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Second Microlocalization and Propagation Theorems for the Wave Front Sets

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Es wird gezeigt, wie die Menge der zweiten analytischen Mikroträger zu definieren ist, um Ausbreitungseigenschaften für differenzierbare, Gevrey- oder analytische Singularitäten von Distributionen ohne zweiten analytischen Träger zu erhalten.

Показывается, как следует определить второй аналитический микроноситель чтобы получить свойства распространения для дифференцируемых, Gevrey- или аналитических сингулярностей обобщенных функций без второго аналитического носителя.

There is showed how to define the second analytic wave front set in order to obtain propagation properties for differentiable, Gevrey, or analytic singularities of distributions without second analytic support.

0. Introduction

M. Kashiwara [4] introduced the notion of a second microlocalization along an involutive manifold of $T^*\mathbf{R}^n$. J. Sjöstrand [10] defined the second analytic wave front set (and even a k-th one) on a lagrangian manifold. Finally, G. Lebeau [8] spread this notion to isotropic manifolds. In this paper, we show how to adapt these definitions in order to obtain propagation properties for differentiable, Gevrey, or analytic singularities. We introduce the second analytic support and identify it to the projection of the second analytic wave front set. The main ideas of this proof are due to G. Lebeau [8]. We finally prove propagation theorems for the first wave front sets of distributions without second analytic support.

1. Second analytic wave front set

The (first) wave front sets have several equivalent definitions [3, 10]. Here we use the following one due to Bros-Iagolnitzer-Sjöstrand, extended to differentiable and Gevrey singularities [2]. Let $(y_0, \eta_0) \in \dot{T}^*\mathbf{R}^n$ and let $\varphi(z, y)$ be an FBI-phase function [10]; that is a holomorphic function in a neighbourhood of $(z_0, y_0) \in \mathbf{C}^n \times \mathbf{R}^n$ such that

$$D_y\varphi(z_0, y_0) = -\eta_0, \quad \mathcal{I} D_y^2\varphi(z_0, y_0) > 0, \quad \operatorname{Det} D_zD_y\varphi(z_0, y_0) \neq 0.$$

Denote by $y(z)$ the unique non-degenerate critical point of $y \in \mathbf{R}^n \to -\mathcal{I}\varphi(z, y)$, $\eta(z) = -D_y\varphi(z, y(z))$, $\varrho(z) = (y(z), \eta(z))$ and $\Phi(z) = -\mathcal{I}\varphi(z, y(z))$. Let $1 \leq s \leq \infty$. If $\mathcal{T} \in \mathcal{E}'(\mathbf{R}^n)$ is a distribution with compact support, then $(y_0, \eta_0) \notin \mathrm{WF}_s\mathcal{T}$ if there are a neighbourhood ω of z_0 and positive constants ε, C_k, $\lambda_0 > 0$ such that

$$|u(z, \lambda)| \leq e^{\frac{\lambda}{2}|\mathcal{I}z|^2} \begin{cases} e^{-\varepsilon\lambda^{1/s}} & \text{if } s < \infty, \\ C_k\lambda^{-k}, \quad k \in \mathbf{N}, & \text{if } s = \infty \end{cases}$$

for all $z \in \omega$, $\lambda > 0$ with

$$u(z, \lambda) = \underset{(y)}{\mathscr{T}}(e^{i\lambda\varphi(z,y)}). \tag{1.1}$$

Notice that $WF_1\mathscr{T} = WF_a\mathscr{T}$ and $WF_\infty\mathscr{T} = WF\mathscr{T}$.

Let $0 < k < n$, V be an involutive submanifold of codimension k of $\dot{T}^*\mathbf{R}^n$ and $\varrho_0 = (y_0, \eta_0) \in V$. Denote by F the isotropic submanifold of V containing ϱ_0 such that $T^\sigma_\varrho F = T_\varrho V$ if σ denotes the usual symplectic 2-form on $T^*\mathbf{R}^n$. Consider $\varphi(z, y)$ an FBI-phase function such that

$$\varrho(z) \in V \quad \text{if} \quad \mathscr{I}z' = 0, \qquad \varrho(z) \in F \quad \text{if} \quad \mathscr{I}z' = 0, z'' = z_0'',$$

$$\varrho(z_0) = \varrho_0, \qquad \Phi(z) = \frac{1}{2}|\mathscr{I}z|^2,$$

where $z = (z', z'') \in \mathbf{C}^k \times \mathbf{C}^{n-k}$. For example, one may take φ as the critical value of $(x, \theta) \to i(z-x)^2/2 + \psi(x, y, \theta)$ if ψ generates a real canonical transformation χ such that $\chi(\mathscr{R}z_0, -\mathscr{I}z_0) = \varrho_0$. Then

$$\Big\{\big((x, D_x\psi(x, y, \theta)), (y, -D_y\psi(x, y, \theta))\big): D_\theta\psi(x, y, \theta) = 0\Big\}$$
$$= \Big\{\big((\mathscr{R}z, -\mathscr{I}z), \varrho(z)\big): z \in \mathbf{C}^n\Big\}.$$

Let $\mathscr{T} \in \mathscr{E}'(\mathbf{R}^n)$ be a distribution with compact support. We write

$$v(z, \mu, \lambda) = \int_{|x'-\mathscr{R}z_0'|<\delta} e^{-\frac{\mu\lambda}{2(1-\mu)}(z'-x')^2} u(x', z'', \lambda)\, dx' \tag{1.2}$$

where $\delta > 0$ is small enough and $u(z, \lambda)$ is given by (1.1).

Before defining the second analytic wave front set of $\mathscr{T}$, we shall give an estimation of v. Let $\chi \in D(\{x' \in \mathbf{R}^k : |x' - \mathscr{R}z_0'| < \delta\})$ be equal to 1 if $|x' - \mathscr{R}z'_0| \leq \alpha$. From the complex shift $x' \to x' + it\chi(x')\,\mathscr{I}z'$, $t > 0$, and the continuity of the distribution $\mathscr{T}$ it follows that $|v(z, \mu, \lambda)|$ is smaller than

$$\lambda^m \left[\exp\left(\frac{\lambda}{2}\left(\frac{\mu}{1-\mu}(1-t)^2 + t^2\right)|\mathscr{I}z'|^2 + \frac{\lambda}{2}|\mathscr{I}z''|^2\right)\right.$$
$$\left. + \exp\left(\frac{\lambda\mu}{2(1-\mu)}|\mathscr{I}z'|^2 + \frac{\lambda}{2}t^2|\mathscr{I}z'|^2 + \frac{\lambda}{2}|\mathscr{I}z''|^2 - \frac{\lambda\mu}{2(1-\mu)}\frac{\alpha^2}{4}\right)\right], \tag{1.3}$$

if $|\mathscr{R}z' - \mathscr{R}z_0'| < \alpha/2$. Taking $t = \mu$, one obtains

$$|v(z, \mu, \lambda)| \leq \lambda^m \exp\left(\frac{\lambda\mu}{2}|\mathscr{I}z'|^2 + \frac{\lambda}{2}|\mathscr{I}z''|^2\right) \tag{1.4}$$

if $|\mathscr{R}z' - \mathscr{R}z_0'| < \alpha/2$, $|z'' - z_0''| < \delta$, $|\mathscr{I}z'| < C$, $\lambda > \lambda_0$, $0 < \mu < \mu_0$, provided that $\mu_0(2-\mu_0) < \alpha^2/4$. For $D_{z_j}v$ $(j = 1, \ldots, k)$ an analogous estimate with a factor $\dfrac{\lambda\mu}{1-\mu}$ holds.

Definition 1.1: If $\mathscr{I}z' = 0$, we denote by $\tau(z, \sigma')$ the class of

$$D_\mu\varrho(z' - i\mu\sigma', z'')|_{\mu=0} \text{ in } N_{\varrho(z)}(V) = T_{\varrho(z)}(T^*\mathbf{R}^n)/T_{\varrho(z)}V.$$

Let $1 \leq s \leq \infty$. The *second analytic wave front set* $WF^{(2)}_{a,s,V}\mathscr{T}$ (resp. $WF^{(2)}_{a,s,F}\mathscr{T}$) *on* V (resp. F) is the closed subset of $N(V)$ (resp. $T^*F \setminus \{0\}$) defined by $(\varrho_0, \tau(z_0, \sigma_0'))$

$\notin \mathrm{WF}^{(2)}_{a,s,V}\mathcal{T}$ (resp. $\mathrm{WF}^{(2)}_{a,s,F}\mathcal{T}$) if there exist some constants ε, r, $\mu_0 > 0$ and a decreasing function f in $(0, \mu_0)$ such that

$$|v(z, \mu, \lambda)| \leqq \exp\left(\frac{\lambda\mu}{2}|\mathcal{J}z'|^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2 - \varepsilon\mu\lambda\right)$$

when

$$0 < \mu < \mu_0; \quad \lambda > f(\mu),$$

$$|z' - (z_0' - i\sigma_0')| < r, \quad |z'' - z_0''| < r \quad (\text{resp. } r\sqrt{\mu})$$

with

$$f(\mu) = \begin{cases} \Lambda_0\mu^{-\sigma}, \ \Lambda_0 > 0, \ \sigma = \dfrac{s}{s-1} & \text{if } 1 < s < \infty, \\ \lambda_0 > 0, \ \lambda\mu \geqq m \ln \lambda, \ m > 0 & \text{if } s = \infty. \end{cases}$$

If Ω is an open subset of $\mathbf{R}^n$ and $\mathcal{T} \in \mathcal{D}'(\Omega)$ is a distribution in Ω, we say that $(\varrho_0, \tau(z_0, \sigma_0')) \notin \mathrm{WF}^{(2)}_{a,s,V}\mathcal{T}$ (resp. $\mathrm{WF}^{(2)}_{a,s,F}\mathcal{T}$) if the same condition holds for $\varphi\mathcal{T}$ where $\varphi \in D(\Omega)$ is equal to 1 in a neighbourhood of y_0.

In the terminology of [5, 9] the case $s = \infty$ defines a *temperate* second wave front set. The *untemperate* one is given by $s = 1$. Of course, the sets $\mathrm{WF}^{(2)}_{a,s,V}$ and $\mathrm{WF}^{(2)}_{a,s,F}$ are increasing with s.

Proposition 1.2: *We have $(\varrho_0, \tau(z_0, \sigma_0')) \notin \mathrm{WF}^{(2)}_{a,s,V}\mathcal{T}$ (resp. $\mathrm{WF}^{(2)}_{a,s,F}\mathcal{T}$) if and only if there exist r, λ_0, $\mu_0 > 0$ and a constant $M > 0$ if $1 < s \leqq \infty$, a decreasing function $\lambda(\eta)$ if $s = 1$, such that*

$$|v(z, \mu, \lambda)| \leqq \exp\left(\frac{\lambda\mu}{2}|\mathcal{J}z'|^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2 - \varepsilon\mu\lambda\right)\begin{cases} e^{\eta\lambda} & \text{if} \quad s = 1, \\ e^{M\lambda^{1/s}} & \text{if} \quad 1 < s < \infty, \\ \lambda^M & \text{if} \quad s = \infty \end{cases} \tag{1.5}$$

when $\lambda > \lambda_0$, $0 < \mu < \mu_0$, $|z' - (z_0' - i\sigma_0')| < r$, $|z'' - z_0''| < r$ (resp. $r\sqrt{\mu}$) and $\lambda > \lambda(\eta)$, $\eta > 0$ if $s = 1$.

Proof: Suppose $(\varrho(z_0), \tau(z_0, \sigma_0')) \notin \mathrm{WF}^{(2)}_{a,s,V}\mathcal{T}$ (resp. $\mathrm{WF}^{(2)}_{a,s,F}\mathcal{T}$). From the continuity of $\mathcal{T}$, it follows that

$$|v(z, \mu, \lambda)| \leqq \lambda^j e^{\frac{\lambda\mu}{2}|\mathcal{J}z'|^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2} e^{-\varepsilon\mu\lambda} e^{\varepsilon\mu\lambda}$$

with

$$\lambda^j e^{\varepsilon\mu\lambda} \leqq \begin{cases} \lambda^{j+\varepsilon m} & \text{if} \quad \lambda\mu \leqq m \ln \lambda, \\ e^{2\varepsilon\Lambda_0\lambda^{1/s}} & \text{if} \quad \lambda\mu^\sigma \leqq \Lambda_0, \ \lambda \geqq \lambda_0, \\ e^{(\eta+\varepsilon\mu)\lambda} & \text{if} \quad \lambda \geqq g(\eta). \end{cases}$$

In the last case, we put $\lambda(\eta) = \sup\left(f(\eta/2\varepsilon), g(\eta/2)\right)$. For $\lambda > \lambda(\eta)$, one has whether $\mu \geqq \eta/2\varepsilon$, thus $\lambda > f(\mu)$, or $\mu \leqq \eta/2\varepsilon$ and $\lambda > g(\eta/2)$ gives $\lambda^j e^{\varepsilon\mu\lambda} \leqq e^{\eta\lambda}$.

On the other hand, assume (1.5). In the case $s = 1$, take $\eta = \mu/2$. In the case $1 < s < \infty$, notice that $M\lambda^{1/s} - \varepsilon\mu\lambda \leqq -\varepsilon\mu\lambda/2$ if $\lambda\mu^\sigma \geqq (2M/\varepsilon)^\sigma$. If $s = \infty$, one has $\lambda^M \leqq e^{\varepsilon\mu\lambda/2}$ if $\mu\lambda \geqq \dfrac{2M}{\varepsilon}\ln\lambda$ ∎

2. Second analytic support

The projections of $\mathrm{WF}^{(2)}_{a,s,V}\mathcal{T}$ and $\mathrm{WF}^{(2)}_{a,s,F}\mathcal{T}$ on V and F are in general smaller than $V \cap \mathrm{WF}_a\mathcal{T}$ and $F \cap \mathrm{WF}_a\mathcal{T}$. They are characterized in the following way.

Definition 2.1: The *second analytic support on* V of a distribution $\mathcal{T} \in \mathcal{E}'(\mathbf{R}^n)$ is the closed subset of V defined by $\varrho_0 \notin [\mathcal{T}]^{(2)}_{a,s,V}$ if there are some constants $\lambda_0, r > 0$, a constant $M > 0$ if $1 < s \leqq \infty$, a decreasing function $\lambda(\eta)$ if $s = 1$, such that

$$|u(z, \lambda)| \leqq e^{\frac{\lambda}{2}|\mathcal{I}z''|^2} \begin{cases} e^{\eta\lambda} & \text{if} \quad s = 1, \\ e^{M\lambda^{1/s}} & \text{if} \quad 1 < s < \infty, \\ \lambda^M & \text{if} \quad s = \infty, \end{cases} \tag{2.1}$$

when $|z - z_0| < r, \lambda > \lambda_0, \lambda > \lambda(\eta)$ if $s = 1$ and $u(z, \lambda)$ is the function given by (1.1). The same definition holds for the *second analytic support on* F, denoted by $[\mathcal{T}]^{(2)}_{a,s,F}$, if $\exp\left(\frac{\lambda}{2}|\mathcal{I}z''|^2\right)$ is replaced in relation (2.1) by $\exp\left(\frac{\lambda}{2}|\mathcal{I}z''|^2 + \lambda C|z'' - z_0''|^2\right)$ for some constant $C > 0$.

We have the following theorem.

Theorem 2.2: *The projection of* $\mathrm{WF}^{(2)}_{a,s,V}\mathcal{T}$ *(resp.* $\mathrm{WF}^{(2)}_{a,s,F}\mathcal{T}$*) on* V *(resp.* F*) is exactly* $[\mathcal{T}]^{(2)}_{a,s,V}$ *(resp.* $[\mathcal{T}]^{(2)}_{a,s,F}$*).*

Proof: Assume (2.1). We estimate the function v given by (1.2) for $|\mathcal{R}z' - \mathcal{R}z_0'| < r_1$, $|\mathcal{I}z' + \sigma_0'| < r_1$, $|z'' - z_0''| < r_1$ (resp. $r_1\sqrt{\mu}$). To do this, we use the complex shift

$$y' \to y' + it\chi(y')\,\mathcal{I}z'$$

where $\chi \in D(\{y' \in \mathbf{R}^k : |y' - \mathcal{R}z_0'| < r\})$ is equal to 1 if $|y' - \mathcal{R}z_0'| < r/2$. Consider two integration domains. For $y' \in [\chi]$ the assumption gives the exponential behaviour

$$-\frac{\lambda\mu}{2(1-\mu)}(\mathcal{R}z' - y')^2 - \frac{\lambda\mu}{2(1-\mu)}(1 - t\chi(y'))^2|\mathcal{I}z'|^2 + \frac{\lambda}{2}|\mathcal{I}z''|^2 + \begin{Bmatrix} 0 \\ \lambda C|z'' - z_0''|^2 \end{Bmatrix}$$

$$\leqq \frac{\lambda\mu}{2}|\mathcal{I}z'|^2 + \frac{\lambda}{2}|\mathcal{I}z''|^2 - \frac{\lambda\mu}{2(1-\mu)}(\mathcal{R}z' - y')^2$$

$$-\frac{\lambda\mu}{2}\left(\left(1 - \frac{(1 - t\chi(y'))^2}{1-\mu}\right)|\mathcal{I}z'|^2 - \begin{Bmatrix} 0 \\ 2Cr_1^2 \end{Bmatrix}\right),$$

where the upper (resp. lower) part of the expression appearing in brackets concerns the second analytic support on V (resp. F). Fix $t > 0$ and choose $\mu_0 > 0$ such that $1 - (1-t)^2/(1-\mu_0) > t$. If $\chi(y') = 1$, there exists a $c > 0$ such that

$$\left(1 - \frac{(1 - t\chi(y'))^2}{1-\mu}\right)|\mathcal{I}z'|^2 - \begin{Bmatrix} 0 \\ 2Cr_1^2 \end{Bmatrix} \geqq ct - \begin{Bmatrix} 0 \\ 2Cr_1^2 \end{Bmatrix} \geqq \frac{ct}{2}$$

if r_1 is small enough and $0 < \mu < \mu_0$. If $0 \leqq \chi(y') < 1$, notice that $|\mathcal{R}z' - y'| \geqq r/4$ if $r_1 \leqq r/4$. Since $\frac{(1 - t\chi(y'))^2}{1-\mu} - 1 \leqq \frac{\mu}{1-\mu}$, we obtain the announced behaviour.

For $y' \notin [\chi]$ the exponential behaviour is

$$-\frac{\lambda\mu}{2(1-\mu)}|\mathcal{R}z' - y'|^2 + \frac{\lambda\mu}{2(1-\mu)}|\mathcal{J}z'|^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2$$
$$\leqq \frac{\lambda\mu}{2}|\mathcal{J}z'|^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2 + \frac{\lambda\mu}{2(1-\mu)}(\mu|\mathcal{J}z'|^2 - |\mathcal{R}z' - y'|^2).$$

We may conclude in the same way as above since $|\mathcal{R}z' - y'| \geqq r/4$.

Now assume $(\varrho_0, \tau_0) \notin \mathrm{WF}^{(2)}_{a,s,V}\mathcal{T}$ (resp. $\mathrm{WF}^{(2)}_{a,s,F}\mathcal{T}$) for all $\tau_0 \in N_{\varrho_0}(V)$. This means there exist constants $r_1, \lambda_0, \mu_0, \varepsilon > 0$, a constant $M > 0$ if $s > 1$, a decreasing function $\lambda(\eta)$ if $s = 1$ such that

$$|v(z,\mu,\lambda)| \leqq \exp\left(\frac{\lambda\mu}{2}|\mathcal{J}z'|^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2 - \varepsilon\mu\lambda\right)\begin{cases} e^{\eta\lambda} & \text{if} \quad s = 1, \\ e^{M\lambda^{1/s}} & \text{if} \quad 1 < s < \infty, \\ \lambda^M & \text{if} \quad s = \infty, \end{cases} \tag{2.2}$$

when $\lambda > \lambda_0$, $0 < \mu < \mu_0$, $|\mathcal{R}z' - \mathcal{R}z_0'| < r_1$, $||\mathcal{J}z'| - 1| < r_1$, $|z'' - z_0''| < r_1$ (resp. $r_1\sqrt{\mu}$) and $\lambda > \lambda(\eta)$ if $s = 1$. In order to show that $\varrho_0 \notin [\mathcal{T}]^{(2)}_{a,s,V}$ (resp. $[\mathcal{T}]^{(2)}_{a,s,F}$) we need the following lemmas.

Lemma 2.3: *Let ω be a bounded open subset of $\mathbf{R}^n$ and $a > 0$. If u is a holomorphic function in a neighbourhood of $\overline{\omega}$, there are some constants $C, \delta > 0$ such that*

$$\left|\int_\omega e^{i(x-y)\cdot(\xi + ia(x-y)|\xi|)}\left(1 + ia(x-y)\cdot\frac{\xi}{|\xi|}\right)u(y)\,dy\right|$$
$$\leqq C\exp\left((-\delta d^2(\mathcal{R}x, \complement\omega) + 3|\mathcal{J}x|)|\xi|\right)$$

if $\xi \in \mathbf{R}^n \setminus \{0\}$, $\mathcal{R}x \in \omega$. Moreover,

$$u(x) = \frac{1}{(2\pi)^n}\int d\xi \int_\omega e^{i(x-y)\cdot(\xi + ia(x-y)|\xi|)}\left(1 + ia(x-y)\cdot\frac{\xi}{|\xi|}\right)u(y)\,dy \tag{2.3}$$

if $\mathcal{R}x \in \omega$, $|\mathcal{J}x| < \dfrac{\delta}{3}d^2(\mathcal{R}x, \complement\omega)$.

Proof: Let $\psi \in D(\{x \in \mathbf{R}^n : |x| < 1\})$ be equal to 1 in $\{x \in \mathbf{R}^n : |x| \leqq 1/2\}$. There exists an $r > 0$ such that u is holomorphic in $\{z \in \mathbf{C}^n : \mathcal{R}z \in \omega, |\mathcal{J}z| < r\}$. Moreover, for each $\mathcal{R}x \in \omega$, there exists a $\delta > 0$ such that $\delta\varrho^2 < \sup(r, 1/2a)$ if $\varrho = d(\mathcal{R}x, \complement\omega)$. After the complex shift

$$y \to y - it\chi(y)\frac{\xi}{|\xi|}, \qquad t = \delta\varrho^2, \quad \chi(y) = \psi\left(\frac{y - \mathcal{R}x}{\varrho}\right),$$

the real part of the phase function in (2.2) is equal to

$$-a|\mathcal{R}x - y|^2|\xi| - t\chi(y)|\xi| - \mathcal{J}x\cdot\xi + a|\mathcal{J}x|^2|\xi| + 2at\chi(y)\,\mathcal{J}x\cdot\xi + at^2\chi^2(y)|\xi|$$
$$\leqq -a|\mathcal{R}x - y|^2|\xi| - \frac{\delta}{2}\varrho^2\chi(y)|\xi| + 3|\mathcal{J}x||\xi|$$

if $|\mathcal{J}x| < 1/a$. If $\chi(y) \neq 1$, notice that $|\mathcal{R}x - y| \geqq \varrho/2$, thus

$$-a|\mathcal{R}x - y|^2|\xi| \leqq -a\varrho^2|\xi|/4 \leqq -\delta\varrho^2|\xi|/2$$

if δ is small enough. Hence integral (2.3) converges absolutely if $|\mathcal{J}x| < \delta d^2(\mathcal{R}x, \mathbf{C}\omega)/3$. To verify (2.3) we may assume x real. If $\varphi \in D(\mathbf{R}^n)$ is equal to 1 in a neighbourhood of x, it follows from [3] that

$$u(x) = \frac{1}{(2\pi)^n} \int d\xi \int\limits_{\omega} e^{i(x-y)\cdot(\xi + ia(x-y)|\xi|)} \left(1 + ia(x-y) \cdot \frac{\xi}{|\xi|}\right) \varphi(y)\, u(y)\, dy$$

$$= \frac{1}{(2\pi)^n} \int d\xi \int\limits_{\omega} e^{i(x-y)\cdot(\xi + ia(x-y)|\xi|)} \left(1 + ia(x-y) \cdot \frac{\xi}{|\xi|}\right) u(y)\, dy$$

since

$$\int e^{i(x-y)\cdot(\xi + ia(x-y)|\xi|)} \left(1 + ia(x-y) \cdot \frac{\xi}{|\xi|}\right) d\xi = 0 \qquad \text{if} \quad x \neq y \ \blacksquare$$

Lemma 2.4: *If u is given by (1.1) and v by (1.2), there exists an $r > 0$ such that*

$$u(z, \lambda) = \frac{1}{2} \left(\frac{\lambda}{2\pi}\right)^k \int e^{-\lambda|\xi'|/2} \left(1 - \frac{i}{\lambda\,|\xi'|} \frac{\xi'}{|\xi'|} \cdot D_{z'}\right) v\left(z' - i\frac{\xi'}{|\xi'|},\ z'',\ \mu(\xi'),\ \lambda\right) d\xi'$$

if $|\mathcal{R}z' - \mathcal{R}z_0'| < \delta$, $|\mathcal{J}z'| < r$, with $\mu(\xi') = |\xi'|/(1 + |\xi'|)$.

Proof: Indeed, the term on the right-hand side is equal to

$$\left(\frac{\lambda}{2\pi}\right)^k \int e^{-\lambda|\xi'|/2} d\xi' \int\limits_{|y-\mathcal{R}z_0'|<\delta} \exp\left(i\lambda(z'-y') \cdot \left(\xi' + \frac{i}{2}(z'-y')\,|\xi'|\right)\right)$$

$$\times \left(1 + \frac{i}{2} \frac{\xi'}{|\xi'|} \cdot (z'-y')\right) u(y', z'', \lambda)\, dy' = u(z, \lambda)$$

if $|\mathcal{R}z' - \mathcal{R}z_0'| < \delta$, $|\mathcal{J}z'|$ is small enough $\blacksquare$

Lemma 2.5: *Let ω be an open subset of $\mathbf{C}^n$ and $f(x, \mu, \Lambda)$ a holomorphic function in ω, for all $\Lambda > c > 0$, $0 < \mu < \mu_0$. If*

$$|f(x, \mu, \Lambda)| \leqq e^{\frac{\Lambda}{2}|\mathcal{J}x|^2}, \qquad x \in \omega,\ \Lambda > c,\ 0 < \mu < \mu_0,$$

then

$$|D_{x_j} f(x, \mu, \Lambda)| \leqq \left(\sqrt{\Lambda} + |\mathcal{J}x_j|\, \Lambda\right) e^{3/2}\, e^{\frac{\Lambda}{2}|\mathcal{J}x|^2}$$

for $d(x, \mathbf{C}\omega) > c^{-1/2}$, $\Lambda > c$, $0 < \mu < \mu_0$.

Proof: From Cauchy's inequality for holomorphic functions, it follows that

$$|D_{x_j} f(x, \mu, \Lambda)| \leqq \frac{1}{\delta}\, e^{\frac{\Lambda}{2}(|\mathcal{J}x_j| + \delta)^2} \prod_{k \neq j} e^{\frac{\Lambda}{2}|\mathcal{J}x_k|^2}.$$

If we put $\delta = 2/\left(a\Lambda + \sqrt{a^2\Lambda^2 + 4\Lambda}\right)$, $a = |\mathcal{J}x_j|$, we obtain

$$\delta < c^{-1/2}, \qquad \frac{\Lambda}{2}(a + \delta)^2 \leqq \frac{\Lambda a^2}{2} + \frac{3}{2}, \qquad \frac{1}{\delta} \leqq a\Lambda + \sqrt{\Lambda},$$

which proves the lemma $\blacksquare$

End of the proof of Theorem 2.2: It remains to estimate the integral given by Lemma 2.4 for $|\mathcal{R}z' - \mathcal{R}z_0'| < r$, $|\mathcal{J}z'| < r$, $|z'' - z_0''| < r$, $\lambda > \lambda_0$. We shall split the integration domain in three parts.

First of all, assume $|\xi'| \geqq \mu_0/(1 - \mu_0)$. From (1.3) we deduce the following exponential behaviour. On the one hand, there is

$$-\frac{\lambda}{2}|\xi'|^2 + \frac{\lambda}{2}\left(|\xi'|(1-t)^2 + t^2\right)\left|\mathcal{J}z' - \frac{\xi'}{|\xi'|}\right|^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2$$
$$\leqq -\frac{\lambda}{2}\left(|\xi'|\left(t(2-t) - |\mathcal{J}z'|(2 + |\mathcal{J}z'|)\right) - t^2(1 + |\mathcal{J}z'|)^2\right) + \frac{\lambda}{2}|\mathcal{J}z''|^2$$
$$\leqq \frac{\lambda}{2}|\mathcal{J}z''|^2$$

because of

$$|\xi'|\left(t(2-t) - |\mathcal{J}z'|(2 + |\mathcal{J}z'|)\right) - t^2(1 + |\mathcal{J}z'|)^2 \geqq t|\xi'| - Ct^2 \geqq \frac{t}{2}|\xi'|$$

if we choose $|\mathcal{J}z'| < r$ small enough and $t < \mu_0/2C(1 - \mu_0)$. On the other hand, we have

$$-\frac{\lambda|\xi'|}{2} + \frac{\lambda}{2}(|\xi'| + t^2)\left|\mathcal{J}z' - \frac{\xi'}{|\xi'|}\right|^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2 - \frac{\lambda}{8}\alpha^2|\xi'|$$
$$= -\frac{\lambda}{2}\left[|\xi'|\left(\frac{\alpha^2}{4} - |\mathcal{J}z'|^2\right) + 2\mathcal{J}z'\cdot\xi' - t^2|\mathcal{J}z'|^2 + 2t\frac{2\mathcal{J}z'\cdot\xi'}{|\xi'|} - t^2\right] + \frac{\lambda}{2}|\mathcal{J}z''|^2$$
$$\leqq -\frac{\lambda}{2}\left(|\xi'|\left(\frac{\alpha^2}{4} - |\mathcal{J}z'|(2 + |\mathcal{J}z'|)\right) - t^2(1 + |\mathcal{J}z'|)^2\right) + \frac{\lambda}{2}|\mathcal{J}z''|^2 \leqq \frac{\lambda}{2}|\mathcal{J}z''|^2$$

since

$$|\xi'|\left(\frac{\alpha^2}{4} - |\mathcal{J}z'|(2 + |\mathcal{J}z'|)\right) - t^2(1 + |\mathcal{J}z'|)^2$$
$$\geqq \frac{\alpha^2}{8}|\xi'| - t^2(1 + |\mathcal{J}z'|)^2 \geqq \frac{\alpha^2}{8}|\xi'| - Ct^2 \geqq 0$$

if r and t are small enough.

Assume $\lambda\mu(\xi') \leqq \Lambda_0$, $\mu(\xi') < \mu_0$. Using (1.4) we obtain

$$\lambda^m \int_{\substack{\lambda\mu(\xi')<\Lambda_0\\ |\xi'|<\mu_0/(1-\mu_0)}} \exp\left(-\frac{\lambda|\xi'|}{2} + \frac{\lambda\mu(\xi')}{2}\left|\mathcal{J}z' - \frac{\xi'}{|\xi'|}\right|^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2\right)$$
$$\times\left(1 + k\frac{\mu(\xi')}{|\xi'|\left(1 - \mu(\xi')\right)}\right)d\xi'$$
$$\leqq (1 + k)\lambda^m \int_{\substack{\lambda\mu(\xi')<\Lambda_0\\ |\xi'|<\mu_0/(1-\mu_0)}} \exp\left(-\frac{\lambda}{2}|\xi'| + \frac{\Lambda_0}{2}(|\mathcal{J}z'| + 1)^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2\right)d\xi'$$
$$\leqq \lambda^{m'} e^{\frac{\lambda}{2}|\mathcal{J}z''|^2}$$

if $\lambda \geqq e^{\frac{\Lambda_0}{2}(1+r)^2}$.

It remains the case $\mu(\xi') < \mu_0$, $\lambda\mu(\xi') > \Lambda_0$. The relation (2.2) and Lemma 2.5 applied with $\Lambda = \lambda\mu$ show that the corresponding part of the integral is smaller than

$$\left.\begin{matrix} e^{\eta\lambda} \\ e^{M\lambda^{1/s}} \\ \lambda^M \end{matrix}\right\} \int\limits_{\substack{\mu(\xi')<\mu_0 \\ \lambda\mu(\xi')>\Lambda_0}} \exp\left(-\frac{\lambda|\xi'|}{2} + \frac{\lambda\mu(\xi')}{2}\left|\mathcal{J}z' - \frac{\xi'}{|\xi'|}\right|^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2 - \varepsilon\mu(\xi')\,\lambda\right)\left(1 + c\frac{\sqrt{\mu(\xi')}}{|\xi'|}\right)d\xi'$$

if $|z'' - z_0''| \leqq r_1$ (resp. $r_1\sqrt{\mu(\xi')}$). The exponential behaviour is

$$-\frac{\lambda}{2}\frac{|\xi'|}{1+|\xi'|}\left(|\xi'| - |\mathcal{J}z'|\,(|\mathcal{J}z'| + 2) + \varepsilon\right) + \frac{\lambda}{2}|\mathcal{J}z''|^2 \leqq \frac{\lambda}{2}|\mathcal{J}z''|^2$$

if we choose r such that $r(2 + r) < \varepsilon$. When studying the second wave front set on F, it remains the case $|z'' - z_0''| \geqq r_1\sqrt{\mu(\xi')}$. From (1.4) we deduce the exponential behaviour

$$-\frac{\lambda}{2}|\xi'| + \frac{\lambda\mu(\xi')}{2}\left|\mathcal{J}z' - \frac{\xi'}{|\xi'|}\right|^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2$$
$$\leqq \frac{\lambda}{2}\frac{|z'' - z_0''|^2}{r_1^2}(|\mathcal{J}z'| + 1)^2 + \frac{\lambda}{2}|\mathcal{J}z''|^2 \leqq \frac{\lambda}{2}|\mathcal{J}z''|^2 + \lambda C\,|z'' - z_0''|^2$$

since $\mu(\xi') \leqq |z'' - z_0''|^2/2r_1^2$. This completes the proof of Theorem 2.2 ∎

3. Propagation of the wave front sets

We now reach the main result of this paper: we prove propagation theorems for the wave front sets of distributions without second analytic support. We need the following lemmas.

Lemma 3.1: *Let $1 < s < \infty$ and $s' < s$. Let $f(x, \lambda)$ be a holomorphic function in a neighbourhood of $x_0 = (x_0', x_0'') \in \mathbf{R}^k \times \mathbf{C}^{n-k}$. If there exist some constants $c, \varepsilon, M, \lambda_0 > 0$ such that*

$$|f(x,\lambda)| \leqq \begin{cases} \lambda^M e^{\frac{\lambda}{2}|\mathcal{J}z''|^2} & \left(\text{resp. } e^{\frac{\lambda}{2}|\mathcal{J}z''|^2 + M\lambda^{1/s}}\right), \\ C_k\lambda^{-k} e^{\frac{\lambda}{2}|\mathcal{J}z|^2}, \quad k \in \mathbf{N} & \left(\text{resp. } e^{\frac{\lambda}{2}|\mathcal{J}z|^2 - c\lambda^{1/s'}}\right) \end{cases}$$

if $|x' - x_0'| \leqq \varepsilon$, $|x'' - x_0''| \leqq \varepsilon$, $\lambda > \lambda_0$, then

$$|f(x,\lambda)| \leqq C_k'\lambda^{-k} e^{\frac{\lambda}{2}|\mathcal{J}z''|^2}, \quad k \in \mathbf{N} \quad \left(\text{resp. } e^{\frac{\lambda}{2}|\mathcal{J}z''|^2 - c'\lambda^{1/s'}}\right)$$

if $|x' - x_0'| \leqq \varepsilon/2$, $|x'' - x_0''| \leqq \varepsilon$, $\lambda > \lambda_0$.

Proof: We may assume $k = 1$ and $x = (x_1, x'')$. The function

$$h(x_1) = \frac{2}{\pi}\left(\arg\frac{x_1 - \varepsilon}{x_1 + \varepsilon} - \frac{\pi}{2}\right)$$

is harmonic in $D = \complement\{x_1 \in \mathbf{C}\colon \mathcal{J}x_1 = 0,\ |\mathcal{R}x_1| \geqq \varepsilon\}$. If $\mathcal{J}x_1 = 0$ and $|x_1| < \varepsilon$, then $h(x_1) = 1$ and if $\mathcal{J}x_1 > 0$, $|x_1| = \varepsilon$, then $h(x_1) = 0$. The subharmonic function

$$v(x,\lambda) = |f(x_1 + x_{01}, x'', \lambda)| - \frac{\lambda}{2}|\mathcal{J}z''|^2 - M\begin{cases}\ln\lambda \\ \lambda^{1/s}\end{cases}$$

verifies

$$v(x, \lambda) \leqq \begin{Bmatrix} \ln C_k - k \ln \lambda \\ -c\lambda^{1/s'} \end{Bmatrix} h(x_1)$$

if $\mathcal{J}x_1 = 0$, $|x_1| < \varepsilon$ or $\mathcal{J}x_1 > 0$, $|x_1| = \varepsilon$. From the maximum principle it follows that

$$v(x, \lambda) \leqq \begin{Bmatrix} \ln C_k - k \ln \lambda \\ -c\lambda^{1/s'} \end{Bmatrix} h(x_1) \leqq \begin{Bmatrix} \ln C_k - k \ln \lambda \\ -c\lambda^{1/s'} \end{Bmatrix} \delta$$

if $\mathcal{J}x_1 > 0$, $|x_1| \leqq \varepsilon/2$. Thus

$$|f(x, \lambda)| \leqq e^{\frac{\lambda}{2}|\mathcal{J}x''|^2} \begin{cases} C_l'\lambda^{-l} \\ e^{-c'\lambda^{1/s}}, \end{cases} \quad |x_1 - x_{01}| < \varepsilon/2, \ \mathcal{J}x_1 > \mathcal{J}x_{01},$$

if one chooses $k \geqq (M + l)/\delta$ (resp. λ large enough). The same argumentation holds for $\mathcal{J}x_1 < \mathcal{J}x_{01}$ ∎

Lemma 3.2: *Let $\omega = \omega' \times \omega''$ be an open subset of $\mathbf{C}^n$, $x_0 = (x_0', x_0'') \in \omega$ and φ a real continuous function on ω''. Consider $r > 0$ such that*

$$K = \{x' \in \mathbf{C}^k : |x_j - x_{0j}| \leqq r, j = 1, \ldots, k\} \subset \omega'.$$

Let $f(x, \lambda)$ be a holomorphic function in ω such that, for each compact subset K_0 of ω, there exist some constants M, $\lambda_0 > 0$ if $s > 1$, a decreasing function $\lambda(\eta)$ if $s = 1$, such that

$$|f(x, \lambda)| \leqq e^{\lambda\varphi(x'')} \begin{cases} e^{\eta\lambda} & \text{if} \quad s = 1, \\ e^{M\lambda^{1/s}} & \text{if} \quad 1 < s < \infty, \\ \lambda^M & \text{if} \quad s = \infty, \end{cases}$$

for $x \in K_0$, $\lambda > \lambda_0$ if $s > 1$, $\lambda > \lambda(\eta)$ if $s = 1$. If there exist $\varepsilon, \delta', \delta'' > 0$ such that

$$|f(x, \lambda)| \leqq e^{\lambda\varphi(x'')} \begin{cases} e^{-\varepsilon\lambda} & & \text{if} \quad s = 1, \\ e^{-\varepsilon\lambda^{1/s'}}, & s' < s, & \text{if} \quad 1 < s < \infty, \\ C_k\lambda^{-k}, & k \in \mathbf{N}, & \text{if} \quad s = \infty, \end{cases}$$

for $|x_j - x_{0j}| \leqq \delta'$, $j = 1, \ldots, k$, $|x'' - x_0''| \leqq \delta''$, $\lambda > \lambda_0$, there exist ε', $\delta > 0$ such that

$$|f(x, \lambda)| \leqq e^{\lambda\varphi(x'')} \begin{cases} e^{-\varepsilon'\lambda} & & \text{if} \quad s = 1, \\ e^{-\varepsilon'\lambda^{1/s'}} & & \text{if} \quad 1 < s < \infty, \\ C_k'\lambda^{-k}, & k \in \mathbf{N}, & \text{if} \quad s = \infty, \end{cases}$$

when $d(x, K) \leqq \delta$, $\lambda > \lambda_0$.

Proof: Consider

$$g(s, \lambda) = \ln \sup_{\substack{|x'' - x_0''| \leqq \delta'', |x_j - x_{0j}| \leqq s \\ j = 1, \ldots, k}} |f(x, \lambda) e^{-\lambda\varphi(x'')}|.$$

We may assume $\delta' < r$. Choose $a > 0$ such that

$$\{x' \in \mathbf{C}^k : |x_j - x_{0j}| \leqq r + a, j = 1, \ldots, k\} \subset \omega'.$$

Write

$$\ln(r + a/2) = \theta \ln \delta' + (1 - \theta) \ln(r + a), \qquad \theta \in (0, 1).$$

From Hadamard's three circles theorem it follows that

$$g\left(r + \frac{a}{2}, \lambda\right) \leqq \theta g(\delta', \lambda) + (1 - \theta)\, g(r + a, \lambda)$$

$$\leqq \begin{cases} -\varepsilon\theta\lambda + (1 - \theta)\,\eta\lambda & \text{if} \quad s = 1, \\ -\varepsilon\theta\lambda^{1/s'} + (1 - \theta)\, M\lambda^{1/s} & \text{if} \quad 1 < s < \infty, \\ \theta(\ln C_k - k \ln \lambda) + (1 - \theta)\, M \ln \lambda & \text{if} \quad s = \infty, \end{cases}$$

$$\leqq \begin{cases} -\varepsilon\theta\lambda/2 & \text{if} \quad \eta \leqq \varepsilon\theta/2(1 - \theta), \\ -\varepsilon\theta\lambda^{1/s'}/2 & \text{if} \quad \lambda \text{ is large enough}, \\ \ln C_k^0 - l \ln \lambda & \text{if} \quad k \geqq \dfrac{(1 - \theta)\, M + l}{\theta} \end{cases} \blacksquare$$

Theorem 3.3: *Let ω be an open connected subset of F.*
If $[\mathcal{T}]^{(2)}_{a,1,F} \cap \omega = \emptyset$, then $\omega \cap \mathrm{WF}_a\mathcal{T} = \omega$ or $\omega \cap \mathrm{WF}_a\mathcal{T} = \emptyset$.
If $[\mathcal{T}]^{(2)}_{a,s,V} \cap \omega = \emptyset$, then $\omega \cap \mathrm{WF}_{s'}\mathcal{T} = \omega$ or $\omega \cap \mathrm{WF}_{s'}\mathcal{T} = \emptyset$, $s' \in [1, s)$.
If $[\mathcal{T}]^{(2)}_{a,\infty,V} \cap \omega = \emptyset$, then $\omega \cap \mathrm{WF}\mathcal{T} = \omega$ or $\omega \cap \mathrm{WF}\mathcal{T} = \emptyset$.

Proof: There exists an open connected subset ω' of $\mathbf{R}^k$ such that $\omega = \{\varrho(z): z' \in \omega', \mathcal{J}z' = 0, z'' = z_0''\}$. Let $z_0' \in \omega'$ and assume $\varrho(z_0) \notin \mathrm{WF}_{s'}\mathcal{T}$. Hence, $E = \{z' \in \omega' : \varrho(z', z_0'') \notin \mathrm{WF}_{s'}\mathcal{T}\}$ is a non-empty subset of ω'. If $E \neq \omega'$, there exists a $y_0' \in \omega' \cap \dot{E}$, where $\dot{E}$ is the boundary of E. If $u(z, \lambda)$ is the function given by (1.1), there exist constants $\lambda_0, r > 0$, a constant $M > 0$, a decreasing function $\lambda(\eta)$ if $s = 1$ such that

$$|u(z, \lambda)| \leqq e^{\frac{\lambda}{2}|\mathcal{J}z''|^2} \begin{cases} e^{\eta\lambda}\, e^{\lambda M|z'' - z_0''|^2} & \text{if} \quad s = 1, \\ e^{M\lambda^{1/s}} & \text{if} \quad 1 < s < \infty, \\ \lambda^M & \text{if} \quad s = \infty, \end{cases}$$

when $|z_j - y_{0j}| < r$, $j = 1, \ldots, k$, $|z'' - z_0''| < r$, $\lambda > \lambda_0$; $\lambda > \lambda(\eta)$ if $s = 1$.

Since $y_0' \in \dot{E}$, there exists a $y_1' \in E$ such that $|z_j - y_{1j}| < r/4$, $j = 1, \ldots, k$. Since $\varrho(y_1', z_0'') \notin \mathrm{WF}_{s'}\mathcal{T}$, there exist constants $\varepsilon, C_k, \delta', \delta'' > 0$ such that

$$|u(z, \lambda)| \leqq e^{\frac{\lambda}{2}|\mathcal{J}z|^2} \begin{cases} e^{-\varepsilon\lambda} & \text{if} \quad s' = 1, \\ e^{-\varepsilon\lambda^{1/s'}} & \text{if} \quad 1 < s' < \infty, \\ C_k\lambda^{-k}, \quad k \in \mathbf{N}, & \text{if} \quad s' = \infty, \end{cases}$$

if $|z' - y_1'| \leqq \delta'$, $|z'' - z_0''| \leqq \delta''$. From Lemma 3.1 it follows that

$$|u(z, \lambda)| \leqq e^{\frac{\lambda}{2}|\mathcal{J}z''|^2} \begin{cases} e^{\lambda M|z'' - z_0''|^2 - \varepsilon\lambda/2} & \text{if} \quad s' = 1, \\ e^{-\varepsilon\lambda^{1/s'}} & \text{if} \quad 1 < s' < \infty, \\ C_k\lambda^{-k}, \quad k \in \mathbf{N}, & \text{if} \quad s' = \infty. \end{cases}$$

Lemma 3.2 gives

$$|u(z, \lambda)| \leqq e^{\frac{\lambda}{2}|\mathcal{J}z''|^2} \begin{cases} e^{\lambda M|z'' - z_0''|^2 - \varepsilon'\lambda} & \text{if} \quad s' = 1, \\ e^{-\varepsilon'\lambda^{1/s'}} & \text{if} \quad 1 < s' < \infty, \\ C_k'\lambda^{-k}, \quad k \in \mathbf{N}, & \text{if} \quad s' = \infty, \end{cases}$$

for $|z_j - y_{1j}| < r/2$, $|z'' - z_0''| \leqq \delta''$, thus for $|z_j - y_{0j}| < r/4$. Hence, $\varrho(y_0', z_0'') \notin \mathrm{WF}_{s'}\mathcal{T}$, which contradicts $y_0' \in \dot{E}$ $\blacksquare$

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