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Operational Calculus and Boundary Value Problem for an Abstract Differential Equation

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Es wird die abstrakte Differentialgleichung

$$S^2x = f \quad \text{mit} \quad s_qSx = x_{1q} \quad \text{und} \quad Bx = x_B$$

betrachtet, wobei $B: L^2 \to \operatorname{Ker} S$, $x \in L^2$, $f \in L^0$ und $x_{1q}, x_B \in \operatorname{Ker} S$ ist.

Рассматривается абстрактное дифференциальное уравнение

$$S^2x = f \quad \text{с} \quad s_qSx = x_{1q} \quad \text{и} \quad Bx = x_B,$$

где $B: L^2 \to \operatorname{Ker} S$, $x \in L^2$, $f \in L^0$ и $x_{1q}, x_B \in \operatorname{Ker} S$.

The abstract differential equation

$$S^2x = f \quad \text{with} \quad s_qSx = x_{1q} \quad \text{and} \quad Bx = x_B$$

is considered, where $B: L^2 \to \operatorname{Ker} S$, $x \in L^2$, $f \in L^0$ and $x_{1q}, x_B \in \operatorname{Ker} S$.

Suppose we are given the operational calculus $CO(L^0, L^1, S, T_q, s_q, Q)$, where L^0 and L^1 are linear spaces, S, T_q, s_q are linear operations called derivative, integral and limit condition, respectively, so that $S: L^1 \to L^0$ (onto), $T_q: L^0 \to L^1$ and $s_q: L^1 \to \operatorname{Ker} S$ ($q \in Q$), where Q is the set of indices. Let us assume that the operations S, T_q, s_q satisfy the following properties:

$$ST_qf = f \quad \text{for} \quad f \in L^0, q \in Q,$$

$$T_qSg = g - s_qg \quad \text{for} \quad g \in L^1, q \in Q \text{ (see [2–4, 7, 11])}.$$

For $L^1 \subset L^0$, $L^2 = \{x \in L^1 : Sx \in L^1\}$ (see [3, 4]) and $B: L^2 \to \operatorname{Ker} S$ a linear operation, let us consider the abstract differential equation

$$S^2x = f \quad \text{with} \quad s_qSx = x_{1q} \quad \text{and} \quad Bx = x_B, \tag{1}$$

where $x \in L^2$, $f \in L^0$ and $x_{1q}, x_B \in \operatorname{Ker} S$.

Theorem 1: *Problem* (1) *has*

a) *at least one solution if* $B|_{\operatorname{Ker} S}$ *is a surjection onto* $\operatorname{Ker} S$,
b) *at most one solution if* $B|_{\operatorname{Ker} S}$ *is an injection,*
c) *exactly one solution if* $B|_{\operatorname{Ker} S}$ *is a bijection onto* $\operatorname{Ker} S$.

Proof: Operating on the equation $S^2x = f$ with the operation T_q^2 and applying the axioms of the operational calculus and the condition $s_qSx = x_{1q}$, we obtain $x = T_q^2f + T_qx_{1q} + c$, $c \in \operatorname{Ker} S$. The condition $Bx = x_B$ leads to $Bx = BT_q^2f + BT_qx_{1q} + Bc = x_B$. Thus we have obtained an equation

$$Bc = g, \quad x_B - BT_qx_{1q} - BT_q^2f = g \in \operatorname{Ker} S$$

with an unknown c. Now the thesis of the theorem follows directly from this equation ∎

Corollary 1: *If $B|_{\mathrm{Ker}\,S}$ is a bijection onto* Ker S, *then Problem* (1) *has only one solution which is given by the formula*

$$x = T_q^2 f + T_q x_{1q} + (B|_{\mathrm{Ker}\,S})^{-1} g, \; g = x_B - BT_q x_{1q} - BT_q^2 f.$$

Theorem 2: *Let the following assumptions be satisfied:*

(i_1) L^0, L^1, L^2 *are commutative algebras with unity* $e \in \mathrm{Ker}\, S$,

(i_2) $S(cx) = c(Sx)$, $T_q(cf) = c(T_q f)$, $s_q(cx) = c(s_q x)$, *where* $c \in \mathrm{Ker}\, S$, $x \in L^1$ *and* $f \in L^0$,

(i_3) $Be = e$,

(i_4) $B(cg) = c(Bg)$ *for* $c \in \mathrm{Ker}\, S$ *and* $g \in L^2$.

Then the operations $\hat{S}$, $\hat{T}_q$, $\hat{s}_q$ *defined by the formulas*

$$\left.\begin{aligned} \hat{S}u &= S^2 u, \; u \in L^2, \\ \hat{T}_q f &= T_q^2 f - B(T_q^2 f), \; f \in L^0, \\ \hat{s}_q u &= (s_q S u)\,[T_q e - B(T_q e)] + Bu, \; u \in L^2 \end{aligned}\right\} \tag{2}$$

satisfy the axioms of operational calculus: $\hat{S}$ *is a derivative,* $\hat{T}_q$ *an integral and* $\hat{s}_q$ *a limit condition.*

Proof: $\hat{S}$, $\hat{T}_q$, $\hat{s}_q$ are linear operations. We must show that

$$\hat{S}\hat{T}_q f = f \; (f \in L^0) \quad \text{and} \quad \hat{T}_q \hat{S} u = u - \hat{s}_q u \; (u \in L^2).$$

Indeed, from the fact that the operations S, T_q, s_q satisfy the axioms of the operational calculus and from the assumptions, we have

$$\hat{S}\hat{T}_q f = S^2 T_q^2 f - S^2 B(T_q^2 f) = f$$

and

$$\begin{aligned} \hat{T}_q \hat{S} u &= T_q^2 S^2 u - B(T_q^2 S^2 u) \\ &= u - s_q u - T_q s_q S u - Bu + B(s_q u) + B(T_q s_q S u) \\ &= u - (s_q S u)\, T_q e - Bu + (s_q S u)\, B(T_q e) = u - \hat{s}_q u \; \blacksquare \end{aligned}$$

Theorem 3: *If the assumptions* (i_1), (i_3), (i_4) *of Theorem* 2 *are satisfied, then Problem* (1) *has only one solution which is given by the formula*

$$x = T_q^2 f + T_q x_{1q} + x_B - BT_q x_{1q} - BT_q^2 f. \tag{3}$$

Proof: From the assumptions it follows that B transforms every element from Ker S onto itself; so $B|_{\mathrm{Ker}\,S}$ is a bijection onto Ker S. The application of Corollary 1 ends the proof ∎

Remark: If $B = s_q$, then we obtain an initial value problem.

The operational calculus obtained in Theorem 2 enables us for instance to solve abstract differential equations of the type

$$\sum_{i=0}^{n} R_i S^{2i} u = f$$

with

$$s_q S^{2i+1} u = u_{1iq} \quad \text{and} \quad B(S^{2i} u) = u_{iB} \quad (i = 0, 1, \dots, n-1),$$

applying the methods presented in [1–4, 7, 11, 12]. The coefficients of the equation can be scalars (numbers), commutative or non-commutative operations with derivative S, integral T_q and operation B.

Examples: A) The differential equation

$$y'' + 2py' + (p^2 + p')\, y = \{f(t)\}$$

with the conditions

$$y'(t_0) + p(t_0)\, y(t_0) = \alpha \quad \text{and} \quad \int\limits_{t_0}^{t_1} y(\tau)\, d\tau = \beta,$$

where $y \in C^2(\langle t_0, t_1\rangle, \mathbf{R})$, $p \in C^1(\langle t_0, t_1\rangle, \mathbf{R})$, $f \in C^0(\langle t_0, t_1\rangle, \mathbf{R})$ and $\alpha, \beta \in \mathbf{R}$, has only one solution, because the operation B,

$$By := \left(\int\limits_{t_0}^{t_1} y(\tau)\ d\tau\right) \exp\left(-\int\limits_{t_0}^{t_1} p(\tau)\ d\tau\right),$$

and $B\Big|_{\mathrm{Ker}\left(\frac{d}{dt}+p\right)}$ is a bijection onto Ker $(d/dt + p)$.

B) If Ker $S \neq \{0\}$, then the abstract differential equation

$$S^2x = 0 \quad \text{with} \quad s_q Sx = 0 \quad \text{and} \quad s_{q_1}Sx = 0,$$

where $x \in L^2$ has, apart from a zero solution, the solution $x = c$, $c \in$ Ker S. The operation $B|_{\mathrm{Ker}\, S} := s_{q_1}S|_{\mathrm{Ker}\, S}$ is not an injection.

C) In the case of operational calculus with the derivative

$$S\{u(x, y)\} = \left\{b\,\frac{\partial u(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y}\right\},$$

the integral

$$T_{y_0}\{f(x, y)\} = \left\{\int\limits_{y_0}^{y} f(x - b(y - \tau), \tau)\ d\tau\right\}$$

and the limit condition

$$s_{y_0}\{u(x, y)\} = \{u(x - b(y - y_0), y_0)\},$$

where $u \in L^1 = C^2(\mathbf{R} \times \langle y_1, y_2\rangle, \mathbf{R})$, $f \in L^0 = C^1(\mathbf{R} \times \langle y_1, y_2\rangle, \mathbf{R})$, $y_0 \in \langle y_1, y_2\rangle$, $b \in \mathbf{R}$ (the case for the function of n variables is presented in [6]), the derivative $\hat{S}$, integral $\hat{T}_{y_0}$ and limit condition $\hat{s}_{y_0}$ are defined by the formulas

$$\hat{S}\{u(x, y)\} = \left\{b^2\,\frac{\partial^2 u(x, y)}{\partial x^2} + 2b\,\frac{\partial^2 u(x, y)}{\partial x\, \partial y} + \frac{\partial^2 u(x, y)}{\partial y^2}\right\},$$

$$\hat{T}_{y_0}\{f(x, y)\} = \left\{\int\limits_{y_0}^{y} (y - \tau)\, f(x - b(y - \tau), \tau)\ d\tau\right\} - B\left\{\int\limits_{y_0}^{y} (y - \tau)\, f(x - b(y - \tau), \tau)\, d\tau\right\},$$

$$\hat{s}_{y_0}\{u(x, y)\} = \left(s_{y_0}\left\{b\,\frac{\partial u(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y}\right\}\right)(\{y - y_0\} - B\{y - y_0\}) + B\{u(x, y)\},$$

where $u \in L^2$ and $f \in L^0$ if the operation $B : L^2 \to \mathrm{Ker}\,(b\ \partial/\partial x + \partial/\partial y)$ satisfies assumptions (i_3), (i_4) of Theorem 2. The partial differential equation

$$\left\{b^2\,\frac{\partial^2 u(x, y)}{\partial x^2} + 2b\,\frac{\partial^2 u(x, y)}{\partial x\, \partial y} + \frac{\partial^2 u(x, y)}{\partial y^2}\right\} = \{f(x, y)\}$$

with the conditions

$$\left\{b\,\frac{\partial u(x, y_0)}{\partial x} + \frac{\partial u(x, y_0)}{\partial y}\right\} = \{\varphi(x)\} \quad \text{and} \quad \left\{\int\limits_{y_1}^{y_2} u(x - b(y - \tau), \tau)\ d\tau\right\} = \psi,$$

where $u \in L^2$, $f \in L^0$, $\varphi \in C^2(\mathbf{R}, \mathbf{R})$ and $\psi \in \operatorname{Ker}(b\, \partial/\partial x + \partial/\partial y)$ has only one solution which is given by formula (3). In this case B,

$$Bu := \left\{(1/(y_2 - y_1)) \int_{y_1}^{y_2} u(x - b(y - \tau), \tau)\, d\tau\right\},$$

satisfies the assumptions of Theorem 2.

D) Into the space $C(\mathbf{N})$ of real sequences $x = \{x_k\}$ let us introduce the derivative $S = \Delta$ according to the formula $\Delta\{x_k\} = \{x_{k+1} - x_k\}$. The limit condition s_{k_0} corresponding to Δ has the form $s_{k_0}\{x_k\} = \{x_{k_0}\}$. The integral T_{k_0} corresponding to Δ and the limit condition s_{k_0} has the form (see [5, 9])

$$T_{k_0}\{x_k\} = \begin{cases} 0 & \text{for } k = k_0 \\ \sum_{i=k_0}^{k-1} x_i & \text{for } k_0 < k \\ -\sum_{i=k}^{k_0-1} x_i & \text{for } k_0 > k. \end{cases}$$

The operation

$$B\{x_k\} = \left\{\sum_{i=0}^{\infty} \alpha_i x_i \Big/ \sum_{i=0}^{\infty} \alpha_i\right\},$$

where $\{\alpha_k\} \in C(\mathbf{N})$, $\{\alpha_k\}$ is a sequence in which the finite amount of elements is different from zero and $\sum \alpha_i \neq 0$, satisfies the assumptions of Theorem 2. On the basis of this theorem we can define the derivative $\hat{S} = \Delta^2$, the integral $\hat{T}_{k_0}$ and the limit condition $\hat{s}_{k_0}$. The difference equation

$$\Delta^2\{x_k\} = \{f_k\} \quad \text{with} \quad x_{k_0+1} - x_{k_0} = a \quad \text{and} \quad \sum_{i=0}^{\infty} \alpha_i x_i = b,$$

where $a, b \in \mathbf{R}$, $\{x_k\}, \{f_k\}, \{\alpha_k\} \in C(\mathbf{N})$, $\{\alpha_k\}$ is a sequence in which only a finite amount of elements is different from zero and $\sum \alpha_i \neq 0$, has only the one solution given by formula (3).

E) For $L^1 \subset L^0$, $L^m := \{x \in L^{m-1} : Sx \in L^{m-1}\}$, $m = 2, 3, \ldots$ (see [3, 4]). If L^i $(i = 0, 1, \ldots, 2n)$ are commutative algebras with unity $e \in \operatorname{Ker} S$ and if the assumptions (i_2)–(i_4) of Theorem 2 are satisfied, then the abstract differential equation

$$\begin{aligned} &S^{2n}x = f, \\ &s_q S^{2i+1}x = x_{1iq} \quad \text{and} \quad B(S^{2i}x) = x_{iB}\ (i = 0, 1, \ldots, n-1), \end{aligned} \tag{4}$$

where $x \in L^{2n}$, $f \in L^0$ and $x_{1iq}, x_{iB} \in \operatorname{Ker} S$ has only one solution, which is given by the formula

$$x = \sum_{i=0}^{n-1} (T_q^2 - BT_q^2)^i\, (x_{1iq}[T_q e - B(T_q e)] + x_{iB}) + (T_q^2 - BT_q^2)^n f. \tag{5}$$

Let us observe that problem (4) is equivalent to the problem

$$\hat{S}^n x = f \quad \text{with} \quad \hat{s}_q \hat{S}^i x = x_{1iqB}, \quad i = 0, 1, \ldots, n-1,$$

where $x \in L^{2n}$, $f \in L^0$ and $x_{1iqB} \in \operatorname{Ker} \hat{S}$. Applying R. BITTNER [4: Theorem 6] and Theorem 2 we will get formula (5) for the only solution of problem (4).

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