constructions of a category of idempotents $\mathcal{I}(M)$, also known as the Karoubi envelope of M, considered a one-object category. In Section 4, we consider the case when M is a left zero monoid. We completely describe all the points and localisations of the topoi $\mathscr{S}\text{ets}_M$ and ${}_M\mathscr{S}\text{ets}$. Section 5 contains our main result concerning the points of the topos $\mathscr{S}\text{ets}_M$ for finite M: it says that the categories $\mathcal{I}(M)$ and $\mathbf{Pts}(\mathscr{S}\text{ets}_M)$ are contravariantly equivalent. Finally, Section 6 deals with the description of the localisations of $\mathscr{S}\text{ets}_M$ in terms of the idempotent ideals of M and some of its consequences.

As pointed out by the referee, some of the results obtained in this paper were already known and written either directly or indirectly. This was not known to me previously and so, the proofs often differ as well. The aim of this paragraph is to give an account of these results. More details on the exact relation to the previous work can be found before each of the mentioned postulates: (1) Proposition 5.5.2; (2) Proposition 5.5.3 (i); (3) Theorem 5.6.1; (4) Proposition 6.1.2; (5) Corollary 6.3.2.

2. Preliminaries

The aim of this section is to recall some classical facts and definitions related to topoi and their localisations and points. It contains a small subsection (Section 2.3), following [17], which explains the link between the prime ideals of a commutative monoid M and the points of $\mathscr{S}\text{ets}_M$.

2.1. General facts on localisations

Recall that a *localisation* $\mathscr{T}$ of a Grothendieck topos $\mathscr{E}$ is a full subcategory of $\mathscr{E}$, such that the following conditions hold:

(i) If x belongs to $\mathscr{T}$ and $y \in \mathscr{E}$ is isomorphic to x, then y belongs to $\mathscr{T}$.

(ii) The inclusion $\iota : \mathscr{T} \to \mathscr{E}$ has a left adjoint $\rho : \mathscr{E} \to \mathscr{T}$, called the *localisation functor*.

(iii) The localisation functor ρ respects finite limits.

It is well known that in this case, $\mathscr{T}$ is also a Grothendieck topos. We denote by $\mathfrak{Loc}(\mathscr{E})$ the poset of all localisations of $\mathscr{E}$.

Let M be a monoid. The category of left (resp., right) M-sets is denoted by ${}_M\mathscr{S}\mathrm{ets}$ (resp., $\mathscr{S}\mathrm{ets}_M$). Both of them are topoi. The subobject classifier Ω^M of the topos $\mathscr{S}\mathrm{ets}_M$ is the collection of all right ideals of M [14]. The action of M on Ω^M is given by $(\mathfrak{m}, a) \mapsto (\mathfrak{m} : a)$, where for a right ideal $\mathfrak{m}$ and an element $a \in M$, we set

$$(\mathfrak{m} : a) = \{x \in M \mid ax \in \mathfrak{m}\}.$$

The "truth" map $t : 1 \to \Omega^M$ is given by $t(1) = M \in \Omega^M$.

Let us recall the notion of a Grothendieck topology on a monoid and that of a sheaf over said Grothendieck topology. A *Grothendieck topology* (or simply *topology*) on a monoid M is a collection of right ideals $\mathcal{F}$, such that

(T1) $M \in \mathcal{F}$,

(T2) if $\mathfrak{a} \in \mathcal{F}$ and $m \in M$, then $(\mathfrak{a} : m) \in \mathcal{F}$,

(T3) if $\mathfrak{b} \in \mathcal{F}$ and $\mathfrak{a}$ is a right ideal of M with $(\mathfrak{a} : b) \in \mathcal{F}$ for any $b \in \mathfrak{b}$, then $\mathfrak{a} \in \mathcal{F}$.

From these conditions, we get the following result (see, for example, [18, 23]):

(i) If $\mathfrak{a} \subseteq \mathfrak{b}$ are right ideals and $\mathfrak{a} \in \mathcal{F}$, then $\mathfrak{b} \in \mathcal{F}$.

(ii) If $\mathfrak{a}, \mathfrak{b} \in \mathcal{F}$, then $\mathfrak{a} \cap \mathfrak{b} \in \mathcal{F}$.

(iii) If $\mathfrak{a}, \mathfrak{b} \in \mathcal{F}$, then $\mathfrak{a}\mathfrak{b} \in \mathcal{F}$.

(iv) $\mathcal{F}$ is an M-subset of Ω^M.

A right M-set A is called an $\mathcal{F}$*-sheaf* if the restriction map

$$A \to \mathsf{Hom}_M(\mathfrak{a}, A), \quad a \mapsto f_a,$$

is a bijection for every $\mathfrak{a} \in \mathcal{F}$. Here, $a \in A$ and $f_a \in \mathsf{Hom}_M(\mathfrak{a}, A)$, where $f_a(x) = xa$, $x \in \mathfrak{a}$. We let $\mathbf{Sh}(M, \mathcal{F})$ denote the topos of $\mathcal{F}$-sheaves.

The following obvious facts are well known: for any monoid M, there is a minimal $\mathcal{F}_{\min}$ and a maximal $\mathcal{F}_{\max}$ among all Grothendieck topologies, where

$$\mathcal{F}_{\min} = \{M\}, \quad \mathcal{F}_{\max} = \Omega^M.$$

We also have $\mathbf{Sh}(M, \mathcal{F}_{\min}) = \mathscr{S}\mathrm{ets}_M$ and $\mathbf{Sh}(M, \mathcal{F}_{\max}) \cong \{\mathbf{1}\}$, where $\{\mathbf{1}\}$ is the trivial category with one object and one morphism.

It is well known (see, for example, [18, Lemma 2.4.1] and references therein) that there is an order-reversing bijection between the set of localisations of $\mathfrak{Loc}(\mathscr{S}\mathrm{ets}_M)$ and all Grothendieck topologies defined on the one object category associated with M. Under this bijection, the localisation corresponding to a topology $\mathcal{F}$ is the category $\mathbf{Sh}(M, \mathcal{F})$ of sheaves on $\mathcal{F}$.

2.2. Points and filtered M-sets

Recall that a *point* of a Grothendieck topos $\mathscr{E}$ is a geometric morphism $\mathsf{p} = (\mathsf{p}_*, \mathsf{p}^*) : \mathscr{S}\mathrm{ets} \to \mathscr{E}$ from the topos of sets $\mathscr{S}\mathrm{ets}$ to $\mathscr{E}$. The inverse image functor $\mathsf{p}^* : \mathscr{E} \to \mathscr{S}\mathrm{ets}$ preserves colimits and finite limits. It is also well known that conversely, for any functor $\mathsf{f} : \mathscr{E} \to \mathscr{S}\mathrm{ets}$ which preserves colimits and finite limits, one has $\mathsf{f} = \mathsf{p}^*$ for a uniquely defined point p. We let $\mathbf{Pts}(\mathscr{E})$ be the category of points of $\mathscr{E}$ and $\mathsf{F}(\mathscr{E})$ the isomorphism classes of the category $\mathbf{Pts}(\mathscr{E})$.

Instead of $\mathbf{Pts}(\mathscr{S}\mathrm{ets}_M)$ and $\mathsf{F}(\mathscr{S}\mathrm{ets}_M)$, we write $\mathbf{Pts}(M)$ and F_M. By Diaconescu's theorem [14], the category $\mathbf{Pts}(M)$ is equivalent to the category of filtered left M-sets. Recall that a left M-set A is called *filtered*, provided the functor

$$(-) \otimes_M A : {}_M\mathscr{S}\mathrm{ets} \to \mathscr{S}\mathrm{ets}$$

commutes with finite limits. The topos point of $\mathscr{S}\mathrm{ets}_M$, corresponding to a filtered left M-set A, is denoted by $\mathsf{p}_A = (\mathsf{p}_{A*}, \mathsf{p}_A^*)$. The inverse image functor $\mathsf{p}_A^* : \mathscr{S}\mathrm{ets}_M \to \mathscr{S}\mathrm{ets}$ is given by

$$\mathsf{p}_A^*(X) = X \otimes_M A,$$

while the direct image functor $\mathsf{p}_{A*} : \mathscr{S}\mathrm{ets} \to \mathscr{S}\mathrm{ets}_M$ sends a set Y to $\mathsf{Hom}_{\mathscr{S}\mathrm{ets}}(A, Y)$. The latter is considered a right M set via

$$(\alpha m)(a) := \alpha(ma).$$

Here, $\alpha \in \mathsf{Hom}_{\mathscr{S}\mathrm{ets}}(A, X)$, $a \in A$ and $m \in M$.

The following well-known fact [15, p. 24] is a very useful tool for checking whether a given M-set is filtered.

Lemma 2.2.1. *A left M-set A is filtered if and only if the following three conditions hold:*

(F1) $A \neq \emptyset$.

(F2) *If $m_1, m_2 \in M$ and $a \in A$ satisfies the condition*

$$m_1 a = m_2 a,$$

there exist $m \in M$ and $\tilde{a} \in A$, such that $m\tilde{a} = a$ and $m_1 m = m_2 m$.

(F3) *If $a_1, a_2 \in A$, there are $m_1, m_2 \in M$ and $a \in A$, such that $m_1 a = a_1$ and $m_2 a = a_2$.*

Example 2.2.2. Clearly, $A = M$ is always filtered and corresponds to the canonical point, denoted by p_M. Thus, p_M^* is the forgetful functor $\mathscr{S}\mathrm{ets}_M \to \mathscr{S}\mathrm{ets}$.

For a localisation $\mathscr{T}$ of $\mathscr{E}$ and $\mathsf{p} = (\mathsf{p}_*, \mathsf{p}^*) : \mathscr{S}\mathrm{ets} \to \mathscr{E}$ a topos point of $\mathscr{E}$, we write $\mathsf{p} \pitchfork \mathscr{T}$ if $\mathsf{p}_*(S) \in \mathscr{T}$ for every set $S \in \mathscr{S}\mathrm{ets}$.

Lemma 2.2.3. *Let $\mathcal{F}$ be a topology on a monoid M and A a filtered left M-set. Then,*

$$\mathsf{p}_A \pitchfork \mathbf{Sh}(M, \mathcal{F})$$

if and only if for any $\mathfrak{a} \in \mathcal{F}$, the canonical map $\mathfrak{a} \otimes_M A \to A$ is an isomorphism. Here, p_A denotes the point of $\mathscr{S}\mathrm{ets}_M$ corresponding to A.

Proof. Since the direct image functor $\mathscr{S}\mathrm{ets} \to \mathscr{S}\mathrm{ets}_M$ corresponding to the point p_A is given by $S \mapsto \mathsf{Hom}_{\mathscr{S}\mathrm{ets}}(A, S)$, we see that $\mathsf{p}_A \pitchfork \mathbf{Sh}(M, \mathcal{F})$ if and only if $\mathsf{Hom}_{\mathscr{S}\mathrm{ets}}(A, S)$ is an $\mathfrak{a}$-sheaf. This means that for all $\mathfrak{a} \in \mathcal{F}$, the canonical map

$$\mathsf{Hom}_{\mathscr{S}\mathrm{ets}}(A, S) \to \mathsf{Hom}_{\mathscr{S}\mathrm{ets}_M}\big(\mathfrak{a}, \mathsf{Hom}_{\mathscr{S}\mathrm{ets}}(A, S)\big)$$

is an isomorphism. The map in question is the same as

$$\mathsf{Hom}_{\mathscr{S}\mathrm{ets}}(A, S) \to \mathsf{Hom}_{\mathscr{S}\mathrm{ets}}(\mathfrak{a} \otimes_M A, S).$$

Since S is any set, Yoneda's lemma implies that this happens if and only if $\mathfrak{a} \otimes_M A \to A$ is an isomorphism. ■

2.3. Points and prime ideals

If M is commutative and $\mathfrak{p}$ is a prime ideal, then the localisation $M_\mathfrak{p}$ is filtered. In this way, one obtains an injective map

$$\mathsf{Spec}(M) \to \mathsf{F}_M.$$

Moreover, if M is finitely generated, the map is a bijection [17]. The inverse image functor corresponding to the point, associated with the filtered M-set $M_\mathfrak{p}$, sends a right M-set X to the localisation $X_\mathfrak{p}$, considered a set.

For a monoid M, denote by M^{com} the maximal commutative, and by M^{sl} the maximal semilattice quotient, respectively. As a semilattice is commutative by definition, we have natural surjective homomorphisms $M \to M^{\mathsf{com}} \to M^{\mathrm{sl}}$.

According to [16], for any commutative monoid M, the induced map

$$\mathsf{Spec}(M^{\mathrm{sl}}) \xrightarrow{\cong} \mathsf{Spec}(M)$$

is bijective, and furthermore, there is an injective map [16]

$$M^{\mathrm{sl}} \to \mathsf{Spec}(M^{\mathrm{sl}}) \cong \mathsf{Spec}(M).$$

This map is bijective if M is commutative and finitely generated.

Corollary 2.3.1. *If M is a finitely generated commutative monoid, then*

$$|\mathsf{F}_M| = |M^{\mathrm{sl}}|.$$

2.4. Induced points

Recall the following well-known fact.

Lemma 2.4.1. *Let $f : M \to M'$ be a monoid homomorphism. For any filtered left M-set A, the left M'-set $M' \otimes_M A$ is filtered.*

The M'-set constructed in the lemma is said to be *induced from A via the homomorphism f*. In this way, one obtains a functor

$$\mathbf{Pts}(f) : \mathbf{Pts}(M) \to \mathbf{Pts}(M'),$$

which induces the map

$$\mathsf{F}_f : \mathsf{F}_M \to \mathsf{F}_{M'}.$$

3. Idempotents of M and points of $\mathscr{S}\mathrm{ets}_M$

In this section, we show that any idempotent of a monoid M gives rise to a point of the topos $\mathscr{S}\mathrm{ets}_M$. This fact was recently observed in [10, 19]. One of our main results claims that if M is finite, any point of $\mathscr{S}\mathrm{ets}_M$ comes in this way; see Theorem 5.6.1 below. We will also recall the category of idempotents (see, for example, [22]) as it will play an important role in this paper.

3.1. Points corresponding to idempotents

First, consider the case when $M = \{1, t\}$, $t^2 = t$. Thanks to Corollary 2.3.1, we see that F_M has only two elements: one corresponds to the filtered M-set M and the other to the singleton, which is easily seen to be a filtered M-set.

Take an arbitrary monoid M. Any idempotent $e \in M$ induces a homomorphism of monoids $\eta : \{1, t\} \to M$, where $t^2 = t$, given by $\eta(t) = e$. The singleton, which is filtered over $\{1, t\}$, induces a left filtered M-set; see Section 2.4. One easily sees that this is isomorphic to Me.

As a reminder, M^{sl} is the associated semilattice of M. That is, $M^{\mathrm{sl}} = M/\sim$, where for all $a, b \in M$, $ab \sim ba$ and $a^2 \sim a$. Denote by $q : M \to M^{\mathrm{sl}}$ the canonical quotient map. As we said, the induced map

$$\mathsf{F}_q : \mathsf{F}_M \to \mathsf{F}_{M^{\mathrm{sl}}}$$

is bijective if M is finitely generated and commutative. In the noncommutative setting, this induced map is not a bijection in general, even if M is finite. However, we will show that $|\mathsf{F}_M|$ is finite if M is finite; see Theorem 5.2.2 below. We also have the following.

Proposition 3.1.1. *Let M be a finite monoid. The canonical map*

$$\mathsf{F}_q : \mathsf{F}_M \to \mathsf{F}_{M^{\mathrm{sl}}}$$

is surjective. In particular, $|\mathsf{F}_M| \geq |M^{\mathrm{sl}}|$.

Proof. Denote by $\mathsf{Idem}(M)$ the set of all idempotents of M. Then, $\mathsf{Idem}(M) \ni e \mapsto Me$ yields the map $\mathsf{Idem}(M) \to \mathsf{F}_M$. Since $\mathsf{Idem}(M^{\mathrm{sl}}) = M^{\mathrm{sl}}$, the functoriality yields the following commutative diagram:

$$\begin{array}{ccc} \mathsf{Idem}(M) & \longrightarrow & \mathsf{F}_M \\ \downarrow\!\!\downarrow & & \downarrow \\ M^{\mathrm{sl}} & \xrightarrow{\sim} & \mathsf{F}_{M^{\mathrm{sl}}}. \end{array}$$

The bottom arrow is a bijection and the left vertical map is surjective, thanks to [22, Lemma 1.6]. It follows that the right vertical map is surjective as well. The last statement follows from Corollary 2.3.1. ∎

3.2. Category $\mathcal{I}(M)$

Let $m \in M$ be an element. We have a natural left ideal Mm, which can also be considered a left M-set. We have the following well-known fact; see, for example, [22, Proposition 1.8].

Lemma 3.2.1. *Let $e \in M$ be an idempotent. For any left M-set X, we have a bijection*

$$eX \cong \mathsf{Hom}_M(Me, X),$$

which sends an element $ex \in eX$ to the homomorphism $\alpha_x : Me \to X$, given by $\alpha_x(me) = mex$.

Proof. Take any morphism of M-sets $\beta : Me \to X$. We have $\beta(e) = \beta(ee) = e\beta(e)$. Thus, $\beta(e) \in eX$ and $\beta \mapsto \beta(e)$ defines a map $\mathsf{Hom}_M(Me, X) \to eX$, which is obviously inverse to the map $ex \mapsto \alpha_x$. ∎

Corollary 3.2.2. *Let $e, f \in M$ be idempotents. We have $\mathsf{Hom}_M(Me, Mf) = eMf$.*

We define the category $\mathcal{I}(M)$ as follows: objects are idempotents of M and a morphism from e to f is an element of the form $fme \in M$, $m \in M$. Another way of saying that is morphisms are equivalence classes of elements of M, where $m \sim n$ if $fme = fne$. We have

$$\mathsf{Hom}_{\mathcal{I}(M)}(e, f) = fMe.$$

The composition is given by the multiplication in M; i.e., $(gnf) \circ (fme) = gnfme$. That is, the composite of arrows

$$e \xrightarrow{fme} f \xrightarrow{gnf} g$$

is equal to $e \xrightarrow{gnfme} g$. The identity morphism of e is just $e = e1e$. It is clear that

$$\mathcal{I}(M^{\mathrm{op}}) = \big(\mathcal{I}(M)\big)^{\mathrm{op}}.$$

Recall the Green relations $\mathscr{L}, \mathscr{R}, \mathscr{J}$. By definition, we have $e\mathscr{L}e'$ provided $Me = Me'$, $e\mathscr{R}e'$ if $eM = e'M$ and $e\mathscr{J}e'$ if $MeM = Me'M$. We let $\mathsf{Idem}_{\mathscr{K}}(M)$ be the corresponding quotient set, where $\mathscr{K} \in \{\mathscr{L}, \mathscr{R}, \mathscr{J}\}$.

It is clear that if $e\mathscr{L}e'$ and $f\mathscr{R}f'$, then

$$\mathsf{Hom}_{\mathcal{I}(M)}(e, f) = \mathsf{Hom}_{\mathcal{I}(M)}(e', f').$$

Lemma 3.2.3. *The assignment $e \mapsto Me$ induces a contravariant equivalence between the category $\mathcal{I}(M)$ and the full subcategory of ${}_M\mathscr{S}$ets, consisting of objects of the form Me.*

Proof. This is a direct consequence of Corollary 3.2.2. ∎

Lemma 3.2.4. *Idempotents e and f are isomorphic in $\mathcal{I}(M)$ if and only if there are $a, b \in M$, such that $ab = e$ and $ba = f$. This happens if and only if $e\mathscr{J}f$. Thus, the set of iso-classes of the category $\mathcal{I}(M)$ is bijective to the set $\mathsf{Idem}_{\mathscr{J}}(M)$.*

Proof. A consequence of [22, Theorem 1.11]. ∎

4. Left zero monoids

In this section, we investigate the points and localisations of the topoi $\mathscr{S}\mathrm{ets}_M$ and ${}_M\mathscr{S}$ets, where M is a left zero monoid. Let us recall that we can define the product in any set S by

$$st = s, \quad s, t \in S.$$

In this way, we obtain a semigroup and we let S_+ be the corresponding monoid, which is obtained by adding the unite element 1 to S. Thus, $S_+ = S \cup \{1\}$. Such monoids are known as *left zero monoids*. It is immediate that S_+ is commutative if and only if $|S| = 0, 1$. Hence, we will assume that $|S| \geq 2$. The opposite monoids are known as *right zero* monoids.

4.1. Points of $\mathscr{S}\mathrm{ets}_{S_+}$

Each element of S_+ is an idempotent. If $s \in S$, then clearly $S_+s = S$. It follows that $\mathsf{Idem}_{\mathscr{J}}(S_+)$ is a two-element set given by 1 and any $s \in S$. Thus, we have at least two non-isomorphic points of $\mathscr{S}\mathrm{ets}_{S_+}$, corresponding to the left filtered S_+-sets S_+ and $S = S_+s$.

Our immediate goal is to prove that there are no other non-isomorphic points of the topos $\mathscr{S}\mathrm{ets}_{S_+}$. The proof uses the following fact.

Lemma 4.1.1. *Let M be a monoid and $M^\times$ the subgroup of all invertible elements of M. We set*

$$I := M \setminus M^\times.$$

Any filtered left M-set A with $A \neq IA$ is isomorphic to M as a left M-set.

Proof. Choose an element $a \in A \setminus IA$ and define a morphism of left M-sets $f : M \to A$ by

$$f(m) = ma.$$

We claim that f is an isomorphism. For surjectivity, take an element $b \in A$. By (F3) (see Lemma 2.2.1), there exist elements $c \in A$ and $m, n \in M$, such that $a = mc$ and $b = nc$. Since $a \notin IA$, we see that $m \notin I$, and hence, it is invertible. So, $c = m^{-1}a$ and $b = nm^{-1}a = f(nm^{-1})$. This proves the surjectivity of f. For injectivity, assume $f(m) = f(n)$; that is, $ma = na$. By (F2), there are $\tilde{m} \in M$ and $b \in A$, such that $a = \tilde{m}b$ and $m\tilde{m} = n\tilde{m}$. Again, $\tilde{m}$ is invertible since $a \notin IA$. Hence, we have $m = n$. ■

Theorem 4.1.2. *If A is a filtered left S_+-set, then A is isomorphic to either S_+ or S.*

Proof. We set

$$\widehat{A} = A \setminus \bigcup_{s \in S} sA.$$

Claim 1. $|\widehat{A}| \leq 1$. In fact, take $x, y \in \widehat{A}$. By condition (F3) of Lemma 2.2.1, there exist $z \in A$, $m_1 \in S_+$ and $m_2 \in S_+$, such that $x = m_1 z$ and $y = m_2 z$. Since $x, y \in \widehat{A}$, we must have $m_1 = 1 = m_2$. It follows that $x = z = y$, which implies the claim.

Claim 2. If $sA \cap tA \neq \emptyset$, then $s = t$. Assume $sa = tb$, for some $a, b \in A$. By (F3), we have $a = m_1 c$, $b = m_2 c$ and thus

$$sc = sm_1 c = sa = tb = tm_2 c = tc.$$

By (F2), there exist $m \in S_+$ and $d \in A$, such that $md = c$ and $sm = tm$. So, we have $s = sm = tm = t$, and the claim follows.

Claim 3. For any $a, b \in A$, one has $sa = sb$. Once again, (F3) says that there are $m_1, m_2 \in S_+$ and $c \in A$, such that $a = m_1 c$ and $b = m_2 c$. Hence, we have

$$sa = sm_1 c = sc = sm_2 c = sb.$$

Claim 4. If $\widehat{A} = \emptyset$, then A is isomorphic to S. By assumption, $A = \bigcup_{s \in S} sA$. Claim 3 implies that each sA consists of exactly 1 element and Claim 2 states that the union must be disjoint. This implies the fourth claim.

To finish the proof of the theorem, observe that if A is not isomorphic to S, then $\widehat{A}$ is not empty, by Claim 4. It follows from Claim 1 that $\widehat{A}$ is a set with one element. We can apply Lemma 4.1.1 to finish the proof. ■

Corollary 4.1.3. *We have*

$$\mathsf{F}(\mathscr{S}\mathrm{ets}_{S_+}) \cong \mathsf{Idem}_{\mathscr{J}}(S_+).$$

In particular, both sets have exactly two elements.

4.2. Points of ${}_{S_+}\mathscr{S}\mathrm{ets}$

In this case, the singleton is a filtered right S_+-set: while conditions (F1) and (F3) hold for any monoid, condition (F2) is easily verified in this circumstance; see also Lemma 5.3.2 below. We will show that, conversely, any filtered right S_+-set is isomorphic to either S_+ or the singleton. Since $sS_+ = \{s\}$ and $S_+sS_+ = S_+tS_+$ for all $s, t \in S$, it follows that any point of the topos of left S_+-sets comes from an idempotent. In particular, the topos ${}_{S_+}\mathscr{S}\mathrm{ets}$ has exactly two non-isomorphic points, and we still have a bijection

$$\mathsf{F}({}_{S_+}\mathscr{S}\mathrm{ets}) \cong \mathsf{Idem}_{\mathscr{J}}(S_+).$$

Lemma 4.2.1. *Any filtered right S_+-set A is either isomorphic to S_+ or a singleton.*

Proof. Let A be a filtered right S_+-set containing at least two distinct elements $a_1 \neq a_2 \in A$. By Lemma 4.1.1, it suffices to show that $A \neq AS$. Assume $A = AS$. By (F3), we can write $a_i = bm_i$, $i = 1, 2$.

Assuming $m_1 = 1$, then $a_2 = a_1m_2$. Since $a_1 \in AS$, we have $a_1 = ct$ for some $t \in S$. We obtain $a_2 = a_1m_2 = ctm_2 = ct = a_1$, contradicting our assumption and, hence, implying $m_1 \in S$. The same reasoning implies $m_2 \in S$. Since $b \in A = AS$, we can write $b = ds$ for some $s \in S$. It follows that

$$a_1 = bm_1 = dsm_1 = ds = dsm_2 = bm_2 = a_2,$$

once again giving a contradiction. The result follows. ∎

4.3. Reformulation of the results

Recall that for a monoid M, the maximal commutative quotient is denoted by M^{com}. We have the induced map

$$\mathsf{F}_M \to \mathsf{F}_{M^{\mathrm{com}}},$$

which is bijective if M is a left zero monoid. As we will see, this map is not always injective; see Example 6.1.5 (ii).

4.4. Localisations of $\mathscr{S}\mathrm{ets}_{S_+}$ and ${}_{S_+}\mathscr{S}\mathrm{ets}$

Recall that there is a one-to-one correspondence between the localisations of $\mathscr{S}\mathrm{ets}_M$ and Grothendieck topologies on M. Here, M is any monoid. Recall also that there are the minimal and maximal Grothendieck topologies on M, denoted, respectively, by $\mathcal{F}_{\min}$ and $\mathcal{F}_{\max}$ and defined as $\mathcal{F}_{\min} = \{M\}$, $\mathcal{F}_{\max} = \Omega^M$. Their corresponding sheaf categories are $\mathscr{S}\mathrm{ets}_M$ and the trivial category (with one object and morphism).

We wish to understand the Grothendieck topologies $\mathcal{F}$, with $\mathcal{F} \neq \mathcal{F}_{\min}$ and $\mathcal{F} \neq \mathcal{F}_{\max}$. Throughout this section, we will assume $M = S_+$. For any set B, we can consider the right M-act

$$i_*(B) := \mathsf{Maps}(S, B),$$

where the action of S_+ on $i_*(B)$ is given as follows: take $i_*(B) \ni \alpha : S \to B$ and $s \in S$. Define the map $\alpha \cdot s : S \to B$ by

$$(\alpha \cdot s)(t) = \alpha(st).$$

Since $\alpha(st) = \alpha(s)$, we have

$$\alpha \cdot s = \mathrm{c}_{\alpha(s)}.$$

where c_b denotes the constant function $S \to B$ with value b. This is known as the *cofree M-set cogenerated by B*. We can extend this to a functor through composition in the usual way. We will not need it, but let us mention that $i_* : \mathscr{S}\mathrm{ets} \to \mathscr{S}\mathrm{ets}_M$ is the push-forward part of a topos point $i = (i_*, i^*) : \mathscr{S}\mathrm{ets} \to \mathscr{S}\mathrm{ets}_M$, corresponding to the filtered S_+-set S; see Section 4.1. The next lemma claims that the functor $i_* : \mathscr{S}\mathrm{ets} \to \mathscr{S}\mathrm{ets}_M$ is full and faithful.

Lemma 4.4.1. *Let B, C be sets and $\psi : i_*(B) \to i_*(C)$ a morphism of S_+-sets. There exists a unique map $f : B \to C$, such that $\psi = i_*(f)$. In other words, for any $\alpha \in i_*(B) = \mathsf{Maps}(S, B)$, one has $\psi(\alpha) = f \circ \alpha$.*

Proof. Since ψ is a morphism of S_+-sets, we have

$$\psi(\alpha \cdot s) = \psi(\alpha) \cdot s = \mathrm{c}_{\psi(\alpha)(s)}$$

for all $\alpha : S \to B$ and $s \in S$. By setting $\alpha = \mathrm{c}_b$, we see that ψ sends constant functions to constant ones. It follows that there is a unique map $f : B \to C$ such that

$$\psi(\mathrm{c}_b) = \mathrm{c}_{f(b)}.$$

In fact, one must have

$$f(b) = \psi(\mathrm{c}_b)(s),$$

for any $s \in S$ and any $b \in B$.

For any $\alpha : S \to B$ and any $s \in S$, we have

$$\mathrm{c}_{f(\alpha(s))} = \psi(\mathrm{c}_{\alpha(s)}) = \psi(\alpha \cdot s) = \psi(\alpha) \cdot t = \mathrm{c}_{\psi(\alpha)(t)}.$$

It follows that $f \circ \alpha = \psi(\alpha)$, and we are done. ■

Before we state our next result, observe that if X is a subset of S, then $XS \subset X$. Thus, X is a right ideal of $M = S_+$ and

$$\Omega^M = \{M\} \cup \mathfrak{B}(S). \tag{4.4.1}$$

Here, $\mathfrak{B}(S)$ is the set of all subsets of S.

Proposition 4.4.2. *Let $\mathcal{F}$ be a Grothendieck topology on $M = S_+$, such that $\mathcal{F} \neq \mathcal{F}_{\min}$, $\mathcal{F} \neq \mathcal{F}_{\max}$. Then,*

$$\mathcal{F} = \{M, S\}.$$

A right S_+-module A is a sheaf over the topology $\mathcal{F}$ if and only if A is a cofree S_+-set and the functor i_* induces an equivalence of categories

$$\mathscr{S}\text{ets} \cong \mathsf{Sh}(M, \mathcal{F}).$$

Proof. The last statement is a consequence of the previous lemma. Next, we observe that $\{M, S\}$ is a Grothendieck topology, thanks to Lemma 6.1.1, because S is a two-sided idempotent ideal of S_+. If $\mathcal{F} \neq \mathcal{F}_{\min}$, $\mathcal{F} \neq \mathcal{F}_{\max}$ and $\mathcal{F} \neq \{M, S\}$, there must exist an $X \in \Omega^M$, such that X is a proper subset of S. Choose any $s \in S \setminus X$. Then, $\emptyset = (X : s) \in \mathcal{F}$. It follows that $\mathcal{F} = \mathcal{F}_{\max}$, and we obtained a contradiction. Hence, $\mathcal{F} = \{M, S\}$ and the first assertion is proved.

Observe that an S_+-set A is a sheaf if and only if the restriction map

$$A = \mathsf{Hom}_{S_+}(S_+, A) \to \mathsf{Hom}_{S_+}(S, A)$$

is bijective. Set

$$B = \{a \in A \mid sa = a,\ s \in S\}.$$

For any morphism $f : S \to A$ of S_+-sets, we have $f(st) = f(s)t$. Equivalently, $f(s) = f(s)t$ for all $s, t \in S$. If follows that the image of f lies in B. This shows that

$$\mathsf{Hom}_{S_+}(S, A) = \mathsf{Maps}(S, B) = i_*(B),$$

and hence, A is a sheaf if and only if $A \cong i_*(B)$, finishing the proof. ∎

Consider the category ${}_{S_+}\mathscr{S}\text{ets}$. Denote by ${}_+S$ the opposite of the monoid S_+. Clearly, ${}_{S_+}\mathscr{S}\text{ets} = \mathscr{S}\text{ets}_{+S}$.

Proposition 4.4.3. *The following hold:*

(i) *We have*

$$\Omega^{+S} = \{\emptyset, S, {}_+S\}.$$

(ii) *Let $\mathcal{F}$ be a Grothendieck topology on ${}_+S$, such that $\mathcal{F} \neq \mathcal{F}_{\min}$, $\mathcal{F} \neq \mathcal{F}_{\max}$. Then,*

$$\mathcal{F} = \{{}_+S, S\}.$$

(iii) *A left S_+-set A is a sheaf on the topology $\mathcal{F}$ if and only if the action of S_+ on A is trivial; that is, $sa = a$ for all $a \in A$ and $s \in S$. Thus,*

$$\mathsf{Sh}({}_+S, \mathcal{F}) \cong \mathscr{S}\text{ets}.$$

Proof. (i) Assume I is a non-empty left ideal of S_+. For any $t \in I$, we have $S = St \subset I$. Thus, either $I = S$ or $I = S_+$.

(ii) Since $\mathcal{F}$ is not maximal, $\emptyset \notin \mathcal{F}$, as otherwise by the first property of Grothendieck topologies, it would have to contain everything. We also have $S \in \mathcal{F}$; otherwise, $\mathcal{F}$ would be $\mathcal{F}_{\min}$. Using property (i) of the Grothendieck topologies again yields $\mathcal{F} = \{{}_+S, S\}$. The fact that this really is a Grothendieck topology follows from the fact that S is a two-sided idempotent ideal of $_+S$, thanks to Lemma 6.1.1.

(iii) Observe that for any left S_+-set A, the left S_+-set structure on $\mathsf{Hom}_{+S}(S, A)$ is given by $(s \cdot \alpha)(t) = \alpha(ts)$, where $s, t \in S$ and $\alpha \in \mathsf{Hom}_{+S}(S, A)$. Since $\alpha(ts) = \alpha(t)$, we see that $s \cdot \alpha = \alpha$. Thus, $\mathsf{Hom}_{+S}(S, A)$ is always a trivial left S_+-set. Recall that A being a sheaf over the topology $\mathcal{F}$ means that the canonical map

$$A = \mathsf{Hom}_{+S}(_+S, A) \to \mathsf{Hom}_{+S}(S, A)$$

is an isomorphism of left S_+-sets. Hence, if A is a sheaf, the action of S_+ on A must be trivial.

Conversely, assume the action of S_+ on A is trivial. The map $\alpha : S \to A$ is a morphism of left S_+-sets if and only if $\alpha(st) = s\alpha(t)$. Since $st = s$ and $sa = a$ for all $a \in A$, we see that $\alpha(s) = \alpha(t)$. Thus, α is constant, and it follows that $A \to \mathsf{Hom}_{+S}(S, A)$ is an isomorphism, implying the proposition. ∎

5. Points of $\mathscr{S}\text{ets}_M$, for finite M

5.1. M-congruences

For an equivalence relation ρ on a set S, we denote by q the canonical map $q : S \to S/\rho$.

Let M be a monoid and A a left M-set. An equivalence relation $\sim_\rho$ on A is called an *M-congruence* if $a \sim b$ implies $ma \sim mb$ for all $m \in M$. It is clear that in this case, the quotient A/ρ has a unique left M-set structure such that $q : A \to A/\rho$ is an M-set map.

We will use this terminology to distinguish between congruences on a monoid M in the world of monoids and congruences on M, considered a left M-set, using the multiplication in M.

Lemma 5.1.1. *Let $\sim_\rho$ be an M-congruence on a left M-set A. For any $a \in A$, the subset*

$$K_\rho(a) = \{x \in M \mid a \sim xa\}$$

is a submonoid of M.

Proof. Since $\sim$ is an equivalence relation, we have $a \sim a = 1 \cdot a$. Thus, $1 \in K$. Assume $x, y \in K(a)$. That is, $a \sim ya$ and $a \sim xa$. Since $\sim$ is M-congruence, we have $xa \sim xya$. It follows that $a \sim xa \sim xya$. Hence, $xy \in K(a)$. ∎

5.2. Finiteness of points

We start with the following observation.

Lemma 5.2.1. *Let A be a filtered left M-set and $a_1, \dots, a_k$ elements of A. There are $m_1, \dots, m_k \in M$ and $c \in A$, such that $a_i = m_i c$ for all $1 \le i \le k$.*

Proof. The case $n = 1$ is clear and $n = 2$ is just condition (F3). We proceed by induction. Assume there are $n_1, \dots, n_{k-1} \in M$ and $b \in A$, such that $a_i = n_i b$ for all $1 \leq i \leq n-1$. By the case $n = 2$, we can choose $c \in A$ and $n', m_k \in M$, such that $n'c = b$ and $m_k c = a_k$. We put $m_i = n' n_i$ for $i = 1, \dots, k-1$ to obtain $m_i c = a_i$ for all $1 \leq i \leq k$. ∎

Theorem 5.2.2. *Let M be a finite monoid.*

(i) *Any filtered M-set A is cyclic, that is, generated by a single element. In particular, we have $|A| \leq |M|$.*

(ii) *The set F_M is finite.*

Proof. (i) Assume there are k distinguished elements of A, say $a_1, \dots, a_k$. By Lemma 5.2.1, we can find $m_1, \dots, m_k \in M$ and $c \in A$, such that $a_i = m_i c$, for all $1 \leq i \leq k$. If $|M| < k$, we see that there are $i \neq j$, such that $m_i = m_j$. It follows that

$$a_i = m_i c = m_j c = a_j.$$

This contradicts our assumptions on the a_i's. Hence, $k \leq |M|$. This implies $|A| \leq |M|$. Moreover, by taking $a_1, \dots, a_k$ to be all the elements of A, we see that A is generated by c, and hence, it is cyclic.

(ii) This is an obvious consequence of (i). ∎

5.3. Right-collapsible monoids and right-collapsible submonoids

Definition 5.3.1. We call a monoid M right-collapsible monoid if for any $m, n \in M$, there exists an $x \in M$, such that

$$mx = nx.$$

For more on this subject, see [12].

Lemma 5.3.2. *Let M be a monoid. The terminal object of the category ${}_M\mathscr{S}\mathrm{ets}$ (i.e., the singleton) is filtered if and only if M is a right-collapsible monoid.*

Proof. For such a set, conditions (F1) and (F3) always hold. Condition (F2) in this case says that for any $m_1, m_2 \in M$, there exists an $m \in M$, such that $m_1 m = m_2 m$. This means that M is a right-collapsible monoid. ∎

Clearly, if M is a semilattice, then M is a right-collapsible monoid (we can take $x = mn$). Another class of right-collapsible monoids (which is a generalisation of semilattices in the finite case) are monoids that have a right zero, that is, an element ϱ, such that $x\varrho = \varrho$ for all $x \in K$. Clearly, if such a ϱ exists, it is unique and an idempotent. Our next goal is to show that the converse is also true if M is finite. We will need the following lemma.

Lemma 5.3.3. *Let K be a right-collapsible monoid. For any finite collection of elements $m_1, \dots, m_k$ of K, there exists an $x \in K$, such that $m_i x = x$ for all $i = 1, \dots, k$.*

Proof. We proceed by induction. Let $k = 1$. In this case, the assertion follows directly from the definition of a right-collapsible monoid, by taking $m = m_1$ and $n = 1$, where $m, n \in M$ are as in Definition 5.3.1. Next, consider the case $k = 2$. Since the assertion is true for $k = 1$, we can find y_i, $i = 1, 2$, such that $m_i y_i = y_i$, $i = 1, 2$. As K is a right-collapsible monoid, there exists a $z \in K$, such that $y_1 z = y_2 z$. Now, for $x = y_1 z = y_2 z$, we have

$$m_i x = m_i y_i z = y_i z = x, \quad i = 1, 2.$$

Let $k > 2$. By the induction assumption, there exists a $y \in K$, for which $m_i y = y$, for all $i = 1, \dots, k - 1$. Since the result is also true for $k = 2$, we can apply it for y and m_k and conclude that there exists an $x \in K$, for which $m_k x = x$ and $yx = x$. We have $m_i x = m_i yx = yx = x$ for all $1 \leq i \leq k$. This finishes the proof. ∎

Corollary 5.3.4. *A finite monoid is a right-collapsible monoid if and only if it has a right zero element.*

A submonoid K of a monoid M is called a *right-collapsible submonoid* if K is a right-collapsible monoid.

5.4. Saturated submonoids

A submonoid K of a monoid M is called *saturated* if for any $m \in M$ for which $mx = x$ for some $x \in K$, one has $m \in K$. It is clear that to any submonoid K, there is a smallest saturated submonoid containing K. This is the intersection of all saturated submonoids containing K. (The fact that the intersection of saturated submonoids is again saturated is readily checked.) We denote this associated saturated submonoid by $\widehat{K}$ and sometimes refer to it as the *saturation* of K.

Example 5.4.1. Let $M = \{1, 0, a, b, ab\}$, where $a^2 = a$, $b^2 = b$ and $ba = 0$. The only saturated right-collapsible submonoids of M are $\{1\}, \{1, a\}, \{1, b\}$ and M itself. On the other hand, $\{1, 0\}, \{1, 0, a\}, \{1, 0, b\}$ and $\{1, 0, ab\}$ are non-saturated right-collapsible submonoids.

5.5. Quotient by right-collapsible submonoids

Let $H \subseteq A$ be a submonoid of a monoid A. We can define a relation $\sim_H$ on A by setting $a \sim_H b$ if and only if there exist $x, y \in H$, such that $ax = by$. This relation, however, need not be an equivalence relation as the transitivity property need not hold. If it does, however, then it is an M-congruence and the quotient, denoted by M/K, is a natural M-set.

Lemma 5.5.1. *Let K be a right-collapsible submonoid of M and $m, n \in M$. The induced relation $\sim_K$ is an M-congruence and $m \sim_K n$ if and only if there exists an $x \in K$, such that $mx = nx$.*

Proof. First, we show that the two relations are the same. One side is clear by letting $x = y$. For the other side, let $x, y \in K$ be elements for which $mx = ny$. Since K is a right-collapsible submonoid, we can find a $z \in K$, such that $xz = yz$. As $mx = ny \Rightarrow mxz = nyz$, the result follows.

To see that $\sim_K$ is an M-congruence, let $m \sim_K n$ and $n \sim_K k$. That is, we have $x, y \in K$, such that $mx = mx$ and $ny = ky$. As K is a right-collapsible submonoid, there exists a $z \in K$, such that $xz = yz$. We have

$$m(xz) = (mx)z = (nx)z = n(xz) = n(yz) = (ny)z = (ky)z = k(yz).$$

Hence, $m \sim_K k$ with the first definition, and so, transitivity holds. ∎

In general, the M-congruence $\sim_K$ is not a congruence (that is, the quotient M/K need not be a monoid), even if K is a right-collapsible submonoid. In fact, take once again $M = \{1, 0, a, b, ab\}$, where $a^2 = a$, $b^2 = b$, $ba = 0$, and let $K = \{1, b\}$. In this case, we have three equivalence classes $1 \sim_K b$, $a \sim_K ab$ and 0. Since $1 \cdot a = a \not\sim_K ba = 0$, we see that $\sim_K$ is not a congruence relation.

We have shown that M/K is a left M-set. More is true, however.

The following can already be inferred from [10, 12]. *In* [12, Lemma 14.13 (ii) and Proposition 16.6], *it is shown that condition (F2) (called condition E in the said monograph) is satisfied by M/K and that this condition agrees with "pullback-flatness" for cyclic M-sets. Further, it is shown in* [10, Proposition 1.10 (iii)] *that in this case, pullback-flatness is the same as being filtered.*

Proposition 5.5.2. *Let K be a right-collapsible submonoid of M. The quotient M/K is a filtered left M-set.*

Proof. The fact that M/K is a left M-set is just Lemma 5.5.1. To see that M/K is in addition filtered, consider the canonical surjective map $q : M \to M/K$. Since M is a monoid, M/K is non-empty and condition (F1) holds.

To show (F2), assume $m_1 q(a) = m_2 q(a)$, with $a, m_1, m_2 \in M$. We have to find $m, \tilde{a} \in M$, such that $m_1 m = m_2 m$ and $mq(\tilde{a}) = q(a)$. To this end, observe that

$$q(m_1 a) = m_1 q(a) = m_2 q(a) = q(m_2 a).$$

Hence, $m_1 ax = m_2 ax$ for an element $x \in K$. Since K is a right-collapsible submonoid, there exists an element $y \in K$, such that $xy = y$. So, we can take $m = ax$ and $\tilde{a} = q(1)$. We have $m_1 m = m_1 ax = m_2 ax = m_2 m$ and $mq(1) = q(ax) = q(a)$ because $axy = ay$ and $ax \sim_K a$.

To show (F3), take $q(a_1), q(a_2) \in M/K$. Then, $a_1 q(1) = q(a_1)$ and $a_2 q(1) = q(a_2)$. Thus, (F3) holds with $a = q(1)$. ∎

We have the following "inverse statement" to the above:

Part (i) of this Proposition can be found in [12, Lemma 14.14].

Proposition 5.5.3. *Let ρ be an M-congruence on M, such that $M/\sim_\rho$ is a filtered left M-set, and define*

$$K_\rho = \{m \in M \mid 1 \sim_\rho m\}.$$

(i) *The subset K_ρ is a saturated right-collapsible submonoid of M.*

(ii) *We have $M/\sim_\rho \simeq M/K_\rho$.*

Proof. (i) By Lemma 5.1.1, K_ρ is a submonoid of M. Take $x, y \in K_\rho$. We have $x \sim_\rho 1 \sim_\rho y$, by the definition of K_ρ. Thus, $q(x) = q(y)$, where $q : M \to M/\sim_\rho$ is the canonical map. It follows that $xq(1) = yq(1)$. Since $M/\sim_\rho$ is filtered, there are $m, \tilde{a} \in M$ such that $xm = ym$ and $mq(\tilde{a}) = q(1)$. The last condition implies that $z := m\tilde{a} \in K$. We have $xz = xm\tilde{a} = ym\tilde{a} = yz$. This shows that K_ρ is a right-collapsible submonoid.

Take $m \in M$ and $x \in K_\rho$, such that $mx = x$. Since $1 \sim_\rho x$, it follows that $m \sim_\rho mx = x \sim_\rho 1$. Thus, $m \in K_\rho$ and K_ρ is saturated.

(ii) Assume $m \sim_\rho n$. It follows that $mq(1) = nq(1)$. By (F2), there are $x, a \in M$, such that $mx = nx$ and $xq(a) = q(1)$. The last condition implies $y = xa \in K$. Since $my = mxa = nxa = ny$, we see that $m \sim_K n$. Conversely, let $m \sim_K n$. That is, $mx = nx$ for $x \in K$. We have $q(mx) = mq(x) = mq(1) = q(m)$ and $q(nx) = q(n)$. This yields $q(m) = q(n)$, which implies $m \sim_\rho n$. This finishes the proof. ∎

Corollary 5.5.4. *Let K be a right-collapsible submonoid of M and ρ the congruence $\sim_K$, corresponding to K. Then,*

(i) *$K_\rho = \{m \in M \mid mx = x$, for an element $x \in K\} = \hat{K}$, where $\hat{K}$ is the saturation of K,*

(ii) *$\hat{K}$ is a right-collapsible submonoid,*

(iii) *if K is saturated, we have $\hat{K} = K$.*

Proof. (i) By definition, $m \in K_\rho$ if and only if $1 \sim_\rho m$. This happens exactly if there exists an $x \in K$, such that $mx = x$. This proves the first equality. Next, take $m \in K$. Since K is a right-collapsible monoid, there exists an $x \in K$, such that $mx = x$. Hence, $1 \sim_\rho m$ and $m \in K_\rho$. This shows that $K \subseteq K_\rho$. Since K_ρ is saturated by Proposition 5.5.3, we have $\hat{K} \subseteq K_\rho$. To show the opposite inclusion, take any $m \in K_\rho$. Thus, $1 \sim_\rho m$. So, $m = mx$ for some $x \in K$. Since $\hat{K}$ is saturated and $x \in K \subseteq \hat{K}$, we see that $m \in \hat{K}$. It follows that $K_\rho = \hat{K}$.

(ii) This follows from part (i) by virtue of Proposition 5.5.3 and part (iii) holds by definition. ∎

Theorem 5.5.5. *Let M be a finite monoid. Any filtered left M-set is isomorphic to M/K, for a suitable saturated right-collapsible submonoid K of M.*

Below (see Theorem 5.6.1), we will prove a much stronger result.

Proof. Any filtered M-set is cyclic by Theorem 5.2.2. Hence, it is isomorphic to M/ρ, where ρ is an M-congruence. Part (ii) of Proposition 5.5.3 says that $K = K_\rho$ is a saturated right-collapsible submonoid of M. It remains to show that

$$M/\sim_\rho \simeq M/K_\rho,$$

which we already did in Proposition 5.5.3. ∎

Corollary 5.5.6. *Let K and L be right-collapsible submonoids of M. Then, $M/K \simeq M/L$ if and only if $\widehat{K} = \widehat{L}$. In particular, $M/K \simeq M/\widehat{K}$.*

5.6. Some examples

(i) Assume M is a right-collapsible monoid and take $K = M$. In this case, M/M is a singleton. So, the terminal object of $\mathscr{S}\mathrm{ets}_M$ is filtered. Conversely, if the terminal object of $\mathscr{S}\mathrm{ets}_M$ is filtered, then M is a right-collapsible monoid. In fact, the terminal object is $M/\sim_\rho$, where $x \sim_\rho y$ for all $x, y \in M$. In this case, $K = M$, and hence, M is a right-collapsible monoid, thanks to the proof of Theorem 5.5.5. In particular, the single element set is filtered if M has a right zero or M is a semilattice.

The above example (i) *can already be found in* [12, Exercise 14.3 (4)].

(ii) Let $e \in M$ be an idempotent. Then, $K = \{1, e\}$ is a right-collapsible submonoid. In this case, $\sim_K$ is the equivalence relation on M, defined by $a \sim_K b$ if and only if $ae = be$. The left M-set $M/K \simeq Me$ is filtered.

The following was indirectly shown in [1, Exercise 3.12 (b)] *and* [19]. *In the first, it is shown that every point of the topos of M-sets is essential, for a finite monoid M. In the second, it is shown that essential points correspond directly to idempotents.*

Theorem 5.6.1. *Let M be a finite monoid. Any filtered M-set is of the form Me, for an idempotent $e \in M$. Thus, $e \mapsto Me$ yields an equivalence of categories*

$$\big(\mathcal{I}(M)\big)^{\mathrm{op}} \cong \mathbf{Pts}(M).$$

Proof. We have already proven (see Theorem 5.5.5) that any filtered M-set is of the form M/K, where K is a saturated right-collapsible submonoid of M. By Corollary 5.3.4, K has a right zero ϱ. Take $L = \{1, \varrho\}$. We have $L \subseteq K$ and $K \subseteq \widehat{L}$. Since K is saturated, we have $K = \widehat{L}$. By Corollary 5.5.6, we have $\sim_K = \sim_L$. Thus, we can take $e = \varrho$. ∎

Corollary 5.6.2. *Let M be a finite monoid. Then,*

(i) $|\,\mathbf{Pts}(M)| = |\,\mathbf{Pts}(M^{\mathrm{op}})|$,

(ii) *if p is a point corresponding to an idempotent e, one has an isomorphism of monoids*

$$\mathrm{End}(\mathsf{p}) \cong (eMe)^{\mathrm{op}},$$

(iii) *there is a bijection*

$$\mathsf{F}_M \cong \mathsf{Idem}_{\mathscr{J}}(M).$$

6. Localisations of $\mathscr{S}\mathrm{ets}_M$

In this section, we assume that M is finite. We will show that there is one-to-one correspondence between the localisations of $\mathscr{S}\mathrm{ets}_M$ and idempotent ideals of M. This fact allows us to prove that the topology on F_M defined by the localisations of the topos $\mathscr{S}\mathrm{ets}_M$ is homeomorphic to the order topology of the poset of principal idempotent ideals.

6.1. Idempotent ideals

Lemma 6.1.1. *Let $\mathfrak{m}$ be a two-sided ideal of a monoid M, such that $\mathfrak{m} = \mathfrak{m}^2$. The set*

$$\mathcal{F}_{\mathfrak{m}} = \{\mathfrak{a} \mid \mathfrak{m} \subseteq \mathfrak{a}\}$$

of right ideals containing $\mathfrak{m}$ is a Grothendieck topology on M.

Proof. The condition (T1) is obvious. For (T2), assume $\mathfrak{m} \subseteq \mathfrak{a}$ and $m \in M$. For any $x \in \mathfrak{m}$, $mx \in M\mathfrak{m} = \mathfrak{m} \subseteq \mathfrak{a}$. Hence, $x \in (\mathfrak{a} : m)$. It follows that $\mathfrak{m} \subseteq (\mathfrak{a} : m)$ and $(\mathfrak{a} : m) \in \mathcal{F}_{\mathfrak{m}}$. So, (T2) holds. For (T3), take $\mathfrak{b} \in \mathcal{F}_{\mathfrak{m}}$. Assume $\mathfrak{a}$ is a right ideal, such that $(\mathfrak{a} : b) \in \mathcal{F}_{\mathfrak{m}}$, for any $b \in \mathfrak{b}$. By assumption, $\mathfrak{m} \subseteq \mathfrak{b}$ and $\mathfrak{m} \subseteq (\mathfrak{a} : b)$ for all $b \in \mathfrak{b}$. Take any $x \in \mathfrak{m}$. Since $\mathfrak{m} = \mathfrak{m}^2$, we can write $x = yz$, where $y, z \in \mathfrak{m}$. Since $z \in \mathfrak{m} \subseteq (\mathfrak{a} : y)$, we see that $x = yz \in \mathfrak{a}$. Thus, $\mathfrak{m} \subseteq \mathfrak{a}$ and $\mathfrak{a} \in \mathcal{F}_{\mathfrak{m}}$, from which (T3) follows. ■

Proposition 6.1.2. *Let $\mathcal{F}$ be a Grothendieck topology on a finite monoid M. There exists a two-sided ideal $\mathfrak{m}$, such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\mathcal{F} = \mathcal{F}_{\mathfrak{m}}$.*

Proof. As M is finite, $\mathcal{F}$ is finite as well. We can also see that $\mathcal{F}$ contains a smallest element $\mathfrak{m}$ since $\mathcal{F}$ is closed with respect to finite intersection. Take $x \in M$. Since $(\mathfrak{m} : x) \in \mathcal{F}$, it follows that $\mathfrak{m} \subseteq (\mathfrak{m} : x)$. Equivalently, $x\mathfrak{m} \subseteq \mathfrak{m}$, which implies that $\mathfrak{m}$ is a two-sided ideal of M. We also know that $\mathcal{F}$ is closed with respect to the product. Hence, $\mathfrak{m}^2 \in \mathcal{F}$. By minimality of $\mathfrak{m}$, we have $\mathfrak{m} \subseteq \mathfrak{m}^2$, and hence, $\mathfrak{m}^2 = \mathfrak{m}$. ■

It should be pointed out that in [1, Exercise 9.1.12 (c)], *it was shown that for finite M, every localisation is essential. In* [11, Theorem 4.4], *it was shown that this in turn corresponds to idempotent ideals.*

We will use the following well-known fact; see [22, Proposition 1.23].

Lemma 6.1.3. *Any two-sided idempotent ideal of a finite monoid M has the form*

$$\bigcup_{i \in I} M e_i M,$$

for a (finite) family of idempotents $(e_i)_{i \in I}$.

Lemma 6.1.4. *Let e be an idempotent of M and $\mathfrak{m} \subseteq M$ a two-sided ideal, such that $\mathfrak{m} = \mathfrak{m}^2$. Then,*

$$\mathsf{p}_{Me} \pitchfork \mathbf{Sh}(M, \mathcal{F}_{\mathfrak{m}})$$

if and only if $e \in \mathfrak{m}$.

Proof. Assume $\mathsf{p}_{Me} \pitchfork \mathbf{Sh}(M, \mathcal{F}_{\mathfrak{m}})$. The canonical map

$$\mu_{\mathfrak{m}} : \mathfrak{m} \otimes_M Me \to Me$$

is an isomorphism by Lemma 2.2.3. The surjectivity of this map implies that $e = xme$, for some $x \in \mathfrak{m}$, $m \in M$. So, $e \in \mathfrak{m}M = \mathfrak{m}$. Conversely, assume $e \in \mathfrak{m}$. We have to show that for any right ideal $\mathfrak{a}$, containing $\mathfrak{m}$, the canonical map $\mu_{\mathfrak{a}} : \mathfrak{a} \otimes_M Me \to Me$ is an isomorphism. For any $m \in M$, we have $me \in \mathfrak{m} \subseteq \mathfrak{a}$. Since $\mu(me \otimes e) = me$, we see that μ is surjective. The injectivity of μ follows directly from the fact that $- \otimes_M Me$ commutes with finite limits, hence respecting monomorphisms. ∎

Example 6.1.5. (i) Let M once again be the monoid $\{1, 0, a, b, ab\}$, $a^2 = a$, $b^2 = b$ and $ba = 0$. It has 4 idempotents $1, 0, a, b$. Since $M1 = M$, $M0 = \{0\}$, $Ma = \{0, a\}$ and $Mb = \{0, b, ab\}$, we see that they all define non-isomorphic idempotents. Thus, M has 4 non-isomorphic points corresponding to the filtered left M-sets $\{0\}, \{0, a\}, \{0, b, ab\}$ and M. Since $M1M = M$, $M0M = \{0\}$, $MaM = \{0, a, ab\}$, $MbM = \{0, b, ab\}$, we see that there are 6 idempotent ideals

$$\emptyset, \{0\}, \{0, a, ab\}, \{0, b, ab\}, \{0, a, b, ab\}, M.$$

(ii) Let $M = T_3$ be the monoid of all endomorphic maps $\{1, 2, 3\} \to \{1, 2, 3\}$. It has $3^3 = 27$ elements. If f is such a map, the cardinality of the image of f is known as the rank of f and is denoted by $rk(f)$. There are 3 maps of rank 1 (i.e., constant maps) and all of them are idempotents. There are 6 idempotent elements of rank 2 and only one idempotent of rank 3. All together, we have 10 idempotents. However, $\mathsf{Idem}_{\mathfrak{J}}(M)$ has only three elements, as two idempotents are $\mathfrak{J}$-equivalent if and only if they have the same rank. All two-sided ideals of T_3 are idempotent, and they are

$$\emptyset \subseteq I_1 \subseteq I_2 \subseteq I_3 = M,$$

where $I_k = \{f \in T_3 \mid rk(f) \leq k\}$, $k = 1, 2, 3$. Observe that in this case M^{com} is isomorphic to the multiplicative monoid $\{0, 1, -1\}$. In fact, for a map $f : \{1, 2, 3\} \to \{1, 2, 3\}$ denote by $[f]$ the class of f in M^{com}. Since nonbijective self maps form an ideal of M and thus we have a well-defined surjective homomorphism $M^{\mathsf{com}} \to \{0, 1, -1\}$ given by $[f] \mapsto 0$ if f is a nonbijective self map and $[f] \mapsto \mathsf{sign}(f)$ if f is bijective. It remains to show that all nonbijective maps define the same element in M^{com}. Denote by c_i the constant map with value i. Then, for all i, j, we have

$$[\mathrm{c}_i] = [\mathrm{c}_i \circ \mathrm{c}_j] = [\mathrm{c}_j \circ \mathrm{c}_i] = [\mathrm{c}_j].$$

This shows that all constant maps have the same image in M^{com}. Moreover, since $f \circ \mathrm{c}_i = \mathrm{c}_{f(i)}$ and $\mathrm{c} \circ f = \mathrm{c}$, we see that the class of constant maps define the zero element of M^{com}. Furthermore, denote by $s, t \in T_3$ the maps given by $s(1) = 1 = s(2)$, $s(3) = 2$ and $t(1) = 1, t(2) = 2 = t(3)$. Then, we have

$$[\mathrm{c}_1] = [s \circ t] = [t \circ s] = [s].$$

Since any nonconstant and nonbijective map can be written as $f \circ s \circ g$ with bijective f and g, we see that all nonbijective maps define the same classes in M^{com}. Since $\{0, 1, -1\}$ has two idempotents, we see that in this case, $3 = |\mathsf{F}(\mathscr{S}\mathrm{ets}_M)| \neq |\mathsf{F}(\mathscr{S}\mathrm{ets}_{M^{\mathsf{com}}})| = 2$.

6.2. Distributivity of the lattice $\mathfrak{Loc}(\mathscr{S}\mathrm{ets}_M)$

Denote by $\mathscr{I}\mathscr{I}(M)$ the set of two-sided idempotent ideals of a monoid M. Our interest in $\mathscr{I}\mathscr{I}(M)$ comes from the bijection

$$\mathfrak{Loc}(\mathscr{S}\mathrm{ets}_M) \cong \mathscr{I}\mathscr{I}(M),$$

which is true for all finite M. This follows from Lemmas 6.1.1 and 6.1.2. In particular, $|\mathfrak{Loc}(\mathscr{S}\mathrm{ets}_M)|$ is finite and

$$\left|\mathfrak{Loc}(\mathscr{S}\mathrm{ets}_M)\right| = \left|\mathfrak{Loc}(\mathscr{S}\mathrm{ets}_{M^{\mathrm{op}}})\right|$$

holds.

Since the union of two-sided idempotent ideals of M is again a two-sided idempotent ideal, we see that $\mathscr{I}\mathscr{I}(M)$ is a join-semilattice, with $I \vee J = I \cup J$. Its greatest element is M and least element is $\emptyset$. In general, the intersection of two-sided idempotent ideals is not an idempotent ideal (see (i) of Example 6.1.5).

It is well known that any finite join-semilattice L with greatest element is a lattice, where

$$a \wedge b = \bigvee_x x,$$

with $a, b \in L$ and x running through all the elements of the set

$$\{x \in L \mid x \leq a \text{ and } x \leq b\}.$$

It follows that for finite M, the set $\mathscr{I}\mathscr{I}(M)$ is a lattice.

Lemma 6.2.1. *Let I and J be two-sided idempotent ideals. Then,*

$$I \wedge J = \bigcup_e MeM,$$

where e runs through all the idempotents of $I \cap J$.

Proof. The right-hand side is a two-sided ideal, generated by idempotents. Hence, it is an element of $\mathscr{I}\mathscr{I}(M)$. If $e \in I \cap J$, it follows that $MeM \subseteq I$ and $MeM \subseteq J$. Thus, $\bigcup_e MeM \subseteq I \wedge J$. Conversely, assume K is a two-sided idempotent ideal, such that $K \subseteq I$ and $K \subseteq J$. By Lemma 6.1.3, we can write $K = \bigcup_{i \in I} Me_iM$ for some idempotents e_i. We have $e_i \in I \cap J$ by assumption, which implies that $I \wedge J$ is also a subset of $\bigcup_e MeM$. The result follows. ∎

Corollary 6.2.2. *If M is finite, then $\mathscr{I}\mathscr{I}(M)$ is a distributive lattice.*

Proof. Take $I, J, K \in \mathscr{I}\mathscr{I}(M)$. We need to show that

$$I \wedge (J \cup K) = (I \wedge J) \cup (I \wedge K).$$

By Lemma 6.2.1, we see that $I \wedge (J \cup K)$ is generated as a two-sided ideal by the idempotent elements of $I \cap (J \cup K)$, while $(I \wedge J) \cup (I \wedge K)$ is generated (as a two-sided ideal) by the idempotent elements of $I \cap J$ and $I \cap K$. The result follows. ∎

6.3. Topology on F_M

We start by recalling the well-known relationship between (finite) distributive lattices, posets and topologies.

Any poset P has a natural topology, called the *order topology*, where a subset $S \subseteq P$ is open if $y \in P$ and $x \leq y$ imply $x \in P$. Thus, $\mathsf{Off}(P)$ is a distributive lattice and it is finite if P is finite. It is well known that any finite distributive lattice L is of this form, for a uniquely defined P (see, for example, [21, p. 106]), specifically, for $P = \mathsf{Irr}(L)$, the subset of irreducible elements of L (an element $x \in L$ is irreducible if $x = y \vee z$ implies $x = y$ or $x = z$).

The poset of our interest is F_M, the iso-classes of the topos points of M, where M is finite. According to Corollary 5.6.2, we have

$$\mathsf{F}_M \cong \mathsf{Idem}_{\mathscr{J}}(M).$$

This allows us to work with $\mathsf{Idem}_{\mathscr{J}}(M)$ instead. This set has a canonical order

$$e \leq_{\mathscr{J}} f \quad \text{if } MeM \subseteq mfM.$$

Thus, we have a canonical order topology on $\mathsf{Idem}_{\mathscr{J}}(M)$ and, as such, on F_M.

Proposition 6.3.1. *One has a bijection*

$$\mathscr{I}\mathscr{I}(M) \to \mathsf{Off}\big(\mathsf{Idem}_{\mathscr{J}}(M)\big).$$

Proof. It suffices to show that the irreducible elements of $\mathscr{I}\mathscr{I}(M)$ are exactly the ideals of the form MeM, where e is an idempotent. This follows from the fact that for any element $I \in \mathscr{I}\mathscr{I}(M)$, one has $I = \bigcup_{e \in I} MeM$, where e is an idempotent. If $MeM = J \cup K$, then $e \in K$, or $e \in J$. We get that $MeM = J$, or $MeM = K$. ∎

The following corollary can already be found as [1, Exercise 9.1.12 (c)].

Corollary 6.3.2. *For finite M, the topology on F_M defined using localisations is homeomorphic to the order topology of the poset of principal idempotent ideals.*

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Ilia Pirashvili
Institut für Mathematik, Universität Augsburg, Universitätsstraße 14, 86159 Augsburg, Germany;
ilia.pirashvili@math.uni-augsburg.de