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Abelian Theorem for the Distributional Stieltjes Transform

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Unter Verwendung des Begriffes des quasiasymptotischen Verhaltens der temperierten Distributionen im Unendlichen wird ein Satz vom Abelschen Typ für die distributionentheoretische Stieltjes-Transformation gegeben. Dieser umfaßt sowohl alle bekannten Ergebnisse als auch einige neue.

Пользуясь понятием квазиасимптотического поведения темперированных дистрибуций в бесконечности доказывается теорема Абеля о дистрибуционной трансформации Стильтьеса. Она включает в себе и все известные результаты и некоторые новые.

Using the notion of quasi-asymptotic behaviour at infinity of tempered distributions, we give an Abelian theorem for the distributional Stieltjes transform. It includes all known results, as well as some new ones.

1. Introduction

It is possible to define the Stieltjes transform of a distribution in different ways. We will mention only the one given by J. Lavoine and O. P. Misra [3], which is related to a subspace of tempered distributions with supports in $[0, \infty)$ and which is used by many authors. We modify the definition of the Stieltjes transform slightly in such a way that it is available for the whole space of tempered distributions defined on $\mathfrak{R}^n$ with supports in $\overline{\mathfrak{R}}_+{}^n$. In the case $n = 1$ this definition includes the mentioned definition from [3]. Using the notion of quasi-asymptotic behaviour at infinity, we prove a theorem of Abelian type. It includes the known results, as well as some new ones.

2. Notations and definitions

$\mathfrak{N}$ is the set of natural numbers and $\mathfrak{N}_0 = \mathfrak{N} \cup \{0\}$. $\mathfrak{R}^n$ is the n-dimensional Euclidean space and $\mathfrak{C}^n$ is the n-dimensional complex space. If $a, b \in \mathfrak{R}^n$ and $x \in \mathfrak{C}^n$, then

$$\langle a, b\rangle = \sum_{i=1}^{n} a_i b_i,\ |a| = \sum_{i=1}^{n} |a_i|,\ x^a = \prod_{i=1}^{n} = x_i^{a_i},$$

$$ax = (a_1 x_1, \ldots, a_n x_n) \text{ and } \|a\|^2 = \langle a, a\rangle;\ a \geqq 0 \text{ means } a_i \geqq 0,$$

$$a > 0 \text{ means } a_i > 0 \text{ and } a \to \infty(0^+) \text{ means } a_i \to \infty(0^+) \text{ for all } i.$$

We set $\mathfrak{R}_+{}^n = \{x \in \mathfrak{R}^n : x > 0\}$, and its closure in $\mathfrak{R}^n$ is denoted by $\overline{\mathfrak{R}}_+{}^n$; $\overline{\mathfrak{R}}_- = (-\infty, 0]$. If $\alpha \in \mathfrak{R}$ and $\varkappa \in \mathfrak{N}$, then $(\alpha + 1)_\varkappa = (\alpha + 1)(\alpha + 2) \ldots (\alpha + \varkappa)$; $(\alpha + 1)_0 = 1$. We set $e = (1, 1, \ldots, 1)$. If $a \in \mathfrak{R}^n$ and $k \in \mathfrak{N}_0{}^n$, then $(a + e)_k = (a_1 + 1)_{k_1} (a_2 + 1)_{k_2} \ldots (a_n + 1)_{k_n}$.

By D^p, $p \in \mathfrak{N}_0^n$, we denote the partial differential operator $\partial^{p_1+\cdots p_n}/\partial x_1^{p_1} \ldots \partial x_n^{p_n}$. The space of $\mathscr{E}^\infty(\mathfrak{R}^n)$-functions ϕ for which all the norms

$$\|\phi\|_m = \sup_{\substack{t\in\mathfrak{R}^n \\ |p|\leq m}} \{(1 + \|t\|^2)^{m/2} |D^p\phi(t)|\}, \qquad m \in \mathfrak{N}_0,$$

are finite is denoted by $\mathscr{S}(\mathfrak{R}^n)$. Its dual $\mathscr{S}'(\mathfrak{R}^n)$ is the space of *tempered distributions*. The completion of $\mathscr{S}(\mathfrak{R}^n)$ with respect to the norm $\|\cdot\|_m$ is denoted by $\mathscr{S}^m$, and its dual by $(\mathscr{S}^m)'$. The pairing between ϕ and f from a testing-function space and its dual is denoted by $(f, \phi) = \langle f, \bar{\phi}\rangle$, $\bar{\phi}$ is the conjugate function for ϕ. The space of tempered distributions f with support supp f contained in $\overline{\mathfrak{R}}_+^n$ is denoted by $\mathscr{S}_+'(\mathfrak{R}^n)$. For a fixed element $s \in \mathfrak{C}^n$ let $\mathscr{A}(s)$ be the space of $\mathscr{E}^\infty(\mathfrak{R}^n)$-functions η such that: $\eta(t) \in [0, 1]$, $t \in \mathfrak{R}^n$; for every $p \in \mathfrak{N}^n$ there is a $c_p > 0$ such that $|D^p\eta(t)| \leqq c_p$, $t \in \mathfrak{R}^n$; there exists an $\varepsilon > 0$, $2\varepsilon < |\mathrm{Re}\, s_i|$ for $i = 1, 2, \ldots, n$, such that $\eta(t) = 1$ if t belongs to the ε-neighbourhood of $\overline{\mathfrak{R}}_+^n$ and $\eta(t) = 0$ outside the 2ε-neighbourhood of $\overline{\mathfrak{R}}_+^n$.

We introduce a family $\{f_a : a \in \mathfrak{R}^n\} \subset \mathscr{S}_+'(\mathfrak{R}^n)$. Firstly, for $\alpha \in \mathfrak{R}$ we set [9: p. 85]

$$f_\alpha(\tau) = \begin{cases} H(\tau)\, \tau^{\alpha-1}/\Gamma(\alpha) & \text{for } \alpha > 0 \\ D^m f_{\alpha+m}(\tau) & \text{for } \alpha \leqq 0,\ \alpha + m > 0 \end{cases} \qquad (\tau \in \mathfrak{R})$$

where H is the characteristic function of $\mathfrak{R}_+$ and $m \in \mathfrak{N}$. It is easy to see that $f_{-n}(\tau) = \delta^{(n)}(\tau)$, $\tau \in \mathfrak{R}$, $n \in \mathfrak{N}$, where δ is the Dirac distribution. Further, for $a \in \mathfrak{R}^n$ we set

$$f_a(t) = \prod_{i=1}^{n} f_{a_i}(t_i), \qquad t \in \mathfrak{R}^n.$$

Denote by K^a, $a \in \mathfrak{R}^n$, the operator on $\mathscr{S}_+'(\mathfrak{R}^n)$ defined by

$$(K^a f)(x) = \big(f_a(t) * f(t)\big)(x), \qquad x \in \mathfrak{R}^n,$$

where $*$ denotes the convolution in $\mathscr{S}'(\mathfrak{R}^n)$ (see [9: § 5.6]). In case $a \in \mathfrak{N}_0^n$, $K^{-a} = D^a$. It is well known that $\mathscr{S}_+'(\mathfrak{R}^n)$ is a commutative ring with respect to the operation of convolution. The unit element is δ and $K^{-a}(K^a f) = \delta * f$, $f \in \mathscr{S}_+'(\mathfrak{R}^n)$, $a \in \mathfrak{R}^n$. Because of that, for $a \in \mathfrak{N}_0^n$ we set $K^a = D^{-a}$. So we have $D^a(D^{-a}f) = f$, $a \in \mathfrak{N}_0^n$, $f \in \mathscr{S}_+'(\mathfrak{R}^n)$.

A real-valued measurable function l defined on $(0, \infty)$ is called *slowly varying* if $\big(l(u\tau)/l(\tau)\big) \to 1$ when $\tau \to \infty$, for each $u > 0$ [6]. We shall always denote by L a function of the form $L(t) = l_1(t_1)\, l_2(t_2) \ldots l_n(t_n)$, $t \in \mathfrak{R}_+^n$, where $l_1, l_2, \ldots, l_n$ are slowly varying.

Now, we shall define the asymptotic and the quasi-asymptotic at infinity.

Definition 1: Let $F \in \mathscr{L}^1_{\mathrm{loc}}(\mathfrak{R}^n)$. If for some $g \in \mathscr{L}^1_{\mathrm{loc}}(\mathfrak{R}^n)$, $g \neq 0$, $a \in \mathfrak{R}^n$ and L

$$\lim_{k\to\infty} \frac{F(kt)}{k^a L(k)} = g(t) \text{ for a.a. } t \in \overline{\mathfrak{R}}_+^n \qquad (k \in \mathfrak{R}_+^n) \tag{1}$$

and for some $T_0 > 0$ and $M > 0$

$$\left|\frac{F(kt)}{k^a L(k)}\right| \leqq M, \quad t \in \mathfrak{R}_+^n, \quad \|t\| \geqq T_0, \quad k > (1, \ldots, 1), \tag{2}$$

then we say that F has the *asymptotic at* ∞ *with respect to* $k^a L(k)$ *with the limit* g.

Definition 2: Let $f \in \mathcal{S}_+'(\Re^n)$. If for some $g \in \mathcal{S}_+'(\Re^n)$, $g \neq 0$, $a \in \Re^n$ and L

$$\lim_{k\to\infty} \left\langle \frac{f(kt)}{k^a L(k)} \phi(t) \right\rangle = \langle g(t), \phi(t) \rangle \qquad (k \in \Re_+{}^n) \tag{3}$$

for every $\phi \in \mathcal{S}(\Re^n)$, then we say that f has the *quasi-asymptotic at* ∞ *with respect to* $k^a L(k)$ *with the limit* g.

In both definitions a is called the *power* of the asymptotic, resp. quasi-asymptotic, behaviour.

Remarks: 1. For g from Definition 2 we have $g(bt) = b^a g(t)$, $t \in \Re^n$, $b \in \Re^n$ and $b > 0$; if g is continuous, then $g = C f_{a+e}$, $a > 0$, for some $C \neq 0$. Indeed, taking into account the properties of L [6] we have

$$b^a \langle g(t), \phi(t) \rangle = \lim_{k\to\infty} \left\langle b^a \frac{f(kt)}{k^a L(k)}, \phi(t) \right\rangle$$

$$= \lim_{k\to\infty} \left\langle \frac{f(kbt)}{k^a L(k)}, \phi(t) \right\rangle = \langle g(bt), \phi(t) \rangle, \quad \phi \in \mathcal{S}(\Re^n).$$

So for $t > 0$, $g(t) = t^a g(e)$. By using the fact that its support is in $\overline{\Re}_+{}^n$, we have $g = C f_{a+e}$.

2. If we compare the quasi-asymptotic from Definition 2, in the case $n > 1$, with that defined in [2] we see that our definition is slightly more restrictive. This is motivated by the fact that we need in our investigations the exact form of g. If $n = 1$ both, definitions are the same.

Let $s \in \mathfrak{C}^n$, $r \in \Re^n$, $\omega \in \Re_+{}^n$ and $\eta \in \mathcal{A}(s)$. We set

$$\sigma_{s,\omega,r,\eta}(t) = \eta(t) (\exp \langle -\omega, t \rangle) (s + t)^{-r-e}, \qquad t \in \Re^n.$$

Obviously, $\sigma_{s,\omega,r,\eta} \in \mathcal{S}(\Re^n)$. If $f \in \mathcal{S}_+'(\Re^n)$ and $\eta_1, \eta_2 \in \mathcal{A}(s)$, then

$$\langle f, \sigma_{s,\omega,r,\eta_1} \rangle = \langle f, \sigma_{s,\omega,r,\eta_2} \rangle. \tag{4}$$

Definition 3. Let $f \in \mathcal{S}_+'(\Re^n)$, $\omega \in \Re_+{}^n$ and $r \in \Re^n$. If for any $s \in (\mathfrak{C} \setminus \overline{\Re}_-)^n$ the limit

$$\hat{f}^r(s) = \lim_{\omega\to 0^+} \langle f(t), \sigma_{s,\omega,r,\eta}(t) \rangle$$

exists, then the function $s \to \hat{f}^r(s)$, $s \in (\mathfrak{C} \setminus \overline{\Re}_-)^n$, is called the S_r-*transform* of f.

Because of (4) $\hat{f}^r(s)$ does not depend on $\eta \in \mathcal{A}(s)$. For every $f \in \mathcal{S}_+'(\Re^n)$ there exists an r for which the S_r-transform is defined; this follows from the fact that there exists an $m \in \Re$ such that $f \in (\mathcal{S}^m)' \cap \mathcal{S}_+'(\Re^n)$ (see [9: p. 91]).

3. Connection between the asymptotic, quasi-asymptotic and S_r-transform

The proofs of the following two propositions are similar to that of the corresponding assertions in [2], so we give these propositions without proofs. In [2] $L(x) \equiv 1$, $k = (k, \ldots, k)$ and instead of $\Re_+{}^n$ a cone is observed.

Proposition 1: *If $F \in \mathcal{L}^1_{\mathrm{loc}}(\Re^n)$ has the asymptotic at ∞ with respect to $k^a L(k)$, $a > -e$, with the limit $g \neq 0$ and has its support in $\overline{\Re}_+{}^n$, then F has the quasi-asymptotic at ∞ with respect to $k^a L(k)$ with the limit g. Moreover, $F(kt)/(k^a L(k))$ converges to $g(t)$ in $(\mathcal{S}^m)'$ for $m > |a| + n$.*

Proposition 2: *If $f \in \mathcal{S}_+'(\mathfrak{R}^n)$ has the quasi-asymptotic at ∞ with respect to $k^a L(k)$ with the limit $g \in \mathcal{S}_+'(\mathfrak{R}^n)$, then there exists a $p \in \mathfrak{N}_0^n$ such that $p + a > 0$ and $(D^{-p}f)(t)$ has the asymptotic at ∞ with respect to $k^{p+a}L(k)$ with the limit Cf_{a+p+e}, $C \neq 0$; in this case $g = Cf_{a+e}$.*

Note that Remark 1 enables us to give in Proposition 2 the explicit form of g. Before we give a connection between the quasi-asymptotic and S_r-transform, we have to prove the following lemma.

Lemma 1: *Suppose that $f \in \mathcal{S}_+'(\mathfrak{R}^n)$, $f = D^pF$, where $p \in \mathfrak{N}_0^n$, $F \in \mathcal{L}^1_{\text{loc}}(\mathfrak{R}^n)$ and* supp $F \subset \overline{\mathfrak{R}}_+^n$. *If for some $r' \in \mathfrak{R}^n$ and $T_0 > 0$*

$$\int\limits_{\|t\| \geqq T_0} \frac{|F(t)|}{t^{r'+p+e}} \, dt < \infty, \tag{5}$$

then there exists $\hat{f}^r$ for $r \geqq r'$ and

$$\hat{f}^r(s) = (r + e)_p \int\limits_{\overline{\mathfrak{R}}_+^n} \frac{F(t)}{(s + t)^{r+p+e}} \, dt, \qquad s \in (\mathfrak{C} \setminus \overline{\mathfrak{R}}_-)^n. \tag{6}$$

Proof: For a fixed $s \in (\mathfrak{C} \setminus \overline{\mathfrak{R}}_-)^n$ we have

$$\hat{f}^r(s) = (-1)^p \lim_{\omega \to 0^+} \langle F(t), D^p \sigma_{s,\omega,t,\eta}(t) \rangle,$$

where the expression $\langle \cdot, \cdot \rangle$ is a sum with members of the form

$$(r + e)_p \int\limits_{\overline{\mathfrak{R}}_+^n} \frac{F(t) \exp \langle -\omega, t \rangle}{(s + t)^{r+p+e}} \, dt \tag{7}$$

and

$$C_k \omega^{p-k} \int\limits_{\overline{\mathfrak{R}}_+^n} \frac{F(t) \exp \langle -\omega, t \rangle}{(s + t)^{r+k+e}} \, dt, \qquad 0 \leqq k \leqq p, \tag{8}$$

where C_k are suitable constants and for at least one $i = i_0$, $k_{i_0} < p_{i_0}$ holds. When $\omega \to 0^+$, the member (7) converges to the integral in (6) because of the property given in (5).

Obviously, for any $\alpha > 0$, $\beta > 0$, $\max \{u^\alpha e^{-\beta u} : u > 0\} = \alpha^\alpha \beta^{-\alpha} e^{-\alpha}$. This implies that for every $\omega > 0$ and $\varepsilon > 0$ there exists a $T_0 > 0$ such that for $\Omega = \{t \in \overline{\mathfrak{R}}_+^n : \|t\| \geqq T_0\}$

$$\left| \omega^{p-k} \int\limits_{\Omega} \frac{F(t) \exp \langle -\omega, t \rangle}{(s + t)^{r+k+e}} \right| dt$$

$$\leqq C \int\limits_{\Omega} \frac{|F(t)|}{|(s + t)^{r+p+e}|} \, dt \leqq C' \int\limits_{\Omega} \frac{|F(t)|}{t^{r'+p+e}} \, dt < \varepsilon,$$

where $p - k \geqq 0$ and for at least one coordinate $p_{i_0} - k_{i_0} > 0$. This shows that all the members of the form (8) tend to zero when $\omega \to 0^+$ ∎

Proposition 3: *Let $f \in \mathcal{S}_+'(\mathfrak{R}^n)$ have the quasi-asymptotic at ∞ with respect to $k^a L(k)$ with the limit g. Then f has the S_r-transform for $r > a$ and there exists a continuous function F, with the support in $\overline{\mathfrak{R}}_+^n$, and $p \in \mathfrak{N}_0^n$, $p + a > 0$, so that $f = D^pF$ and F*

has the asymptotic at ∞ *with respect to* $k^{p+a}L(k)$ *with the limit* Cf_{a+p+e}, $C \neq 0$; *particularly, we have* $g = Cf_{a+e}$ *and*

$$\hat{f}^r(s) = (r+e)_p \int_{\overline{\mathfrak{R}}_+^n} \frac{F(t)}{(s+t)^{r+p+e}}\, dt, \quad s \in (\mathfrak{C} \setminus \overline{\mathfrak{R}}_-)^n.$$

Proof: The assumptions on f and Proposition 2 imply that there exists a $p \in \mathfrak{N}^n$ such that $p + a > 0$ and that $D^{-p}f = F$ is a continuous function with the support in $\overline{\mathfrak{R}}_+^n$. F has also the asymptotic at ∞ with respect to $k^{a+p}L(k)$ with the limit Cf_{a+p+e}. Now, for $r > a$ and for a suitable $T_0 > 0$ we have that $F(t)\, t^{-r-p-e}$, $\|t\| \geqq T_0$, is an integrable function and $f = D^pF$. Now, the proposition follows from Lemma 1 ∎

Remark: J. Lavoine and O. P. Misra [3] defined the Stieltjes transform in one dimension for distributions belonging to a space $\mathcal{J}'(r)$ of distributions T having supports in $[0, \infty)$ and admitting the decomposition $T = B + D^kF$, $k \in \mathfrak{N}_0$, where F is a function having the support in $[a, \infty)$ for some finite number a $(a > 0)$ such that $F(x)\, x^{-r-k-1} \in \mathcal{L}^1(\mathfrak{R})$, and B is a distribution having the support in $[0, a]$. Every such distribution $T \in \mathcal{J}'(r)$ is a tempered one and has by Lemma 1 the S_r-transform. J. Lavoine and O. P. Misra defined its Stieltjes transform in the following way:

$$S_r(T) = \left\langle B(t), \frac{1}{(\cdot + t)^{r+1}} \right\rangle + (r+1)_k \int_a^{\infty} \frac{F(t)}{(\cdot + t)^{r+k+1}}\, dt, \quad \text{on } \mathfrak{C} \setminus \overline{\mathfrak{R}}_-.$$

They remarked that $T \in \mathcal{J}'(r)$ is equivalent to $T = D^mG$, $m \in \mathfrak{N}_0$, if $G(x) = 0$ for $x < 0$ and if the integral

$$\int_0^{\infty} |G(x)|\, (x+b)^{-r-m-1}\, dx, \quad b > 0$$

exists. It is easy to see that $S_r(T) = \hat{T}^r$.

4. Abelian theorem for the S_r-transform

To prove the next theorem we use the following lemma which is a direct consequence of a theorem from [6: pp. 64—65].

Lemma 2: *If* $\beta > 1$, *then*

$$\int_x^{\infty} L(u)\, u^{-\beta}\, du \sim \frac{1}{\beta - 1}\, x^{1-\beta} L(x), \quad x \to \infty, \quad x \in \mathfrak{R}.$$

Theorem 1: *Suppose that* $f \in \mathcal{S}_+'(\mathfrak{R}^n)$ *has the quasi-asymptotic at* ∞ *with respect to* $k^aL(k)$ *with the limit* $g \in \mathcal{S}_+'(\mathfrak{R}^n)$. *Then* $g = Cf_{a+e}$ *and for* $r > a$, $r_i \neq -m$, $m \in \mathfrak{N}$, $i = 1, \ldots, n$, *we have*

$$\lim_{\substack{s \to \infty \\ s \in \Lambda_\omega}} \frac{\hat{f}^r(s)}{s^{-(r-a)}L(\sigma)} = C \prod_{i=1}^{n} \frac{\Gamma(r_i - a_i)}{\Gamma(r_i + 1)}, \quad \sigma = (|s_1|, \ldots, |s_n|),$$

where $\omega \in \mathfrak{C}^n$, $\|\omega_i\| = 1$, $\arg \omega_i \neq \pi$, $i = 1, \ldots, n$, *and* $\Lambda_\omega \equiv \{s = k\omega : k \in \mathfrak{R}_+^n\}$. *If* $n = 1$ *and* $L(x) \equiv 1$, *the convergence is uniform in the closed domain* $\Omega_\varepsilon = \{s \in \mathfrak{C}: -\pi + \varepsilon \leqq \arg s \leqq \pi - \varepsilon\}$, $\varepsilon > 0$.

Proof: We shall split the proof into two parts. We use the same notations as in Section 3.

The case $r_i - 1 \leqq a_i < r_i$ for at least one i: From Proposition 3 it follows that $\hat{f}^r(s) \to 0$ when $s \to \infty$, $s \in \Lambda_\omega$. Let m be the first integer $\geqq |r + p + e|$. Proposition 1 implies

$$\lim_{k\to\infty} \frac{\hat{f}^{r+e}(ks)}{k^{a-r-e}L(k)} = \lim_{k\to\infty} (r + 2e)_p \left\langle \frac{F(t)}{k^{a-r-e}L(k)}, \frac{\eta(t)}{(ks + t)^{r+p+2e}} \right\rangle$$

$$= \lim_{k\to\infty} (r + 2e)_p \left\langle \frac{F(kt)}{k^{a+p}L(k)}, \frac{\eta(kt)}{(s + t)^{r+p+2e}} \right\rangle$$

$$= (r + 2e)_p \left\langle Cf_{a+p+e}(t), \frac{\eta^x(t)}{(s + t)^{r+p+2e}} \right\rangle$$

$$= C \prod_{i=1}^{n} \frac{\Gamma(r_i - a_i + 1)}{\Gamma(r_i + 2)} s^{a-r-e} \big(\eta, \eta^x \in \mathcal{A}(s)\big).$$

So, if $k_i \geqq k_0$, $i = 1, \ldots, n$, then

$$\hat{f}^{r+e}(ks) = \big(1 + \varepsilon(ks)\big) C \prod_{i=1}^{n} \frac{\Gamma(r_i - a_i + 1)}{\Gamma(r_i + 2)} (ks)^{a-r-e} L(k),$$

where $\varepsilon(ks) \to 0$ when $k \to \infty$ and $s \in \Lambda_\omega$. Taking into account that $\hat{f}^r(s) \to 0$ when $s \to \infty$, $s \in \Lambda_\omega$, we have

$$\hat{f}^r(ks) = s^e(r + e)^e \int_{k_1}^{\infty} \ldots \int_{k_n}^{\infty} \hat{f}^{r+e}(us)\, du$$

$$= C \prod_{i=1}^{n} \frac{\Gamma(r_i - a_i + 1)}{\Gamma(r_i + 1)} s^{a-r} \left\{ \prod_{i=1}^{n} \int_{k_i}^{\infty} u_i^{a_i - r_i - 1} L(u_i)\, du_i \right.$$

$$\left. + \int_{k_1}^{\infty} \ldots \int_{k_n}^{\infty} \varepsilon(su)\, u^{a-r-e} L(u)\, du \right\}.$$

In order to prove the statement of the theorem we only have to use the fact that $\varepsilon(us) \to 0$ when $u \to \infty$, $s \in \Lambda_\omega$ and to apply Lemma 2.

The case $a < r - e$: Let m be an integer such that $|r + p + e| > m \geqq |a + p + e|$. The function F from Proposition 3 belongs to $(\mathcal{S}^m)'$ and the family of functions $\{\eta(t)/(s + t)^{r+p+e} : s \in \Lambda_\omega, \eta \in \mathcal{A}(s)\}$ belongs to $\mathcal{S}^m$. Hence, for $s \in \Lambda_\omega$, $\eta \in \mathcal{A}(s)$, we have

$$\lim_{k\to\infty} \frac{\hat{f}^r(ks)}{k^{a-r}L(k)} = (r + e)_p \lim_{k\to\infty} \left\langle \frac{F(t)}{k^{a+p}L(k)}, \frac{\eta(t)}{(s + t)^{r+p+e}} \right\rangle$$

$$= C(r + e)_p \left\langle f_{a+p+e}(t), \frac{\eta(t)}{(s + t)^{r+p+e}} \right\rangle$$

$$= C \prod_{i=1}^{n} \frac{\Gamma(r_i - a_i)}{\Gamma(r_i + 1)} s^{-(r-a)}.$$

Let us remark that in the proof we used the well-known equality

$$\int_0^\infty \frac{t^{a+p}}{(s+t)^{r+p+2}}\,dt = s^{a-r-1}\frac{\Gamma(a+p+1)\,\Gamma(r-a+1)}{\Gamma(r+p+2)},$$

where $a \in \mathfrak{R}$, $p \in \mathfrak{R}$, $s \in \mathfrak{C} \setminus \overline{\mathfrak{R}}_-$, $a + p > -1$, $r + p + 2 > -1$.

At the end, the uniformity of the limit process follows from Montel's theorem: *If $f(z)$ is regular and bounded in the angle between two rays and $f(z) \to a$ as $z \to \infty$ on one ray in the interior of the angle, then $f(z) \to a$ uniformly in any interior angle* ∎

We shall give several examples to illustrate the advantages which we obtained by introducing the quasi-asymptotic.

1. The functions $F_m(t) \equiv H(t-1)\,t^{-m}$, $t \in \mathfrak{R}$, $m = 2, 3, \ldots$, (H is the characteristic function of $\mathfrak{R}_+ = (0, \infty)$) behave as t^{-m} when $t \to \infty$. But they all have the quasi-asymptotic at ∞ with respect to k^{-1} with the limit $\frac{1}{m-1}\,\delta$. Indeed, if $\phi \in \mathcal{S}(\mathfrak{R})$, then

$$\lim_{k\to\infty} \langle kf(kx), \phi(x)\rangle = \lim_{k\to\infty} k^{1-m} \int_{1/k}^\infty \phi(x)\,x^{-m}\,dx$$

$$= \lim_{k\to\infty} \int_1^\infty \phi\left(\frac{x}{k}\right) x^{-m}\,dx = \frac{1}{m-1}\,\phi(0).$$

From the main theorem follows

$$\lim_{s\to\infty} s^{r+1}\hat{f}\,(s) = \frac{1}{m-1}, \quad r > -1, \quad (s \in \mathfrak{C}\setminus\overline{\mathfrak{R}}_-).$$

We obtain the same result for the classical Stieltjes transform. We see that the power of the asymptotic behaviour of $\hat{f}^r$ does not depend on $m = 2, 3, \ldots$ just as the quasi-asymptotic does. In the case $m = 1$, the function $F_1(t) = H(t-1)\,t^{-1}$, $t \in \mathfrak{R}$, has the quasi-asymptotic at ∞ with respect to $k^{-1} \ln k$ with the limit δ, and our theorem implies

$$\lim_{s\to\infty} \frac{\hat{f}^r(s)}{s^{-(r+1)} \ln |s|} = 1, \quad s \in \mathfrak{C}\setminus\overline{\mathfrak{R}}_-.$$

This result can be checked directly by using the classical definition of the Stieltjes transform.

2. Let $T = D^pF \in \mathcal{S}_+'(\mathfrak{R})$ such that $F \in \mathcal{L}^1(\mathfrak{R}_+)$. Then T has the quasi-asymptotic at ∞ with respect to $k^{-(p+1)}$ with the limit

$$(-1)^p \left(\int_0^\infty F(t)\,dt\right) \delta^{(p)}.$$

Namely, for $\phi \in \mathcal{S}'(\mathfrak{R})$ we have

$$\lim_{k\to\infty} k^{p+1}\langle T(kt), \phi(t)\rangle = \lim_{k\to\infty} (-1)^p\, k\langle F(kt), \phi^{(p)}(t)\rangle$$

$$= (-1)^p \lim_{k\to\infty} \int_0^\infty F(t)\,\phi^{(p)}\left(\frac{t}{k}\right) dt = (-1)^p\,\phi^{(p)}(0) \int_0^\infty F(t)\,dt.$$

This example shows that a whole family of distributions has the same power of the quasi-asymptotic and consequently the asymptotic behaviour of their Stieltjes transform has the same power, too. Let us remark that F need not be of power growth [5].

3. The distribution $PF(1/x^m)_+$, $m = 1, 2, \ldots$, has the quasi-asymptotic at ∞ with respect to $k^{-m}L(k)$ with the limit $(-1)^{m-1}\,\delta^{(m)}/(m-1)!$. From [7: T. I, p. 42] follows

$$\frac{d}{dx}\left(PF(1/x^m)_+\right) = PF(-m/x^{m+1})_+ + (-1)^m \frac{\delta^{(m)}}{m!}.$$

The assertion follows from

$$\lim_{k\to\infty} k^{m+1}\langle \delta^{(m)}(kt), \phi(t)\rangle$$

$$= \lim_{k\to\infty} k(-1)^m \langle \delta(kt), \phi^{(m)}(t)\rangle = (-1)^m \phi^{(m)}(0) = \langle \delta^{(m)}, \phi\rangle$$

and

$$\lim_{k\to\infty} \frac{k}{\ln k} \langle PF(1/tk)_+, \phi(t)\rangle = \frac{1}{\ln k} \langle D_t(H(kt) \ln kt), \phi(t)\rangle$$

$$= -\frac{1}{\ln k} \int_0^\infty \ln (kt)\, \phi'(t)\, dt = \phi(0).$$

Now, we can compare the results of our theorem, in the case $n = 1$, with the known results on Abelian theorems at infinity of other authors. All of them started from the space $\mathcal{J}'(r)$ [3], which is a subspace of $\mathcal{S}_+'(\mathfrak{R})$. If $T \in \mathcal{J}'(r)$, as we remarked, it has not only the Stieltjes transform $S_r(T)$ in the sense of the definition of J. LAVOINE and O. P. MISRA [3]; it has also the S_r-transform $\hat{T}^r$ in the sense of our definition and $S_r(T)(s) = \hat{T}^r(s)$, $s \in \mathfrak{C} \setminus \overline{\mathfrak{R}}_-$.

Using the notations of our theorem, we can establish the following differences: In [3] J. LAVOINE and O. P. MISRA proved the case $L(x) \equiv 1$, $a > -1$, s is real number and $r > -1$. In the next paper [4] they supposed $T = B + g$, where B is a distribution having compact support and $g \in \mathcal{L}^1_{\text{loc}}(\overline{\mathfrak{R}}_+)$ such that $g(x) \sim Ax^a \log^j x$ in the usual sense as $x \to \infty$; s is a real number and $-1 < \operatorname{Re} a < \operatorname{Re} r$. R. D. CARMICHAEL and E. O. MILTON [1] proved their theorem for $L(x) \equiv 1$, $a > -1$, $s \in \equiv Q_K \equiv \{s = u + iv : u > 0, |v| \leqq Ku, K \geqq 0\}$ and $r > -1$. A. TAKAČI [8] generalized this result, omitting the supposition $a > -1$. V. MARIĆ, M. SKENDŽIĆ and A. TAKAČI [5] proved the case $\operatorname{Re} a > -1$, s — real and $\operatorname{Re} r > -1$.

In a forthcoming paper we shall give similar results for the case of the quasi-asymptotic at zero and the corresponding Abelian Theorem for the Stieltjes transform.

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