Zeitschrift für Analysis
und ihre Anwendungen
Bd. 6 (5) 1987, S. 449–458

Studies on Transonic Flow Problems by Nonlinear Variational Inequalities

H.-P. Gittel

Es wird das transsonische Strömungsproblem als Variationsungleichung in einer konvexen Menge behandelt, die durch eine geeignete Entropiebedingung, eine Schranke für die Gasgeschwindigkeit und durch Randbedingungen für das Geschwindigkeitspotential gegeben ist. Diese Variationsgleichung wird unter Verwendung der Katchanov-Methode und der Kompaktheit der konvexen Menge gelöst. Weiterhin wird ein Resultat über Strömungen im Unterschallbereich angegeben.

Рассматривается задача околозвукового течения как вариационное неравенство в некотором выпуклом множестве, которое даётся подходящим условием энтропии, границей для газовой скорости и граничными условиями для потенциала скоростей. Это вариационное неравенство решается с помощью метода Качанова и компактности введенного выпуклого множества. Приводится также один результат о течениях в дозвуковой области.

The transonic flow problem is handled as a variational inequality in a convex set which is given by a suitable entropy condition, by a bound for the gas velocity and by boundary conditions for the velocity potential. Using Katchanov's method and the compactness of the convex set this variational inequality is solved. Furthermore, a result on flows in the subsonic region is given.

1. Introduction

We consider an *irrotational, steady and isentropic flow of a non-viscous, compressible fluid* in a bounded, simply connected domain $\Omega \subset \mathbf{R}^N$ ($N \geqq 2$). This flow can be described by the equation for the *velocity potential* u ($\mathfrak{v} = \nabla u$ — *gas velocity*) in gas dynamics:

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\varrho(|\nabla u|^2) \frac{\partial u}{\partial x_i} \right) = 0. \tag{1.1}$$

Here *pressure* p and *density* ϱ are given by the relations $p = p(\varrho)$ and $\varrho = \varrho(|\nabla u|^2)$, respectively, and Bernoulli's law is used [5: 10–11, 489]. For a *polytropic gas* we have $p/p_0 = (\varrho/\varrho_0)^\varkappa$, and hence

$$\varrho = \varrho(|\nabla u|^2) = \varrho_0 \left(1 - \frac{|\nabla u|^2}{q_{\mathrm{m}}}\right)^{\frac{1}{\varkappa-1}} \tag{1.2}$$

for

$$|\nabla u|^2 < q_{\mathrm{m}} := \frac{2\varkappa}{\varkappa - 1} \frac{p_0}{\varrho_0}. \tag{1.3}$$

with constants $\varrho_0, p_0 > 0$, $\varkappa > 1$. It is well known that (1.1) is

$$\text{elliptic for} \quad |\nabla u|^2 < q_c := \frac{\varkappa - 1}{\varkappa + 1} q_m \quad (\textit{subsonic region}),$$

$$\text{hyperbolic for} \quad |\nabla u|^2 > q_c \quad (\textit{supersonic region}).$$

If we consider a *transonic flow*, then we have to take into account that there exist supersonic regions as well as subsonic ones in Ω and the transitions between them are usually discontinuous. There shocks with jumps in $\mathfrak{v}$, ϱ, p occur where a *entropy condition* must be satisfied. In this paper we confine ourselves to the case that this condition can be formulated in the form

$$-\int_\Omega \nabla u \, \nabla h \, dx \leq \int_\Omega M(x, u(x)) \, h \, dx \quad \text{for all} \quad h \in \left(C_0^\infty(\Omega)\right)_+, \tag{1.4}$$

with

$$\left(C_0^\infty(\Omega)\right)_+ = \{h \in C^\infty(\Omega) \mid \operatorname{supp} h \Subset \Omega, h \geqq 0\}.$$

This is a mathematical generalization of the fact that the entropy condition of the physical model in the one-dimensional case implies $d^2u/dx^2 < +\infty$ [5: 380–385; 3: 213–214]. The condition (1.4) with $M = M(x)$ was used by GLOWINSKI [3] in numerical studies. FEISTAUER and NEČAS denote it for M = const as "simplified form" [1]. A "natural form" of the entropy condition is given in [1] by

$$\int_\Omega \varrho'(|\nabla u|^2) \, |\nabla u|^2 \, \nabla u \, \nabla h \, dx \leq M \int_\Omega h \, dx \quad \text{for all} \quad h \in \left(C_0^\infty(\Omega)\right)_+. \tag{1.5}$$

First of all, FEISTAUER and NEČAS proved existence and uniqueness results for weak solutions of (1.1) by *Katchanov's method* [1]. This method known also as *secant modulus method* especially in elasticity [9] consists in the construction of a suitable functional whose minimization is equivalent to the solution of the boundary value problem considered. In the case of a transonic flow this functional is non-convex, without any compactness properties. Therefore in [1] it was necessary to assume a posteriori conditions for a minimizing sequence to ensure its convergence to a solution. These conditions were a convenient entropy condition ((1.4) with M = const or (1.5)) and some regularity assumptions (for example (1.3)).

In this paper we use the same ideas like in [1] but we minimize the functional over convex sets whose elements satisfy a priori (1.4) and (1.3). It is clear so we get solutions of variational inequalities which are generalizations of (1.1). We are using again Katchanov's method represented for variational inequalities in [10]. The entropy condition (1.5) will not be considered in this paper because its left-hand side in non-convex in u.[1]) It is easy to see that the results of [1] remain valid if instead of the entropy conditions (1.4), (1.5) with M = const those ones with functions $M(x, u)$ are used.

2. Formulation of the problem as a variational inequality

We assume that the boundary $\partial\Omega$ of the (sufficiently large) domain Ω is Lipschitz-continuous and has the representation $\partial\Omega = S_1 \cup S_2 \cup S \cup \mathfrak{N}$ where S_1, S_2 and S are open subsets of $\partial\Omega$ and $\mu_{N-1}(\mathfrak{N}) = 0$, μ_{N-1} the $(N-1)$-dimensional Lebesgue

[1]) Using the entropy conditions (1.5) or (1.4) with M = const the transonic flow problem is handled as a minimum problem by similar methods in [11].

measure on $\partial\Omega$. Then typical *boundary conditions* for the potential u are the following ones:

Case 1: $u = 0$ on S_1,

$$\varrho(|\nabla u|^2)\,\frac{\partial u}{\partial n} = g \quad \text{on} \quad S \cup S_2.$$

Example: If $g = 0$ on S, $g < 0$ on S_2 and Ω is like in Figure 1 then we get the situation corresponding to a channel flow with the inlet S_2 and the outlet S_1.

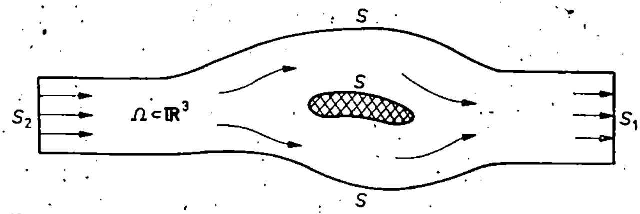

Fig. 1

Case 2: $\mu_{N-1}(S_1) = 0$,

$$\varrho(|\nabla u|^2)\,\frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega, \quad \text{where} \quad \int_{\partial\Omega} g\,do = 0 \quad \text{is assumed.}$$

As *weak formulations* of these boundary value problems we have: There is to find an $u \in V$ such that

$$\int_\Omega \varrho(|\nabla u|^2)\,\nabla u\,\nabla v\,dx = \int_R gv\,do \quad \text{for all} \quad v \in V, \tag{2.1}$$

with

$$V = \{v \in W_1^2(\Omega) \mid v = 0 \quad \text{on } S_1 \text{ in trace sense}\}, \qquad R = S \cup S_2 \tag{2.2}$$

in Case 1 and

$$V = \left\{v \in W_1^2(\Omega) \mid \int_\Omega v\,dx = 0\right\}, \qquad R = \partial\Omega \tag{2.3}$$

in Case 2, respectively.

Remark 2.1: The conditions for v in the above definitions of V can be considered normalizations because the velocity potential is determined only up to a constant. In both cases V is a Hilbert space with norm $\|v\| = \left(\int_\Omega |\nabla v|^2\,dx\right)^{1/2}$ and $l_g(v) = \int_R gv\,do$ is a bounded linear functional on V (we write $l_g \in V^*$) if $g \in L^2(R)$ is assumed.

Equation (2.1) is the Euler-Lagrange-equation for the *variational problem*

$$F(v) := \frac{1}{2}\int_\Omega \left(\int_0^{|\nabla v|^2} \varrho(q)\,dq\right) dx - \int_R gv\,do \to \operatorname*{Min}_{v \in V}.$$

Now, we minimize $F(v)$ over all $v \in V$, which additionally satisfy the constraints (1.3) and (1.4), that means:

$$F(v) \to \operatorname*{Min}_{v \in K} \tag{2.4}$$

with

$$K = K_{a,M} = \{v \in V \mid v \text{ satisfies (1.4)}, |\nabla v| \leqq a \text{ a.e. on } \Omega\}. \quad (2.5)$$

K is a non-empty closed convex subset of V if the following assumptions for the function $M : \Omega \times \mathbf{R} \to \mathbf{R}$ are fulfilled:

i) $M_x(u) := M(x, u)$ is concave in u for a.a. $x \in \Omega$.
ii) $M_x(u)$ is continuous on $\mathbf{R}$ for a.a. $x \in \Omega$,
$M_u(x) := M(x, u)$ is measurable on Ω for all $u \in \mathbf{R}$
(*Carathéodory property*).
iii) There are numbers $B > 0$, $\bar{p} \geqq 1$ and a function $A \in L^2(\Omega)$ such that

$$|M(x, u)| \leqq A(x) + B \cdot |u|^{\bar{p}/2} \quad \text{for a.a. } x \in \Omega \text{ and all } u \in \mathbf{R}$$

(*growth condition*).
iv) $M(x, 0) \geqq 0$ a.e. on Ω.

Remark 2.2: K is a bounded subset of $V \cap W_1^\infty(\Omega) =: V^\infty$ and every $v \in K$ is a.e. equal to a function from $C^{0,1}(\bar{\Omega})$ [8: 25—29]. From ii), iii), the continuity of the *Němyckii operator*

$$\mathcal{N} : L^{\bar{p}}(\Omega) \to L^2(\Omega),$$
$$\mathcal{N}(v)(x) = M(x, v(x))$$

follows [2: 53—54]. Condition iv) just ensures $K \neq \emptyset$ ($0 \in K$).

Remark 2.3: To exclude solutions without physical sense the relation $a^2 < q_m$ is necessary. We will choose $a^2 > q_c$ to admit indeed transonic flows. (The case $a^2 < q_c$ will be briefly handled at the end of Chapter 4.) Furthermore we must take into account that the physical model of the irrotational, isentropic flow is only valid for stream fields with the Mach number $m = |\mathfrak{v}|/c < 1.6$ (c-local sound speed, $c^2 = \partial p/\partial \varrho$). Here the so-called weak shocks occur only so that the changes in entropy and vorticity are negligible [1; 5: 377—380, 488—489].

If $u \in K$ is a minimum point of the variational problem (2.4) then u satisfies for all $v \in K$ the *variational inequality*

$$DF(u, u - v) = \int_\Omega \varrho(|\nabla u|^2) \nabla u \nabla(u - v)\, dx - \int_R g(u - v)\, do \leqq 0. \quad (2.6)$$

It is easy to see that a solution $u \in K$ of it also satisfying equation (2.1) if

$$|\nabla u| \leqq a' < a \quad \text{a.e. on } \Omega \quad (2.7)$$

and

$$-\int_\Omega \nabla u \nabla h\, dx \leqq \int_\Omega M'(x)\, h(x)\, dx \quad \text{for all } h \in (C_0^\infty(\Omega))_+ \quad (2.8)$$

with $M'(x) \leqq M(x, u(x)) - \varepsilon(A(x) + 1)$ a.e. on Ω, $\varepsilon > 0$.

3. Katchanov's method for variational inequalities

Now we are going to represent this method in the abstract formulation due to [10]. Let V be a Hilbert space with norm $\|\cdot\|$ and $F_1 : V \to \mathbf{R}$ a functional with the Gâteaux differential $DF_1(u, \cdot)$ at every $u \in V$. For each $u \in V$ let us consider the bilinear form $B(u; v, w)$, symmetric in $v, w \in V$, with the following properties ($u, v, w \in V$;

c_1, c_2 constants):

$$|B(u; v, w)| \leqq c_1 \|v\| \|w\|, \tag{3.1}$$

$$B(u; v, v) \geqq c_2 \|v\|^2 \qquad (c_2 > 0), \tag{3.2}$$

$$DF_1(u, v) = B(u; u, v), \tag{3.3}$$

$$\frac{1}{2}\left(B(u; v, v) - B(u, u; u)\right) \geqq F_1(v) - F_1(u). \tag{3.4}$$

Moreover, let K be a closed convex subset of V and $l \in V^*$. For $u \in K$ we denote by $w = w(u) \in K$ the unique solution [4: 24] of the linear variational inequality

$$B(u; w, w - v) \leqq l(w - v) \quad \text{for all } v \in K. \tag{3.5}$$

Theorem 3.1: *For the functional* ψ,

$$\psi(u) = F_1(u) - F_1(w(u)) - l(u - w(u)), \tag{3.6}$$

we have:

a) $\psi(u) \geqq \frac{c_2}{2} \|u - w(u)\|^2$ *for all* $u \in K$,

b) $\inf \{\psi(u) \mid u \in K\} = 0$, *and* $\{u_n\}$ *defined by* $u_{n+1} = w(u_n) \in K$, $u_0 \in K$ *arbitrary, is a minimizing sequence.*

c) *Each minimizing sequence* $\{u_n\} \subset K$ *for* ψ *is bounded and* $\|u_n - w(u_n)\| \to 0$.

d) *If for* $v, w \in K$ *the functional* $B_{v,w}(\cdot) := B(\cdot; v, w)$ *is equicontinuous (that means: independent of* v, w*) on* K *then every limit point of a minimizing sequence from* K *for* ψ *is a solution of the problem to find an* $u \in K$ *such that*

$$B(u; u, u - v) \leqq l(u - v) \quad \textit{for all } v \in K. \tag{3.7}$$

Proof: a) According to (3.2) we have

$$c_2 \|u - w\|^2 \leqq B(u; u - w, u - w) = B(u; u, u) + B(u; w, w - 2u). \tag{3.8}$$

For $v \in V$ we introduce the notations

$$F(v) = F_1(v) - l(v) \tag{3.9}$$

and

$$\Pi_u(v) = F_1(u) + \frac{1}{2} B(u; v, v) - \frac{1}{2} B(u; u, u) - l(v).$$

Then (3.4) yields $\Pi_u(v) \geqq F(v)$. On the other hand we have

$$\begin{aligned} \psi(u) &= F(u) - F(w(u)) \\ &\geqq F(u) - \Pi(w(u)) = l(w - u) - \frac{1}{2} B(u; w, w) + \frac{1}{2} B(u; u, u). \end{aligned} \tag{3.10}$$

Hence, using (3.5) we obtain

$$\psi(u) \geqq B(u; w, w - u) - \frac{1}{2} B(u; w, w) + \frac{1}{2} B(u; u, u),$$

which together with (3.8) gives the assertion.

b) In the first step we prove that F is coercive and bounded from below. From (3.4) with $v = 0$ and (3.2) it follows that

$$F(u) \geqq F_1(0) + \frac{1}{2} B(u; u, u) - l(u) \geqq F_1(0) + \frac{c_2}{2} \|u\|^2 - \|l\|_{V^*} \|u\|.$$

The inequality $2 \|l\|_{V^*} \|u\| \leqq \varepsilon^{-1} \|l\|_{V^*}^2 + \varepsilon \|u\|^2$, for all $\varepsilon > 0$, yields

$$F(u) \geqq F_1(0) + \frac{c_2}{4} \|u\|^2 - \frac{1}{c_2} \|l\|_{V^*}^2 \tag{3.11}$$

if we put $\varepsilon = c_2/2$. Let us consider the sequence $\{u_n\} \subset K$ defined above. From (3.10) and a) it follows that $\{F(u_n)\}$ is monotone decreasing, and hence, this sequence is convergent and $\psi(u_n) \to 0$.

c) There is a constant $\gamma \geqq 0$ such that

$$\|w(u)\| \leqq \gamma \quad \text{for all } u \in K. \tag{3.12}$$

This follows from (3.2), (3.5) and (3.1). Indeed, if we fix $v_0 \in K$ then

$$\begin{aligned} c_2 \|w(u)\|^2 &\leqq B(u; w, w) = B(u; w, w - v_0) + B(u; w, v_0) \\ &\leqq l(w - v_0) + c_1 \|w\| \|v_0\| \leqq \|l\|_{V^*} (\|w\| + \|v_0\|) + c_1 \|w\| \|v_0\| \\ &\leqq \|l\|_{V^*} \|v_0\| + \frac{1}{2\varepsilon} (c_1 \|v_0\| + \|l\|_{V^*})^2 + \frac{\varepsilon}{2} \|w\|^2 \end{aligned}$$

for all $\varepsilon > 0$. The last inequality with a suitable ε implies (3.12). If $\psi(u_n) \to \inf \{\psi(u) \mid u \in K\} = 0$, then a) yields $\|u_n - w(u_n)\| \to 0$. Hence, using (3.12) we have $\|u_n\| \leqq \|u_n - w(u_n)\| + \|w(u_n)\| = 1 + \gamma$ for sufficient large n.

d) Let us consider a sequence $\{u_n\} \subset K$ with $\psi(u_n) \to 0$ and $u_n \to u$ in V which implies $u \in K$. From a) it follows immediately

$$w_n := w(u_n) \to u. \tag{3.13}$$

To pass to the limit $n \to \infty$ in (3.5) with $u = u_n$ we observe

$$\begin{aligned} &|B(u_n; w_n, w_n - v) - B(u; u, u - v)| \\ &\leqq |B(u_n; w_n, w_n - v) - B(u; w_n, w_n - v)| + |B(u; w_n - u, w_n - v)| \\ &+ |B(u; u, w_n - u)|. \end{aligned}$$

The first term on the right hand side of this inequality is less than ε for $n \geqq n_0(\varepsilon)$ because we have

$$|B(u_n; \varphi, \chi) - B(u; \varphi, \chi)| < \varepsilon \quad \text{for all } \varphi, \chi \in K,\ n \geqq n_0(\varepsilon)$$

in view of the assumption layed on B. From (3.1) and (3.13) it follows that the two other terms on the right hand side are converging to 0, too. Using $l(w_n - v) \to l(u - v)$ we obtain finally (3.7) ∎

Corollary: *If additionally the condition*

$$DF_1(v + h, h) - DF_1(v, h) \geqq c_3 \|h\|^2 \quad \text{for all } v, v + h \in K \tag{3.14}$$

holds with a constant $c_3 > 0$ (independent of v, h) then the functional F defined by (3.9) *has a unique minimum over K. The sequence $\{u_n\}$ from part* b) *is converging to the minimum point $u \in K$.*

Proof: From (3.11), (3.14) it follows that F is coercive, bounded from below, weakly lower semi-continuous and strictly convex on K. Hence, the existence and uniqueness of a minimum point $u \in K$ for F is easy to verify [2: 16—19]. Then u is also the solution of (3.7). Using (3.14), (3.3) and (3.7) we obtain

$$\begin{aligned} c_3 \|u_n - u\|^2 &\leqq DF_1(u_n, u_n - u) - DF_1(u, u_n - u) \\ &= B(u_n; u_n, u_n - u) - B(u; u, u_n - u) \\ &\leqq B(u_n; u_n, u_n - u) + l(u - u_n). \end{aligned}$$

Moreover, we have

$$\begin{aligned} &B(u_n; u_n, u_n - u) + l(u - u_n) \\ &= B(u_n; u_n - u_{n+1}, u_n - u + u_{n+1}) + B(u_n; u_{n+1}, u_{n+1} - u) \\ &\quad + l(u_{n+1} - u_n) + l(u - u_{n+1}) \\ &\leqq B(u_n; u_n - u_{n+1}, u_n - u + u_{n+1}) + l(u_{n+1} - u_n) \end{aligned}$$

by virtue of (3.5). From this and (3.1), (3.12) it follows that

$$\begin{aligned} c_3 \|u_n - u\|^2 &\leqq (c_1 \|u_n - u + u_{n+1}\| + \|l\|_{V^*}) \|u_{n+1} - u_n\| \\ &\leqq \big(c_1(\|u\| + 2\gamma) + \|l\|_{V^*}\big) \|u_{n+1} - u_n\| \end{aligned}$$

and c) yields the convergence $u_n \to u$. ∎

4. Application to transonic flow problems

Now, we will apply results of Chapter 3 to the variational inequality (2.6) with $K \subset V$ defined by (2.2) (or (2.3)) and (2.5). Here we put

$$\left.\begin{aligned} F_1(v) &= \frac{1}{2} \int_\Omega \left(\int_0^{|\nabla v|^2} \varrho(q)\, dq \right) dx, \\ l(v) = l_g(v) &= \int_R gv\, do, \qquad B(u; v, w) = \int_\Omega \varrho(|\nabla u|^2)\, \nabla v\, \nabla w\, dx. \end{aligned}\right. \tag{4.1}$$

ϱ is a given function with the following properties:

$$\left.\begin{aligned} &\text{i) } \varrho \text{ and } \varrho' \text{ are continuous in } [0, \infty), \\ &\text{ii) } 0 < \varrho_\infty \leqq \varrho(q) \leqq \varrho_0 < \infty, \\ &\text{iii) } -\varrho_1 \leqq \varrho'(q) \leqq 0 \end{aligned}\right\} \tag{4.2}$$

for all $q \in [0, \infty)$ with constants $\varrho_\infty, \varrho_0, \varrho_1$. It is easy to verify that then the assumptions (3.1)—(3.4) are satisfied. Here $\varrho'(q) \leqq 0$ is the fundamental condition which yields the inequality (3.4) because of the concavity of the function [1: Example 3.16]

$$\Gamma(q) = \int_0^q \varrho(\sigma)\, d\sigma \qquad \big(\Gamma''(q) = \varrho'(q) \leqq 0\big).$$

Such a function ϱ can be obtained in the following way: $\varrho(q)$ is defined by (1.2) for $q \in [0, a^2]$. For $q \in [q_m, \infty)$ we put $\varrho(q) = \text{const} =: \varrho_\infty$ with $0 < \varrho_\infty < \varrho(a^2)$. In the interval $[a^2, q_m]$ the function ϱ is extended suitable smooth and monotone decreasing [1] (cf. Figure 2). It is not necessary that ϱ has the form (1.2) in $[0, a^2]$. Each other function which is continuous differentiable, positive and monotone decreasing in $[0, a^2]$ yields such a ϱ with i)—iii) if we extend the function in an analogous way.

We can use Katchanov's method (3.5) as a linearization of problem (2.6) and we have the assertions a)—d) of Theorem 3.1. Here the assumption in part d) is satisfied because we obtain

$$|B(u; v, w) - B(\overline{u}; v, w)| \leqq \int_\Omega |\varrho(|\nabla \overline{u}|^2) - \varrho(|\nabla u|^2)|\, |\nabla v|\, |\nabla w|\, dx$$

$$\leqq \varrho_1 \int_\Omega ||\nabla u|^2 - |\nabla \overline{u}|^2\, |\nabla v||\, |\nabla w|\, dx$$

$$= \varrho_1 \int_\Omega |\nabla(u - \overline{u})\, \nabla(u + \overline{u})|\, |\nabla v|\, |\nabla w|\, dx$$

by virtue of the mean value theorem and the property iii) of ϱ. Therefore, from the definition (2.5) of K it follows that

$$|B(u; v, w) - B(\overline{u}; v, w)| \leqq 2\varrho_1 a^3 \left(\mu_N(\Omega)\right)^{1/2} \|u - \overline{u}\| \qquad (u, \overline{u}, v, w \in K).$$

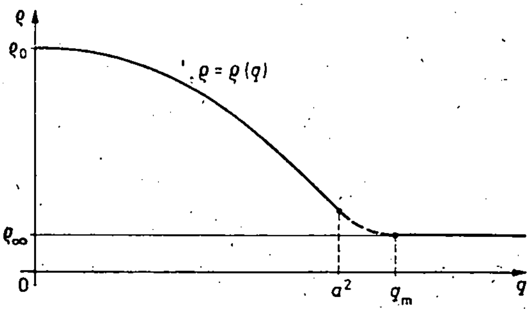

Fig. 2

The Corollary yields immediately a result for subsonic flow problems in Ω without any entropy condition: If we choose $a^2 < q_c$ then $K^0 = \{v \in V \mid |\nabla v| \leqq a \text{ a.e. on } \Omega\}$ is again a non-empty closed convex subset of V. On K^0 the condition (3.14) is satisfied. This follows from the inequality $\varrho(q) + 2q\varrho'(q) > 0$, for $q \in [0, a^2]$ [1: Remark 3.28].

To obtain an existence result for transonic flow problems we will apply Theorem 3.1/d). For this we else need a compactness result for minimizing sequences from K for ψ.

5. A compactness property

Using the same ideas as in [1: Chapter 4] we will prove a result which is really more than we need.

Theorem 5.1: *For $p > 2$, $p \geqq \overline{p}$ let*

$$V^p = V \cap W_1^p(\Omega) \quad \text{and} \quad \|v\|_{V^p} = \left(\int_\Omega |\nabla v|^p\, dx\right)^{1/p}$$

Then the subset $\mathcal{K} = \mathcal{K}_{b,M} := \{v \in V^p \mid v \text{ satisfies } (1.4), \|v\|_{V^p} \leqq b\}$ is compact in V.

Proof: a) Let us consider a sequence $\{v_n\} \subset \mathcal{K}$. Then without loss of generality we can assume that

$$v_n \rightharpoonup v \quad \text{in} \quad V \subset W_1^2(\Omega)^{2)} \tag{5.1}$$

2) $\rightharpoonup$ denotes the weak convergence in the corresponding space.

and $v_n \to v$ in $L^p(\Omega)$, if $n \to \infty$. According to Remark 2.2 we obtain

$$\mathcal{N}(v_n) \to \mathcal{N}(v) \quad \text{in} \quad L^2(\Omega).^{3)} \tag{5.2}$$

For the functionals G_n, $G \in V^*$ defined by

$$G_n(h) = \int_\Omega M(x, v_n(x))\, h\, dx + \int_\Omega \nabla v_n \nabla h\, dx,$$

$$G(h) = \int_\Omega M(x, v(x))\, h\, dx + \int_\Omega \nabla v \nabla h\, dx$$

it follows that

$$G_n \rightharpoonup G \quad \text{in} \quad V^* \quad \text{and in} \quad (\mathring{W}_1^2(\Omega))^*.^{4)}$$

Moreover, we have $G_n(h) \geqq 0$ for $h \in (C_0^\infty(\Omega))_+$ by virtue of (1.4), so that the Theorem of MURAT [7: Theorem 1] yields $G_n \to G$ in $(\mathring{W}_1^q(\Omega))^*$ for all $q > 2$. From (5.2) it follows that $\mathcal{N}(v_n) \to \mathcal{N}(v)$ in $(\mathring{W}_1^q(\Omega))^*$, and hence, $H_n \to H$ in $(\mathring{W}_1^q(\Omega))^*$, where

$$H_n(h) := \int_\Omega \nabla v_n \nabla h\, dx \quad \text{and} \quad H(h) := \int_\Omega \nabla v \nabla h\, dx.$$

b) We also have $H_n \to H$ in $(V^p)^*$. The prove of this assertion is contained in a part of the prove of Theorem 4.30 in [1]. There the Meyers' results from [6] are used. Our prove is fully the same one and we omit the details.[5])

c) By virtue of the definitions of H_n, H it follows that

$$\begin{aligned} |H_n(v_n) - H(v)| &\leqq |(H_n - H)(v_n)| + |H(v_n - v)| \\ &\leqq \|H_n - H\|_{(V^p)^*} \|v_n\|_{V^p} + \left| \int_\Omega \nabla v\, \nabla (v_n - v)\, dx \right|. \end{aligned}$$

Using b) and $\|v_n\|_{V^p} \leqq b$ (see the definition of $\mathcal{K}$) we obtain that the first term on the right hand side is converging to 0. Since (5.1) yields the same fact for the second term we have $\|v_n\|^2 = H_n(v_n) \to H(v) = \|v\|^2$. Hence, using (5.1) again we obtain $v_n \to v$ in V, if $n \to \infty$ ∎

Theorem 5.2 (Existence theorem for the variational inequality (2.6)): *Each minimizing sequence from K for ψ (with the notations (2.2) (or (2.3)), (2.5), (3.6), (4.1)) has a limit point which is a solution of (2.6).*

Proof: We have $K = K_{a,M} \subset \mathcal{K}_{b,M}$ for a suitable b. Therefore, Theorem 5.1 implies the compactness of a minimizing sequence $\{u_n\} \subset K$ in V and Theorem 3.1/d) yields immediately the assertion ∎

Remark 5.1: The existence result is also a consequence of the fact that $F = F_1 - l$ is a continuous functional on V. Since K is closed and compact in V according to Theorem 5.1 it is easy to see that F has a minimum on K in $u^{(0)} \in K$ which satisfies (2.6). But in this paper we were just representing an existence proof based on Katchanov's method because in this way we got additionally an approximation method for the solution of (2.6) by the linear variational inequalities (3.5). With the same arguments as for the minimum we obtain that F has a maximum on K

---

3) The condition $p \geqq \bar{p}$ can be weakened with the help of imbedding theorems for Sobolev spaces (see for example [8: 25—31]).

4) Cf. footnote 2.

5) Applying Lemma 2.1 in [11] to the functionals G_n, $G \in (W_1^2(\Omega))^*$ defined above and using (5.2) we can also get the same result.

in $u^{(1)} \in K$, too. Hence, $u^{(1)}$ is a solution of the variational inequality

$$\int_{\Omega} \varrho(|\nabla u|^2) \nabla u \nabla(u - v) dx \geqq \int_{R} g(u - v) do \quad \text{for all } v \in K. \tag{5.3}$$

If $u^{(1)}$ satisfies (2.7), (2.8) then $u^{(1)}$ is also a solution of equation (2.1). However, it is not possible to solve (5.3) by Katchanov's method because the function $-\varrho$ does not satisfy the necessary assumptions (4.2).

Remark 5.2: It is clear, that we have also solutions of the variational problem (2.4) and of the variational inequality (2.6), respectively, if we use $\mathcal{K}$ instead of K because $\mathcal{K}$ is again a non-empty, closed, convex and compact subset of V. But, in this case it is possible that solutions without physical sense (that means: they are not satisfying (1.3)) occur.

Acknowledgement: We are grateful to Prof. Dr. J. NEČAS from the Charles University inPrague, who brought us to this interesting problem. We had been inspired by his lectures and by helpful discussions with him.

REFERENCES

[1] FEISTAUER, M., and J. NEČAS: On the Solvability of Transonic Potential Flow Problems. Z. Anal. Anw. 4 (1985), 305–329.
[2] FUČIK, S., NEČAS, J., and V. SOUČEK: Einführung in die Variationsrechnung. Leipzig: BSB B. G. Teubner Verlagsgesellschaft 1977.
[3] GLOWINSKI, R.: Lectures on numerical methods for nonlinear variational problems. Berlin–Heidelberg–New York: Springer-Verlag 1980.
[4] KINDERLEHRER, D., and G. STAMPACCHIA: An introduction to variational inequalities and their applications. New York–London–Toronto–Sydney–San Francisco: Academic Press 1980.
[5] LANDAU, L. D., and E. M. LIFSCHITZ: Lehrbuch der Theoretischen Physik, Bd. VI: Hydrodynamik. Berlin: Akademie-Verlag 1966.
[6] MEYERS, N. G.: An L^p-estimate for the gradient of solutions of second order elliptic divergence equations. Ann. Scuola Norm. Sup. Pisa (3) 17 (1963), 189–206.
[7] MURAT, F.: L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$. J. Math. Pures Appl. 60 (1981), 309–322.
[8] NEČAS, J.: Introduction to the theory of nonlinear elliptic equations. Leipzig: BSB B. G. Teubner Verlagsgesellschaft 1983.
[9] NEČAS, J., and I. HLAVAČEK: Mathematical theory of elastic and elasto-plastic bodies: An introduction. Amsterdam–Oxford–New York: Elsevier 1981.
[10] NEČAS, J., and I. HLAVAČEK: Solution of Signorini's contact problem in the deformation theory of plasticity by secant modules method. Apl. Mat. 28 (1983), 199–214.
[11] NEČAS, J., MANDEL, J., and M. FEISTAUER: Entropy regularization of the transonic potential flow problem. Comment. Math. Univ. Carolinae 25 (1984), 431–443.

Manuskripteingang: 31. 07. 1985

VERFASSER:

Dr. HANS-PETER GITTEL
Sektion Mathematik der Karl-Marx-Universität
Karl-Marx-Platz 10
DDR-7010 Leipzig