Zeitschrift für Analysis
und ihre Anwendungen
Bd. 6 (4) 1987, S. 289–300

Nonuniform Conditions for Periodic Solutions of Forced Liénard Systems with Distinct Delays

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Mit Hilfe der klassischen Leray-Schauder-Theorie und des Koinzidenzabbildungsgrades wird die Existenz periodischer Lösungen Liénardscher Systeme mit verschiedenen Verzögerungen im Falle nichtgleichmäßiger Bedingungen bewiesen. Es werden Resonanz- und Nichtresonanzfälle in der Umgebung des Eigenwertes Null behandelt.

С помощью классической теории Лерейя-Шаудера и степени совпадения доказывается существование периодических решений возбужденных Лиенардских систем с различными запаздываниями под неравномерными условиями. Рассматриваются случаи резонанса и нерезонанса относительно собственного значения нуль.

We use classical Leray-Schauder techniques and the coincidence degree in order to obtain the existence of periodic solutions of forced Liénard systems with distinct delays, under nonuniform conditions. The resonance and nonresonance case with respect to the eigenvalue zero are considered.

1. Introduction

In this paper we study the existence of 2π-periodic solutions for the following Liénard systems with distinct delays $\tau_1, \ldots, \tau_n \in [0, 2\pi)$

$$x''(t) + \frac{d}{dt}\left[\operatorname{grad} F\big(x(t)\big)\right] + g\big(t, x_1(t-\tau_1), \ldots, x_n(t-\tau_n)\big) = e(t) \qquad (1.1)$$

a.e. on $J = [0, 2\pi]$, where $F : \mathbf{R}^n \to \mathbf{R}^n$ is of class C^2, $g : J \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies Carathéodory conditions and $e : J \to \mathbf{R}^n$ is integrable.

The existence of 2π-periodic solutions for (1.1) has been recently studied by de Pascale and Iannacci [3] and Iannacci and Nkashama [9] in the scalar case under resonance and nonresonance conditions, respectively. We recall that in the case of effective delay (i.e. $\tau \neq 0$), if $\tau/\pi \in \mathbf{Q}$, then the eigenvalues of the problem

$$x''(t) + \lambda x(t-\tau) = 0, \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \qquad (1.2)$$

are not contained in the nonnegative part of the real axis (see [3: Rem. 1]). In [9] is given an existence theorem of solutions for the problem (1.1) with $n = 1$ under the assumption

$$\gamma(t) \leqq \liminf_{|x|\to+\infty} x^{-1}g(t, x) \leqq \limsup_{|x|\to+\infty} x^{-1}g(t, x) \leqq \Gamma(t) \leqq 1$$

uniformly for a.e. $t \in J$, where γ can cross the eigenvalue zero of the problem (1.2) on a suitable subset of J of positive measure. Moreover, in [3], dealing with resonant situations at the eigenvalue zero, the assumption

$$\limsup_{|x|\to+\infty} x^{-1}g(t, x) \leqq \Gamma(t) \leqq 1$$

was still considered.

Employing a technique due to GUPTA and MAWHIN [6] for $\tau = 0$, we consider the case when, for every $i = 1, \ldots, n$, the expressions $\limsup_{|x_i| \to +\infty} x_i^{-1} g_i(t, x)$ can cross suitably any number of either positive or negative eigenvalues of the problem (1.2). Notice that these improvements are given in the context of systems with distinct delays. The paper is divided into two sections. The first section is devoted to nonresonance conditions at the two first eigenvalues and in the second section we deal with resonant situations at the eigenvalue zero. Each section is divided into three subsections.

In Theorem 1 and Theorem 4 we assume that for every i

$$\gamma_i(t) \leqq \liminf_{|x_i| \to +\infty} x_i^{-1} g_i(t, x) \leqq \limsup_{|x_i| \to +\infty} x_i^{-1} g_i(t, x) \leqq \Gamma_i(t)$$

and

$$\limsup_{|x_i| \to +\infty} x_i^{-1} g_i(t, x) \leqq \Gamma_i(t),$$

respectively, uniformly for a.e. $t \in J$, where γ_i can cross the eigenvalue zero on a suitable subset of J of positive measure and Γ_i can cross the positive eigenvalues of the problem (1.2) on a subset of J of sufficiently small positive measure.

In Theorem 2 and Theorem 5 we assume that for every i

$$\Gamma_i(t) \leqq \liminf_{|x_i| \to +\infty} x_i^{-1} g_i(t, x) \leqq \limsup_{|x_i| \to +\infty} x_i^{-1} g_i(t, x) \leqq \gamma_i(t)$$

and

$$\Gamma_i(t) \leqq \liminf_{|x_i| \to +\infty} x_i^{-1} g_i(t, x),$$

respectively, uniformly for a.e. $t \in J$, where Γ_i can cross the negative eigenvalues of the problem (1.2) and γ_i can cross zero in a suitable way.

Finally, in Theorem 3 and Theorem 6 we prove the existence of 2π-periodic solutions of the system (1.1) under the assumption that a part of the equations of this system satisfies conditions of Theorem 1 and Theorem 4, respectively, and the remaining equations satisfy conditions of Theorem 2 and Theorem 5, respectively.

Our results are based on lemmas giving *a priori* estimates and degree arguments. Let us mention that for other existence results concerning either unforced or forced Liénard functional differential equations or systems but not directly related to the results introduced here, see e.g. HALE [8], GRAFTON [5], MAWHIN [12], INVERNIZZI and ZANOLIN [11] and the bibliography therein.

2. Notations, definitions and preliminary results

Let us set $J = [0, 2\pi]$. We will use the symbol $x = \mathrm{col}\,(x_1, \ldots, x_n) \in \mathbf{R}^n$ and the symbol $\|\cdot\|$ for the Euclidean norm in $\mathbf{R}^n$. Further, we will use the following spaces:

1. $L^p(J, \mathbf{R}^n)$ are the usual Lebesgue spaces, $1 \leqq p \leqq \infty$. We denote their norm by $\|\cdot\|_{L^p}$ if $n > 1$ and by $|\cdot|_{L^p}$ if $n = 1$.
2. $H^1(J, \mathbf{R}^n) = \{x : J \to \mathbf{R}^n : x$ absolutely continuous, $x' \in L^2(J, \mathbf{R}^n)$, $x(0) - x(2\pi) = 0\}$ with the norm

$$\|x\|_{H^1} = \left\{ \sum_{i=1}^{n} \left[\frac{1}{(2\pi)} \int_J x_i(t)\, dt \right]^2 + \frac{1}{2\pi} \int_J (x_i'(t))^2\, dt \right\}^{1/2}.$$

We use the symbol $|\cdot|_{H^1}$ if $n = 1$.

3. $\tilde{H}^1(J, \mathbf{R}^n) = \left\{ x \in H^1(J, \mathbf{R}^n) : \int_J x(t)\, dt = 0 \right\}.$

4. $W^{2,1}(J, \mathbf{R}^n) = \{x : J \to \mathbf{R}^n : x$ and x' absolutely continuous, $x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0\}$ with the norm

$$\|x\|_{W^{2,1}} = \left\{ \sum_{i=1}^{n} \left[\frac{1}{2\pi} \sum_{k=0}^{2} \int_J |x_i^{(k)}(t)| \, dt \right]^2 \right\}^{1/2}.$$

5. $C(J, \mathbf{R}^n)$ is the Banach space of continuous functions with the norm

$$\|x\|_C = \max \{\|x(t)\| : t \in J\}.$$

For the sake of simplicity in the notations of the spaces we will omit $\mathbf{R}^n$ when $n = 1$. Clearly if $x = \operatorname{col}(x_1, \ldots, x_n) \in H^1(J, \mathbf{R}^n)$, then $x_i \in H^1(J)$ for $i = 1, \ldots, n$. We recall that every $x_i \in H^1(J)$ can be written in the form

$$x_i = \bar{x}_i + \tilde{x}_i \quad \text{with} \quad \tilde{x}_i \in \tilde{H}^1(J) \quad \text{and} \quad \bar{x}_i = \frac{1}{2\pi} \int_J x_i(t) \, dt.$$

Moreover,

$$|x_i|_{H^1} = \left\{ \bar{x}_i^2 + \frac{1}{2\pi} \int_J (x_i'(t))^2 \, dt \right\}^{1/2} \text{ so that } \|x\|_{H^1} = \left\{ \sum_{i=1}^{n} |x_i|_{H^1}^2 \right\}^{1/2}.$$

In the sequel we will use the following result.

Lemma 1 (Mawhin and Ward [13]): *Let $\Gamma \in L^1(J)$ such that $\Gamma(t) \leqq 1$ a.e. on J, with strict inequality on a subset of positive measure. Then, there exists $\delta = \delta(\Gamma) > 0$ such that for all $\tilde{x} \in H^1(J)$ one has*

$$\frac{1}{2\pi} \int_J \left((\tilde{x}'(t))^2 - \Gamma(t) \, (\tilde{x}(t))^2 \right) dt \geqq \delta \, |\tilde{x}|_{H^1}^2.$$

For the sake of simplicity we will write $x(t - \tau)$ instead of $\left(x_1(t - \tau_1), \ldots, x_n(t - \tau_n)\right)$. A function $x \in W^{2,1}(J, \mathbf{R}^n)$ will be called a *solution* of (1.1) if it satisfies (1.1) almost everywhere. Let $h : J \to \mathbf{R}$ be a real function. We define $h^+(t) = \max \{h(t), 0\}$ and $h^-(t) = \min \{h(t), 0\}$, $t \in J$. If $h \in L^1(J)$ we denote by $\bar{h}$ the mean value of h in J.

3. Nonresonance conditions

3.1. The case of assumption B$^+$

The following lemma extends Lemma 3 of [9], when τ is an effective delay, to the case when Γ crosses the positive eigenvalues of problem (1.2) suitably in subsets of J of positive measure.

Lemma 2: *Let $\gamma, \Gamma \in L^1(J)$ be such that $\gamma(t) \leqq \Gamma(t)$ a.e. on J and $\Gamma = {}^0\Gamma + {}^1\Gamma + {}^\infty\Gamma$ with*

${}^1\Gamma \in L^1(J)$, ${}^1\Gamma(t) \geqq 0$ a.e. on J, and ${}^\infty\Gamma \in L^\infty(J)$, ${}^\infty\Gamma(t) \geqq 0$ a.e. on J,
${}^0\Gamma \in L^1(J)$, ${}^0\Gamma(t) \not\equiv 0$ and ${}^0\Gamma(t) \leqq 1$ a.e. on J with strict inequality on a subset of positive measure,

and

$$\delta({}^0\Gamma^+) - \frac{\pi^2}{3} |{}^1\Gamma|_{L^1} - |{}^\infty\Gamma|_{L^\infty} + \frac{2\pi^2}{3} \bar{\gamma}^- > 0, \quad \bar{\gamma}^+ > -4\bar{\gamma}^-. \tag{3.1}$$

Then, there exist $\varepsilon = \varepsilon(\gamma, \Gamma) > 0$ *and* $\mu = \mu(\gamma, \Gamma) > 0$ *such that for all* $p \in L^1(J)$ *satisfying* $\gamma(t) - \varepsilon \leqq p(t) \leqq \Gamma(t) + \varepsilon$ *a.e. on* J, *all continuous functions* $f : \mathbf{R} \to \mathbf{R}$ *and all* $x \in W^{2,1}$ *we have*

$$\frac{1}{2\pi} \int_J (\bar{x} - \tilde{x}(t)) \big(x''(t) + f(x(t))\, x'(t) + p(t)\, x(t - \tau)\big)\, dt \geqq \mu\, |x|^2_{H^1}.$$

Proof: Assume the contrary. Then there exist sequences $\{p_n\} \subset L^1(J)$, $\{x_n\} \subset W^{2,1}(J)$, $|x_n|_{H^1} = 1$, with $\gamma(t) - 1/n \leqq p_n(t) \leqq \Gamma(t) + 1/n$ such that

$$D^+_{P_n}(x_n) = \frac{1}{2\pi} \int_J \big\{(x_n'(t))^2 + p_n(t)\, x_n(t - \tau)\, (\bar{x}_n - \tilde{x}_n(t))\big\}\, dt < \frac{1}{n}.$$

By means of the arguments used in Lemma 3 of [9], we have, taking subsequences if it is necessary, that there exist $x \in H^1(J)$, $p \in L^1(J)$ such that $x_n \xrightarrow{C} x$, $x_n \xrightarrow{H^1} x$ and $p_n \xrightarrow{L^1} p$, with $\gamma(t) \leqq p(t) \leqq \Gamma(t)$ for a.e. $t \in J$ and $D_p{}^+(x) \leqq 0$.

We claim that

$$D_p{}^+(x) \geqq \left(\delta({}^0\Gamma^+) - |{}^\infty\Gamma|_{L^\infty} - \frac{\pi^2}{3}\, |{}^1\Gamma|_{L^1} + \frac{2\pi^2}{3}\, \bar{\gamma}^-\right) |\tilde{x}|^2_{H^1} + \frac{\bar{x}^2}{2}\, (4\bar{\gamma}^- + \bar{\gamma}^+).$$

Indeed we can write

$$\begin{aligned} D_p{}^+(x) &= \frac{1}{2\pi} \left[\int_J \left(x'^2(t) - \frac{1}{2}\, p(t)\, (\tilde{x}^2(t) + \tilde{x}^2(t - \tau))\right) dt \right. \\ &\quad \left. + \int_J \frac{1}{2}\, p(t)\, (x(t - \tau) - \tilde{x}(t))^2\, dt\right] + \frac{1}{2}\, \bar{p}\bar{x}^2 \\ &\geqq \frac{1}{2\pi} \int_J \frac{1}{2}\, p^-(t) \left\{-\tilde{x}^2(t) - \tilde{x}^2(t - \tau) + (x(t - \tau) - \tilde{x}(t))^2\right\} dt \\ &\quad + \frac{1}{2\pi} \int_J \left(x'^2(t) - \frac{1}{2}\, p^+(t)\, (\tilde{x}^2(t) + \tilde{x}^2(t - \tau)\right) dt + \frac{1}{2}\, \bar{p}\bar{x}^2. \end{aligned}$$

Since $-\tilde{x}^2(t) - \tilde{x}^2(t - \tau) + (x(t - \tau) - \tilde{x}(t))^2 \leqq 3\bar{x}^2 + (\tilde{x}(t) - \tilde{x}(t - \tau))^2$, we have, using the inequality

$$|\tilde{x}(t) - \tilde{x}(t - \tau)| \leqq \frac{2\pi}{\sqrt{3}}\, |\tilde{x}|_{H^1}, \tag{3.2}$$

$$\begin{aligned} &\frac{1}{2\pi} \int_J p^-(t) \left(-\tilde{x}^2(t) - \tilde{x}^2(t - \tau) + (x(t - \tau) - \tilde{x}(t))^2\right) dt \\ &\geqq \frac{1}{2\pi} \int_J p^-(t)\, (\tilde{x}(t) - \tilde{x}(t - \tau))^2\, dt + \frac{3}{2}\, \overline{p^-}\bar{x}^2 \geqq \frac{2\pi^2}{3}\, |\tilde{x}|^2_{H^1}\, \overline{p^-} + \frac{3}{2}\, \overline{p^-}\bar{x}^2. \end{aligned}$$

Since $p^+(t) \leqq {}^0\Gamma^+(t) + {}^1\Gamma(t) + {}^\infty\Gamma(t)$, we have

$$\frac{1}{2\pi}\int_J x'^2(t) - \frac{1}{2}\, p^+(t)\left(\tilde{x}^2(t) + \tilde{x}^2(t-\tau)\right) dt$$

$$\geqq \frac{1}{2\pi}\int_J x'^2(t) - \frac{1}{2}\, {}^0\Gamma^+(t)\left(\tilde{x}^2(t) + \tilde{x}^2(t-\tau)\right) dt$$

$$- \frac{1}{2\pi}\int_J \frac{1}{2}\left({}^\infty\Gamma + {}^1\Gamma\right)(t)\left(\tilde{x}^2(t-\tau)\right) dt$$

$$\geqq |\tilde{x}|^2_{H^1}\,\delta({}^0\Gamma^+) - \frac{1}{2\pi}\int_J \frac{1}{2}\left({}^\infty\Gamma + {}^1\Gamma\right)(t)\left(\tilde{x}^2(t) + \tilde{x}^2(t-\tau)\right) dt,$$

where $\delta({}^0\Gamma^+)$ is the best constant associated to ${}^0\Gamma^+$ by Lemma 1. Using (see e.g. [15: p. 208])

$$|\tilde{x}|_{L^2} \leqq |\tilde{x}'|_{L^2} = |\tilde{x}|_{H^1} \quad \text{and} \quad |\tilde{x}|_{L^\infty} \leqq \frac{\pi}{\sqrt{3}}\,|\tilde{x}|_{L^2} = \frac{\pi}{\sqrt{3}}\,|\tilde{x}|_{H^1}, \tag{3.3}$$

we obtain

$$\frac{1}{2\pi}\int_J \frac{1}{2}\left(\tilde{x}^2(t) + \tilde{x}^2(t-\tau)\right)\left({}^1\Gamma + {}^\infty\Gamma\right)(t)\, dt \leqq \left(\frac{\pi^2}{3}\,|{}^1\Gamma|_{L^1} + |{}^\infty\Gamma|_{L^\infty}\right)|\tilde{x}|^2_{H^1},$$

so that, since $p(t) \geqq \gamma(t)$ for a.e. t,

$$0 \geqq D_p{}^+(x) \geqq \left(\delta({}^0\Gamma^+) - |{}^\infty\Gamma|_{L^\infty} + \frac{\pi^2}{3}\left(2\bar{\gamma}^- - |{}^1\Gamma|_{L^1}\right)\right)|\tilde{x}|^2_{H^1} + \frac{\bar{x}^2}{2}\left(4\bar{\gamma}^- + \bar{\gamma}^+\right).$$

From (3.1) it follows that $\tilde{x} = 0$ and $\bar{x} = 0$, which yields a contradiction since $x_n \xrightarrow{C} x$ and $|x_n|_{H^1} = 1$ ∎

We are now in a position to prove the following existence theorem of 2π-periodic solutions for system (1.1).

Theorem 1: *Let $F \in C^2(\mathbf{R}^n, \mathbf{R})$ and $g: J \times \mathbf{R}^n \to \mathbf{R}^n$, $g(t, x) = \left(g_1(t, x), \ldots, g_n(t, x)\right)$, such that*

(i) *$g(\cdot, x)$ is measurable on J for each $x \in \mathbf{R}^n$,*
(ii) *$g(t, \cdot)$ is continuous on $\mathbf{R}^n$ for a.e. $t \in J$;*
(iii) *for each real number $r > 0$ and for any $i = 1, \ldots, n$ there exist a_i, $b_i \in L^1(J)$ (which depend also on r) such that*

$$|g_i(t, x)| \leqq a_i(t) + b_i(t)\left(\sum_{k=1}^n |x_k|^\alpha\right)$$

for a.e. $t \in J$, all $x \in \mathbf{R}^n$ with $|x_i| \leqq r$ and $0 \leqq \alpha < 1$.
Assume that for every $i = 1, \ldots, n$ the inequalities

$$(\mathbf{B}^+) \qquad \gamma_i(t) \leqq \liminf_{|x_i|\to\infty} x_i^{-1}\, g_i(t, x) \leqq \limsup_{|x_i|\to\infty} x_i^{-1} g_i(t, x) \leqq \Gamma_i(t)$$

hold uniformly for a.e. $t \in J$ and $x_j \in \mathbf{R}$, $j \neq i$, and *γ_i, Γ_i satisfy the conditions of Lemma 2.*

Then, system (1.1) has at least one 2π-periodic solution for each $e \in L^1(J, \mathbf{R}^n)$.

Proof: Let ε_i, μ_i be positive constants associated to γ_i and Γ_i by Lemma 2. From assumption $\mathbf{B}^+$ there exists $r_i > 0$ such that for a.e. $t \in J$ and for every $x \in \mathbf{R}^n$ with $|x_i| \geqq r_i$ we have $\gamma_i(t) - \varepsilon_i \leqq x_i^{-1} g_i(t, x) \leqq \Gamma_i(t) + \varepsilon_i$. Put $\varepsilon = \min \varepsilon_i$, $\mu = \min \mu_i$ and $r = \max r_i$. Define $\tilde{\gamma}_i : J \times \mathbf{R}^n \to \mathbf{R}$ by

$$\tilde{\gamma}_i(t, x) = \begin{cases} x_i^{-1} g_i(t, x) & \text{if } |x_i| \geqq r \\ r^{-1} g_i(t, x_1, \ldots, x_{i-1}, r, x_{i+1}, \ldots, x_n)(x_i r^{-1}) & \\ \qquad + \Gamma_i(t)(1 - x_i r^{-1}) & \text{if } 0 \leqq x_i < r \\ r^{-1} g_i(t, x_1, \ldots, x_{i-1}, -r, x_{i+1}, \ldots, x_n)(x_i r^{-1}) & \\ \qquad + \Gamma_i(t)(1 + x_i r^{-1}) & \text{if } -r \leqq x_i < 0. \end{cases}$$

We have

$$\gamma_i(t) - \varepsilon \leqq \tilde{\gamma}_i(t, x) \leqq \Gamma_i(t) + \varepsilon \quad \text{for a.e. } t \in J, \qquad x \in \mathbf{R}^n. \tag{3.4}$$

Moreover, the function $\tilde{g}_i : J \times \mathbf{R}^n \to \mathbf{R}$ defined by $\tilde{g}_i(t, x) = \tilde{\gamma}_i(t, x)\, x_i$ satisfies assumptions (i)—(iii) and $\tilde{g}_i(t, x) = g_i(t, x)$ for all $x \in \mathbf{R}^n$ with $|x_i| \geqq r_i$. Let $h_i = g_i - \tilde{g}_i$ and denote by $\left(\frac{d}{dt} \frac{\partial F}{\partial x_i}\big(x(t)\big)\right)$ the i-th component of $\frac{d}{dt}\big(\operatorname{grad} F(x(t))\big)$. By construction and from the assumptions on g_i it follows that

$$|h_i(t, x)| \leqq 2\left(a_i(t) + b_i(t)\left(\sum_{k=1}^{n} |x_k|^\alpha\right)\right) \text{ for a.e. } t \in J,\ x \in \mathbf{R}^n. \tag{3.5}$$

Our system (1.1) is equivalent to

$$x_i''(t) + \left(\frac{d}{dt} \frac{\partial F}{\partial x_i}\big(x(t)\big)\right) + \tilde{\gamma}_i\big(t, x(t - \tau)\big)\, x_i(t - \tau_i) + h_i\big(t, x(t - \tau)\big) = e_i(t), \quad i = 1, \ldots, n.$$

By the same degree argument as in the proof of Theorem 1 of [10], it suffices to show that the set of 2π-periodic solutions for the system

$$\begin{aligned} &x_i''(t) + (1 - \lambda)\, \Gamma_i(t)\, x_i(t - \tau_i) + \lambda \left(\frac{d}{dt} \frac{\partial F}{\partial x_i}\big(x(t)\big)\right) + \lambda h_i\big(t, x(t - \tau)\big) \\ &+ \lambda \tilde{\gamma}_i\big(t, x(t - \tau)\big)\, x_i(t - \tau_i) = \lambda e_i(t),\ i = 1, \ldots, n \end{aligned} \tag{3.6}$$

is bounded for every $\lambda \in [0, 1]$. Let $x \in W^{2,1}(J, \mathbf{R}^n)$ be a 2π-periodic solution of system (3.6). Multiplying each equation of this system by $\big(\bar{x}_i - \tilde{x}_i(t)\big)$ and taking into account the assumptions on Γ_i, (3.4) and Lemma 2, one gets for each $i = 1, \ldots, n$

$$\begin{aligned} 0 &= \frac{1}{2\pi} \int_J \big(\bar{x}_i - \tilde{x}_i(t)\big) \Big(x_i''(t) + \lambda \left(\frac{d}{dt} \frac{\partial F}{\partial x_i}\big(x(t)\big)\right) + (1 - \lambda)\, \Gamma_i(t)\, x_i(t - \tau_i) \\ &\quad + \lambda \gamma_i\big(t, x(t - \tau)\big)\, x_i(t - \tau_i) + \lambda h_i\big(t, x(t - \tau)\big) - \lambda e_i(t)\Big)\, dt \\ &\geqq \mu\, |x_i|_{H^1}^2 - |\bar{x}_i - \tilde{x}_i|_C\, |h_i|_{L^1} - |\bar{x}_i - \tilde{x}_i|_C\, |e_i|_{L^1} \\ &\quad + \frac{\lambda}{2\pi} \int_J \big(\bar{x}_i - \tilde{x}_i(t)\big) \frac{d}{dt} \frac{\partial F}{\partial x_i}\big(x(t)\big)\, dt. \end{aligned}$$

From inequalities (3.2) and (3.5) one has

$$0 \geqq \mu\, |x_i|_{H^1}^2 - \frac{2\pi}{\sqrt{3}}\, |x_i|_{H^1} \left\{|e_i|_{L^1} + 2\,|a_i|_{L^1} + 2\,|b_i|_{L^1} \left(\sum_{k=1}^{n} |x_k|_C^\alpha\right)\right\}$$

$$+ \frac{\lambda}{2\pi} \int_J \left(\bar{x}_i - \tilde{x}_i(t)\right) \frac{d}{dt} \frac{\partial F}{\partial x_i} \left(x(t)\right) dt.$$

Let

$$L = \max_i \left(|e_i|_{L^1} + 2\,|a_i|_{L^1}\right), \quad M = 2 \max_i |b_i|_{L^1},$$

$$p_i' = \frac{\lambda}{2\pi} \int_J \left(\bar{x}_i - \tilde{x}_i(t)\right) \frac{d}{dt} \frac{\partial F}{\partial x_i} \left(x(t)\right) dt.$$

One gets

$$0 \geqq \mu\, |x_i|_{H^1}^2 - \frac{2\pi}{\sqrt{3}}\, |x_i|_{H^1} \left\{L + M \left(\sum_{k=1}^{n} |x_k|_C^\alpha\right)\right\} + p_i.$$

On the other hand, for each $i = 1, \ldots, n$ one has

$$|x_i|_{H^1} \leqq \left(\sum_{j=1}^{n} |x_j|_{H^1}^2\right)^{1/2} = \|x\|_{H^1} \text{ and } |x_k|_C^\alpha \leqq \frac{2}{\sqrt{3}}\, \|x\|_{H^1}^\alpha,$$

hence

$$\sum_{k=1}^{n} |x_k|_C^\alpha \leqq \frac{2\pi}{\sqrt{3}}\, n\, \|x\|_{H^1}^\alpha \text{ and } 0 \geqq \mu\, |x_i|_{H^1}^2 - \frac{2\pi}{\sqrt{3}}\, \|x\|_{H^1} \left(L + \frac{2\pi}{\sqrt{3}}\, nM\, \|x\|_{H^1}^\alpha\right) + p_i.$$

Now, adding for $i = 1, \ldots, n$ the last inequalities we derive

$$0 \geqq \mu\, \|x\|_{H^1}^2 - 2\left(\sqrt{3}\right)^{-1} \pi\, \|x\|_{H^1}\, n\left(L + 2\left(\sqrt{3}\right)^{-1} \pi n M\, \|x\|_{H^1}^\alpha\right).$$

Hence

$$0 \geqq \mu\, \|x\|_{H^1}^2 - 2\left(\sqrt{3}\right)^{-1} \pi n L\, \|x\|_{H^1} - 4\left(\sqrt{3}\right)^{-1} \pi^2 n^2 M\, \|x\|_{H^1}^{\alpha+1}.$$

Hence it is clear that there exists $\beta_1 > 0$ such that $\|x\|_{H^1} < \beta_1$; from this and the fact that $H^1(J, \mathbf{R}^n) \subset C(J, \mathbf{R}^n)$ compactly, we have that $\|x\|_C < \beta_2$ for some $\beta_2 > 0$ and the proof is complete ∎

3.2. The case of assumption B⁻

To get the (in some sense) dual version of Theorem 1 we premise

Lemma 3: *Let $\gamma, \Gamma \in L^1(J)$ be such that $\gamma(t) \geqq \Gamma(t)$ a.e. on J and $\Gamma = {}^0\Gamma + {}^1\Gamma + {}^\infty\Gamma$ with*

${}^1\Gamma \in L^1(J)$, $0 \geqq {}^1\Gamma(t)$ a.e. on J, and ${}^\infty\Gamma \in L^\infty(J)$, $0 \geqq {}^\infty\Gamma(t)$ a.e. on J,

${}^0\Gamma \in L^1(J)$, ${}^0\Gamma(t) \not\equiv 0$ and ${}^0\Gamma(t) \geqq -1$ a.e. on J with strict inequality on a subset of positive measure,

and

$$\delta(-{}^0\Gamma^-) - \frac{\pi^2}{3}\, |{}^1\Gamma|_{L_1} - |{}^\infty\Gamma|_{L^\infty} - \frac{2\pi^2}{3}\, (\bar{\gamma}^+) > 0, \quad \bar{\gamma}^- < -4\bar{\gamma}^+.$$

Then, there exist $\varepsilon = \varepsilon(\gamma, \Gamma) > 0$ *and* $\mu = \mu(\gamma, \Gamma) > 0$ *such that for all* $p \in L^1(J)$ *satisfying* $\Gamma(t) - \varepsilon \leqq p(t) \leqq \gamma(t) + \varepsilon$ *a.e. on* J, *all* $f \in C(\mathbf{R}, \mathbf{R})$ *and all* $x \in W^{2,1}(J)$ *we have*

$$\frac{1}{2\pi} \int_J -x(t) \big(x''(t) + f(x(t))\, x'(t) + p(t)\, x(t - \tau)\big)\, dt \geqq \mu\, |x|_{H^1}^2.$$

Proof: Using the same arguments as in the proof of Lemma 2, it is easy to see that it suffices to show that

$$D_p^-(x) \doteq \frac{1}{2\pi} \int_J \big(x'^2(t) - p(t)\, x(t)\, x(t - \tau)\big)\, dt \geqq 0.$$

We have

$$D_p^-(x) \geqq \frac{1}{2\pi} \int_J \left(x'^2(t) + \frac{1}{2}\, p(t) \big(\tilde{x}^2(t) + \tilde{x}^2(t - \tau)\big)\right) dt$$

$$- \frac{1}{2\pi} \int_J \frac{1}{2}\, p(t) \big(x(t - \tau) - \tilde{x}(t)\big)^2\, dt - \frac{1}{2}\, \overline{p}\overline{x}^2.$$

Since $p^-(t) \geqq \Gamma^-(t)$, using (3.2) we obtain

$$D_p^-(x) \geqq \frac{1}{2\pi} \int_J \left(x'^2(t) + \frac{1}{2}\, \Gamma^- \big(\tilde{x}^2(t) + \tilde{x}^2(t - \tau)\big)\right) dt$$

$$- \frac{2\pi^2}{3}\, |\tilde{x}|_{H^1}^2 \overline{p}^+ - \frac{3}{2}\, \overline{x}^2 \overline{p}^+ - \frac{1}{2}\, \overline{p}^- \overline{x}^2$$

$$\geqq \left(\delta(-{}^0\Gamma^-) - |{}^\infty\Gamma|_{L^\infty} - |{}^1\Gamma|_{L^1} \frac{\pi^2}{3} - \frac{2\pi^2}{3}\, \overline{p}^+\right) |\tilde{x}|_{H^1}^2 - \frac{\overline{x}^2}{2}\, (4\overline{p}^+ - \overline{p}^-)$$

and the proof is complete ∎

Theorem 2: *Let* $F \in C^2(\mathbf{R}^n, \mathbf{R})$ *and* $g : J \times \mathbf{R}^n \to \mathbf{R}^n$ *satisfy assumptions* (i)—(iii) *of Theorem* 1. *Assume that for each* $i = 1, \ldots, n$ *the inequalities*

(B) $$\Gamma_i(t) \leqq \liminf_{|x_i| \to \infty} x_i^{-1} g_i(t, x) \leqq \limsup_{|x_i| \to \infty} x_i^{-1} g_i(t, x) \leqq \gamma_i(t)$$

hold uniformly for a.e. $t \in J$ *and* $x_j \in \mathbf{R}$, $j \neq i$, *and* γ_i, Γ_i *satisfy the conditions of Lemma* 3.
Then, system (1.1) *has at least one* 2π*-periodic solution for each* $e \in L^1(J, \mathbf{R}^n)$.

Proof: It is similar to that of Theorem 1, using the same notations. Let $x \in H^1(J, \mathbf{R}^n)$ be a 2π-periodic solution for system (3.6). Multiplying each equation of this system by $\big(-\overline{x}_i - \tilde{x}_i(t)\big)$ and taking into account the assumptions and Lemma 3, one gets, for each $i = 1, \ldots, n$,

$$0 = \frac{1}{2\pi} \int_J \big(-\overline{x}_i - \tilde{x}_i(t)\big) \left(x_i''(t) + \lambda \left(\frac{d}{dt} \frac{\partial F}{\partial x_i} \big(x(t)\big)\right) + (1 - \lambda)\, \Gamma_i(t)\, x_i(t - \tau)\right.$$

$$\left. + \lambda \gamma_i\big(t, x(t - \tau)\big)\, x_i(t - \tau_i) + \lambda h_i\big(t, x(t - \tau)\big) - \lambda e_i(t)\right) dt$$

$$\geqq \frac{\mu}{2}\, |x_i|_{H^1}^2 - |\overline{x}_i - \tilde{x}_i|_C\, (|h_i|_{L^1} + |e_i|_{L^1}) - \frac{\lambda}{2\pi} \int_J x_i(t)\, \frac{d}{dt} \frac{\partial F}{\partial x_i} \big(x(t)\big)\, dt.$$

With a technique similar to that used in Theorem 1 we obtain that there exists a constant $\beta > 0$ such that $\|x\|_C < \beta$ and the proof is complete ∎

3.3. The mixed case

Theorem 3: *Let $F \in C^2(\mathbf{R}^n, \mathbf{R})$ and $g : J \times \mathbf{R}^n \to \mathbf{R}^n$ satisfy assumptions* (i)–(iii) *of Theorem* 1. *Let n_0 be an integer such that $1 \leqq n_0 \leqq n$. Assume that for every $i = 1, \ldots, n_0$ condition $\mathbf{B}^+$ holds uniformly for a.e. $t \in J$ and $x_j \in \mathbf{R}$, $j \neq i$, where γ_i and Γ_i are as in Theorem* 1. *Moreover assume that for every $i = n_0 + 1, \ldots, n$ condition $\mathbf{B}^-$ holds uniformly for a.e. $t \in J$ and $x_j \in \mathbf{R}$, $j \neq i$, where γ_i and Γ_i are as in Theorem* 2.

Then, system (1.1) *has at least one 2π-periodic solution for each $e \in L^1(J, \mathbf{R}^n)$.*

Proof: For each $i = 1, \ldots, n_0$ we proceed as in the proof of Theorem 1 and for $i = n_0 + 1, \ldots, n$ we proceed as in the proof of Theorem 2 to get the boundedness of the solutions for system (3.6) ∎

4. Resonance conditions

4.1. The case of assumption $\mathbf{E}^+$

The following lemma extends Lemma 2 of [3], when τ is an effective delay, to the case when Γ crosses the positive eigenvalues of problem (1.2) in some subset of J of positive measure.

Lemma 4: *Let $\varepsilon > 0$ and $\Gamma \in L^1(J)$ be such that $\Gamma = {}^0\Gamma + {}^1\Gamma + {}^\infty\Gamma$ with*

${}^1\Gamma \in L^1(J)$, ${}^1\Gamma(t) \geqq 0$ a.e. on J, and ${}^\infty\Gamma \in L^\infty(J)$, ${}^\infty\Gamma(t) \geqq 0$ a.e. on J,

${}^0\Gamma \in L^1(J)$, $0 \leqq {}^0\Gamma(t) \leqq 1$ a.e. on J, with ${}^0\Gamma(t) < 1$ on a subset of positive measure.

Then, for all $p \in L^1(J)$ satisfying $0 \leqq p(t) \leqq \Gamma(t) + \varepsilon$ a.e. on J, all continuous functions $f : \mathbf{R} \to \mathbf{R}$ and all $x \in W^{2,1}(J)$, we have

$$\frac{1}{2\pi} \int_J \left(\bar{x} - \tilde{x}(t)\right)\left(x''(t) + f(x(t))\, x'(t) + p(t)\, x(t - \tau)\right) dt$$

$$\geqq \left[\delta({}^0\Gamma) - \frac{\pi^2}{3} |{}^1\Gamma|_{L^1} - |{}^\infty\Gamma|_{L^\infty} - \varepsilon\right] |\tilde{x}|_{H^1}^2,$$

where $\delta({}^0\Gamma)$ is associated to ${}^0\Gamma$ by Lemma 1.

Proof: Integrating by parts and following the proof of Lemma 2 of [3], one gets

$$D_p^+(x) \geqq \frac{1}{2\pi} \int_J \left[(x'(t))^2 - \Gamma(t) \frac{\tilde{x}^2(t) + \tilde{x}^2(t - \tau)}{2}\right] dt$$

$$- \frac{\varepsilon}{2\pi} \int_J \frac{\tilde{x}^2(t) + \tilde{x}^2(t - \tau)}{2}\, dt$$

$$+ \frac{1}{2\pi} \int_J \frac{p(t)}{2} \left[\left(x(t - \tau) - \tilde{x}(t)\right)^2 + \bar{x}^2\right] dt.$$

Therefore, by Lemma 1 and inequalities (3.2), we have

$$D_p^+(x) \geqq \delta({}^0\Gamma)\, |\tilde{x}|_{H^1}^2 - |{}^1\Gamma|_{L^1}\, |\tilde{x}|_{L^\infty}^2 - |{}^\infty\Gamma|_{L^\infty}\, |\tilde{x}|_{L^2}^2 - \varepsilon\, |\tilde{x}|_{H^1}^2$$

$$\geqq \left[\delta({}^0\Gamma) - \frac{\pi^2}{3} |{}^1\Gamma|_{L^1} - |{}^\infty\Gamma|_{L^\infty}\right] |\tilde{x}|_{H^1}^2 - \varepsilon\, |\tilde{x}|_{H^1}^2$$

and the proof is complete ∎

Theorem 4: *Let $F \in C^2(\mathbf{R}^n, \mathbf{R})$ and $g : J \times \mathbf{R}^n \to \mathbf{R}^n$ satisfy assumptions* (i)—(iii) *of Theorem* 1 *and be such that for any $i = 1, \ldots, n$*

($\mathbf{E}_1^+$) *there exists $R_i > 0$ such that $g_i(t, x)\, x_i \geqq 0$ for all $x \in \mathbf{R}^n$ with $|x_i| \geqq R_i$,*

($\mathbf{E}_2^+$) $\limsup_{|x_i| \to +\infty} x_i^{-1} g_i(t, x) \leqq \Gamma_i(t)$ *uniformly for a.e. $t \in J$ and uniformly for $x_j \in \mathbf{R}$, $j \neq i$.*

Moreover assume that for every $i = 1, \ldots, n$, Γ_i satisfies the conditions of Lemma 4 *with*

$$\delta({}^0\Gamma_i) - \pi^2/3\, |{}^1\Gamma_i|_{L^1} - |{}^\infty\Gamma_i|_{L^\infty} > 0.$$

Then system (1.1) *has at least one 2π-periodic solution for each $e \in L^1(J, \mathbf{R}^n)$ with $\bar{e} = 0$.*

Proof: Let $0 < \varepsilon < \min_i \{\delta({}^0\Gamma_i) - \pi^2/3\, |{}^1\Gamma_i|_{L^1} - |{}^\infty\Gamma_i|_{L^\infty}\}$. Then there exists $r_i > 0$ such that for a.e. $t \in J$ and for all $x \in \mathbf{R}^n$ with $|x_i| \geqq r_i$ one has $0 \leqq x_i^{-1} g_i(t, x) \leqq \Gamma_i(t) + \varepsilon$. Proceeding like in the proof of Theorem 1, we can write the system (1.1) in the equivalent form

$$x_i''(t) + \left(\frac{d}{dt} \frac{\partial F}{\partial x_i}\big(x(t)\big)\right) + \tilde{\gamma}_i\big(t, x(t - \tau)\big)\, x_i(t - \tau_i) + h_i\big(t, x(t - \tau)\big) = e_i(t),$$
$$i = 1, \ldots, n.$$

Degree arguments will imply the existence of a 2π-periodic solution for (1.1) if the set of possible 2π-periodic solutions of the system

$$x_i''(t) + (1 - \lambda)\, \Gamma_i(t)\, x_i(t - \tau_i) + \lambda \left(\frac{d}{dt} \frac{\partial F}{\partial x_i}\big(x(t)\big)\right) + \lambda h_i\big(t, x(t - \tau)\big)$$
$$+ \lambda \tilde{\gamma}_i\big(t, x(t - \tau)\big)\, x_i(t - \tau_i) = \lambda e_i(t),\ i = 1, \ldots, n \tag{4.1}$$

is a priori bounded independently of $\lambda \in (0, 1]$. Let $x \in W^{2,1}(J, \mathbf{R}^n)$ be a 2π-periodic solution of (4.1). Multiplying each equation of this system by $\big(\bar{x}_i - \tilde{x}_i(t)\big)$ and integrating on J, we obtain, using Lemma 4,

$$0 = \frac{1}{2\pi} \int_J \big(\bar{x}_i - \tilde{x}_i(t)\big) \left(x_i'' + (1 - \lambda)\, \Gamma_i(t)\, x_i(t - \tau_i) + \lambda \left(\frac{d}{dt} \frac{\partial F}{\partial x_i}\big(x(t)\big)\right)\right.$$
$$\left. + \lambda h_i\big(t, x(t - \tau)\big) + \lambda \tilde{\gamma}_i\big(t, x(t - \tau)\big)\, x_i(t - \tau_i) - \lambda e_i(t)\right) dt$$
$$\geqq \mu\, |\tilde{x}_i|_{H^1}^2 - (|h_i|_{L^1} + |e_i|_{L^1})\, |\bar{x}_i - \tilde{x}_i|_{L^\infty} + p_i$$

with $0 < \mu < \min_i \{\delta({}^0\Gamma_i) - \pi^2/3\, |{}^1\Gamma_i|_{L^1} - |{}^\infty\Gamma_i|_{L^\infty} - \varepsilon\}$. By means of the arguments used in the proof of Theorem 2 of [2], there exists a constant $d > 0$ such that $\|x\|_{H^1} < d$ independently of $\lambda \in (0, 1]$ and we can complete the proof like in Theorem 1 ∎

Corollary: *Theorem* 4 *remains valid if we suppose, instead of $\mathbf{E}_1^+$ and $\bar{e} = 0$, that there exist constants a_i, b_i, R_i such that $a_i \leqq \bar{e}_i \leqq b_i$ and*

$$g_i(t, x) \geqq b_i \text{ for a.e. } t \in J \text{ and } x \in \mathbf{R}^n \text{ with } x_i > R_i > 0,$$
$$g_i(t, x) \leqq a_i \leqq b_i \text{ for a.e. } t \in J \text{ and } x \in \mathbf{R}^n \text{ with } x_i < -R_i.$$

Proof: The system (1.1) is equivalent to

$$x_i''(t) + \frac{d}{dt} \frac{\partial F}{\partial x_i}\big(x(t)\big) + g_i^1\big(t, x(t)\big) = e_i^1(t), \qquad i = 1, \ldots, n,$$

where

$$g_i^1(t, x) = g_i(t, x) - \frac{1}{2}(a_i + b_i) \text{ and } e_i^1(t) = e_i(t) - \frac{1}{2}(a_i + b_i).$$

Observe that $g_i^1(t, x)$ verifies $\mathbf{E}_1^+$; therefore, the arguments used in the proof of Theorem 4 yield the conclusion ∎

4.2. The case of assumption E⁻

The following lemma extends Lemma 3 of [3].

Lemma 5: *Let $\varepsilon > 0$ and let $\Gamma \in L^1(J)$ be such that $\Gamma = {}^0\Gamma + {}^1\Gamma + {}^\infty\Gamma$ with*

${}^1\Gamma \in L^1(J)$, $0 \geqq {}^1\Gamma(t)$ a.e. on J, and ${}^\infty\Gamma \in L^\infty(J)$, $0 \geqq {}^\infty\Gamma(t)$ a.e. on J,

${}^0\Gamma \in L^1(J)$, $0 \geqq {}^0\Gamma(t) \geqq -1$ a.e. on J with ${}^0\Gamma(t) > -1$ on a subset of positive measure.

Then for all $p \in L^1(J)$ satisfying $\Gamma(t) - \varepsilon \leqq p(t) \leqq 0$ a.e. on J, all continuous functions $f: \mathbf{R} \to \mathbf{R}$ and all $x \in W^{2,1}(J)$, we have

$$\frac{1}{2\pi}\int_J \big(-x(t)\big)\big(x''(t) + f(x(t))\,x'(t) + p(t)\,x(t-\tau)\big)\,dt$$

$$\geqq \left[\delta(-{}^0\Gamma) - \frac{\pi^2}{3}|{}^1\Gamma|_{L^1} - |{}^\infty\Gamma|_{L^\infty} - \varepsilon\right]|\tilde{x}|^2_{H^1},$$

where $\delta(-{}^0\Gamma)$ is associated to $-{}^0\Gamma$ by Lemma 1.

Proof: Integrating by parts and using the proofs of Lemma 3 of [3: p. 155] and Lemma 4 herein, one gets the conclusion ∎

Theorem 5: *The assertion of Theorem 4 holds true if assumptions $\mathbf{E}_1^+$ and $\mathbf{E}_2^+$ are replaced respectively by*

($\mathbf{E}_1^-$) *there exists $R_i > 0$ such that $g_i(t, x)\,x_i \leqq 0$ for all $x \in \mathbf{R}^n$ with $|x_i| \geqq R_i$,*

($\mathbf{E}_2^-$) $\liminf_{|x_i| \to +\infty} x_i^{-1} g_i(t, x) \geqq \Gamma_i(t)$ *uniformly for a.e. $t \in J$ and uniformly for $x_j \in \mathbf{R}$, $j \neq i$, where Γ_i satisfies the conditions of Lemma 5, and, moreover,*

$$\delta(-{}^0\Gamma) - \frac{\pi^2}{3}|{}^1\Gamma|_{L^1} - |{}^\infty\Gamma|_{L^\infty} > 0.$$

Proof: It is similar to that of Theorem 4 herein and we omit it for the sake of brevity. Let us mention the required *a priori* estimates are obtained using Lemma 5 above ∎

4.3. The mixed case

Theorem 6: *Let $F \in C^2(\mathbf{R}^n, \mathbf{R})$ and $g: J \times \mathbf{R}^n \to \mathbf{R}^n$ satisfy assumptions* (i)–(iii) *of Theorem 1. Let n_0 be an integer such that $1 \leqq n_0 \leqq n$. Assume that for every $i = 1, \ldots, n_0$, conditions $\mathbf{E}^+$ of Theorem 4 hold uniformly a.e. on J and for all $x_j \in \mathbf{R}$, $j \neq i$, where Γ_i are as in Theorem 4. Moreover, assume that for every $i = n_0 + 1, \ldots, n$, conditions $\mathbf{E}^-$ of Theorem 5 hold uniformly a.e. on J and for all $x_j \in \mathbf{R}$, $j \neq i$, where Γ_i are as in Theorem 5. Then* (1.1) *has at least one 2π-periodic solution for each $e \in L^1(J, \mathbf{R}^n)$ with $\bar{e} = 0$.*

Proof: For each $i = 1, \ldots, n_0$, we proceed as in the proof of Theorem 4 and for $i = n_0 + 1, \ldots, n$, we proceed as in the proof of Theorem 5 for getting the required *a priori* estimates ∎

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Manuskripteingang: 19. 12. 1985

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