(1)

Operational Calculus and Boundary Value Problem for an **Abstract Differential Equation**

E. MIELOSZYK

Es wird die abstrakte Differentialgleichung

 $S^2x = f$ mit $s_a Sx = x_{1a}$ und $Bx = x_B$

betrachtet, wobei $B: L^2 \to \text{Ker } S$, $x \in L^2$, $f \in L^0$ und x_{1q} , $x_B \in \text{Ker } S$ ist.

Рассматривается абстрактное дифференциальное уравнение

 $S^2x = f$ c $s_o Sx = x_{1g}$ ii $Bx = x_B$,

где $B: L^2 \to \text{Ker } S$, $x \in L^2$, $f \in L^0$ и $x_{1a}, x_B \in \text{Ker } S$.

The abstract differential equation

 $S^2x = f$ with $s_qSx = x_{1q}$ and $Bx = x_B$ is considered, where $B: L^2 \to \text{Ker } S$, $x \in L^2$, $f \in L^0$ and x_{1q} , $x_B \in \text{Ker } S$.

Suppose we are given the operational calculus $CO(L^0, L^1, S, T_q, s_q, Q)$, where L^0 and L^1 are linear spaces, S, T_q , s_q are linear operations called derivative, integral and limit condition, respectively, so that $S: L^1 \to L^0$ (onto), $T_q: L^0 \to L^1$ and $s_q: L^1 \to \text{Ker } S$. $(q \in Q)$, where Q is the set of indices. Let us assume that the operations S, T_a , s_a satisfy the following properties:

 $S T_q f = f$ for $f \in L^0, q \in Q$, $T_a S g \doteq g' - s_a g$ for $g \in L^1$, $q \in Q$ (see [2-4, 7, 11]).

For $L^1 \subset L^0$, $L^2 = \{x \in L^1 : Sx \in L^1\}$ (see [3, 4]) and $B: L^2 \to \text{Ker } S$ a linear operastion, let us consider the abstract differential equation

 $S^2x = f$ with $s_qSx = x_{1q}$ and $Bx = x_B$,

where $x \in L^2$, $f \in L^0$ and x_{1q} , $x_B \in \text{Ker } S$.

Theorem 1: Problem (1) has

a) at least one solution if $B|_{\text{Ker }S}$ is a surjection onto Ker S,

b) at most one solution if $B|_{\text{Ker }S}$ is an injection,

c) exactly one solution if $B|_{\text{Ker }S}$ is a bijection onto Ker S.

Proof: Operating on the equation $S^2x = f$ with the operation T_g^2 and applying the axioms of the operational calculus and the condition $s_q S x = x_{1q}$, we obtain $x = T_q^2 f$ $+T_qx_{1q}+c, c \in \text{Ker } S.$ The condition $Bx=x_B$ leads to $Bx=\ddot{B}T_q^2f+B T_qx_{1q}+\ddot{B}c$ $x_{\rm B}$. Thus we have obtained an equation

$$
Bc = g, x_B - BT_q x_{1q} - BT_q^2 f = g \in \text{Ker } S
$$

with an unknown c. Now the thesis of the theorem follows directly from this equation **I**

I!

Corollary 1: If $B_{\text{Ker }S}$ *is a bijection onto* Ker *S*, then Problem (1) has only one solu*tion which is given by the formula*

$$
x = T_q^2 f + T_q x_{1q} + (B|_{\text{Ker }S})^{-1} g, g = x_B - BT_q x_{1q} - BT_q^2 f.
$$

The ore in *2: Let the following assumptions be satisfied:*

- (i₁) L^0 , L^1 , L^2 are commutative algebras with unity $e \in \text{Ker } S$,
- $(i_2) S(cx) = c(Sx), T_q(cf) = c(T_qf), s_q(cx) = c(s_qx), where c \in \text{Ker } S, x \in L^1$ and */€L°, Theorem 2. Let the follows

(i₁)* L^0 *,* L^1 *,* L^2 *are commu

(i₂)* $S(cx) = c(Sx)$ *,* $T_q(cf)$ *
* $f \in L^0$ *,

(i₃)* $Be = e$ *,

(i₄)* $B(cg) = c(Bg)$ *for* $c \in$ *

<i>Then the operations S,* T_q *, s*
 $S_u = S^2u$, $u \in L^2$,
	- (i_3) $Be = e$,

$$
(i_4) B(cg) = c(Bg) \text{ for } c \in \text{Ker } S \text{ and } g \in L^2.
$$

q defined by the formulas

Corollary 1: If
$$
B|_{K_{eff}}
$$
 is a bijection onto Ker S, then Problem (1) has only one solu-
\n*m which is given by the formula*
\n $x = T_q^3 f + T_q x_{1q} + (B|_{K_{eff}})^{-1} g, g = x_B - BT_q x_{1q} - BT_q^2 f$.
\nheorem 2: Let the following assumptions be satisfied:
\n(i₁) L^0, L^1, L^2 are commutative algebras with unity $e \in \text{Ker } S$,
\n(i₂) $S(cx) = c(Sx), T_q(cf) = c(T_qf), s_q(cx) = c(s_qx), where c \in \text{Ker } S, x \in L^1$ and
\n $f \in L^0$,
\n(i₃) $Be = e$,
\n(i₄) $B(cg) = c(Bg)$ for $c \in \text{Ker } S$ and $g \in L^2$.
\nthen the operations $\hat{S}, \hat{T}_q, \hat{s}_q$ defined by the formulas
\n $\hat{S}u = S^2u, u \in L^2$,
\n $\hat{T}_q f = T_q^2 f - B(T_q^2 f), f \in L^0$,
\n $\hat{s}_q u = (s_q Su) [T_qe - B(T_qe)] + Bu, u \in L^2$
\n $\hat{s}_q u = (s_q Su) [T_qe - B(T_qe)] + Bu, u \in L^2$
\n $\hat{s}_q u = (s_q Su) [T_qe - B(T_qe)] + Bu, u \in L^2$
\n $\hat{S}f_q f = f(f \in L^0)$ and $\hat{T}_q \hat{S}u = u - \hat{s}_q u (u \in L^2)$.

satisfy the axioms of operational calculus: \hat{S} is a derivative, \hat{T}_a an integral and \hat{s}_a a limit *condition.*

$$
\hat{S}\hat{T}_{q}f = f(f \in L^{0}) \text{ and } \hat{T}_{q}\hat{S}u = u - \hat{s}_{q}u^{'}(u \in L^{2}).
$$

Indeed, from the fact that the operations S, T_q, s_q satisfy the axioms of the operational calculus and from the assumptions, we have

$$
\hat{S}\hat{T}_{q}f=S^{2}T_{q}^{2}f-S^{2}B(T_{q}^{2}f)=f
$$

- and

$$
Su = S-u, u \in L,
$$

\n
$$
\hat{T}_q f = T_q^2 f - B(T_q^2 f), f \in L^0,
$$
\n
$$
\hat{s}_qu = (s_q S u) [T_q e - B(T_q e)] + Bu, u \in L^2
$$
\nsatisfy the axioms of operational calculus: \hat{S} is a derivative, \hat{T}_q an integral and \hat{s}_q a limit condition.
\nProof: \hat{S} , \hat{T}_q , \hat{s}_q are linear operations. We must show that
\n
$$
\hat{S}T_q f = f (f \in L^0)
$$
 and $\hat{T}_q S u = u - \hat{s}_q u (u \in L^2).$
\nIndeed, from the fact that the operations S, T_q, s_q satisfy the axioms of the operational
\ncalculus and from the assumptions, we have
\n
$$
\hat{S}T_q f = S^2 T_q^2 f - S^2 B (T_q^2 f) = f
$$
\nand
\n
$$
\hat{T}_q S u = T_q^2 S^2 u - B(T_q^2 S^2 u)
$$
\n
$$
= u - s_q u - T_q s_q S u - Bu + B(s_q u) + B(T_q s_q S u)
$$
\n
$$
= u - (s_q S u) T_q e - Bu + (s_q S u) B (T_q e) = u - \hat{s}_q u
$$
\nTheorem 3: If the assumptions (i), (i), (i), (i), (j), (j), (j), (j), (j), (j), (k), (l) Theorem 2 are satisfied, then Problem
\n(1) has only one solution which is given by the formula
\n
$$
x = T_q^2 f + T_q x_{1q} + x_B - BT_q x_{1q} - BT_q^2 f.
$$
\n(3)
\nProof: From the assumptions it follows that B transforms every element from
\nKer S onto itself; so $B|_{Ker S}$ is a bijection onto Ker S. The application of Corollary 1
\nHemark: If $B = s_q$, then we obtain an initial value problem.
\nThe operational calculus obtained in Theorem 2 enables us for instance to solve

Theorem 3: If the assumptions (i₁), (i₃), (i₄) of Theorem 2 are satisfied, then Problem (1) *has only one solution which is given by the formula*

$$
x = T_q^2 f + T_q x_{1q} + x_B - B T_q x_{1q} - B T_q^2 f. \tag{3}
$$

Proof: From the assumptions it follows that *B* transforms every element from Ker S onto itself; so $B|_{\text{Ker }S}$ is a bijection onto Ker S. The application of Corollary $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Theorem 3: If the assumptions (i₁), (i₃), (i₄) of Theorem 2 are satisfied, then Proble

(1) has only one solution which is given by the formula
 $x = T_q^3f + T_qx_{1q} + x_B - BT_qx_{1q} - BT_q^2f$.

Proof: From the assumptions it fo

r

The operational calculus obtained'in Theorem *2* enables us for instance to solve abstract differential equations of the type

$$
\sum_{i=0}^{n} R_i S^{2i} u = f
$$

$$
s_o S^{2i+1}u = u_{1io}
$$
 and $B(S^{2i}u) = u_{iB}$ $(i = 0, 1, ..., n - 1),$

applying the methods presented in $[1-4, 7, 11, 12]$. The coefficients of the equation can be scalars (numbers), commutative or non-commutative operations with-derivative *S*, integral T_q and operation *B*.

V
V
V V V V V
V V V V V V

S ^S

Conserversity Conserversity Conservant Caler
 Examples: A) The differential equation
 $y'' + 2py' + (p^2 + p')y = \{f(t)\}$

Operational Calculus for

\nes: A) The differential equation

\n
$$
y'' + 2py' + (p^2 + p')y = \{f(t)\}
$$
\nconditions

with the conditions

 \mathbf{r}

Operational Calculus

\nless: A) The differential equation

\n
$$
y'' + 2py' + (p^2 + p')y = \{f(t)\}
$$
\nconditions

\n
$$
y'(t_0) + p(t_0) y(t_0) = \alpha \quad \text{and} \quad \int_t^t y(\tau) d\tau = \beta,
$$
\n
$$
C^2(\langle t_0, t_1 \rangle, \mathbf{R}), p \in C^1(\langle t_0, t_1 \rangle, \mathbf{R}), f \in C^0(\langle t_0, t_1 \rangle, \mathbf{R})
$$
\nthe operation B

where $y \in C^2(\langle \tilde{t}_0, t_1 \rangle, \mathbf{R})$, $p \in C^1(\langle t_0, t_1 \rangle, \mathbf{R})$, $f \in C^0(\langle t_0, t_1 \rangle, \mathbf{R})$ and $\alpha, \beta \in \mathbf{R}$, has only one solution, because the operation B , **Examples:** A) The differential equality $y'' + 2py' + (p^2 + p')y =$
with the conditions
 $y'(t_0) + p(t_0) y(t_0) = \alpha$ and
where $y \in C^2((t_0, t_1), \mathbf{R}), p \in C^1((t_0, t_1), \mathbf{R})$
because the operation B,
 $By := \left(\int_0^t y(\tau) d\tau\right) \exp\left(-\frac{1}{\tau}\right)$

with the conditions
\n
$$
y'(t_0) + p(t_0) y(t_0) = \alpha \text{ and } \int_t^t y(\tau) d\tau = \beta
$$
\nwhere $y \in C^2((t_0, t_1), \mathbf{R}), p \in C^1((t_0, t_1), \mathbf{R}), f \in C^0((t_0, t_1), \beta)$
\nbecause the operation \hat{B} ,
\n
$$
By := \left(\int_t^t y(\tau) d\tau\right) \exp\left(-\int_t^t p(\tau) d\tau\right),
$$
\nand
\n
$$
B|_{\text{Ker}\left(\frac{d}{dt} + p\right)} \text{ is a bijection onto } \text{Ker } (d/dt + p).
$$
\n**B)** If $\text{Ker } S = \{0\}$, then the abstract differential eq
\n
$$
S^2x = 0 \quad \text{with} \quad s_q S x = 0 \quad \text{and} \quad s_q S x = 0
$$
\nwhere $x \in L^2$ has, apart from a zero solution, the
\n $B|_{\text{Ker } S} := s_q S|_{\text{Ker } S} \text{ is not an injection.}$
\n**C)** In the case of operational calculus with the deri
\n
$$
S\{u(x, y)\} = \left\{b \frac{\partial u(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y}\right\},
$$
\nthe integral
\n
$$
T_{y_0} \{f(x, y)\} = \left\{\int_t^y f(x - b(y - \tau), \tau) d\tau\right\}.
$$

and *B d* is a bijection onto Ker $(d/dt + p)$. (t_0, t_1, t_2) is a bijection onto Ker $(d/dt + p)$.
 \neq {0}, then the abstract differential equation

B) If Ker $S = \{0\}$, then the abstract differential equation

$$
S^2x = 0 \quad \text{with} \quad s_q S x = 0 \quad \text{and} \quad s_{q_1} S x = 0,
$$

where $x \in L^2$ has, apart from a zero solution, the solution $x = c$, $c \in \text{Ker } S$. The operation $B|_{\text{Ker } S} := s_{q_1} S|_{\text{Ker } S}$ is not an injection. $x = c, \ c \in \text{Ker } S.$ $By := \left(\int_{t_0}^{t_1} y(\tau) d\tau\right) \exp\left(-\int_{t_0}^{t_1} p(\tau) d\tau\right),$

nd $B\Big|_{\text{Ker}\Big(\frac{d}{dt} + p\Big)}$ is a bijection onto Ker $(d/dt + p)$.
 B) If Ker $S = \{0\}$, then the abstract differential equation
 $S^2x = 0$ with $s_qSx = 0$ and $s_qSx =$ where $x \in L^2$ has, apart from a zero solution
 *i*_{lKerS}: = $s_{q_1}S|_{KerS}$ is not an injection.
 C) In the case of operational calculus with $S\{u(x, y)\} = \left\{b \frac{\partial u(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y}\right\}$

the integral $\begin{pmatrix} x \\ y \end{pmatrix}$,
 $\begin{pmatrix} t + p \\ x \end{pmatrix}$, the soluti
the derivativ,
 $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *- S*. The operation $B\left[\ker\left(\frac{d}{dt}+p\right)\right]$ is a bijection onto Ker $(d/dt + p)$.

If Ker $S = \{0\}$, then the abstract differential equation
 $S^2x = 0$ with $s_qSx = 0$ and $s_q.Sx = 0$,
 $x \in L^2$ has, apart from a zero solution, the solution $x = c$,

$$
\begin{aligned}\n\text{er}\left(\frac{d}{dt}+p\right) &\text{is a bijection onto Ker } (d/dt) \\
\text{fer } S &+ \{0\}, \text{ then the abstract difference} \\
S^2x &= 0 \quad \text{with} \quad s_q S x = 0 \quad \text{and} \quad s_q. \\
L^2 \text{ has, apart from a zero solution} \\
s_q.S|\text{Kers is not an injection.} \\
\text{he case of operational calculus with } t \\
S\{u(x, y)\} &= \left\{b\frac{\partial u(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y}\right\}, \\
\text{all} \\
\end{aligned}
$$

the integral

•

$$
T_{y_0}\{f(x, y)\} = \left\{\int_{y_0}^y f(x - b(y - \tau), \tau) d\tau\right\}.
$$

and the limit condition

$$
s_{y_0}\{u(x, y)\} = \{u(x - b(y - y_0), y_0)\},\
$$

L² has, apart from a zero solution, the solution $x = c$, $c \in \text{Ker } S$. Th
 $s_q, S|_{\text{Ker } S}$ is not an injection.

he case of operational calculus with the derivative
 $S\{u(x, y)\} = \left\{b \frac{\partial u(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y}\right\},$

al
 where $u \in L^{\Gamma} = C^2 (\mathbf{R} \times \langle y_1, y_2 \rangle, \, \mathbf{R}),$ $f \in L^0 = C^1 (\mathbf{R} \times \langle y_1, y_2 \rangle, \, \mathbf{R}),$ $y_0 \in \langle y_1, y_2 \rangle, \, b \in \mathbf{R}$ (the case for and the limit condition
 $s_{y_o}\{u(x, y)\} = \{u(x - b(y - y_o), y_o)\},$

where $u \in L^r = C^2(\mathbf{R} \times \langle y_1, y_2 \rangle, \mathbf{R}), f \in L^0 = C^1(\mathbf{R} \times \langle y_1, y_2 \rangle, \mathbf{R}), y_o \in \langle y_1, y_2 \rangle, b \in \mathbf{R}$ (the case for

the function of *n* variables is presented $S\{u(x, y)\} = \left\{b\frac{\partial u(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y}\right\}$
the integral
 $T_{y_0}\{f(x, y)\} = \left\{\int_{y_0}^{y} f(x - b(y - y_0))\right\}$
and the limit condition
 $s_{y_0}\{u(x, y)\} = \left\{u(x - b(y - y_0),\right.$
where $u \in L^1 = C^2(\mathbf{R} \times \langle y_1, y_2 \rangle, \mathbf{R}), f \in L^0$
the fun $u(x, y_1, y_2), R$, $f \in L^0 = C^1(R \times \langle y_1, y_2 \rangle, R)$, y_0
 a n variables is presented in [6]), the derivative \hat{S} , inted by the formulas
 $u(x, y)$ } = $\left\{ b^2 \frac{\partial^2 u(x, y)}{\partial x^2} + 2b \frac{\partial^2 u(x, y)}{\partial x \partial y} + \frac{\partial^2 u(x, y)}{\partial y^2} \right\}$,

$$
T_{y_1}f(x, y) = \left\{\int_{y_2}^{x} f(x - b(y - \tau), \tau) d\tau\right\}
$$

and the limit condition

$$
s_{y_2}[u(x, y)] = \left\{u(x - b(y - y_0), y_0)\right\},
$$

where $u \in L^T = C^2(\mathbb{R} \times \langle y_1, y_2), \mathbb{R})$, $f \in L^0 = C^1(\mathbb{R} \times \langle y_1, y_2), \mathbb{R})$, $y_0 \in \langle y_1, y_2 \rangle$, $b \in \mathbb{R}$ (the case for
the function of n variables is presented in [6]), the derivative S, integral \hat{T}_{y_2} and limit condition

$$
\hat{S}_{y_1}
$$
 are defined by the formulas

$$
\hat{S}_{\{u(x, y)\}} = \left\{b^2 \frac{\partial^2 u(x, y)}{\partial x^2} + 2b \frac{\partial^2 u(x, y)}{\partial x \partial y} + \frac{\partial^2 u(x, y)}{\partial y^2}\right\},
$$

$$
\hat{T}_{y_2}\{f(x, y)\} = \left\{\int_{y_2}^{y} (y - \tau) f(x - b(y - \tau), \tau) d\tau\right\} - B\left\{\int_{y_2}^{y} (y - \tau) f(x - b(y - \tau), \tau) d\tau\right\},
$$

$$
S_{y_1}\{u(x, y)\} = \left\{\int_{y_2}^{y} (y - \tau) f(x - b(y - \tau), \tau) d\tau\right\} - B\left\{\int_{y_2}^{y} (y - \tau) f(x - b(y - \tau), \tau) d\tau\right\},
$$

$$
\hat{S}_{y_2}\{u(x, y)\} = \left\{\int_{y_2}^{y} (y - \tau) f(x - b(y - \tau), \tau) d\tau\right\} - B\left\{\int_{y_1}^{y} (y - \tau) f(x - b(y - \tau), \tau) d\tau\right\},
$$

$$
\hat{S}_{y_2}\{u(x, y)\} = \left\{\int_{y_2}^{y} (y - \tau) f(x - b(y - \tau), \tau) d\tau\right\} - B\left\{\int_{y_1}^{y} (y - \tau) f(x - b(y - \tau), \tau) d\tau\
$$

where $u \in L^2$ and $f \in L^0$ if the operation $B: L^2 \to \text{Ker } (b \partial/\partial x + \partial/\partial y)$ satisfies assumptions

$$
\left\{b^2\,\frac{\partial^2 u(x,\,y)}{\partial x^2}+2b\,\frac{\partial^2 u(x,\,y)}{\partial x\,\partial y}+\frac{\partial^2 u(x,\,y)}{\partial y^2}\right\}=\,\{f(x,\,y)\}
$$

L² and
$$
f \in L^0
$$
 if the operation $B: L^2 \to \text{Ker}(b \partial/\partial x + \partial/\partial y)$ satisfies assumption
\nTheorem 2. The partial differential equation
\n
$$
\left\{b^2 \frac{\partial^2 u(x, y)}{\partial x^2} + 2b \frac{\partial^2 u(x, y)}{\partial x \partial y} + \frac{\partial^2 u(x, y)}{\partial y^2}\right\} = \left\{f(x, y)\right\}
$$
\nand
\nand
\n
$$
\left\{b \frac{\partial u(x, y_0)}{\partial x} + \frac{\partial u(x, y_0)}{\partial y}\right\} = \left\{\varphi(x)\right\} \text{ and } \left\{\int_{y_1}^{y_2} u(x - b(y - \tau), \tau) \, d\tau\right\} = \psi,
$$

where $u \in L^2$, $f \in L^0$, $\varphi \in C^2(\mathbf{R}, \mathbf{R})$ and $\psi \in \mathbf{Ker}$ *(b* $\partial/\partial x + \partial/\partial y$ *)* has only one solution which is 254 E. MIELOSZYK
where $u \in L^2$, $f \in L^0$, $\varphi \in C^2(\mathbf{R}, \mathbf{R})$ and $\psi \in \mathbf{Q}$ and $\psi \in \mathbf{Q}$ and ψ and

$$
B u := \left\{ (1/(y_2 - y_1)) \int_{y_1}^{y_1} u(x - b(y - \tau), \tau) d\tau \right\},
$$

Satisfies the assumptions of Theorem 2.

Satisfies the assumptions of Theorem 2.

D) Into the space $C(N)$ of real sequences $x = \{x_k\}$ let us introduce the derivative $S = \Delta$

The form s_k . Into the formula $\Delta(x_k) = \{x_{k+1$ D) Into the space $C(N)$ of real sequences $x = \{x_k\}$ let us introduce the derivative $S = \Delta$ the form s_k , $\{x_k\} = \{x_k\}$. The integral T_k , corresponding to Δ and the limit condition s_k , has the form (see $[5, 9]$) $(y_2 - y_1)$

ons of The
 $C(N)$ of re
 $C(N)$ of re

uula $\Delta\{x_k\}$
 \downarrow , The in
 $\begin{cases} 0 & k-1 \\ \sum x_i & k \end{cases}$ satisfies the assumptions of Theorem 2.

D) Into the space $C(N)$ of real sequences $x = \{x_k\}$ let us introduce the

according to the formula $\Delta\{x_k\} = \{x_{k+1} - x_k\}$. The limit condition s_k corres

the form $s_k \{x_k\} = \{$

$$
\begin{aligned}\n\text{given by formula (3). In this case } B, \\
Bu &:= \begin{cases}\n(1/(y_2 - y_1)) \int u(x - b(y - y_1) \, dx + b(y - y_1) \, dx\end{cases} \\
\text{satisfies the assumptions of Theorem 2.} \\
\text{and a complex form } \begin{cases}\nD \text{ into the space } C(\mathbb{N}) \text{ of real sequences}\n\end{cases} \\
\text{according to the formula } \Delta \{x_k\} = \{x_{k+1} - x_k\} \\
\text{the form (see [5, 9])} \\
\text{the form (see [5, 9])} \\
\text{Theorem } \begin{cases}\n0 & \text{for } k = k_0 \\
k-1 \\
\sum_{i=k}^{k-1} x_i \quad \text{for } k_0 < k\n\end{cases} \\
\text{The operation} \\
B\{x_k\} &:= \begin{cases}\n\sum_{i=k}^{\infty} \alpha_i x_i \Big/ \sum_{i=0}^{\infty} \alpha_i x_i \Big/ \dots \\
\sum_{i=0}^{\infty} \alpha_i x_i \Big/ \dots \\
\text{where } \{\alpha_k\} \in C(\mathbb{N}), \|\alpha_k\| \text{ is a sequence in which}\n\end{cases}\n\end{aligned}
$$

$$
B\{x_k\} = \left\{\sum_{i=0}^{\infty} \alpha_i x_i / \sum_{i=0}^{\infty} \alpha_i\right\},\,
$$

• $i = k$
 $\begin{cases} \sum_{i=1}^{n} x_i & \text{for } k_0 > k. \end{cases}$
 $B\{x_k\} = \left\{ \sum_{i=0}^{\infty} \alpha_i x_i \middle/ \sum_{i=0}^{\infty} \alpha_i \right\},$

where $\{\alpha_k\} \in C(\mathbb{N})$, $\{\alpha_k\}$ is a sequence in which the finite amount of elements is different from

zero and $\$ The operation
 $B\{x_k\} = \left\{\sum_{i=0}^{\infty} \alpha_i x_i \middle/ \sum_{i=0}^{\infty} \alpha_i\right\},$

where $\{\alpha_k\} \in C(\mathbb{N})$, $\{\alpha_k\}$ is a sequence in which the finite amount of elements is different from

zero and $\sum \alpha_i = 0$, satisfies the assumptions can define the derivative $\hat{S} = \Delta^2$, the integral \hat{T}_{k_0} and the limit condition \hat{s}_{k_0} . The difference equation $\begin{aligned} \n\langle x_k \rangle & = \left\{ \sum_{i=0}^{\infty} \alpha_i x_i \middle/ \sum_{i=0}^{\infty} x_i \right\}, \\ \n\langle C(\mathbf{N}), \{\alpha_k\} \text{ is a sequence in which the finite amount of } \alpha_i \neq 0, \text{ satisfies the assumptions of Theorem 2. On the derivative } \hat{S} = \Delta^2, \text{ the integral } \hat{T}_{k_0} \text{ and the limit of } \Delta^2 |x_k| = \{f_k\} \quad \text{with} \quad x_{k_0+1} - x_{k_0} = a \quad \text{and} \quad \sum_{i=0}^{\infty} \alpha_i x_i = b, \\ \n\$

$$
\Delta^2\{x_k\} = \{f_k\} \text{ with } x_{k_0+1} - x_{k_0} = a \text{ and } \sum_{i=0}^{\infty} \alpha_i x_i = b,
$$

where $a, b \in \mathbf{R}, \{x_k\}, \{f_k\}, \{\alpha_k\} \in C(\mathbf{N}), \{\alpha_k\}$ is a sequence in which only a finite amount of elements is different from zero and $\sum \alpha_i = 0$, has only the one solution given by formula (3).

E) For $L^1 \subset L^0$, $L^m := \{x \in L^{m-1} : Sx \in L^{m-1}\}$, $m = 2, 3, ...$ (see [3, 4]). If L^i $(i = 0, 1, ..., 2n)$ are satisfied, then the abstract differential equation 1. $\{x_k\}, \{y_k\}, \{\alpha_k\} \in C(\mathbb{N}), \{\alpha_k\}$ is a sequence in which only a

im zero and $\sum \alpha_i \neq 0$, has only the one solution given

1. L^0 , $L^m := \{x \in L^{m-1} : Sx \in L^{m-1}\}, m = 2, 3, ...$ (see [3, ive algebras with unity $e \in \text{Ker }$

E) For
$$
L^2 \subset L^p
$$
, $L^m := \{x \in L^{m-1} : x \in L^{m-1}\}$, $m = 2, 3, \ldots$ (see [3, 4]). If L^i $(i = 0, 1, \ldots, 2n)$ are commutative algebras with unity $e \in \text{Ker } S$ and if the assumptions $(i_2) - (i_4)$ of Theorem 2 are satisfied, then the abstract differential equation\n
$$
S^{2n}x = f,
$$
\n
$$
s_q S^{2i+1}x = x_{1iq}
$$
 and $B(S^{2i}x) = x_{iB} \ (i = 0, 1, \ldots, n-1),$ \nwhere $x \in L^{2n}$, $f \in L^0$ and x_{1iq} , $x_{iB} \in \text{Ker } S$ has only one solution, which is given by the formula\n
$$
x = \sum_{i=0}^{n-1} (T_q^2 - BT_q^2)^i (x_{1iq} [T_q e - B(T_q e)] + x_{iB}) + (T_q^2 - BT_q^2)^n f.
$$
\nLet us observe that problem (4) is equivalent to the problem\n
$$
\hat{S}^n x = f \text{ with } s_q \hat{S}^i x = x_{1iqB}, \quad i = 0, 1, \ldots, n-1,
$$
\nwhere $x \in L^{2n}$, $f \in L^0$ and $x_{1iqB} \in \text{Ker } \hat{S}$. Applying R. Brrr. [4: Theorem 6] and Theorem 2, we will get formula (5) for the only solution of problem (4).

where $x \in L^{2n}$, $f \in L^0$ and x_{1iq} , $x_{iB} \in \text{Ker } S$ has only one solution, which is given by the formula 'I. San Angeles (1988)
'I. San Angeles (1988)
'I. San Angeles (1988)

$$
x = \sum_{i=0}^{n-1} (T_q^2 - BT_q^2)^i (x_{1iq} [T_q e - B(T_q e)] + x_{iB}) + (T_q^2 - BT_q^2)^n / . \tag{5}
$$

Let us observe that problem (4) is equivalent to the problem

 $\hat{S}^n x = f$ with $\partial_q \hat{S}^i x = x_{1iqB}$, $i = 0, 1, ..., n-1$,

where $x \in L^{2n}$, $f \in L^0$ and $x_{1iqB} \in \text{Ker } \hat{S}$. Applying R. BITTNER [4: Theorem 6] and Theorem 2 we will get formula (5) for the only solution of problem (4). $S^nx = f$ with $\delta_q \hat{S}^i x = a$
where $x \in L^{2n}$, $f \in L^0$ and $x_{iqB} \in K$ e
we will get formula (5) for the only
REFERENCES
[1] BERG, L.: Operatorenrechnung
[2] BITTNER, R.: Operational calcu
[3] BITTNER, R.: Algebraic and a

REFERENCES -

- [1] BERG, L.: Opera torenrechnung. I: Algebraische Methoden. Berlin: Dt.Verlag Wiss. 1972.
- [2] BITTNER, R.: Operational calculus in linear spaces. Studia Math. 20 (1961), $1-18$.
- REFERENCES

[1] BERG, L.: Operatorenrechnung. I: Algebraische Methoden. Berlin: Dt. Verlag Wiss. 1972

[2] BITTNER, R.: Operational calculus in linear spaces. Studia Math. 20 (1961), 1 18.

[3] BITTNER, R.: Algebraic and [3] BITTNER, R.: Algebraic and analytic properties of solution of abstract differential equations. Rozprawy Mat. 41 (1964), $1-63$.
	- [4] BITTNER, R.: Rachunek operatorów w przestrzeniach liniowych. Warszawa: Polish Sci.

 $\frac{1}{\sqrt{2}}\sum_{i=1}^{n} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^2$

- [5] BITTNER, R., and E. MIELOSZYK: Properties of eigenvalues and eigenelements of some difference equations in a given operational calculus. Zeszyty Naukowe UG w Gdańsku, Matematyka 5 (1981), $5-18$.
- [6] BTTTNER, R., and E. MrELoszx: Application of the operational calculus to solving non-homogeneous linear partial differential equations of the first order with real coefficients. Zeszyty Naukowe PG w Gdañsku, Matematyka 12 (1982), 33-45.
- [7] DIMOVSKI, I.: Convolutional calculus. Sofia: Bulg. Acad. Sci. 1982.
- *[8] MIELOSZYK,* E.: Operational calculus in algebras. PubI. Math. Debrecen 34 (1987).
- [9] MIELOSZYK, E.: Partial difference equation. Acts, Math. Hungarica (in print).
- [10] MIELOSZYK, E.: Operation $T_{x_0}^k$ and its application (in preparation).
- [11] PRZEWORSKA-ROLE\VICZ, D.: Shifts and periodicity for right invertible operators (Res. Notes in Mathematics 43). Boston (Mass.): Pitman Adv. PubI. Program 1980.
- [12] TASCHE, M.: Funktionalanalytische Methoden in der Operatorenrechnung. Nova Acta Leopoldina 231 (1978), $1-95$. VSKI, I.: Convolutional calculus. Sofia: Bulg. Acad. Sci. 1982.

0SZYK, E.: Operational calculus in algebras. Publ. Math. Debrecen 34

0SZYK, E.: Partial difference equation. Acta Math. Hungarica (in pri

0SZYK, E.: Operat

Manuskripteingang: 10. 01. 1986; in revidierter Fassung 03. 06. 1986

Dr. ELI0rusz MIELOSZYK

- Institute of Mathematics of the Technical University,-
- Majakowskiego 11
- P-80-952 Cdañsk