Nonuniform Conditions for Periodic Solutions of Forced Liénard Systems with **Distinct Delays**

G. CONTI, R. LANNACCI and M. N. NKASHAMA

Mit Hilfe der klassischen Leray-Schauder-Theorie und des Koinzidenzabbildungsgrades wird die Existenz periodischer Lösungen Liénardscher Systeme mit verschiedenen Verzögerungen im Falle nichtgleichmäßiger Bedingungen bewiesen. Es werden Resonanz- und Nichtresonanzfälle in der Umgebung des Eigenwertes Null behandelt.

С помощью классической теории Лерейя-Шаудера и степени совпадения доказывается существование периодических решений возбужденных Лиенардских систем с различными запаздываниями под неравномерными условиями. Рассматриваются случаи резонанса и нерезонанса односительно собственного значения нуль.

We use classical Leray-Schauder techniques and the coincidence degree in order to obtain the existence of periodic solutions of forced Liénard systems with distinct delays, under nonuniform conditions. The resonance and nonresonance case with respect to the eigenvalue zero are considered.

1. Introduction

In this paper we study the existence of 2π -periodic solutions for the following Liénard systems with distinct delays $\tau_1, ..., \tau_n \in [0, 2\pi)$

$$
x''(t) + \frac{d}{dt} [\text{grad } F(x(t))] + g(t, x_1(t - \tau_1), \ldots, x_n(t - \tau_n)] = e(t) \qquad (1.1)
$$

a.e. on $J = [0, 2\pi]$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^2 , $g : J \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies Cara-
théodory conditions and $e : J \to \mathbb{R}^n$ is integrable.

The existence of 2π -periodic solutions for (1.1) has been recently studied by DE PASCALE and IANNACCI [3] and IANNACCI and NKASHAMA [9] in the scalar case under resonance and nonresonance conditions, respectively. We recall that in the case of effective delay (i.e. $\tau = 0$), if $\tau/\pi \in \mathbb{Q}$, then the eigenvalues of the problem

$$
x''(t) + \lambda x(t - \tau) = 0, \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \tag{1.2}
$$

are not contained in the nonnegative part of the real axis (see [3: Rem. 1]). In [9] is given an existence theorem of solutions for the problem (1.1) with $n = 1$ under the assumption

$$
\gamma(t) \leq \liminf_{|x| \to +\infty} \dot{x}^{-1} g(t, x) \leq \limsup_{|x| \to +\infty} x^{-1} g(t, x) \leq \Gamma(t) \leq 1
$$

uniformly for a.e. $t \in J$, where γ can cross the eigenvalue zero of the problem (1.2) on a suitable subset of J of positive measure. Moreover, in [3], dealing with resonant situations at the eigenvalue zero, the assumption

$$
\limsup_{|x|\to+\infty}x^{-1}g(t,x)\leq \underline{\Gamma}(t)\leq 1
$$

was still considered.

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Employing a technique due to GUPTA and MAWHIN [6] for $\tau = 0$, we consider the case when, for every $i = 1, ..., n$, the expressions $\limsup x_i^{-1}g_i(t, x)$ can cross suitably $|x_i| \rightarrow +\infty$

any number of either positive or negative eigenvalues of the problem (1.2). Notice that these improvements are given in the context of systems with distinct delays. The paper is divided into *two* sections. The first section is devoted to nonresonance conditions at the two first eigenvalues and in the second section we deal with resonant situations at the eigenvalue zero. Each section is divided into three subsections. The paper is divided into two sections. The first section is dev

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in the streament situations at the eigenvalue zero. Each section is divided in

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In Theorem 1 and Theorem 4 we assume that for every
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$$

\n $\gamma_i(t) \leq \liminf_{|x_i| \to +\infty} x_i^{-1}g_i(t, x) \leq \limsup_{|x_i| \to +\infty} x_i^{-1}g_i(t, x) \leq \Gamma_i(t)$
\nd
\n $\limsup_{|x_i| \to +\infty} x_i^{-1}g_i(t, x) \leq \Gamma_i(t)$,

and

$$
\limsup_{|x_i|\to+\infty} x_i^{-1}g_i(t,x) \leq \Gamma_i(t),
$$

respectively, uniformly for a.e. $t \in J$, where γ_i can cross the eigenvalue zero on a suitable subset of J of positive measure and Γ_i can cross the positive eigenvalues of the problem (1.2) on a subset of *J* of sufficiently small positive measure, formly for a.e. $t \in J$, where
formly for a.e. $t \in J$, where
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and Theorem 5 we assume there is $|x_t| \to +\infty$
 $|x_t| \to +\infty$
 $\liminf x_i^{-1}q_i(t, x)$.

In Theorem 2 and Theorem 5 we assume that for every
$$
i
$$

\n
$$
\Gamma_i(t) \leq \lim_{|x_i| \to +\infty} \inf_{x_i^{-1}g_i(t, x)} \leq \lim_{|x_i| \to +\infty} \sup_{|x_i| \to +\infty} x_i^{-1}g_i(t, x) \leq \gamma_i(t)
$$
\n
$$
\Gamma_i(t) \leq \lim_{|x_i| \to +\infty} \inf_{x_i^{-1}g_i(t, x)} \dots
$$

and

$$
\Gamma_i(t) \leq \liminf_{|x_i| \to +\infty} x_i^{-1} g_i(t,x),
$$

respectively, uniformly for a.e. $t \in J$, where Γ_i can cross the negative eigenvalues of the problem (1.2) and γ_i can cross zero in a suitable way.

Finally, in Theorem 3 and Theorem 6 we prove the existence of 2π -periodic solutions of the system (1.1) under the assumption that a part of the equations of this system satisfies conditions of Theorem 1 and Theorem 4, respectively, and the remaining equations satisfy conditions of Theorem 2 and Theorem 5, resjectively.

Our results are basedon lemmas giving a *priori* estimates and degree arguments. Let us mention that for other existence' results concerning either unforced or forced Liénard functional differential equations or systems but not directly related to the results introduced here, see e.g. HALE [8], GRAFTON [5], MAWHIN [12], INVERNIZZI and ZANOLIN [11] and the bibliography therein. maining equations satisfy conditions of Theorem 2 and

Our results are based on lemmas giving a priori estimates at

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HALE [8], GRAFTON [5], MAWHIN [12], INVERN

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Notations, definition

Let us set $J = [0, 2\pi]$. We will use the symbol $x = col(x_1, ..., x_n) \in \mathbb{R}^n$ and the symbol $\Vert \cdot \Vert$ for the Euclidean norm in \mathbb{R}^n . Further, we will use the following spaces:

- **1.** $L^p(J, \mathbb{R}^n)$ are the usual Lebesgue spaces, $1 \leq p \leq \infty$. We denote their norm by $\|\cdot\|_{L^p}$ if $n > 1$ and by $\|\cdot\|_{L^p}$ if $n = 1$.
- 2. $H^1(J, \mathbb{R}^n) = \{x: J \to \mathbb{R}^n : x \text{ absolutely continuous, } x' \in L^2(J, \mathbb{R}^n), x(0) x(2\pi) \}$

ions, definitions and preliminary results
\n
$$
tJ = [0, 2\pi]
$$
. We will use the symbol $x = \text{col}(x_1, ..., x_n)$
\nne Euclidean norm in \mathbb{R}^n . Further, we will use the follow
\n \mathbb{R}^n) are the usual Lebesgue spaces, $1 \leq p \leq \infty$. We d
\n $f n > 1$ and by $|\cdot|_{L^p}$ if $n = 1$.
\n \mathbb{R}^n) = $\{x : J \to \mathbb{R}^n : x$ absolutely continuous, $x' \in L^2(\$
\nwith the norm
\n $\|x\|_{H^1} = \left\{\sum_{i=1}^n \left[\frac{1}{(2\pi)} \int x_i(t) dt\right]^2 + \frac{1}{2\pi} \int (x_i'(t))^2 dt\right\}^{1/2}$.
\nse the symbol $|\cdot|_{H^1}$ if $n = 1$.

We use the symbol $|\cdot|_{H^1}$ if $n = 1$. 3. $\tilde{H}^{1}(J, \mathbf{R}^{n}) = \left\{x \in H^{1}(J, \mathbf{R}^{n}) : \int x(t) dt = 0\right\}.$

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 $\mathcal{L}^{1}(J, \mathbb{R}^{n}) = \{x: J \to \mathbb{R}^{n}: x \text{ and } x' \text{ absolutely continuous, } x(0) - x(2\pi) = x'(0)\}$ 4. $W^{2,1}(J, \mathbb{R}^n) = \{x : J \to \mathbb{R}^n : x \text{ and } x' \text{ absolutely continuous, } x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0\}$ with the norm

Nonuniform Condi
\n
$$
\mathbf{R}^{n} = \{x : J \to \mathbf{R}^{n} : x \text{ and } x' \text{ absolutely}
$$
\n
$$
\begin{aligned}\n|x|_{W^{1,1}} &= \left\{ \sum_{i=1}^{n} \left[\frac{1}{2\pi} \sum_{k=0}^{2} \int_{j} |x_{i}(k)(t)| dt \right]^{2} \right\}^{1/2}, \\
\text{where } \mathbf{R}^{n} \text{ is the Banach space of continuous functions.}\n\end{aligned}
$$

5. $C(J, \mathbb{R}^n)$ is the Banach space of continuous functions with the norm

$$
||x||_C = \max \{||x(t)|| : t \in J\}.
$$

For the sake of simplicity in the notations of the spaces we will omit \mathbb{R}^n when $n = 1$. Clearly if $x = \text{col}(x_1, ..., x_n) \in H^1(J, \mathbb{R}^n)$, then $x_i \in H^1(J)$ for $i = 1, ..., n$. We recall that every $x_i \in H^1(J)$ can be written in the form $|c| = \max \{||x(t)|| : t, \in J\}.$

e of simplicity in the notations of the spaces we will omit
 $=$ col $(x_1, ..., x_n) \in H^1(J, \mathbb{R}^n)$, then $x_i \in H^1(J)$ for $i = 1$
 $x_i \in H^1(J)$ can be written in the form
 $= \overline{x}_i + \overline{x}_i$ with $\overline{x}_i \in \$

$$
x_i = \overline{x}_i + \tilde{x}_i
$$
 with $\tilde{x}_i \in \tilde{H}^1(J)$ and $\overline{x}_i = \frac{1}{2\pi} \int x_i(t) dt$.

Moreover,

By
$$
x_i \in H^1(J)
$$
 can be written in the form

\n
$$
x_i = \bar{x}_i + \tilde{x}_i \quad \text{with} \quad \tilde{x}_i \in \tilde{H}^1(J) \quad \text{and} \quad \bar{x}_i = \frac{1}{2\pi} \int x_i(t) \, dt.
$$
\nr,

\n
$$
|x_i|_{H^1} = \left\{ \bar{x}_i^2 + \frac{1}{2\pi} \int (x_i'(t))^2 \, dt \right\}^{1/2} \text{ so that } ||x||_{H^1} = \left\{ \sum_{i=1}^n |x_i|_{H^1}^2 \right\}^{1/2}.
$$
\nsequel we will use the following result.

\n1a 1 (MAWHIN and WARD [13]): Let $\Gamma \in L^1(J)$ such that $\Gamma(t) \leq 1$ it inequality on a subset of positive measure. Then, there exists $\delta =$ for all $\tilde{x} \in H^1(J)$ one has

\n
$$
\frac{1}{2\pi} \int \left((\tilde{x}'(t))^2 - \Gamma(t) \left(\tilde{x}(t) \right)^2 \right) dt \geq \delta \left| \tilde{x} \right|_{H^1}^2.
$$
\ne sake of simplicity we will write $x(t - \tau)$ instead of $(x_1(t - \tau_1), \ldots, x_n)$ on $x \in W^{2,1}(J, \mathbb{R}^n)$ will be called a solution of (11) if it satisfies (1).

In the sequel we will use the following result.

Lemma 1 (MAWHIN and WARD [13]): Let $\Gamma \in L^1(J)$ such that $\Gamma(t) \leq 1$ a.e. on J, with strict inequality on a subset of positive measure. Then, there exists $\delta = \delta(\Gamma) > 0$ *such that for all* $\tilde{x} \in H^1(J)$ *one has*

$$
\frac{1}{2\pi}\int\limits_I \left((\tilde x'(t))^2-\varGamma(t)\ (\tilde x(t))^2\right)dt\geqq \delta\ |\tilde x|_H^2.
$$

For the sake of simplicity we will write $x(t - \tau)$ instead of $(x_1(t - \tau_1), \ldots, x_n(t - \tau_n))$. A function $x \in W^{2,1}(\bar{J}, \mathbb{R}^n)$ will be called a *solution* of (1.1) if it satisfies (1.1) almost everywhere. Let $h: J \to \mathbf{R}$ be a real function. We define $h^+(t) = \max \{h(t), 0\}$ and For the sake of simplicity we will write $x(t - \tau)$ instead of $(x_1(t - \tau_1), ..., x_n(t - \tau_n))$
A function $x \in W^{2,1}(J, \mathbb{R}^n)$ will be called a *solution* of (1.1) if it satisfies (1.1) almost
everywhere. Let $h : J \to \mathbb{R}$ be a rea

3., Nonresonance conditions

3.1. The case of assumption B

The following lemma extends Lemma 3 of [9], when τ is an effective delay, to the case positive measure. *A* function $x \in W^{2,1}(J)$

everywhere. Let $h : J$
 $h^-(t) = \min \{h(t), 0\}, t$

3. Nonresonance cond

3.1. The case of assum

The following lemma a

when Γ crosses the positive measure.

Lemma 2: Let γ , I
 \rightarrow $\Gamma \in L^1(J$

when
$$
\Gamma
$$
 crosses the positive eigenvalues of problem (1.2) suitably in subsets of J of positive measure.
\nLemma 2: Let $\gamma, \Gamma \in L^1(J)$ be such that $\gamma(t) \leq \Gamma(t)$ a.e. on J and $\Gamma = \mathfrak{D} + 1\Gamma$
\n $+ \mathfrak{D} \Gamma$ with
\n ${}^{1}\Gamma \in L^1(J), {}^{1}\Gamma(t) \geq 0$ a.e. on J , and ${}^{\infty}\Gamma \in L^{\infty}(J), {}^{\infty}\Gamma(t) \geq 0$ a.e. on J ,
\n ${}^{0}\Gamma \in L^1(J), {}^{0}\Gamma(t) \equiv 0$ and ${}^{0}\Gamma(t) \leq 1$ a.e. on J with strict inequality on a
\nsubset of positive measure,
\nand
\n
$$
\delta({}^{0}\Gamma^{+}) - \frac{\pi^2}{3} |{}^{1}\Gamma|_{L^1} - |{}^{\infty}\Gamma|_{L^{\infty}} + \frac{2\pi^2}{3} \bar{\gamma}^{-} > 0, \quad \bar{\gamma}^{+} > -4\bar{\gamma}^{-}.
$$
\n(3.1)

and

$$
\delta({}^{0}T^{+})-\frac{\pi^{2}}{3}|{}^{1}\Gamma|_{L^{1}}-|{}^{\infty}\Gamma|_{L^{\infty}}+\frac{2\pi^{2}}{3}\bar{\nu}^{-}>0,\quad\bar{\nu}^{+}>-4\bar{\nu}^{-}.
$$
\n(3.1)

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:

Then, there exist $\varepsilon = \varepsilon(\gamma, \Gamma) > 0$ *and* $\mu = \mu(\gamma, \Gamma) > 0$ *such that for all* $p \in L^{1}(J)$ *satisfying* $\gamma(t) - \varepsilon \leq p(t) \leq \Gamma(t) + \varepsilon$ *a.e. on J., all continuous functions* $f: \mathbf{R} \to \mathbf{R}$ and $all x \in W^{2,1}$ we have

$$
\frac{1}{2\pi} \int (\bar{x} - \tilde{x}(t)) (x''(t) + f(x(t)) x'(t) + p(t) x(t - \tau)) dt \ge \mu |x|_{H^1}^2.
$$

Proof: Assume the contrary. Then there exist sequences $\{p_n\} \subset H^{2,1}(J), |x_n|_{H^1} = 1$, with $\gamma(t) - 1/n \le p_n(t) \le \Gamma(t) + 1/n$ such that

 $L^{1}(J), \; \{x_{n}\}$ $\sigma \subset W^{2,1}(J), |x_n|_{H^1} = 1$, with $\gamma(t) - 1/n \leq p_n(t) \leq \Gamma(t) + 1/n$ such that

$$
\lim_{t \to 0} \int f(x) \, dx
$$
\n
$$
\therefore \text{ Assume the contrary. Then there exist sequences } \{p_n\} \subset \mathbb{R}, |x_n|_{H^1} = 1, \text{ with } \gamma(t) - 1/n \leq p_n(t) \leq \Gamma(t) + 1/n \text{ such that}
$$
\n
$$
D_{P_n}^+(x_n) = \frac{1}{2\pi} \int \left\{ (x_n'(t))^2 + p_n(t) \, x_n(t - \tau) \left(\overline{x}_n - \overline{x}_n(t) \right) \right\} dt < \frac{1}{n}.
$$

 $\begin{aligned}\n &= \varepsilon(\gamma, \Gamma) > 0 \text{ and } \mu = \mu(\gamma, \Gamma) > 0 \text{ such that for a } p(t) \leq \Gamma(t) + \varepsilon \text{ a.e. on } J, \text{ all continuous functions } f: \\
 &\hat{x}(t) \big(x''(t) + f(x(t)) \, x'(t) + p(t) \, x(t-\tau) \big) \, dt \geq \mu \, |x|_H^2, \\
 &\text{the contrary. Then there exist sequences } \{p_n\} \subset \mathbf{1}, \text{ with } \gamma(t) - 1/n \leq p_n(t) \leq \Gamma(t) + 1/n \text{ such that } \\
 &\frac{1$ By means of the arguments used in Lemma 3 of [9], we have, taking subsequences if $D_{P_n}^+(x_n) = \frac{1}{2\pi} \int \left\{ (x'_n(t))^2 + p_n(t) x_n(t - \tau) (\bar{x}_n - \bar{x}_n(t)) \right\} dt < \frac{1}{n}.$
By means of the arguments used in Lemma 3 of [9], we have, taking subsequences if
it is necessary, that there exist $x \in H^1(J), p \in L^1(J)$ such that x By means of the arguments used in Lemma 3 of [9], we have,
it is necessary, that there exist $x \in H^1(J)$, $p \in L^1(J)$ such that
 $p_n \xrightarrow{L^1} p$, with $\gamma(t) \leq p(t) \leq \Gamma(t)$ for a.e. $t \in J$ and $D_p^+(x) \leq 0$.
We claim that $p_n \xrightarrow{L^1} p$, with $\gamma(t) \leq p(t) \leq \Gamma(t)$ for a.e. $t \in J$ and $D_p^+(x) \leq 0$.
We claim that $D_{P_n}^+(x_n) = \frac{1}{2\pi} \int \{(x'_n(t))^2 + 1\}$
so f the arguments used in Lassary, that there exist $x \in E$
with $\gamma(t) \leq p(t) \leq \Gamma(t)$ for a.
im that
 $D_p^+(x) \geq \left(\delta(^0T^+) - |\infty\Gamma|_{L^\infty}\right)$
c can write *PHOD* 3 of [9], we have, takin
 P(*J*), $p \in L^1(J)$ such that $x_n \stackrel{f}{=}$
 e. $t \in J$ and $D_p^+(x) \leq 0$.
 $-\frac{\pi^2}{3} |{}^1\Gamma|_{L^1} + \frac{2\pi^2}{3} \bar{\gamma}^- \bigg) |\tilde{x}|_{H^1}^2$

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$$
D_{p}^{+}(x) \geq \left(\delta(0T^{+})-|\infty T|_{L^{\infty}}-\frac{\pi^{2}}{3}|^{1}\Gamma|_{L^{1}}+\frac{2\pi^{2}}{3}\bar{\gamma}^{-}\right)|\tilde{x}|_{H^{1}}^{2}+\frac{\bar{x}^{2}}{2}(4\bar{\gamma}^{-}+\bar{\gamma}^{+}).
$$

Indeed we can write

:

$$
D_p^{+}(x) \geq \left(\delta(0T^{+}) - |\infty T|_{L^{\infty}} - \frac{\pi^2}{3} |T|_{L^1} + \frac{2\pi^2}{3} \bar{\gamma}^{-}\right) |\tilde{x}|_{H^1}^2 + \frac{\bar{x}^2}{2} (4\bar{\gamma}^{-} + \bar{\gamma}^{+}).
$$

\nIndeed we can write
\n
$$
D_p^{+}(x) = \frac{1}{2\pi} \left[\int_{J} \left(x'^2(t) - \frac{1}{2} p(t) \left(\tilde{x}^2(t) + \tilde{x}^2(t - \tau) \right) \right) dt + \int_{J} \frac{1}{2} p(t) \left(x(t - \tau) - \tilde{x}(t) \right)^2 dt \right] + \frac{1}{2} \bar{p} \bar{x}^2
$$
\n
$$
\geq \frac{1}{2\pi} \int \frac{1}{2} p^{-}(t) \left\{ -\tilde{x}^2(t) - \tilde{x}^2(t - \tau) + \left(x(t - \tau) - \tilde{x}(t) \right)^2 \right\} dt + \frac{1}{2\pi} \int \left(x'^2(t) - \frac{1}{2} p^{+}(t) \left(\tilde{x}^2(t) + \tilde{x}^2(t - \tau) \right) dt + \frac{1}{2} \bar{p} \bar{x}^2 \right]
$$
\nSince $-\tilde{x}^2(t) - \tilde{x}^2(t - \tau) + \left(x(t - \tau) - \tilde{x}(t) \right)^2 \leq 3\bar{x}^2 + \left(\tilde{x}(t) - \tilde{x}(t - \tau) \right)^2$, we have,
\nusing the inequality
\n
$$
|\tilde{x}(t) - \tilde{x}(t - \tau)| \leq \frac{2\pi}{\sqrt{3}} |\tilde{x}|_{H^1},
$$
\n
$$
\frac{1}{2\pi} \int \left(\tilde{x}^2(t) - \tilde{x}^2(t - \tau) + \left(x(t - \tau) - \tilde{x}(t) \right)^2 \right) dt
$$
\n(3.2)

))2, we have, Since $-\bar{x}^2(t) - \bar{x}^2(t-\tau) + (x(t-\tau))$
using the inequality

$$
\begin{aligned}\n\tilde{x}^{2}(t) - \tilde{x}^{2}(t-\tau) + (x(t-\tau) - \tilde{x}(t))^{2} &\leq 3\bar{x}^{2} + (\tilde{x}(t) - \tilde{x}(t-\tau))^{2}, \text{ we have,} \\
\text{inequality} \\
|\tilde{x}(t) - \tilde{x}(t-\tau)| &\leq \frac{2\pi}{\sqrt{3}} |\tilde{x}|_{H^{1}}, \\
\frac{1}{2\pi} \int p^{-}(t) \left(-\tilde{x}^{2}(t) - \tilde{x}^{2}(t-\tau) + (x(t-\tau) - \tilde{x}(t))^{2} \right) dt \\
&\geq \frac{1}{2\pi} \int p^{-}(t) \left(\tilde{x}(t) - \tilde{x}(t-\tau) \right)^{2} dt + \frac{3}{2} \bar{p}^{-} \tilde{x}^{2} \geq \frac{2\pi^{2}}{3} |\tilde{x}|_{H^{1}}^{2} \bar{p}^{-} + \frac{3}{2} \bar{p}^{-} \bar{x}^{2}.\n\end{aligned}
$$
\n(3.2)

Since $p^+(t) \leq \frac{0}{t}(t) + \frac{1}{t}(t) + \frac{1}{t}(t)$, we have

Since
$$
p^+(t) \leq {}^0T^+(t) + {}^1\Gamma(t) + {}^{\infty}\Gamma(t)
$$
, we have
\n
$$
\frac{1}{2\pi} \int x'^2(t) - \frac{1}{2} p^+(t) (\tilde{x}^2(t) + \tilde{x}^2(t - \tau)) dt
$$
\n
$$
\geq \frac{1}{2\pi} \int x'^2(t) - \frac{1}{2} {}^0\Gamma^+(t) (\tilde{x}^2(t) + \tilde{x}^2(t - \tau)) dt
$$
\n
$$
- \frac{1}{2\pi} \int \frac{1}{2} ({}^{\infty}\Gamma + {}^1\Gamma) (t) (\tilde{x}^2(t - \tau)) dt
$$
\n
$$
\geq |\tilde{x}|_{H^1}^2 \delta({}^0\Gamma^+) - \frac{1}{2\pi} \int \frac{1}{2} ({}^{\infty}\Gamma + {}^1\Gamma) (t) (\tilde{x}^2(t) + \tilde{x}^2(t - \tau)) dt,
$$
\nwhere $\delta({}^0\Gamma^+)$ is the best constant associated to ${}^0\Gamma^+$ by Lemma 1. Using
\np. 208])
\n
$$
|\tilde{x}|_{L^1} \leq |\tilde{x}'|_{L^1} = |\tilde{x}|_{H^1}
$$
 and $|\tilde{x}|_{L^{\infty}} \leq \frac{\pi}{\sqrt{3}} |\tilde{x}|_{L^1} = \frac{\pi}{\sqrt{3}} |\tilde{x}|_{H^1},$
\nwe obtain
\n
$$
\frac{1}{2\pi} \int \frac{1}{2} (\tilde{x}^2(t) + \tilde{x}^2(t - \tau)) ({}^1\Gamma + {}^{\infty}\Gamma) (t) dt \leq \left(\frac{\pi^2}{3} |{}^1\Gamma|_{L^1} + |{}^1\Gamma|_{L^1
$$

• where $\delta({}^0T^+)$ is the best constant associated to ${}^0T^+$ by Lemma 1. Using (see e.g. [15:

$$
-\frac{1}{2\pi} \int \frac{1}{2} (\infty \Gamma + 1 \Gamma) (t) (\tilde{x}^2(t - \tau)) dt
$$

\n
$$
\geq |\tilde{x}|_{H^1}^2 \delta(0 \Gamma^+) - \frac{1}{2\pi} \int \frac{1}{2} (\infty \Gamma + 1 \Gamma) (t) (\tilde{x}^2(t) + \tilde{x}^2(t - \tau)) dt,
$$

\n(0^{T+}) is the best constant associated to 0^{T+} by Lemma 1. Using (see e.g. [15:
\n
$$
|\tilde{x}|_{L^1} \leq |\tilde{x}'|_{L^1} = |\tilde{x}|_{H^1} \text{ and } |\tilde{x}|_{L^{\infty}} \leq \frac{\pi}{\sqrt{3}} |\tilde{x}|_{L^1} = \frac{\pi}{\sqrt{3}} |\tilde{x}|_{H^1},
$$
 (3.3)

 $\mathbf{A}^{(n)}$

$$
\geq |\tilde{x}|_{H}^{2}, \delta(2T^{+}) - \frac{1}{2\pi} \int \frac{1}{2} (\infty T + {}^{1}T) (t) (\tilde{x}^{2}(t) + \tilde{x}^{2}(t - \tau)) dt,
$$
\nwhere $\delta(2T^{+})$ is the best constant associated to ${}^{0}T^{+}$ by Lemma 1. Using (see e.g. [15:
\np. 208])
\n
$$
|\tilde{x}|_{L^{1}} \leq |\tilde{x}'|_{L^{1}} = |\tilde{x}|_{H^{1}} \text{ and } |\tilde{x}|_{L^{\infty}} \leq \frac{\pi}{\sqrt{3}} |\tilde{x}|_{L^{1}} = \frac{\pi}{\sqrt{3}} |\tilde{x}|_{H^{1}}, \qquad (3.3)
$$
\nwe obtain
\n
$$
\frac{1}{2\pi} \int \frac{1}{2} (\tilde{x}^{2}(t) + \tilde{x}^{2}(t - \tau)) ({}^{1}T + {}^{\infty}T) (t) dt \leq (\frac{\pi^{2}}{3} |{}^{1}T|_{L^{1}} + |{}^{\infty}T|_{L^{\infty}}) |\tilde{x}|_{H^{1}},
$$
\nso that, since $p(t) \geq \gamma(t)$ for a.e. t ,
\n $0 \geq \tilde{D}_{p}^{+}(x) \geq (\delta(2T^{+}) - |{}^{\infty}T|_{L^{\infty}} + \frac{\pi^{2}}{3} (2\overline{\gamma}^{-} - |{}^{1}T|_{L^{1}})) |\tilde{x}|_{H^{1}}^{2} + \frac{\overline{x}^{2}}{2} (4\overline{\gamma}^{-} + \overline{\gamma}^{+}).$
\nFrom (3.1) it follows that $\tilde{x} = 0$ and $\tilde{x} = 0$, which yields a contradiction since
\n $x_{n} \stackrel{\epsilon}{\longrightarrow} x$ and $|x_{n}|_{H^{1}} = 1$
\nWe are now in a position to prove the following existence theorem of 2 π -periodic.
\nsolutions for system (1.1).
\nTheorem 1: Let $F \in C^{2}(\mathbb{R}^{n}, \mathbb{R})$ and $g : J \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}, g(t, x) = (g_{1}(t, x), ..., g_{n}($

$$
\frac{1}{2\pi} \int \frac{1}{2} \left(\tilde{x}^2(t) + \tilde{x}^2(t-\tau) \right) \left({}^{1}\Gamma + {}^{\infty}\Gamma \right) (t) dt \leq \left(\frac{\pi^2}{3} |{}^{1}\Gamma|_{L^1} + |{}^{\infty}\Gamma|_{L^{\infty}} \right)
$$

So that, since $p(t) \geq \gamma(t)$ for a.e. t ,
 $0 \geq \tilde{D}_p{}^{+}(x) \geq \left(\delta({}^{0}\Gamma^{+}) - |{}^{\infty}\Gamma|_{L^{\infty}} + \frac{\pi^2}{3} (2\overline{\gamma}{}^{-} - |{}^{1}\Gamma|_{L^1}) \right) |\tilde{x}|_{H^1}^2 + \frac{\overline{x}^2}{2} (4\overline{\gamma}{}^{-} + \overline{\gamma}{}^{+}).$
From (3.1) it follows that $\tilde{x} = 0$ and $\overline{x} = 0$, which yields a contradiction
 $x_n \xrightarrow{C} x$ and $|x_n|_{H^1} = 1$
We are now in a position to prove the following existence theorem of 2π -pe
solutions for system (1.1).
Theorem 1: Let $F \in C^2(\mathbf{R}^n, \mathbf{R})$ and $g : J \times \mathbf{R}^n \to \mathbf{R}^n$, $g(t, x) = \left(g_1(t, x), ..., g_n(t) \right)$
such that
(i) $g(\cdot, x)$ is measurable on J for each $x \in \mathbf{R}^n$,
(ii) $g(t, \cdot)$ is continuous on \mathbf{R}^n for a.e. $t \in J$;
(iii) for each real number $r > 0$ and for any $i = 1, ..., n$ there exist $a_i, b_i \in$

From (3.1) it follows that $\bar{x} = 0$ and $\bar{x} = 0$, which yields a contradiction since $x_n \xrightarrow{C} x$ and $|x_n|_{H^1} = 1$

We are now in a position to prove the following existence theorem of 2π -periodic. solutions for system (1.1) .

 Theorem 1: *Let* $F \in C^2(\mathbf{R^n},\, \mathbf{R})$ *and* $g: J \times \mathbf{R^n} \rightarrow \mathbf{R^n}, g(t,x) = \big(g_1(t,x),\, \dots,\, g_n(t,x)\big),$ $\begin{aligned} &id \ g: J \times \mathbf{R}^n \rightarrow \mathbf{R}^n, g(t, \theta) \ &\text{each} \ x \in \mathbf{R}^n, \ &\text{a.e.} \ t \in J, \ &\text{and} \ for \ any \ i=1, \ldots, \ &\text{that} \ &\begin{aligned} &|x_k|^a \\ &\text{if} \ s &\text{and} \ 0 \leq \alpha < 1. \end{aligned} \end{aligned}$

- *(i)* $g(\cdot, x)$ *is measurable on J for each* $x \in \mathbb{R}^n$,
- (ii) $g(t, \cdot)$ is continuous on \mathbb{R}^n for a.e. $t \in J$;
- (iii) *for each real number r > 0 and for any i = 1, ..., n there exist* a_i *,* $b_i \in L^1(J)$

are now in a position to prove the fo
\nis for system (1.1).
\norem 1: Let
$$
F \in C^2(\mathbb{R}^n, \mathbb{R})
$$
 and $g : J \neq g(\cdot, x)$ is measurable on J for each $x \in$
\n $g(t, \cdot)$ is continuous on \mathbb{R}^n for a.e. $t \in$
\nfor each real number $r > 0$ and for (which depend also on r) such that
\n $|g_i(t, x)| \leq a_i(t) + b_i(t) \left(\sum_{k=1}^n |x_k|^{\alpha}\right)$
\nfor a.e. $t \in J$, all $x \in \mathbb{R}^n$ with $|x_i| \leq r$ an
\ne that for every $i = 1, ..., n$ the inequality
\n $\gamma_i(t) \leq \lim_{|x_i| \to \infty} \inf x_i^{-1} g_i(t, x) \leq \lim_{|x_i| \to \infty} \sup_{|x_i| \to \infty}$
\n*informly for a.e.* $t \in J$ and $x_j \in \mathbb{R}$, $j \geq 2$.

(1) $g(\cdot, x)$ is measurable on *J* for each $x \in \mathbb{R}^n$,

(ii) $g(t, \cdot)$ is continuous on \mathbb{R}^n for a.e. $t \in J$;

(iii) for each real number $r > 0$ and for any $i = 1, ...,$

(which depend also on r) such that
 $|g_i(t, x)| \le$ Assume that for every $i = 1, \ldots, n$ the inequalities such that

(i) $g(\cdot, x)$ is meass

(ii) $g(t, \cdot)$ is contin

(iii) for each real ι

(which depend
 $|g_i(t, x)| \leq \iota$

for a.e. $t \in J$, a

Assume that for every

(B⁺) $\gamma_i(t) \leq \lim_{|x_i| \to \infty}$

hold uniformly for a

Lemma

$$
|g_i(t, x)| \le a_i(t) + b_i(t) \left(\sum_{k=1}^n |x_k|^{\alpha}\right)
$$

for a.e. $t \in J$, all $x \in \mathbb{R}^n$ with $|x_i| \le r$ and $0 \le \alpha < 1$.
Assume that for every $i = 1, ..., n$ the inequalities

$$
(\mathbf{B}^+) \qquad \gamma_i(t) \le \lim_{|x_i| \to \infty} \inf x_i^{-1} g_i(t, x) \le \lim_{|x_i| \to \infty} x_i^{-1} g_i(t, x) \le \Gamma_i(t)
$$

hold uniformly for a.e. $t \in J$ *and* $x_j \in \mathbb{R}$, $j \neq i$, and γ_i , Γ_i *satisfy the conditions of Lemma 2.*

Then, system (1.1) has at least one 2π *-periodic solution for each* $e \in L^1(J, \mathbb{R}^n)$ *.*

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Proof: Let ε_i , μ_i be positive constants associated to γ_i and Γ_i by Lemma 2. From assumption \mathbf{B}^+ there exists $r_i > 0$ such that for a.e. $t \in J$ and for every $x \in \mathbf{R}^n$ with $|x_i| \ge r_i$ we have $\gamma_i(t) - \varepsilon_i \le x_i^{-1}g_i(t, x) \le \Gamma_i(t) + \varepsilon_i$. Put $\varepsilon = \min \varepsilon_i$, $\mu = \min \mu_i$ and \vec{r} = max $\vec{r_i}$. Define $\tilde{\gamma}_i : \vec{J} \times \mathbb{R}^n \to \mathbb{R}$ by

G. CONT, R. IANNACCI and M. N. NKASHAMA
\n
$$
\begin{aligned}\n\therefore \text{Let } \varepsilon_i, \mu_i \text{ be positive constants associated to } \gamma_i \text{ and } \Gamma_i \text{ by Lemma 2. From} \\
\text{on } \mathbf{B}^+ \text{ there exists } r_i > 0 \text{ such that for a.e. } t \in J \text{ and for every } x \in \mathbf{R}^n \text{ with} \\
\text{we have } \gamma_i(t) - \varepsilon_i \leq x_i^{-1}g_i(t, x) \leq \Gamma_i(t) + \varepsilon_i. \text{ Put } \varepsilon = \min \varepsilon_i, \ \mu = \min \mu_i \\
\text{max } r_i. \text{ Define } \tilde{\gamma}_i : J \times \mathbf{R}^n \to \mathbf{R} \text{ by} \\
\begin{cases}\nx_i^{-1}g_i(t, x) > \text{if } |x_i| \geq r \\
r^{-1}g_i(t, x_1, \ldots, x_{i-1}, r, x_{i+1}, \ldots, x_n) (x_ir^{-1}) \\
+ \Gamma_i(t) (1 - x_ir^{-1}) & \text{if } 0 \leq x_i < r \\
r^{-1}g_i(t, x_1, \ldots, x_{i-1}, -r, x_{i+1}, \ldots, x_n) (x_ir^{-1}) \\
+ \Gamma_i(t) (1 + x_ir^{-1}) & \text{if } -r \leq x_i < 0.\n\end{cases} \\
\gamma_i(t) - \varepsilon \leq \tilde{\gamma}_i(t, x) \leq \Gamma_i(t) + \varepsilon \text{ for a.e. } t \in J, \quad x \in \mathbf{R}^n. \tag{3.4} \\
\gamma_i(t) - \varepsilon = \tilde{\gamma}_i(t, x) \leq \Gamma_i(t) + \varepsilon \text{ for a.e. } t \in J, \quad x \in \mathbf{R}^n. \tag{3.5} \\
\gamma_i(t) - \varepsilon = \tilde{\gamma}_i(t, x) \leq \Gamma_i(t) + \varepsilon \text{ for a.e. } t \in J, \quad x \in \mathbf{R}^n. \tag{3.4} \\
\gamma_i(t) - \varepsilon = \tilde{\gamma}_i(t, x) \leq \Gamma_i(t) + \varepsilon \text{ for a.e. } t \in J, \quad x \in \mathbf{R}^n. \tag{3.5} \\
\gamma_i(t) - \varepsilon = \tilde{\gamma}_i(t,
$$

We have

• -

•

 $\frac{1}{2}$

$$
\gamma_i(t) - \varepsilon \leq \tilde{\gamma}_i(t, x) \leq \Gamma_i(t) + \varepsilon \quad \text{for a.e. } t \in J, \qquad x \in \mathbf{R}^n. \tag{3.4}
$$

Moreover, the function $\tilde{g}_i : J \times \mathbb{R}^n \to \mathbb{R}$ defined by $\tilde{g}_i(t, x) = \tilde{\gamma}_i(t, x) x_i$ satisfies assumptions (i)–(iii) and $\tilde{g}_i(t, x) = g_i(t, x)$ for all $x \in \mathbb{R}^n$ with $|x_i| \geq r_i$. Let $h_i = g_i$ \tilde{g}_i and denote by $($ $\begin{bmatrix} i_1(t,x) \\ \text{on} & \tilde{g} \\ \text{and} & \tilde{g} \\ \frac{d}{dt} & \frac{\partial F}{\partial x} \\ \text{on} & \text{as} \\ \end{bmatrix}$ struction and from the assumptions on g_i it follows that $\left\{\n\begin{aligned}\n&+ \Gamma_i(t) \ (1-x_i r^{-1}) &\text{if } 0 \leq x_i < r \\
&-r, x_{i+1}, \ldots, x_n) \ (x_i r^{-1}) \\
&+ \Gamma_i(t) \ (1+x_i r^{-1}) &\text{if } -r \leq x_i < 0.\n\end{aligned}\n\right.$

for a.e. $t \in J, \quad x \in \mathbb{R}^n.$ (3.4)
 R defined by $\tilde{g}_i(t, x) = \tilde{\gamma}_i(t, x) \ x_i$ satisfies

, x)

denote by
$$
\left(\frac{u}{dt} \frac{\partial u}{\partial x_i} (x(t))\right)
$$
 the *i*-th component of $\frac{u}{dt}$ (grad $F(x(t))$). By con-
and from the assumptions on g_i it follows that
 $|h_i(t, x)| \leq 2 \left(a_i(t) + b_i(t) \left(\sum_{k=1}^n |x_k|^a\right)\right)$ for a.e. $t \in J$, $x \in \mathbb{R}^n$. (3.5)

Our system (1.1).is equivalent to

a and from the assumptions on
$$
g_i
$$
 it follows that
\n
$$
|h_i(t, x)| \leq 2 \left(a_i(t) + b_i(t) \left(\sum_{k=1}^n |x_k|^s \right) \right) \text{ for a.e. } t \in J, x \in \mathbb{R}^n.
$$
\n(3.5)
\n
$$
x_i''(t) + \left(\frac{d}{dt} \frac{\partial F}{\partial x_i} (x(t)) \right) + \tilde{y}_i(t, x(t - \tau)) x_i(t - \tau_i) + h_i(t, x(t - \tau)) = e_i(t),
$$
\n
$$
i = 1, ..., n.
$$

By the same degree argument as in the proof of Theorem 1 of [10], it suffices to show that the set of 2π -periodic solutions for the system

Our system (1.1) is equivalent to
\n
$$
x_i''(t) + \left(\frac{d}{dt} \frac{\partial F}{\partial x_i}(x(t))\right) + \bar{\gamma}_i(t, x(t-\tau)) x_i(t-\tau_i) + h_i(t, x(t-\tau)) = e_i(t),
$$
\n
$$
i = 1, ..., n.
$$
\nthe same degree argument as in the proof of Theorem 1 of [10], it suffices to show
\nt the set of 2π -periodic solutions for the system
\n
$$
x_i''(t) + (1-\lambda) \Gamma_i(t) x_i(t-\tau_i) + \lambda \left(\frac{d}{dt} \frac{\partial F}{\partial x_i}(x(t))\right) + \lambda h_i(t, x(t-\tau))
$$
\n
$$
+ \lambda \tilde{\gamma}_i(t, x(t-\tau_i)) x_i(t-\tau_i) = \lambda e_i(t), i = 1, ..., n
$$
\n(3.6)
\nounded for every $\lambda \in [0, 1]$. Let $x \in W^{2,1}(J, \mathbb{R}^n)$ be a 2π -periodic solution of system
\n(i). Multiplying each equation of this system by $(\bar{x}_i - \tilde{x}_i(t))$ and taking into account

is bounded for every $\lambda \in [0, 1]$. Let $x \in W^{2,1}(J, \mathbb{R}^n)$ be a 2π -periodic solution of system (3.6). Multiplying each equation of this system by $(\bar{x}_i - \tilde{x}_i(t))$ and taking into account the assumptions on Γ_i , (3.4) and Lemma 2, one gets for each $i = 1, \ldots$,

ame degree argument as in the proof of Theorem 1 of [10], it suffices to show
\nset of
$$
2\pi
$$
-periodic solutions for the system
\n $x_i''(t) + (1 - \lambda) \Gamma_i(t) x_i(t - \tau_i) + \lambda \left(\frac{d}{dt} \frac{\partial F}{\partial x_i}(x(t))\right) + \lambda h_i(t, x(t - \tau))$
\n $+ \lambda \tilde{\gamma}_i(t, x(t - \tau)) x_i(t - \tau_i) = \lambda e_i(t), i = 1, ..., n$ (3.6)
\nled for every $\lambda \in [0, 1]$. Let $x \in W^{2,1}(J, \mathbb{R}^n)$ be a 2π -periodic solution of system
\nultiplying each equation of this system by $(\bar{x}_i - \tilde{x}_i(t))$ and taking into account
\nmptions on Γ_i , (3.4) and Lemma 2, one gets for each $i = 1, ..., n$
\n $0 = \frac{1}{2\pi} \int (\bar{x}_i - \tilde{x}_i(t)) \left(x_i''(t) + \lambda \left(\frac{d}{dt} \frac{\partial F}{\partial x_i}(x(t))\right) + (1 - \lambda) \Gamma_i(t) x_i(t - \tau_i) + \lambda \gamma_i(t, x(t - \tau)) x_i(t - \tau_i) + \lambda h_i(t, x(t - \tau)) - \lambda e_i(t) \right) dt.$
\n $\geq \mu |x_i|^2_{H^1} - |\bar{x}_i - \tilde{x}_i|_c |h_i|_{L^1} - |\bar{x}_i - \tilde{x}_i|_c |e_i|_{L^1}$
\n $+ \frac{\lambda}{2\pi} \int (\bar{x}_i - \tilde{x}_i(t)) \frac{d}{dt} \frac{\partial F}{\partial x_i}(x(t)) dt.$

Nonuniform Conditions for Periodic Solutions

From inequalities (3.2) and (3.5) one has

$$
0 \geq \mu |x_i|_{H^1}^2 - \frac{2\pi}{\sqrt{3}} |x_i|_{H^1} \left\{ |e_i|_{L^1} + 2 |a_i|_{L^1} + 2 |b_i|_{L^1} \left(\sum_{k=1}^n |x_k|_C^k \right) \right\}
$$

$$
+ \frac{\lambda}{2\pi} \int (\overline{x}_i - \overline{x}_i(t)) \frac{d}{dt} \frac{\partial F}{\partial x_i} (x(t)) dt.
$$

Let

$$
L = \max_{i} (|e_i|_{L^1} + 2 |a_i|_{L^1}), \quad M = 2 \max_{i} |b_i|_{L^1}
$$

$$
p_i' = \frac{\lambda}{2\pi} \int (\bar{x}_i - \bar{x}_i(t)) \frac{d}{dt} \frac{\partial F}{\partial x_i} (x(t)) dt.
$$

One gets

$$
0 \geq \mu |x_i|^2_{H^1} - \frac{2\pi}{\sqrt{3}} |x_i|_{H^1} \left\{ L + M\left(\sum_{k=1}^n |x_k|^2_{C}\right) \right\} + p_i.
$$

On the other hand, for each $i = 1, ..., n$ one has

$$
|x_i|_{H^1} \le \left(\sum_{j=1}^n |x_j|_{H^1}^2\right)^{1/2} = ||x||_{H^1}
$$
 and $|x_k|_C^2 \le \frac{2}{\sqrt{3}} ||x||_{H^1}^2$,

hence

$$
\sum_{k=1}^n |x_k|_{C}^* \leqq \frac{2\pi}{\sqrt{3}} n ||x||_{H^1}^* \text{ and } 0 \geqq \mu |x_i|_{H^1}^2 - \frac{2\pi}{\sqrt{3}} ||x||_{H^1} \left(L + \frac{2\pi}{\sqrt{3}} n M ||x||_{H^1}^* \right) + p_i.
$$

Now, adding for $i = 1, ..., n$ the last inequalities we derive

$$
0 \geq \mu \, ||x||_{H^1}^2 - 2(|\sqrt{3}|^{-1} \pi \, ||x||_{H^1} \, n(L + 2(|\sqrt{3}|^{-1} \, \pi n \, M \, ||x||_{H^1}^2).
$$

Hence

$$
0 \geq \mu \|x\|_{H^1}^2 - 2(\sqrt{3})^{-1} \pi n L \|x\|_{H^1} - 4(\sqrt{3})^{-1} \pi^2 n^2 M \|x\|_{H^1}^{*1}.
$$

Hence it is clear that there exists $\beta_1 > 0$ such that $||x||_{H^1} < \beta_1$; from this and the fact that $H^1(J, \mathbb{R}^n) \subset C(J, \mathbb{R}^n)$ compactly, we have that $||x||_C < \beta_2$ for some $\beta_2 > 0$ and the proof is complete \blacksquare

3.2. The case of assumption B⁻

To get the (in some sense) dual version of Theorem 1 we premise

Lemma 3: Let $\gamma, \Gamma \in L^1(J)$ be such that $\gamma(t) \ge \Gamma(t)$ a.e. on J and $\Gamma = 0 \Gamma + 1 \Gamma + \gamma \Gamma$ $with$

$$
{}^{1}\Gamma \in L^{1}(J), 0 \geq {}^{1}\Gamma(t) \text{ a.e. on } J, \text{ and } {}^{\infty}\Gamma \in L^{\infty}(J), 0 \geq {}^{\infty}\Gamma(t) \text{ a.e. on } J,
$$

 ${}^{\circ}\Gamma \in L^1(J)$, ${}^{\circ}\Gamma(t) \not\equiv 0$ and ${}^{\circ}\Gamma(t) \geq -1$ a.e. on J with strict inequality on a subset of positive measure,

and

$$
\delta(-^0\varGamma^-)-\frac{\pi^2}{3}\,|^1\varGamma|_{L_1}-|^{\infty}\varGamma|_{L^{\infty}}-\frac{2\pi^2}{3}\,(\vec{\mathrm{p}}^+)>0\,,\ \ \bar{\mathrm{p}}^-<-4\bar{\mathrm{p}}^+.
$$

Then, there exist $\varepsilon = \varepsilon(\gamma, \Gamma) > 0$ and $\mu = \mu(\gamma, \Gamma) > 0$ such that for all $p \in L^1(J)$ satisfying $\Gamma(t) - \varepsilon \leq p(t) \leq \gamma(t) + \varepsilon$ a.e. on J, all $f \in C(\mathbf{R}, \mathbf{R})$ and all $x \in W^{2,1}(\mathcal{J})$ we have $- x(t) (x''(t) + f(x(t)) x'(t) + p(t) x(t - \tau)) dt \geq \mu |x|_{H^1}^2.$

Proof: Using the same arguments as in the proof of Lemma 2, it is easy to see that it suffices to show that

$$
D_{p}^{-}(x) \doteq \frac{1}{2\pi} \int \left(x'^{2}(t) - p(t) x(t) x(t - \tau) \right) dt \geq 0.
$$

We have

$$
D_{p}^{-}(x) \geq \frac{1}{2\pi} \int \left(x'^{2}(t) + \frac{1}{2} p(t) \left(\tilde{x}^{2}(t) + \tilde{x}^{2}(t-\tau) \right) dt \right. \\ - \frac{1}{2\pi} \int \frac{1}{2} p(t) \left(x(t-\tau) - \tilde{x}(t) \right)^{2} dt - \frac{1}{2} \overline{p} \overline{x}^{2}
$$

Since $p^-(t) \geq \Gamma^-(t)$, using (3.2) we obtain

$$
D_{p}^{-}(x) \geq \frac{1}{2\pi} \int \left(x'^{2}(t) + \frac{1}{2} \Gamma \left(\tilde{x}^{2}(t) + \tilde{x}^{2}(t - \tau) \right) \right) dt
$$

$$
- \frac{2\pi^{2}}{3} |\tilde{x}|_{H'}^{2} \bar{p}^{+} - \frac{3}{2} \tilde{x}^{2} \bar{p}^{+} - \frac{1}{2} \bar{p}^{-} \bar{x}^{2}
$$

$$
\geq \left(\delta(-{}^{0}\Gamma^{-}) - |\^{\infty}\Gamma|_{L^{\infty}} - |\Gamma|_{L^{1}} \frac{\pi^{2}}{3} - \frac{2\pi^{2}}{3} \bar{p}^{+} \right) |\tilde{x}|_{H}^{2} - \frac{\bar{x}^{2}}{2} (4\bar{p}^{+} - \bar{p}^{-})
$$

and the proof is complete \blacksquare

Theorem 2: Let $F \in C^2(\mathbf{R}^n, \mathbf{R})$ and $g: J \times \mathbf{R}^n \to \mathbf{R}^n$ satisfy assumptions (i) -(iii) of Theorem 1. Assume that for each $i = 1, ..., n$ the inequalities

 $\Gamma_i(t) \leqq \liminf_{|x_i| \to \infty} x_i^{-1} g_i(t, x) \leqq \limsup_{|x_i| \to \infty} x_i^{-1} g_i(t, x) \leqq \gamma_i(t)$ (\mathbf{B}^{\dagger})

hold uniformly for a.e. $t \in J$ and $x_i \in \mathbb{R}$, $j \neq i$, and γ_i , Γ_i satisfy the conditions of Lemma 3. Then, system (1.1) has at least one 2π -periodic solution for each $e \in L^1(J, \mathbb{R}^n)$.

Proof: It is similar to that of Theorem 1, using the same notations. Let $x \in H^1(J, \mathbb{R}^n)$ be a 2π -periodic solution for system (3.6). Multiplying each equation of this system by $(-\bar{x}_i - \bar{x}_i(t))$ and taking into account the assumptions and Lemma 3, one gets, for each $i=1,\ldots,n$,

$$
0 = \frac{1}{2\pi} \int \left(-\overline{x}_i - \overline{x}_i(t)\right) \left(x_i''(t) + \lambda \left(\frac{d}{dt} \frac{\partial F}{\partial x_i} (x(t))\right) + (1 - \lambda) \Gamma_i(t) x_i(t - \tau) + \lambda \gamma_i(t, x(t - \tau)) x_i(t - \tau_i) + \lambda h_i(t, x(t - \tau)) - \lambda e_i(t) \right) dt
$$

$$
\geq \frac{\mu}{2} |x_i|_{H^1}^2 - |\overline{x}_i - \overline{x}_i|_{C} (|h_i|_{L^1} + |e_i|_{L^1}) - \frac{\lambda}{2\pi} \int x_i(t) \frac{d}{dt} \frac{\partial F}{\partial x_i} (x(t)) dt.
$$

With a technique similar to that used in Theorem 1 we obtain that there exists a constant $\beta > 0$ such that $||x||_C < \beta$ and the proof is complete \blacksquare

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3.3. The mixed ease

Theorem 3: Let $F \in C^2(\mathbb{R}^n, \mathbb{R})$ and $g: J \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy assumptions (i)-(iii) of *Theorem* 1. Let n_0 be an integer such that $1 \leq n_0 \leq n$. Assume that for every $i = 1, ..., n_0$ *condition* \mathbf{B}^+ *holds* uniformly for a.e. $t \in J$ and $\overline{x}_i \in \mathbf{R}$, $j \neq i$, where γ_i and Γ_i are as in *Theorem 1. Moreover assume that for every* $i = n_0 + 1, ..., n$ *condition B⁻ holds uniformly for a.e.* $t \in J$ *and* $x_i \in \mathbb{R}$, $j \neq i$, *where* γ_i *and* Γ_i *are as in Theorem* 2. **IDENTIFY ASSET THEOREM 3.** Theorem 1. Let n_0 be an integer such that $1 \leq n_0 \leq n$. Assume that for every $i = 1, ..., n$, *n* condition B⁺ holds uni **Solutions for Periodic Solutions**
 3.3. The mixed case
 Theorem 3: Let $F \in C^2(\mathbb{R}^n, \mathbb{R})$ and $g: J \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy assumptions (i)-(iii)
 Theorem 1. Let n_0 be an integer such that $1 \leq n_0 \leq n$

Then, system (1.1) has at least one 2π *-periodic solution for each e* $\in L^1(J, \mathbb{R}^n)$ *.*

Proof: For each $i = 1, ..., n_0$ we proceed as in the proof of Theorem 1 and for $i = n_0 + 1, ..., n$ we proceed as in the proof of Theorem 2 to get the boundedness of Theorem 1. Moreover assume that for every $i = n_0 +$
Theorem 1. Moreover assume that for every $i = n_0 +$
formly for a.e. $t \in J$ and $x_i \in \mathbb{R}$, $j \neq i$, where γ_i and 1
Then, system (1.1) has at least one 2π -periodic

4. Resonance conditions

4.1. The case of assumption E^+

The following lemma extends Lemma 2 of [3], when τ is an effective delay, to the case when Γ crosses the positive eigenvalues of problem (1.2) in some subset of J of positive measure. *OF* € L'(J), *0* **Of(g)**

Lemma 4: Let
$$
\varepsilon > 0
$$
 and $\Gamma \in L^1(J)$ be such that $\Gamma = {}^0\Gamma + {}^1\Gamma + {}^{\infty}\Gamma$ with

$$
{}^{1}\tilde{\Gamma} \in L^{1}(J), {}^{1}\Gamma(t) \geq 0 \text{ a.e. on } J, \text{ and } {}^{\infty}\Gamma \in L^{\infty}(J), {}^{\infty}\Gamma(t) \geq 0 \text{ a.e. on } J,
$$

 $1\ a.e.$ on J , with ${}^0\varGamma(t) < 1$ on a subset of positive measure.

Then, for all p $\in L^{1}(J)$, $T(t) \geq 0$ *a.e. on J*, *and* ∞ T
T $\in L^{1}(J)$, $0 \leq 0$ $T(t) \leq 1$ *a.e. on J*, *with*
Then, for all p $\in L^{1}(J)$ *satisfying* $0 \leq$
functions $f: \mathbb{R} \to \mathbb{R}$ *and all* $p(t) \leq \Gamma(t) + \varepsilon$ a.e. on J, all continuous *functions* $f: \mathbf{R} \to \mathbf{R}$ *and all* $x \in W^{2,1}(J)$ *, we have*

The following lemma extends Lemma 2 of [3], when
$$
\tau
$$
 is an effective delay, to the c
when Γ crosses the positive eigenvalues of problem (1.2) in some subset of J of posit
measure.
Lemma 4: Let $\varepsilon > 0$ and $\Gamma \in L^1(J)$ be such that $\Gamma = {}^0\Gamma + {}^1\Gamma + {}^{\infty}\Gamma$ with
 ${}^1\Gamma \in L^1(J)$, ${}^1\Gamma(t) \ge 0$ a.e. on J , and ${}^{\infty}\Gamma \in L^{\infty}(J)$, ${}^{\infty}\Gamma(t) \ge 0$ a.e. on J ,
 ${}^0\Gamma \in L^1(J)$, $0 \le {}^0\Gamma(t) \le 1$ a.e. on J , with ${}^0\Gamma(t) < 1$ on a subset of positive measure
Then, for all $p \in L^1(J)$ satisfying $0 \le p(t) \le \Gamma(t) + \varepsilon$ a.e. on J , all continuous
functions $f: \mathbb{R} \to \mathbb{R}$ and all $x \in W^{2,1}(J)$, we have

$$
\frac{1}{2\pi} \int (\bar{x} - \bar{x}(t)) (x''(t) + f(x(t)) x'(t) + p(t) x(t - \tau)) dt
$$

$$
\ge \left[\delta({}^0\Gamma) - \frac{\pi^2}{3} |{}^1\Gamma|_{L^1} - |{}^{\infty}\Gamma|_{L^{\infty}} - \varepsilon \right] |\tilde{x}|_H^2,
$$
where $\delta({}^0\Gamma)$ is associated to ${}^0\Gamma$ by Lemma 1.
Proof: Integrating by parts and following the proof of Lemma 2 of [3], one get

$$
D_p^+(x) \ge \frac{1}{2\pi} \int \left[(x'(t))^2 - \Gamma(t) \frac{\tilde{x}^2(t) + \tilde{x}^2(t - \tau)}{\tilde{x}^2(t)} \right] dt
$$

-

$$
\sum_{i} \sum_{j} \left[\delta(0I) - \frac{\pi^2}{3} |^{1} \Gamma|_{L^1} - |^{\infty} \Gamma|_{L^{\infty}} - \varepsilon \right] |\tilde{x}|_H^2,
$$

\nHere $\delta(0I)$ is associated to $0I$ by Lemma 1.
\nProof: Integrating by parts and following the proof of Lemma 2 of [3], one gets
\n
$$
D_p^+(x) \geq \frac{1}{2\pi} \int \left[(x'(t))^2 - \Gamma(t) \frac{\tilde{x}^2(t) + \tilde{x}^2(t - \tau)}{2} \right] dt
$$
\n
$$
- \frac{\varepsilon}{2\pi} \int \frac{\tilde{x}^2(t) + \tilde{x}^2(t - \tau)}{2} dt
$$
\n
$$
+ \frac{1}{2\pi} \int \frac{p(t)}{2} \left[(x(t - \tau) - \tilde{x}(t))^2 + \tilde{x}^2 \right] dt.
$$

\nTherefore, by Lemma 1 and inequalities (3.2), we have
\n
$$
D_p^+(x) \geq \delta(0I) |\tilde{x}|_H^2 - |^1 \Gamma|_{L^1} |\tilde{x}|_{L^{\infty}}^2 - |\tilde{\infty} \Gamma|_{L^{\infty}} |\tilde{x}|_{L^1}^2 - \varepsilon |\tilde{x}|_H^2.
$$
\n
$$
\geq \left[\delta(0I) - \frac{\tilde{x}^2}{3} |^1 \Gamma|_{L^1} - |\tilde{\infty} \Gamma|_{L^{\infty}} \right] |\tilde{x}|_H^2 - \varepsilon |\tilde{x}|_H^2.
$$

Therefore, by Lemma 1 and inequalities (3.2), we have

$$
+\frac{1}{2\pi}\int \frac{F(t)}{2}[(x(t-\tau)-\tilde{x}(t))^2+\bar{x}^2]dt.
$$

\ne, by Lemma 1 and inequalities (3.2), we have
\n
$$
D_p^+(x) \ge \delta({}^0T) |\tilde{x}|_{H^1}^2 - |{}^1\Gamma|_{L^1} |\tilde{x}|_{L^\infty}^2 - |{}^{\infty}\Gamma|_{L^\infty} |\tilde{x}|_{L^1}^2 - \varepsilon |\tilde{x}|_{H^1}^2
$$

\n
$$
\ge \left[\delta({}^0T) - \frac{\pi^2}{3}|{}^1\Gamma|_{L^1} - |{}^{\infty}\Gamma|_{L^\infty}\right] |\tilde{x}|_{H^1}^2 - \varepsilon |\tilde{x}|_{H^1}^2.
$$

and the proof is complete \blacksquare

Theorem 4: Let $F \in C^2(\mathbb{R}^n, \mathbb{R})$ and $g: J \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy assumptions (i)-(iii) *of Theorem 1 and be such that for any* $i = 1, \ldots, n$

- (\mathbf{E}_1^+) there exists $R_i > 0$ such that $g_i(t, x)$ $x_i \geq 0$ for all $x \in \mathbb{R}^n$ with $|x_i| \geq R_i$,
- (E₂⁺⁾ lim sup $x_i^{-1}g_i(t, x) \leq \Gamma_i(t)$ *uniformly for a.e.* $t \in J$ *and uniformly for* $x_j \in \mathbb{R}$ *,* $j = 1, \ldots, j + i$.
 A l $\alpha = 1 + \infty$
 A diversion $i = 1, \ldots, n$ *,* Γ_i *satisfies the conditions of Lemma 4 with*
 $\delta(0,\Gamma$ $|x_i| \rightarrow +\infty$
 $j \neq i$.

Moreover assume that for every i = 1, ..., n, Γ_i *satisfies the conditions of Lemma 4 with*

$$
\delta({}^0\varGamma_i) - \pi^2/3 \, |{}^1\varGamma_i|_{L^1} - |{}^{\infty}\varGamma_i|_{L^{\infty}} > 0,
$$

Then system (1.1) has at least one 2π *-periodic solution for each* $e \in L^1(J, \mathbb{R}^n)$ *with* $\bar{e}=0.$

Proof: Let $0 < \varepsilon < \min \{\delta^0 T_i\} - \frac{\pi^2}{3} |T_i|_{L^1} - |\mathcal{F}_i|_{L^\infty}\}$. Then there exists $r_i > 0$ such that for a.e. $t \in J$ and for all $x \in \mathbb{R}^n$ with $|x_i| \geq r_i$ one has $0 \leq x_i^{-1}g_i(t, x)$ $\Gamma_i(t) + \varepsilon$. Proceeding like in the proof of Theorem 1, we can write the system (1.1) of Theorem 1 and be such that for any $i = 1, ..., n$
 (E_1^+) there exists $R_i > 0$ such that $g_i(t, x) x_i \ge 0$ for all $x \in \mathbb{R}$
 (E_2^+) lines up $x_i^{-1}g_i(t, x) \le \Gamma_i(t)$ uniformly for a.e. $t \in J$ an
 $i = 1, ..., n$, Γ_i satisfies th x for a.e. $t \in J$ and for all $x \in \mathbb{R}^n$ with $|x_i| \ge r_i$ one has $0 \le x_i^{-1}$
 $-\varepsilon$. Proceeding like in the proof of Theorem 1, we can write the system
 $x_i''(t) + \left(\frac{d}{dt} \frac{\partial F}{\partial x_i}(x(t))\right) + \tilde{\gamma}_i(t, x(t - \tau)) x_i(t - \tau_i) + h_i(t, x(t -$ Proof: Let $0 < \varepsilon <$
such that for a.e. $t \in$
 $\leq \Gamma_i(t) + \varepsilon$. Proceedin
in the equivalent form
 $x_i''(t) + \left(\frac{d}{dt}\right)^i$
 $i = 1, ..., n$.
Degree arguments wil

$$
x_i''(t) + \left(\frac{d}{dt}\frac{\partial F}{\partial x_i}\big(x(t)\big)\right) + \tilde{\gamma}_i(t,x(t-\tau))\,x_i(t-\tau_i) + h_i(t,x(t-\tau)) = e_i(t),
$$

\n $i = 1, ..., n.$

Degree arguments will imply the existence of a 2π -periodic solution for (1.1) if the set of possible 2π -periodic solutions of the system

Let
$$
0 < \varepsilon < \min_i \{0^{(1)} - \pi^2/5 \mid 1\} \lfloor 1 - \lfloor -1 \rfloor \lfloor \lfloor 1 \rfloor \infty \}
$$
. Then there exists $r_i > 0$ to i for a.e. $t \in J$ and for all $x \in \mathbb{R}^n$ with $|x_i| \ge r_i$ one has $0 \le x_i^{-1} g_i(t, x) + \varepsilon$. Proceeding like in the proof of Theorem 1, we can write the system (1.1) $x_i''(t) + \left(\frac{d}{dt} \frac{\partial F}{\partial x_i}(x(t))\right) + \tilde{\gamma}_i(t, x(t - \tau)) x_i(t - \tau_i) + h_i(t, x(t - \tau)) = e_i(t),$ $i = 1, ..., n$. r $i = 1, ..., n$. r r

is a priori bounded independently of $\lambda \in (0, 1]$. Let $x \in W^{2,1}(J, \mathbb{R}^n)$ be a 2π -periodic solution of (4.1). Multiplying each equation of this system by $(\bar{x}_i - \tilde{x}_i(t))$ and integratsolution of (4.1). Multiplying each equation of this system by $(\bar{x}_i - \tilde{x}_i(t))$ and integrating on *J*, we obtain, using Lemma 4,

$$
+ \lambda \tilde{\gamma}_i(t, x(t-\tau)) x_i(t-\tau_i) = \lambda e_i(t), i = 1, ..., n
$$
\n
$$
+ \lambda \tilde{\gamma}_i(t, x(t-\tau)) x_i(t-\tau_i) = \lambda e_i(t), i = 1, ..., n
$$
\n
$$
+ \lambda \tilde{\gamma}_i(t, \mathbf{R}^n)
$$
\n
$$
= \lambda e_i(t), i = 1, ..., n
$$
\n
$$
= \lambda e_i(t), \mathbf{R}^n
$$
\n
$$
= \lambda e_i(t), \math
$$

with $0 < \mu < \min \{\delta(\mathbf{P}_i) - \pi^2/3 \, |^1 \Gamma_i|_{L^1} - |\mathbf{P}_i|_{L^\infty} - \varepsilon\}$. By means of the arguments used in the proof of Theorem 2 of [2], there exists a constant $d > 0$ such that $||x||_{H^1} < d$.

independently of $\lambda \in (0, 1]$ and we can complete the proof like in Theorem 1 **I**
 Gorollary: Theorem 4 remains valid if we suppose, instead of \mathbf{E}_1^+ *and* \bar{e} =
 there exist constants a_i, b_i, R_i *such that Corollary: Theorem 4 remains valid if we suppose, instead of* \mathbf{E}_1^+ *and* $\bar{e} = 0$ *, that there exist constants* a_i *,* b_i *,* R_i *such that* $a_i \leq \bar{e}_i \leq b_i$ *and* **•**
 •
 •
 • $\begin{array}{l} \text{Corollary: Theorem}\ r\text{e exist constants }a_i,\ b\ g_i(t,x)\geqq b_i\ \text{for}\ g_i(t,x)\leqq a_i\leqq \ \text{Proof: The system (1}\ x_i''(t)+\frac{d}{dt}\frac{\partial F}{\partial x_i} \end{array}$

$$
q_i(t, x) \geq b
$$
, for a.e. $t \in J$ and $x \in \mathbb{R}^n$ with $x_i > R_i > 0$,

$$
g_i(t, x) \leq a_i \leq b_i \text{ for a.e. } t \in J \text{ and } x \in \mathbb{R}^n \text{ with } x_i < -R_i.
$$

Proof: The system (1.1) is equivalent to

/

$$
g_i(t, x) \leq a_i \leq b_i \text{ for a.e. } t \in J \text{ and } x \in \mathbb{R}^n \text{ with } x_i < -R_i.
$$

\n
$$
\therefore \text{ The system (1.1) is equivalent to}
$$

\n
$$
x_i''(t) + \frac{d}{dt} \frac{\partial F}{\partial x_i} (x(t)) + g_i^{(1)}(t, x(t)) = e_i^{(1)}(t), \qquad i = 1, ..., n,
$$

,

where

Nonuniform Conditions for Periodic Solutions
\n
$$
g_i^{1}(t,x) \doteq g_i(t,x) - \frac{1}{2} (a_i + b_i)^2 \text{ and } e_i^{1}(t) = e_i(t) - \frac{1}{2} (a_i + b_i).
$$

Observe that $g_i^1(t, x)$ verifies \mathbf{E}_1^+ ; therefore, the arguments used in the proof of Theo**rem 4 yield the conclusion I 4.2.** The case of assumption E-
 $\int f(t, x) \, dt = g_i(t, x) - \frac{1}{2} (a_i + b_i) - \text{and} \quad e_i(t) = e_i(t) - \frac{1}{2} (a_i + b_i).$

Observe that $g_i(t, x)$ verifies \mathbf{E}_i +; therefore, the arguments used in the proof of T

rem 4 yield the conclusion

The following lemma extends Lemma 3 of [3].

- Lemma 5: Let $\varepsilon > 0$ and let $\Gamma \in L^1(J)$ be such that $\Gamma = {}^0\Gamma + {}^1\Gamma + {}^{\infty}\Gamma$ with **Ease of assumption E**
 *N*ing lemma extends Lemma 3 of [3].
 a 5: Let $\varepsilon > 0$ and let $\Gamma \in L^1(J)$ be such that $\Gamma = {}^0\Gamma + {}^1\Gamma + {}^{\infty}\Gamma$ with
 ${}^1\Gamma \in L^1(J), 0 \geq {}^1\Gamma(t)$ a.e. on J, and ${}^{\infty}\Gamma \in L^{\infty}(J), 0 \geq {}^{\infty}\Gamma(t)$
	-
- ${}^1\Gamma \in L^1(J), 0 \geq {}^1\Gamma(t)$ a.e. on J, and ${}^{\infty}\Gamma \in \dot{L}^{\infty}(J), 0 \geq {}^{\infty}\Gamma(t)$ a.e. on J,
 ${}^0\Gamma \in L^1(J), 0 \geq {}^0\Gamma(t) \geq -1$ a.e. on J with ${}^0\Gamma(t) > -1$ on a subset of positive
measure. $g_i^{1}(t, x) \doteq g_i(t, x) - \frac{1}{2}$ (

hat $g_i^{1}(t, x)$ verifies \mathbf{E}_1^+ ;

id the conclusion \blacksquare

case of assumption \mathbf{E}^-
 ving lemma extends Len

a 5: Let $\varepsilon > 0$ and let Γ
 $\Gamma \in L^1(J), 0 \geq \Gamma(t)$ a.e.
 $\Gamma \in L^1$

Then for all $p \in L^1(J)$ *satisfying* $\Gamma(t) - \varepsilon \leq p(t) \leq 0$ *a.e. on J, all continuous functions* $f: \mathbf{R} \to \mathbf{R}$ and all $x \in W^{2,1}(J)$, we have

Lemma 5: Let
$$
\varepsilon > 0
$$
 and let $\Gamma \in L^1(J)$ be such that $\Gamma = {}^0\Gamma + {}^1\Gamma + {}^{\infty}\Gamma$ with
\n ${}^1\Gamma \in L^1(J)$, $0 \geq {}^1\Gamma(t)$ a.e. on J, and ${}^{\infty}\Gamma \in \tilde{L}^{\infty}(J)$, $0 \geq {}^{\infty}\Gamma(t)$ a.e. on J,
\n ${}^0\Gamma \in L^1(J)$, $0 \geq {}^0\Gamma(t) \geq -1$ a.e. on J with ${}^0\Gamma(t) > -1$ on a subset of γ
\nmeasure.
\nThen for all $p \in L^1(J)$ satisfying $\Gamma(t) - \varepsilon \leq p(t) \leq 0$ a.e. on J, all continuous fu
\n $f : \mathbb{R} \to \mathbb{R}$ and all $x \in W^{2,1}(J)$, we have
\n
$$
\frac{1}{2\pi} \int (-x(t)) (x''(t) + f(x(t)) x'(t) + p(t) x(t - \tau)) dt
$$
\n
$$
\geq [\delta(-{}^0\Gamma) - \frac{\pi^2}{3} |{}^1\Gamma|_{L^1} - |{}^{\infty}\Gamma|_{L^{\infty}} - \varepsilon] |{}^5\tilde{E}|_H^2;
$$
\nwhere $\delta(-{}^0\Gamma)$ is associated to $-{}^0\Gamma$ by Lemma 1.
\nProof: Integrating by parts and using the proofs of Lemma 3 of [3: p. 15
\nLemma 4 herein, one gets the conclusion
\nTheorem 5: The assertion of Theorem 4 holds true if assumptions \mathbb{E}_1^+ and
\nreplaced respectively by
\n (\mathbb{E}_1^-) there exists $\mathbb{R}_i > 0$ such that $g_i(t, x) x_i \leq 0$ for all $x \in \mathbb{R}^n$ with $|x_i| \geq R_i$,
\n (\mathbb{E}_2^-) $\liminf x_i^{-1}g_i(t, x) \geq \Gamma_i(t)$ uniformly for a.e. $t \in J$ and uniformly for

Proof: Integrating by parts and using the proofs of Lemma 3 of [3: p. 155] and Lemma *4* herein, one gets the conclusion I

Theorem 5: The assertion of Theorem 4 holds true if assumptions $\mathbf{E_1}^+$ **and** $\mathbf{E_2}^+$ **are (Example 1) (End) for all x f for all x f for all x f** *f n* **f** *f f*

-
- **(E₁**) **i** *C*_E i and **E₂** *t* are the creplaced respectively by
 (E₁) **there** exists $R_i > 0$ such that $g_i(t, x) x_i \leq 0$ for all $x \in \mathbb{R}^n$ with $|x_i| \geq R_i$,
 (E₂) $\liminf_{|x_i| \to +\infty} x_i^{-1} g_i(t, x) \geq \Gamma_i(t)$ unifo **IXPREDITE:** The distribution of \mathbf{I} is dissolved to $\mathbf{I} = \mathbf{I}$ by parts and using the p
 IXPREDITE: The assertion of Theorem 4 hold,
 IXPREDITE: The assertion of Theorem 4 hold,
 IXPREDITE: The same of $\$ $j \neq i$, where Γ_i satisfies the conditions of Lemma 5, and, moreover, $\inf_{x_i} x_i^{-1}g_i(t, x)$
 $\qquad i, where \Gamma_i$ sal
 $\delta(-^0\Gamma) - \frac{\pi^2}{3}$

$$
\delta (-^0 \varGamma) = \frac{\pi^2}{3} \, |{}^1\varGamma|_{L^1} = |{}^{\infty}\varGamma|_{L^{\infty}} > 0.
$$

Proof: It is similar to that of Theorem 4 herein and we omit it for the sake of brevity. Let us mention the required *a priori* estimates **are obtained using Lemma 5** Theorem 5: The assertion of Theorem 4 holds true if assumption-

replaced respectively by
 (E_1^-) there exists $R_i > 0$ such that $g_i(t, x) x_i \leq 0$ for all $x \in \mathbb{R}^n$ with
 (E_2^-) $\liminf_{|x_i| \to +\infty} x_i^{-1}g_i(t, x) \geq \Gamma_i(t)$ un

4.3. The mixed case

Theorem 6: Let $F \in C^2(\mathbf{R}^n, \mathbf{R})$ and $g: J \times \mathbf{R}^n \to \mathbf{R}^n$ satisfy assumptions (i)-(iii) of Theorem 1. Let n_0 be an integer such that $1 \leq n_0 \leq n$. Assume that for every $i = 1, ..., n_0$, *conditions* **E**^{t} *of Theorem 4 hold uniformly a.e. on J and for all* $x_j \in \mathbb{R}$ *,* $j \neq i$ *, where* Γ_i *,* Γ_i conditions E^+ of Theorem 4 hold uniformly a.e. on J and for all $x_i \in \mathbf{R}$, $j \neq i$, where Γ_i
are as in Theorem 4. Moreover, assume that for every $i = n_0 + 1, ..., n$, conditions E^- of
Theorem 5 hold uniformly a.e. on *Theorem* 5 hold uniformly a.e. on *J* and for all $x_i \in \mathbf{R}$, $j \neq i$, where Γ_i are as in Theorem 5.
Then (1.1) has at least one 2π -periodic solution for each $e \in L^1(J, \mathbf{R}^n)$ with $\bar{e} = 0$.

Proof: For each $i = 1, ..., n_0$, we proceed as in the proof of Theorem 4 and for $i = n₀ + 1, ..., n$, we proceed as in the proof of Theorem 5 for getting the required *a priori* estimates I **Formularity:** n_0 , we proceed as in the proof of Theorem 4
 $n_0 + 1, ..., n$, we proceed as in the proof of Theorem 5 for getting the

ori estimates \blacksquare

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G. CONTI

Istituto Matematico ,,U. Dini", Università degli Studi di Firenze Viale Morgagni 67/A, Firenze (Ttalia)

R. IANNACCI Dipartimento di Matematica, Università della Calabria-Arcavacata Cosenza (Italia)

M. N. NKASHAMA

Tnstitut de Mathématique, Université Catholique de Louvain Chemin du Cyclotron 2, B -1348 Louvain-la-Neuve