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On the Optimality of Methods for Ill-Posed Problems

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Es wird die Optimalität einer Klasse von Regularisierungsverfahren für lineare inkorrekte Aufgaben untersucht. Die allgemeinen Resultate werden für die Lawrentjewsche und Tichonowsche Methode sowie für eine Klasse von Iterationsverfahren und ihre stetigen Versionen präzisiert.

Изучается оптимальность класса метода регуляризации линейных некорректных задач. Общие результаты иллюстрируются применением к методам Лавреньева и Тихонова, классу итерационных методов и их непрерывным аналогам.

The optimality of a class of regularization methods for linear ill-posed problems is investigated. The general results are applied to Lavrentjew's and Tikhonov's methods and to a class of iteration methods and their continuous versions.

1. Introduction

Consider an ill-posed problem (see [15])

$$Au = f \tag{1.1}$$

where $A \in \mathcal{L}(H, F)$ is a linear bounded operator between Hilbert spaces H and F. Any mapping: $P : F \to H$ can be treated as a method to solve equation (1.1) — the approximate solution is given by Pf, or by Pf_δ if only a polluted value of f is given ($\|f_\delta - f\| \leqq \delta$). For a set $M \subset H$ let us introduce the function

$$\Delta(\delta; M; P; A) = \sup_{\substack{u \in M, f_\delta \in F \\ \|Au - f_\delta\| \leqq \delta}} \|Pf_\delta - u\|, \qquad 0 < \delta \leqq \delta_0,$$

which indicates the maximal error of the method P provided that the exact solution of (1.1) varies in M and the corresponding right-hand term $f \in AM$ has the accuracy δ (see [10]). A method $P_\delta : F \to H$ is called *optimal* on M if

$$\Delta(\delta; M; P_\delta; A) = \inf_P \Delta(\delta; M; P; A) \tag{1.2}$$

where the infimum is taken over all methods $P : F \to H$. It is well known (see [10]) that

$$\inf_P \Delta(\delta; M; P; A) \geqq \frac{1}{2} \Omega(2\delta; M; A) \tag{1.3}$$

where $\Omega(\varepsilon; M; A) = \sup \{\|u_1 - u_2\| : u_1, u_2 \in M \text{ and } \|Au_1 - Au_2\| \leqq \varepsilon\}$.

Below we shall consider methods of special type. Let's take a family of continuous functions $g_r : [0, a] \to \mathbf{R}$ depending on a positive parameter r. In case $H = F$, $A = A^* \geqq 0$, $\|A\| \leqq a$ we define the approximate solution via the formula (see [1, 2, 6–9]

$$u_r = (I - Ag_r(A))\, u_0 + g_r(A)\, f_\delta \tag{1.4}$$

where $u_0 \in H$ is a given initial approximation (e.g. $u_0 = 0$). In case of a non-self-adjoint operator $A \in \mathcal{L}(H, F)$, $\|A\|^2 \leqq a$, we first symmetrize the problem ($A^*Au = A^*f$) and then apply a method similar to (1.4):

$$u_r = \big(I - A^*Ag_r(A^*A)\big)\, u_0 + g_r(A^*A)\, A^*f_\delta. \tag{1.5}$$

Let us introduce the set $M_{p\varrho u_0} \subset H$ of so called *sourcelike* elements,

$$M_{p\varrho u_0} = \{u \in H : u - u_0 = |A|^p v, \|v\| \leqq \varrho\}, \tag{1.6}$$

where $p > 0$ and $\varrho > 0$, $|A| = (A^*A)^{1/2} \in \mathcal{L}(H, H)$. Formula (1.3) takes the form

$$\inf_P \Delta(\delta; M_{p\varrho u_0}; P; A) \geqq \omega(\delta; M_{p\varrho}; A)$$

where $M_{p\varrho} = M_{p\varrho 0}$ and $\omega(\delta; M; A) = \sup\{\|u\| : u \in M, \|Au\| \leqq \delta\}$. It is easy to prove the following assertion (see e.g. [7]):

$$\text{if } (\delta/\varrho)^{1/(p+1)} \in \sigma(|A|), \text{ then } \omega(\delta; M_{p\varrho}; A) = \varrho^{1/(p+1)}\delta^{p/(p+1)}$$

and hence

$$\inf_P \Delta(\delta; M_{p\varrho u_0}; P; A) \geqq \varrho^{1/(p+1)}\delta^{p/(p+1)}. \tag{1.7}$$

It is a typical situation for the ill-posed problems that the range $\mathcal{R}(A)$ of A is non-closed, and then the spectrum $\sigma(|A|)$ of $|A|$ contains at least a sequence of positive numbers λ_k such that $\lambda_k \to 0$ as $k \to \infty$. For the corresponding $\delta = \delta_k = \varrho \lambda_k^{p+1}$, $k = 1, 2, \ldots$, inequality (1.7) holds.

Now it is natural to ask, under which conditions upon the functions $g_r : [0, a] \to \mathbf{R}$ there is a parameter choice $r = r(\delta; M_{p\varrho u_0})$ such that

$$\sup_{\substack{u \in M_{p\varrho u_0}, f_\delta \in F,\\ \|Au - f_\delta\| \leqq \delta}} \|u_r - u\| \leqq \varrho^{1/(p+1)}\delta^{p/(p+1)}. \tag{1.8}$$

(Comparing (1.8) with (1.7) we see that the corresponding method is optimal on $M_{p\varrho u_0}$.) We shall answer this question assuming that

$$g_r(\lambda) = rg(r\lambda), \qquad 0 \leqq \lambda < \infty, \quad r > 0, \tag{1.9}$$

with a given generating function $g : [0, \infty) \to \mathbf{R}$ such that

$$\sup_{0 \leqq \lambda < \infty} \lambda^p |1 - \lambda g(\lambda)| < \infty \qquad (0 \leqq p \leqq p_0, \quad p \in \mathbf{R}). \tag{1.10}$$

Many concrete methods fit in the setting defined by (1.4), (1.9), (1.10) or (1.5), (1.9), (1.10). We shall apply the general results to the **Lavrentiev** and Tikhonov's methods and to continuous versions of iteration methods. Finally we transfer the results to a class of iteration methods. Note that for all those and many other methods, it is easy to propose parameter choices $r = r(\delta; M_{p\varrho u_0})$ for which (see e.g. [7, 8])

$$\sup_{\substack{u \in M_{p\varrho u_0}, f_\delta \in F,\\ \|Au - f_\delta\| \leqq \delta}} \|u_r - u\| \leqq c\varrho^{1/(p+1)}\delta^{p/(p+1)} \tag{1.11}$$

with a constant $c > 1$ (order optimality on $M_{p\varrho u_0}$).

A most practical way for an order optimal choice of r provides the residual (or discrepancy) principle (see [7–9]). For a fixed $f \in \mathcal{R}(A)$ this principle yields a suitable value of r, whereby it is not necessary to know, to which concrete set $M_{p\varrho u_0}$, $p > 0$, $\varrho > 0$, the exact solution of (1.1) belongs. But we fight for $c = 1$ in (1.11), and in order to determine the corresponding value of r, the information about p and ϱ is needed. Note also that it is not needed constructing approximations (1.4) and (1.5); only the choice of the parameter r depends on $M_{p\varrho u_0}$. Another idea is exploited on [4]: the authors construct a special version of Tikhonov's method depending on the set $M \subset H$ on which an optimal method is searched.

2. Formulae for maximal error

Let B be a linear bounded operator from a Hilbert space G into H. Let us introduce the set

$$M_{B\varrho u_0} = \{u \in H : u - u_0 = Bv, \|v\| \leqq \varrho\}, \quad \varrho > 0, \quad u_0 \in H.$$

Lemma 2.1: *If $H = F$, $A = A^* \geqq 0$, $\|A\| \leqq a$ then, for u_r defined in* (1.4),

$$\sup_{\substack{u \in M_{B\varrho u_0}, f_\delta \in F, \\ \|Au - f_\delta\| \leqq \delta}} \|u_r - u\|$$

$$= \inf_{0<t<1} \left\| \frac{\varrho^2}{t} \big(I - Ag_r(A)\big) BB^*\big(I - Ag_r(A)\big) + \frac{\delta^2}{1-t} g_r^2(A) \right\|^{1/2}. \tag{2.1}$$

Proof: For u_r and for $u \in H$

$$u_r - u = \big(I - Ag_r(A)\big)(u_0 - u) + g_r(A)(f_\delta - Au).$$

Hence

$$\sup_{\substack{u \in M_{B\varrho u_0}, f_\delta \in F, \\ \|Au - f_\delta\| \leqq \delta}} \|u_r - u\| = \sup_{\|v\| \geqq \varrho, \|z\| \leqq \delta} \|\big(I - Ag_r(A)\big) Bv + g_r(A) z\|$$

$$= \sup_{\|v\| \leqq 1, \|z\| \leqq 1} \|\varrho\big(I - Ag_r(A)\big) Bv + \delta g_r(A) z\|.$$

According to Melkman and Micchelli [4], for any Hilbert spaces X and X_i and for any operators $C_i \in \mathcal{L}(X, X_i)$, $i = 0, 1, 2$,

$$\sup_{\|C_1 x\| \leqq 1, \|C_2 x\| \leqq 1} \|C_0 x\| = \inf_{0<t<1} \sup_{t\|C_1 x\|^2 + (1-t)\|C_2 x\|^2 \leqq 1} \|C_0 x\|.$$

In our case $X = G \times F$, $X_0 = H$, $X_1 = G$, $X_2 = F (= H)$,

$$C_0 \begin{pmatrix} v \\ z \end{pmatrix} = \varrho\big(I - Ag_r(A)\big) Bv + \delta g_r(A) z, \quad C_1 \begin{pmatrix} v \\ z \end{pmatrix} = v, \quad C_2 \begin{pmatrix} v \\ z \end{pmatrix} = z,$$

$$x = \begin{pmatrix} v \\ z \end{pmatrix} \in X = G \times F,$$

and the Melkman-Micchelli's formula yields

$$\sup_{\substack{u \in M_{B\varrho u_0}, f_\delta \in F, \\ \|Au - f_\delta\| \leqq \delta}} \|u_r - u\|$$

$$= \inf_{0<t<1} \sup_{t\|v\|^2 + (1-t)\|z\|^2 \leqq 1} \|\varrho\big(I - Ag_r(A)\big) Bv + \delta g_r(A) z\|$$

$$= \inf_{0<t<1} \|C_0\|_{\mathcal{L}(G_t \times F_{1-t}, H)} = \inf_{0<t<1} \|C_0 C_0^t\|_{\mathcal{L}(H,H)}^{1/2},$$

where $G_t \times F_{1-t}$ as a set coincides with $G \times F$ but is equipped with the scalar product

$$\left\langle \begin{pmatrix} v_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ z_2 \end{pmatrix} \right\rangle = t\langle v_1, v_2 \rangle_G + (1 - t) \langle z_1, z_2 \rangle_F,$$

and $C_0^t \in \mathcal{L}(H, G_t \times F_{1-t})$ is adjoint to $C_0 \in \mathcal{L}(G_t \times F_{1-t}, H)$,

$$C_0^t u = \begin{pmatrix} t^{-1}\varrho B^*(I - Ag_r(A))\, u \\ (1-t)^{-1}\, \delta g_r(A)\, u \end{pmatrix}, \quad u \in H.$$

This leads to formula (2.1) ∎

Lemma 2.2: *If $A \in \mathcal{L}(H, F)$, $\|A\|^2 \leqq a$, then, for u_r defined in* (1.5),

$$\sup_{\substack{u \in M_{B\varrho u_0}, f_\delta \in F, \\ \|Au - f_\delta\| \leqq \delta}} \|u_r - u\| = \inf_{0<t<1} \left\| \frac{\varrho^2}{t} \big(I - A^*Ag_r(A^*A)\big) BB^*\big(I - A^*Ag_r(A^*A)\big) + \frac{\delta^2}{1-t} g_r^2(A^*A)\, A^*A \right\|^{1/2}. \tag{2.2}$$

The proof is similar to that of Lemma 2.1.

Below we restrict us to the case (1.9) and $B = |A|^p$ (then $M_{B\varrho u_0} = M_{p\varrho u_0}$, see (1.6)). Denoting $h(\lambda) = 1 - \lambda g(\lambda)$, formula (2.1) takes the form

$$\sup_{\substack{u \in M_{p\varrho u_0}, f_\delta \in F, \\ \|Au - f_\delta\| \leqq \delta}} \|u_r - u\| = \inf_{0<t<1} \sup_{\lambda \in \sigma(A)} \left[\frac{\varrho^2}{t} \lambda^{2p}\big(1 - r\lambda g(r\lambda)\big)^2 + \frac{\delta^2}{1-t} r^2 g^2(r\lambda)\right]^{1/2}$$
$$= \inf_{0<t<1} \sup_{\mu \in \sigma(rA)} \left[\frac{\varrho^2}{t} (\mu/r)^{2p}\, h^2(\mu) + \frac{\delta^2}{1-t} r^2 g^2(\mu)\right]^{1/2},$$

and the following result is an immediate consequence from Lemma 2.1.

Lemma 2.3: *Let $H = F$, $A = A^* \geqq 0$ and let* (1.9) *and* (1.10) *hold. Then, for u_r defined in* (1.4), *with*

$$r = d\varrho^{1/(p+1)}\, \delta^{-1/(p+1)}, \quad d > 0, \quad 0 < p \leqq p_0, \tag{2.3}$$

the following formula is true:

$$\sup_{\substack{u \in M_{p\varrho u_0}, f_\delta \in F, \\ \|Au - f_\delta\| \leqq \delta}} \|u_r - u\| = c_{dp\varrho\delta}^{[1]} \varrho^{1/(p+1)} \delta^{p/(p+1)} \tag{2.4}$$

whereby

$$c_{dp\varrho\delta}^{[1]} = \inf_{0<t<1} \sup_{\mu \in \sigma(d\varrho^{1/(p+1)}\delta^{-1/(p+1)}A)} \varphi_p^{[1]}(d, t, \mu), \tag{2.5}$$

$$\varphi_p^{[1]}(d, t, \mu) = \left[\frac{d^{-2p}}{t} \mu^{2p} h^2(\mu) + \frac{d^2}{1-t} g^2(\mu)\right]^{1/2}, \quad h(\mu) = 1 - \mu g(\mu). \tag{2.6}$$

Disregarding the fine structure of the spectrum $\sigma(A)$ we can recommend the choice of d, solving the minimax problem

$$\inf_{d>0} \inf_{0<t<1} \sup_{0 \leqq \mu < \infty} \varphi_p^{[1]}(d, t, \mu) \equiv c_p^{[1]}. \tag{2.7}$$

Instead of equality (2.4) we then obtain the estimate

$$\sup_{\substack{u\in M_{p\varrho u_0}, f_\delta\in F,\\ \|Au-f_\delta\|\leqq\delta}} \|u_r - u\| \leqq c_p^{[1]}\varrho^{1/(p+1)}\delta^{p/(p+1)} \tag{2.8}$$

which is precise for sufficiently small $\delta > 0$ if $\sigma(A) \supset [0, \varepsilon]$, $\varepsilon > 0$. Note that due to (1.10), the function $\varphi_p^{[1]}(d, t, \mu)$ with $p \in (0, p_0]$ is bounded in μ $(0 \leqq \mu < \infty)$.
From Lemma 2.2 we obtain.

Lemma 2.4: *Let $A \in \mathcal{L}(H, F)$ and let (1.9) and (1.10) hold. Then, for u_r defined in (1.5), with*

$$r = d\varrho^{2/(p+1)}\delta^{-2/(p+1)}, \quad d > 0, \quad 0 < p \leqq 2p_0, \tag{2.9}$$

the following formula is true:

$$\sup_{\substack{u\in M_{p\varrho u_0}, f_\delta\in F,\\ \|Au-f_\delta\|\leqq\delta}} \|u_r - u\| = c_{dp\varrho\delta}^{[2]}\varrho^{1/(p+1)}\delta^{p/(p+1)} \tag{2.10}$$

whereby

$$c_{dp\varrho\delta}^{[2]} = \inf_{0<t<1} \sup_{\mu\in\sigma(d\varrho^{2/(p+1)}\delta^{-2/(p+1)}A^*A)} \varphi_p^{[2]}(d, t, \mu), \tag{2.11}$$

$$\varphi_p^{[2]}(d, t, \mu) = \left[\frac{d^{-p}}{t}\, \mu^p h^2(\mu) + \frac{d}{1-t}\, \mu g^2(\mu)\right]^{1/2}, \quad h(\mu) = 1 - \mu g(\mu). \tag{2.12}$$

Disregarding the fine structure of $\sigma(A^*A)$ again, we can recommend the choice of parameter d, solving the minimax problem

$$\inf_{d>0}\ \inf_{0<t<1}\ \sup_{0\leqq\mu<\infty} \varphi_p^{[2]}(d, t, \mu) \equiv c_p^{[2]}. \tag{2.13}$$

Instead of (2.10) we then obtain the estimate

$$\sup_{\substack{u\in M_{p\varrho u_0}, f_\delta\in F,\\ \|Au-f_\delta\|\leqq\delta}} \|u_r - u\| \leqq c_p^{[2]}\varrho^{1/(p+1)}\delta^{p/(p+1)} \tag{2.14}$$

which is precise for small $\delta > 0$ if $\sigma(A^*A) \supset [0, \varepsilon]$, $\varepsilon > 0$. Due to (1.10), the function $\varphi_p^{[2]}(d, t, \mu)$ is bounded in μ $(0 \leqq \mu < \infty)$ if $p \in (0, 2p_0]$.
To solve minimax problems (2.7) and (2.13), it is useful to know stationary points of the functions $\varphi_p^{[1]}$ and $\varphi_p^{[2]}$ introduced in (2.6) and (2.12).

Lemma 2.5: *Let a differentiable function $g: [0, \infty) \to \mathbf{R}$ satisfy (1.10) and let $h(\lambda) = 1 - \lambda g(\lambda)$ decrease, $h'(\lambda) < 0$ for $0 < \lambda < \infty$. Then both of the functions $\varphi_p^{[1]}(d, t, \mu)$ with $0 < p \leqq p_0$ and $\varphi_p^{[2]}(d, t, \mu)$ with $0 < p \leqq 2p_0$ have a unique stationary point in the region $d > 0$, $0 < t < 1$, $\mu > 0$, namely the point, defined by coordinates*

$$d = h^{-1}\left(\frac{1}{p+1}\right), \quad t = \frac{1}{p+1}, \quad \mu = h^{-1}\left(\frac{1}{p+1}\right). \tag{2.15}$$

Proof: The proof is straightforward, equalizing the first derivatives of $\varphi_p^{[1]}(d, t, \mu)$ or $\varphi_p^{[2]}(d, t, \mu)$ to zero and examining the corresponding system of three equations ∎

3. Optimality conditions

Now we are ready to prove the main results of the paper.

Theorem 3.1: *Let $H = F$, $A = A^* \geqq 0$ and let a differentiable function $g : [0, \infty) \to \mathbf{R}$ satisfy* (1.10) *whereby $h(\lambda) = 1 - \lambda g(\lambda)$ decreases, $h'(\lambda) < 0$ for $0 < \lambda < \infty$. If, for a $p \in (0, p_0]$, the inequality*

$$\psi_p^{[1]}(\mu) \equiv (p+1)\left[h^{-1}\left(\frac{1}{p+1}\right)\right]^{-2p} \mu^{2p}h^2(\mu)$$

$$+ (p+1)\, p^{-1}\left[h^{-1}\left(\frac{1}{p+1}\right)\right]^2 g^2(\mu) \leqq 1, \tag{3.1}$$

$0 \leqq \mu < \infty$, holds then, for u_r defined by (1.4), (1.9) *with the parameter choice*

$$r = h^{-1}\left(\frac{1}{p+1}\right) \varrho^{1/(p+1)}\delta^{-1(p+1)} \tag{3.2}$$

the following error estimate is true:

$$\sup_{\substack{u \in M_{p\varrho u_0}, f_\delta \in F, \\ \|Au - f_\delta\| \leqq \delta}} \|u_r - u\| \leqq \varrho^{1/(p+1)}\delta^{p/(p+1)}. \tag{3.3}$$

Conversely, if (3.1) *is violated for a $\mu \in [0, \infty)$, and $\sigma(A) \supset [0, \varepsilon]$, $\varepsilon > 0$, then, for all sufficiently small $\delta > 0$ and all $r > 0$*

$$\sup_{\substack{u \in M_{p\varrho u_0}, f_\delta \in F, \\ \|Au - f_\delta\| \leqq \delta}} \|u_r - u\| > \varrho^{1/(p+1)}\delta^{p/(p+1)}. \tag{3.4}$$

The point $\mu = h^{-1}\big(1/(p+1)\big)$ is a stationary point of $\psi_p^{[1]}$ whereby

$$\psi_p^{[1]}\left(h^{-1}\left(\frac{1}{p+1}\right)\right) = 1.$$

Corollary 3.1 (see Section 1): *Under conditions of Theorem 3.1, if* (3.1) *holds, then method* {(1.4), (1.9), (3.2)} *is optimal on $M_{p\varrho u_0}$ provided that $(\delta/\varrho)^{1/(p+1)} \in \sigma(A)$. If* (3.1) *is violated and $\sigma(A) \supset [0, \varepsilon]$, $\varepsilon > 0$, then method* {(1.4), (1.9)}, *with an arbitrary choice of parameter $r = r(\delta, M_{p\varrho u_0})$ is non-optimal on $M_{p\varrho u_0}$ for all sufficiently small $\delta > 0$.*

Proof of Theorem 3.1: Note first that

$$\psi_p^{[1]}(\mu) = \left[\varphi_p^{[1]}\left(h^{-1}\left(\frac{1}{p+1}\right), \frac{1}{p+1}, \mu\right)\right]^2.$$

with $\varphi_p^{[1]}$ defined in (2.6). It follows from Lemma 2.5 that $\mu = h^{-1}\big(1/(p+1)\big)$ is a stationary point of $\psi_p^{[1]}$. Obviously $\varphi_p^{[1]}\big(d, h(d), d\big) = 1$ for any $d > 0$ and hence this equality holds for point (2.15):

$$\varphi_p^{[1]}\left(h^{-1}\left(\frac{1}{p+1}\right), \frac{1}{p+1}, h^{-1}\left(\frac{1}{p+1}\right)\right) = 1, \qquad \psi_p^{[1]}\left(h^{-1}\left(\frac{1}{p+1}\right)\right) = 1.$$

Fixing $\mu = h^{-1}\big(1/(p+1)\big)$ we obtain from $[\varphi_p^{[1]}(d, t, \mu)]^2$ a strictly convex function in d and t which attains its minimum in its stationary point: for $(d, t) \neq \big(h^{-1}(1/(p+1)), 1/(p+1)\big)$,

$$\varphi_p^{[1]}\left(d, t, h^{-1}\left(\frac{1}{p+1}\right)\right) > \varphi_p^{[1]}\left(h^{-1}\left(\frac{1}{p+1}\right), \frac{1}{p+1}, h^{-1}\left(\frac{1}{p+1}\right)\right) = 1.$$

On the other hand, in view of (3.1)

$$\varphi_p^{[1]}\left(h^{-1}\left(\frac{1}{p+1}\right), \frac{1}{p+1}, \mu\right) = [\psi_p^{[1]}(\mu)]^{1/2} \leqq 1 \qquad (0 \leqq \mu < \infty).$$

Thus the minimax in (2.7) is attainable at the point $d = h^{-1}(1/(p+1))$, $t = 1/(p+1)$, $\mu = h^{-1}(1/(p+1))$ whereby $c_p^{[1]} = 1$. This means that, with parameter choice (3.2), error estimate (2.8) takes form (3.3).

If (3.1) is violated, then (2.7) yields $c_p^{[1]} > 1$. If thereby $\sigma(A) \supset [0, \varepsilon]$, $\varepsilon > 0$, then (see (2.5))

$$\inf_{d>0} c_{dp\varrho\delta}^{[1]} \to c_p^{[1]} \quad \text{as} \quad \delta \to 0,$$

and (3.4) is a consequence of (2.4). This completes the proof of Theorem 3.1 ∎

In a similar way one can prove

Theorem 3.2: *Let $A \in \mathcal{L}(H, F)$ and let a differentiable function $g: [0, \infty) \to \mathbf{R}$ satisfy the same conditions as in Theorem 3.1. If, for a $p \in (0, 2p_0]$ the inequality*

$$\psi_p^{[2]}(\mu) \equiv (p+1)\left[h^{-1}\left(\frac{1}{p+1}\right)\right]^{-p} \mu^p h^2(\mu) + (p+1)\, p^{-1} h^{-1}\left(\frac{1}{p+1}\right) \mu g^2(\mu) \leqq 1, \tag{3.5}$$

$(0 \leqq \mu < \infty)$ holds, then, for u_r defined by (1.5), (1.9), with the parameter choice

$$r = h^{-1}\left(\frac{1}{p+1}\right) \varrho^{2/(p+1)} \delta^{-2/(p+1)} \tag{3.6}$$

error estimate (3.3) holds. Conversely, if (3.5) is violated for a $\mu \in [0, \infty)$, and $\sigma(A^*A) \supset [0, \varepsilon]$, $\varepsilon > 0$, then, for all sufficiently small $\delta > 0$ and all $r > 0$, inequality (3.4) holds.

The point $\mu = h^{-1}(1/(p+1))$ is a stationary point of $\psi_p^{[2]}$ whereby

$$\psi_p^{[2]}\left(h^{-1}\left(\frac{1}{p+1}\right)\right) = 1.$$

Corollary 3.2: *Under conditions of Theorem 3.2, if (3.5) holds, then method {(1.5), (1.9), (3.6)} is optimal on $M_{p\varrho u_0}$ provided that $(\delta/\varrho)^{2/(p+1)} \in \sigma(A^*A)$. If (3.5) is violated and $\sigma(A^*A) \supset [0, \varepsilon]$, $\varepsilon > 0$, then method {(1.5), (1.9)}, with an arbitrary choice of $r = r(\delta, M_{p\varrho u_0})$ is non-optimal on $M_{p\varrho u_0}$ for all sufficiently small $\delta > 0$.*

4. Optimality of Lavrentiev and Tikhonov methods

The Lavrentiev method (see [12])

$$u_\alpha = (\alpha I + A)^{-1} f_\delta \qquad (\alpha = r^{-1}, A = A^* \geqq 0) \tag{4.1}$$

is an example of methods {(1,4), (1.9)} with $u_0 = 0$ and $g(\lambda) = h(\lambda) = (1 + \lambda)^{-1}$ Condition (1.10) is fulfilled with $p_0 = 1$; so is the condition of Theorem 3.1 on $h(\lambda)$. Condition (3.1) takes the form

$$\psi_p^{[1]}(\mu) \equiv (p+1)\,(p^{-2p}\mu^{2p} + p)\,(1+\mu)^{-2} \leqq 1 \qquad (0 \leqq \mu < \infty). \tag{4.2}$$

It is easy to verify that (4.2) is fulfilled for $0 < p \leqq (\sqrt{5} - 1)/2 \approx 0.618$ and is violated for $p > (\sqrt{5} - 1)/2$ (namely, $\psi_p^{[1]}(0) = (p + 1)\, p > 1$). According to Theorem 3.1, the choice of the parameter $\alpha = p^{-1}\varrho^{-1/(p+1)}\delta^{1/(p+1)}$ provides the optimal estimate

$$\sup_{\substack{u \in M_{p\varrho}, f_\delta \in F, \\ \|Au - f_\delta\| \leqq \delta}} \|u_\alpha - u\| \leqq \varrho^{1/(p+1)}\delta^{p/(p+1)} \tag{4.3}$$

for $0 < p \leqq (\sqrt{5} - 1)/2$; on the other hand, for $p > (\sqrt{5} - 1)/2$ there is no parameter choice $\alpha = \alpha(\delta, M_{p\varrho})$ that makes Lavrentiev method optimal on $M_{p\varrho}$. But it is known that, with $\alpha = d^{-1}\varrho^{-1/(p+1)}\delta^{1/(p+1)}$, $d = \text{const} > 0$, Lavrentiev method is order optimal on $M_{p\varrho}$ also for $(\sqrt{5} - 1)/2 < p \leqq 1$; for $p > 1$ there is no parameter choice which could provide even order optimality of the method on $M_{p\varrho}$.

The Tikhonov method (see [14, 15])

$$u_\alpha = (\alpha I + A^*A)^{-1} A^*f_\delta \qquad (\alpha = r^{-1}, A \in \mathcal{L}(H, F)) \tag{4.4}$$

corresponds to the same generating function $g(\lambda) = (1 + \lambda)^{-1}$. Condition (3.5) takes the form

$$\psi_p^{[2]}(\mu) \equiv (p + 1)\,(p^{-p}\mu^p + \mu)\,(1 + \mu)^{-2} \leqq 1 \qquad (0 \leqq \mu < \infty) \tag{4.5}$$

and is fulfilled for $0 < p \leqq 2$. According to Theorem 3.2, Tikhonov method (4.4), with the parameter choice $\alpha = p^{-1}\varrho^{-2/(p+1)}\delta^{2/(p+1)}$ is optimal on $M_{p\varrho}$ ((4.3) holds) for $0 < p \leqq 2$. It is known that, for $p > 2$, there is no parameter choice which provides even order optimality of the method on $M_{p\varrho}$.

5. Optimality of continuous versions of iteration methods

The following two methods can be considered as continuous versions of iteration methods (see [13])

$$u'(t) + Au(t) = f_\delta, \quad u(0) = u_0 \qquad (t = r, A = A^* \geqq 0), \tag{5.1}$$

$$u'(t) + A^*Au(t) = A^*f_\delta, \quad u(0) = u_0 \quad (t = r, A \in \mathcal{L}(H, F)). \tag{5.2}$$

Methods (5.1) and (5.2) belong to the class of methods (1.4), (1.9) and (1.5), (1.9), respectively, whereby $g(\lambda) = \lambda^{-1}(1 - e^{-\lambda})$, $h(\lambda) = e^{-\lambda}$. Condition (1.10) is fulfilled with $p_0 = \infty$. Optimality conditions (3.1) and (3.5) have the form

$$\begin{aligned} \psi_p^{[1]}(\mu) \equiv (p + 1)\,&\{[\ln(1 + p)]^{-2p}\,\mu^{2p}e^{-2\mu} \\ &+ p^{-1}[\ln(1 + p)]^2\,\mu^{-2}(1 - e^{-\mu})^2\} \leqq 1, \\ \psi_p^{[2]}(\mu) \equiv (p + 1)\,&\{[\ln(1 + p)]^{-p}\,\mu^p e^{-2\mu} \\ &+ p^{-1}[\ln(1 + p)]\,\mu^{-1}(1 - e^{-\mu})^2\} \leqq 1 \end{aligned}$$

$(0 \leqq \mu < \infty)$, respectively. An analytical check ot these conditions occured to be complicated and so they were examined numerically. The result is as follows: the first inequality is fulfilled for $0 < p \leqq p_1 \approx 1.043$ (this number appears from the condition $\psi_p^{[1]}(0) = (p + 1)\,p^{-1}[\ln(1 + p)]^2 \leqq 1$); the second inequality is fulfilled for $0 < p \leqq p_2 \approx 7.124$. According to Theorem 3.1, the choice of the parameter $t = [\ln(1 + p)]\,\varrho^{1/(p+1)}\delta^{-1/(p+1)}$ in method (5.1) yields, for $0 < p \leqq p_1$, the optimal estimate

$$\sup_{\substack{u \in M_{p\varrho u_0}, f_\delta \in F, \\ \|Au - f_\delta\| \leqq \delta}} \|u(t) - u\| \leqq \varrho^{1/(p+1)}\delta^{p/(p+1)}; \tag{5.3}$$

there is no parameter choice $t = t(\delta, M_{p\varrho u_0})$ which provides the optimality of method (5.1) on $M_{p\varrho u_0}$ for $p > p_1$. According to Theorem 3.2, the parameter choice $t = [\ln(1+p)]\,\varrho^{2/(p+1)}\delta^{-2/(p+1)}$ in method (5.2) provides, for $0 < p \leqq p_2$, optimal estimate (5.3); there is no parameter choice $t = t(\delta, M_{p\varrho u_0})$ which makes method (5.2) optimal on $M_{p\varrho u_0}$ for $p > p_2$. It is known that method (5.1) with $t = d\varrho^{1/(p+1)}\delta^{-1/(p+1)}$ and method (5.2) with $t = d\varrho^{2/(p+1)}\delta^{-2/(p+1)}$, $d = \text{const} > 0$, are order optimal on $M_{p\varrho u_0}$ for all $p \in (0, \infty)$.

6. Asymptotical optimality of iteration methods

Consider the iteration methods (see [5, 7—9])

$$u_n = u_{n-1} - B(Au_{n-1} - f_\delta), \qquad \big(n = 1, 2, \ldots; B = b(A), A = A^* \geqq 0\big) \tag{6.1}$$

and

$$u_n = u_{n-1} - C(Au_{n-1} - f_\delta), \quad \big(n = 1, 2, \ldots; C = b(A^*A)\,A^*, A \in \mathcal{L}(H, F)\big), \tag{6.2}$$

where $b : [0, a] \to \mathbf{R}$ with $a \geqq \|A\|$ (case (6.1)) or $a \geqq \|A\|^2$ (case (6.2)) is a continuous function such that

$$0 < b(\lambda) < \frac{2}{\lambda} \quad \text{for} \quad 0 \leqq \lambda \leqq a. \tag{6.3}$$

Most usual iterations, introduced in [3, 11—13], correspond to the functions $b(\lambda) \equiv \beta \in (0, 2/a)$ and $b(\lambda) = (\beta + \lambda)^{-1}$, $\beta = \text{const} > 0$. Iteration methods (6.1) and (6.2) present a special case of methods (1.4) and (1.5) in which the parameter $r = n$ takes only integer values and the function $g_r = g_n$ is defined by

$$g_n(\lambda) = \sum_{j=0}^{n} \big(1 - \lambda b(\lambda)\big)^j\, b(\lambda) = \frac{1}{\lambda}\big[1 - \big(1 - \lambda b(\lambda)\big)^n\big]. \tag{6.4}$$

As far as $r = n$ cannot take all positive real values, it is impossible to establish the optimality of iteration methods on a set M. But it is possible to indicate a choice $n = n(\delta, p, \varrho)$ so that iteration method (6.1) or (6.2) occurs to be asymptotically optimal on $M_{p\varrho u_0}$ in the following sense (compare with (1.8)):

$$\overline{\lim_{\delta \to 0}} \frac{\sup\limits_{u \in M_{p\varrho u_0}, f_\delta \in F, \|Au - f_\delta\| \leqq \delta} \|u_{n(\delta, p, \varrho)} - u\|}{\varrho^{1/(p+1)}\delta^{p/(p+1)}} \leqq 1. \tag{6.5}$$

More precisely, such a choice of $n = n(\delta, p, \varrho)$ is possible only for those $p > 0$ for which the corresponding continuous version of iterations ((5.1) or (5.2)) is optimal on $M_{p\varrho u_0}$:

Theorem 6.1: *Let $H = F$, $A = A^* \geqq 0$, $\|A\| \leqq a$ and let a continuous function $b : [0, a] \to \mathbf{R}$ satisfy inequalities* (6.3). *Then, for $0 < p \leqq p_1 \approx 1.043$ the choice*

$$n = n(\delta, p, \varrho) = \operatorname{int}\left\{\frac{\ln(1+p)}{b(0)}\,\varrho^{1/(p+1)}\delta^{-1/(p+1)}\right\} \tag{6.6}$$

in iteration method (6.1) *provides asymptotic error estimate* (6.5).

Theorem 6.2: *Let $A \in \mathcal{L}(H, F)$, $\|A\|^2 \leqq a$ and let a continuous function $b: [0, a] \to \mathbf{R}$ satisfy inequalities* (6.3). *Then, for $0 < p \leqq p_2 \approx 7.124$ the choice*

$$n = n(\delta, p, \varrho) = \operatorname{int}\left\{\frac{\ln(1 + p)}{g(0)}\, \varrho^{2/(p+1)}\delta^{-2/(p+1)}\right\}. \tag{6.7}$$

in iteration method (6.2) *provides asymptotic error estimate* (6.5). (Here int λ denotes the integer part of a real number λ.)

The proof of Theorem 2 is alike that of Theorem 1. For the latter one we establish first

Lemma 6.1: *The following statements are consequences from* (6.3), (6.4) *and the continuity of b:*

a) $g_n(\lambda) > 0, \max\limits_{\tau n^{-1} \leqq \lambda \leqq a} g_n(\lambda) \leqq \frac{1}{\tau}\, n$ *for* $\tau > 0$, $n \geqq 1$;

b) *for any $\varepsilon > 0$ and $p > 0$ there is a $\tau_0 = \tau_0(\varepsilon, p) > 0$ and a $n_0 = n_0(\varepsilon, p)$ such that*

$$\max_{\tau n^{-1} \leqq \lambda \leqq a} \lambda^p\, |1 - \lambda g_n(\lambda)| \leqq \varepsilon n^{-p} \quad \text{for} \quad \tau \geqq \tau_0,\ n \geqq n_0.$$

Proof: Statement a) is obvious. Let us prove b). Choose an $\alpha \in (0, a)$ such that $0 \leqq 1 - \lambda b(\lambda) \leqq 1 - \beta\lambda$ for $0 \leqq \lambda \leqq \alpha$ with a positive constant β. It is clear that $\alpha \leqq \beta^{-1}$ and

$$\theta \equiv \max_{\alpha \leqq \lambda \leqq a} |1 - \lambda b(\lambda)| < 1.$$

The function $\lambda^p(1 - \beta\lambda)^n$ is non-negative and decreasing in λ on $[p/\beta(n + p), 1/\beta]$. Assuming that $\tau > p\beta^{-1}$ (then $\tau/n > p/\beta(n + p)$) and $\tau/n \leqq \alpha$ (that means taking sufficiently great n) we have

$$\begin{aligned}
\max_{\tau n^{-1} \leqq \lambda \leqq a} \lambda^p\, |1 - \lambda g_n(\lambda)| &= \max_{\tau n^{-1} \leqq \lambda \leqq a} \lambda^p\, |1 - \lambda b(\lambda)|^n \\
&\leqq \max\left\{\max_{\tau n^{-1} \leqq \lambda \leqq \alpha} \lambda^p(1 - \beta\lambda)^n,\ \max_{\alpha \leqq \lambda \leqq a} \lambda^p\, |1 - \lambda b(\lambda)|^n\right\} \\
&\leqq \max\left\{\left(\frac{\tau}{n}\right)^p \left(1 - \frac{\beta\tau}{n}\right)^n,\ a^p\theta^n\right\} \\
&\leqq \max\,\{\tau^p e^{-\beta\tau},\ a^p\theta^n n^p\}\, n^{-p}.
\end{aligned}$$

The result is not greater than εn^{-p} if $\tau^p e^{-\beta\tau} \leqq \varepsilon$, $(an)^p\, \theta^n \leqq \varepsilon$ which hold for sufficiently great $\tau \geqq \tau_0$ and $n \geqq n_0$ ∎

Proof of Theorem 6.1: According to Lemma 2.1, for any $n \geqq 1$ we have (putting $t = 1/(p + 1)$)

$$\begin{aligned}
&\sup_{\substack{u \in M_{p\varrho u_0}, f_\delta \in F, \\ \|Au - f_\delta\| \leqq \delta}} \|u_n - u\| \\
&= \inf_{0 < t < 1} \max_{\lambda \in \sigma(A)} \left(\frac{\varrho^2}{t}\, \lambda^{2p}\big(1 - \lambda g_n(\lambda)\big)^2 + \frac{\delta^2}{1 - t}\, g_n{}^2(\lambda)\right)^{1/2} \\
&\leqq \max_{0 \leqq \lambda \leqq a} \left[(p + 1)\, \varrho^2 \lambda^{2p}\big(1 - \lambda g_n(\lambda)\big)^2 + \frac{p + 1}{p}\, \delta^2 g_n{}^2(\lambda)\right]^{1/2} \\
&= \max\,\{s_n(\delta, p, \varrho),\ \sigma_n(\delta, p, \varrho)\}
\end{aligned}$$

where

$$s_n(\delta, p, \varrho) = \max_{0 \geqq \lambda \leqq \tau n^{-1}} [\ldots]^{1/2} \quad \text{and} \quad \sigma_n(\delta, p, \varrho) = \max_{\tau n^{-1} \leqq \lambda \leqq a} [\ldots]^{1/2}.$$

Due to Lemma 6.1,

$$\sigma_n(\delta, p, \varrho) \leqq \left[(p + 1)\, \varrho^2 \varepsilon^2 n^{-2p} + \frac{p+1}{p}\, \delta^2 (n/\tau)^2\right]^{1/2}$$

with an arbitrary $\varepsilon > 0$ assuming that $\tau \geqq \tau_0(\varepsilon, p)$, $n \geqq n_0(\varepsilon, p)$. Taking a sufficiently small $\varepsilon > 0$ and choosing $n = n(\delta, p, \varrho)$ according to (6.6) we obtain

$$\sigma_{n(\delta,p,\varrho)}(\delta, p, \varrho) \leqq \varrho^{1/(p+1)} \delta^{p/(p+1)}.$$

It remains to show that

$$\lim_{\delta \to 0} \frac{s_{n(\delta,p,\varrho)}(\delta, p, \varrho)}{\varrho^{1/(p+1)} \delta^{p/(p+1)}} \leqq 1.$$

By means of substitution $\lambda = \mu/n$ and (6.4) rewrite

$$s_n(\delta, p, \varrho) = \max_{0 \leqq \mu \leqq \tau} \left[(p + 1)\, \varrho^2 n^{-2p} \mu^{2p} \left(1 - \frac{\mu}{n}\, b\left(\frac{\mu}{n}\right)\right)^{2n} + \frac{p+1}{p}\, \delta^2 n^2 \mu^{-2} \left(1 - \left(1 - \frac{\mu}{n}\, b\left(\frac{\mu}{n}\right)\right)^n\right)^2\right]^{1/2}.$$

Note that

$$\max_{0 \leqq \mu \leqq \tau} \left|\left(1 - \frac{\mu}{n}\, g\left(\frac{\mu}{n}\right)\right)^n - e^{-g(0)\mu}\right| \to 0 \quad \text{as} \quad n \to \infty.$$

Choosing $n = n(\delta, p, \varrho)$ according to (6.6) we obtain

$$\lim_{\delta \to 0} \frac{s_{n(\delta,p,\varrho)}(\delta, p, \varrho)}{\varrho^{1/(p+1)} \delta^{p/(p+1)}} = \max_{0 \leqq \mu \leqq \tau} \left\{(p + 1)\, [\ln (1 + p)]^{-2p}\, [g(0)\, \mu]^{2p}\, e^{-2g(0)\mu} + \frac{p+1}{p} [\ln (1 + p)]^2\, [g(0)\, \mu]^{-2}\, (1 - e^{-g(0)\mu})^2\right\}^{1/2}$$

$$= \max_{0 \leqq \mu \leqq \tau} \left[\psi_p^{[1]}(g(0)\, \mu)\right]^{1/2}$$

with $\psi_p^{[1]}$ defined in Section 5. But $\psi_p^{[1]}(\mu) \leqq 1$ for $0 \leqq \mu < \infty$ in so far as $0 < p \leqq p_1 \approx 1.043$ (see Section 5). This completes the proof ∎

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