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Stability Properties of Space Periodic Standing Waves

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Es werden Gleichgewichtslösungen von parabolischen Systemen der Form $\dot{u} = D\Delta u + F(\alpha, u)$ betrachtet, wo D eine 2×2-Diagonalmatrix, α ein Verzweigungsparameter, $u = (u_1, u_2)$ ein Zustandsvektor und F eine polynomiale Nichtlinearität bedeutet. Es wird angenommen, daß ein trivialer Lösungszweig $u(\delta) \in \mathbf{R}^2$, $\delta \in I = (-\varepsilon, \varepsilon)$, vorliegt, d. h. $F(\alpha_0 + \delta, u(\delta)) = 0$ für ein α_0 und alle $\delta \in I$ ist. Eine Periode L wird festgehalten, und unter passenden Voraussetzungen werden Verzweigungsscharen von räumlich L-periodischen, stehenden Wellen konstruiert. Es wird gezeigt, daß diese Scharen für $L \uparrow \infty$ generisch instabil werden, daß sie aber unter der Annahme $d_{uu}F(\alpha_0, u(0)) = 0$ gegenüber nL-periodischen Störungen $(1 < n \in \mathbf{N})$ stabil bleiben, falls dies für kleine $\delta < 0$ auf den trivialen Lösungszweig $u(\delta)$ zutrifft. Physikalische Spezialfälle bilden die sogenannten Landau-Ginzburg-Gleichungen, die in der Landau-Theorie der Phasenübergänge auftreten.

Рассматриваются равновесные решения параболических систем вида $\dot{u} = D\Delta u + F(\alpha, u)$, где D — диагональная 2×2-матрица, α — параметр ветвления, $u = (u_1, u_2)$ — вектор состояний и F — полиномиальная нелинейность. Предполагается существование нетривиальной ветви решения $u(\delta) \in \mathbf{R}^2$, $\delta \in I = (-\varepsilon, +\varepsilon)$, т.е. $F(\alpha_0 + \delta, u(\delta)) = 0$ для некоторого α_0 и всех $\delta \in I$. Тогда фиксируется период L и при подходящих предположениях конструируются разветвленные семейства пространственно L-периодических стоячих волн. Показывается, что эти семейства будут неустойчивыми при $L \uparrow \infty$. Однако, при условии $d_{uu}F(\alpha_0, u(0)) = 0$ они остаются устойчивыми относительно nL-периодических возмущений $(1 < n \in \mathbf{N})$ если это имеет место для тривиальной ветви решения при малых $\delta < 0$. Физические частные случаи — так называемые уравнения Ландау-Гинзбурга, встречаемые в теории фазового перехода Ландау.

Equilibrium solutions of parabolic systems of the form $\dot{u} = D\Delta u + F(\alpha, u)$ are considered, where D designates as 2×2 diagonal matrix, α a bifurcation parameter, $u = (u_1, u_2)$ a state vector and F a polynomial nonlinearity. A trivial solution branch $u(\delta) \in \mathbf{R}^2$, $\delta \in I = (-\varepsilon, \varepsilon)$, is supposed to be given, i.e. $F(\alpha_0 + \delta, u(\delta)) = 0$ for some α_0 and every $\delta \in I$. Then a period L is fixed and under suitable assumptions space-L-periodic bifurcating standing waves are contructed. It is shown that these bifurcating branches become generically unstable as $L \uparrow \infty$. Under the condition of $d_{uu}F(\alpha_0, u(0)) = 0$ however, they will remain stable against nL-periodic perturbations $(1 < n \in \mathbf{N})$, provided that the trivial solution-branch $u(\delta)$ behaves alike for small $\delta < 0$. The so-called Landau-Ginzburg equations arising in Landau's theory of phase transitions constitute a special example in physics.

1. Introduction

1.1 Physical background

In some parts of statistical mechanics, particularly in Landau's theory of phase transitions, one encounters parabolic systems of the form

$$\dot{u}_i = \mathcal{L}_i u_i + F_i(\alpha, u_i, \ldots, u_N), \qquad i = 1, \ldots, N, \tag{1}$$

where $u = (u_1, \ldots, u_N)$ is a state vector, α is a one-dimensional bifurcation parameter. The $\mathscr{L}_i$ are elliptic second-order operators and $F = (F_1, \ldots, F_N)$ designates a nonlinearity, polynomial in both in α and in the $u_1, \ldots, u_N$. The dimension of the space on which the $\mathscr{L}_i$ are operating is $n \leqq 3$ and the $\mathscr{L}_i$ and the F_i are independent of $x_1, \ldots, x_n, t$, i.e. the system is invariant under translations in space and time. The vector $u(x, t)$ describes the state of a large body Ω, at the point $x \in \Omega$, for the time $t \geqq 0$, provided that x is far away from the boundary $\partial\Omega$. It is a widespread belief, sustained by experience, that under these assumptions u has properties being largely independent of the conditions at the boundary. It is customary now to impose periodic boundary conditions on u, i.e. u has to be L-periodic with respect to each space variable and its period L should be small compared with the dimensions of Ω and large compared with molecular distances but unspecified otherwise. One expects the solution u to have properties essentially independent of L or to show a certain asymptotic behaviour as $L \uparrow \infty$. Here was shall investigate a class of bifurcating solutions from this point of view.

1.2 Bifurcation problems

Let be $N = 2$ for simplicity and $\mathscr{L}_i = \tau_i \Delta$ with $\tau_i > 0$, in general, where Δ is designating the Laplacian acting on $\mathbf{R}^2$. The parabolic system (1) gives rise to the elliptic equilibrium system vectorially written as

$$\mathscr{L}u + F(\alpha, u) = 0. \tag{2}$$

Let $I = (-\varepsilon, \varepsilon)$ be an intervall, $u: I \to \mathbf{R}^2$ an analytic mapping and α_0 a parameter value such that $F(\alpha_0 + \delta, u(\delta)) = 0$ for $\delta \in I$. Then the family

$$\{u(\delta)\}_{\delta \in I} \tag{3}$$

will trivially be an L-periodic branch of solutions of (2) for any $L > 0$. Although we are restricting ourselves to even solutions only, there is generically a large number of solution branches bifurcating from the trivial branch (3) under familiar spectral conditions. We shall investigate how the stability of these branches is depending on the period L: e.g. a L-periodic branch being nL-periodic, too, for natural n, one may ask, whether it will remain stable against nL-periodic perturbations if it behaves alike against L-periodic perturbations. It will be shown that all bifurcating branches become generically unstable as $n \uparrow \infty$. Among many other bifurcating branches there is a distinguished set, termed as *standing waves*, being of the form $v(2\pi L^{-1}k \cdot x)$ with v 2π-periodic and $k \cdot x = k_1x_1 + k_2x_2$ for any integers k_1, k_2. For these branches we shall get a positive result saying that the branch will remain stable against large nL-periodic perturbations, provided that some additional assumptions are satisfied. For a precise formulation of these and of further results we refer to the text.

2. Functional analytic background

2.1 Sobolev spaces

Let $\mathbf{C}$ be the complex numbers, $\mathbf{R}$ the real ones and let exp denote the exponential function. For $p = 1, 2$ define $T_p(L)$ to be the set of finite trigonometric sums

$$\sum \zeta_k \exp(i2\pi L^{-1}k \cdot x)$$

where $\zeta_k \in \mathbf{C}^p$ and $k \cdot x = k_1x_1 + k_2x_2$ with $x = (x_1, x_2) \in \mathbf{R}^2$ and $k = (k_1, k_2)$, for integers k_1, k_2. We set

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$$

with $\alpha = (\alpha_1, \alpha_2)$ being a multiindex and $|\alpha| = \alpha_1 + \alpha_2$. For $u, v \in T_p(L)$ we define the scalar product

$$[u, v]_m = \frac{1}{L^2} \int_0^L \int_0^L \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)\, dx^2$$

where $(a, b) = \sum a_j \bar{b}_j$ for $a, b \in \mathbf{C}^p$. It gives rise to the norm $|\cdot|_m = [\cdot, \cdot]^{1/2}$. Now let $(H_m{}^p(L), [\cdot, \cdot]_m)$ be the Hilbert space obtained from the pre-Hilbert space $(T_p(L), [\cdot, \cdot]_m)$ by taking the closure with respect to $|\cdot|_m$. Providing $T_p(L)$ with another scalar product

$$\langle u, v\rangle_m = \sum (\zeta_k, \eta_k) \left(1 + (2\pi L^{-1})^{2m} |k|^{2m}\right)$$

for

$$u = \sum \zeta_k \exp(2\pi L^{-1}k \cdot x), \qquad v = \sum \eta_k \exp(2\pi L^{-1}k \cdot x)$$

with $|k| = (k_1^2 + k_2^2)^{1/2}$ we get a second Hilbert space $(G_m{}^p(L), \langle\cdot, \cdot\rangle_m)$ with norm $\|\cdot\|_m$, by taking the closure of $T_p(L)$ with respect to the norm $\|\cdot\|_m = \langle\cdot, \cdot\rangle_m{}^{1/2}$.

Though $H_m{}^p(L)$ and $G_m{}^p(L)$ are not identical, they contain the same elements and are boundedly isomorphic to each other. Without running the danger of confusion we may identify both of them, thus writing $H_m{}^p(L)$ and considering it, provided with equivalent norms $|\cdot|_m$ and $\|\cdot\|_m$. The $H_m{}^p(L)$ are spaces of vector-valued L-periodic functions having generalized derivatives up to order m. They admit a Fourier-series description as follows: Each $u \in H_m{}^p(L)$ has a Fourier expansion $f(u) = \sum \zeta_k \exp(i2\pi L^{-1}k \cdot x)$ satisfying

$$\sum |\zeta_k|^2 \left(1 + (2\pi L^{-1})^{2m} |k|^{2m}\right) < \infty, \qquad \zeta_k \in \mathbf{C}^p. \tag{4}$$

Conversely, to each Fourier series f having this property there is a unique $u \in H_m{}^p(L)$ with $f = f(u)$. Thus $H_m{}^p(L)$ may be identified with the set of all Fourier series satisfying (4), provided with the scalar product defined by

$$\langle u, v\rangle_m = \sum (\zeta_k, \eta_k) \left(1 + (2\pi L^{-1})^{2m} |k|^{2m}\right)$$

if

$$u = \sum \zeta_k \exp(i2\pi L^{-1}k \cdot x) \quad \text{and} \quad v = \sum \eta_k \exp(i2\pi L^{-1}k \cdot x).$$

The spaces $H_m{}^p(L)$ have some familiar properties. Set $Q = \{(x_1, x_2) \mid 0 < x_1, x_2 < L\}$ and let $C_p{}^{m\lambda}(\bar{Q})$, $p = 1, 2$, be the set of (eventually vector-valued) functions having uniformly bounded and uniformly continuous derivatives up to order m on Q, whose m-th derivatives satisfy a Hölder condition of order λ $(0 < \lambda \leq 1)$ on Q. By introducing a suitable norm, $C_p{}^{m\lambda}(\bar{Q})$ becomes a Banach space, for details see [1: pp. 9, 10]. The properties in question are the following:

(P1) $H_q{}^p(L)$ is *compactly embedded* in $H_m{}^p(L)$ for $m < q$ [2: p. 169].

(P2) $H^p_{j+2}(L)$ is *continuously embedded* in $C_p{}^{j\lambda}(\bar{Q})$ for $0 < \lambda < 1$ [1: Theorem 5.4]. For the space dimension $n = 3$, $0 < \lambda < 1$ has to be replaced by $0 < \lambda \leq 1/2$.

(P3) $H_m{}^1(L)$ is a *Banach algebra* for $m \geq 2$ with respect to the norms $|\cdot|_m, \|\cdot\|_m$ respectively, i.e. if $u, v \in H_m{}^1(L)$, then $uv \in H_m{}^1(L)$ and $|uv|_m \leq K_1 |u|_m |v|_m$, $\|uv\|_m \leq K_2 \|u\|_m \|v\|_m$ for suitable constants K_1, K_2 depending on L [1: Theorem 5.23].

Since we will be working for the most part with the spaces $H_m^2(L)$, we set $H_m(L) = H_m^2(L)$ writing $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_2$.

2.2 Generators of semigroups

As a last preparatory step we show that the elliptic operators which may appear as Fréchet derivatives are wellbehaved. To this end let

$$D = (\delta_{jm}\tau_j) \quad \text{with} \quad \tau_j > 0,$$

$$B_j(x) \quad \text{and} \quad A(x) \quad \text{for} \quad j = 1, 2 \quad \text{and} \quad x = (x_1, x_2)$$

be 2×2 matrices, $A(x)$ as well as $B_j(x)$ having entries in $H_2^1(L)$. We introduce the operators E_0, E_1, E_2 and ∂_j $(j = 1, 2)$ as follows:

$E_2 = D\Delta$ selfadjoint is defined on $H_4(L)$ by the equation

$$E_2\left(\sum \zeta_k \exp(i2\pi L^{-1}k \cdot x)\right) = -\sum |k|^2 (2\pi L^{-1})^2 (D\zeta_k) \exp(i2\pi L^{-1}k \cdot x).$$

∂_j is defined on $H_3(L)$ by the equation

$$\partial_j\left(\sum \zeta_k \exp(i2\pi L^{-1}k \cdot x)\right) = \sum \zeta_k i2\pi L^{-1}k_j \exp(i2\pi L^{-1}k \cdot x).$$

Since $H_2^1(L)$ is a Banach algebra, $B_j(x)$ and $A(x)$ act as bounded operators on $H_2(L)$, and so we may define

$$E_0 = A(x), \qquad E_1 = \sum B_j(x)\, \partial_j.$$

Lemma 1: *The operator $E = E_2 + E_1 + E_0$ is closed and generates a holomorphic semigroup.*

Proof: Set $\varepsilon^2 = \min(\tau_1, \tau_2)$, $\varepsilon > 0$, and rewrite E in the form $E = -E_\varepsilon + M$ with $E_\varepsilon = (2\pi L^{-1})^2 \varepsilon^2 I - E_2$ and $M = E_1 + E_0 + (2\pi L^{-1})^2 \varepsilon^2 I$, where I has the meaning of the identity. Evidently E_ε is positive definite, selfadjoint with Dom $E_\varepsilon = H_4(L)$ and Dom $E_\varepsilon^{1/2} = H_3(L)$. According to [8: Lemmata 2.1 and 2.3] our lemma is proved, if we can show that

(i) Dom $E_\varepsilon^{1/2} \subseteqq$ Dom M and
(ii) $\|Mu\|_2 \leqq c\, \|E_\varepsilon^{1/2}u\|_2$ for $u \in$ Dom $E_\varepsilon^{1/2}$.

Now is Dom $M = H_3(L)$ by definition, whence (i) follows. In order to verify (ii), we set $\tilde{E}_\varepsilon = (2\pi L^{-1})^2 \varepsilon^2 I - \varepsilon^2 I\Delta$. The operator $\tilde{E}_\varepsilon$ is positive definite, selfadjoint with Dom $\tilde{E}_\varepsilon = H_4(L)$ and Dom $\tilde{E}_\varepsilon^{1/2} = H_3(L)$. Moreover, $\|\tilde{E}_\varepsilon^{1/2}u\|_2 \leqq \|E_\varepsilon^{1/2}u\|_2$ for $u \in H_3(L)$. It thus suffices to prove (ii) with $\tilde{E}_\varepsilon^{1/2}$ in place of $E_\varepsilon^{1/2}$. By a straightforward computation we obtain

$$\|\tilde{E}_\varepsilon^{1/2}u\|_2^2 = \sum (\zeta_k, \zeta_k)\, (2\pi L^{-1})^2 \varepsilon^2 (1 + |k|^2) \left(1 + (2\pi L^{-1})^4 |k|^4\right),$$

$$\|\partial_j u\|_2^2 = \sum (\zeta_k, \zeta_k)\, |k_j|^2 (2\pi L^{-1})^2 \left(1 + (2\pi L^{-1})^4 |k|^4\right)$$

for $u = \sum \zeta_k \exp(i2\pi L^{-1}k \cdot x) \in H_3(L)$, entailing the existence of a constant c_0 with $\|\partial_j u\|_2 < c_0 \|\tilde{E}_\varepsilon^{1/2}u\|_2$ for $u \in H_3(L)$. By property (P3) of $H_2^1(L)$ there is another constant c_1 such that

$$\|B_j u\|_2 \leqq c_1 \|u\|_2 \quad \text{and} \quad \left\|\left(A + (2\pi L^{-1})^2 \varepsilon^2 I\right) u\right\|_2 \leqq c_1 \|u\|_2$$

for $u \in H_2(L)$. Combining these results we will find a constant c such that $\|Mu\|_2 \leqq c\, \|\tilde{E}_\varepsilon^{1/2}u\|_2$ for $u \in H_3(L)$ verifying (ii) ∎

Lemma 2: *The resolvents of E are compact.*

Proof: Let λ belong to the (non-empty) resolvent set of E, let for the moment $H_4'(L)$ be $H_4(L)$, but provided with the norm $\|\cdot\|_2$. By Lemma 1, $E - \lambda$ is closed, mapping its domain $H_4'(L)$ onto $H_2(L)$. The operator $E - \lambda$ is then closed as a mapping from $H_4(L)$ onto $H_2(L)$, because $\|u\|_2 \leqq \|u\|_4$. Since $(E - \lambda)^{-1}$ maps $H_2(L)$ onto $H_4(L)$, $(E - \lambda)^{-1}$ is also closed and consequently bounded, i.e. $\|(E - \lambda)^{-1} u\|_4 < c\|u\|_2$ for a suitable c. Now $H_4(L)$ is compactly embedded in $H_2(L)$, by property (P1) in Section 1.1. By a combination of these facts the lemma will be proved ∎

$F(\alpha, u)$ being polynomial in α, u_1, u_2 and $H_2(L)$ being a Banach algebra, known results on local existence and uniqueness apply to the evolution equation

$$\dot{u} = Eu + F(\alpha, u) \tag{5}$$

(see e.g. [9: Theorem 1.5/p. 187]). For $v \in H_2(L)$, let $d_uF(\alpha, v)$ be the Jacobian of F with respect to u at v. The operator $E' = E + d_uF(\alpha, v)$ being of the same type as E consequently has the properties described in Lemmata 1 and 2. Therefore the principle of linearized stability (see [6: Chapter 5]) applies to (5), i.e. if $u_0 \in H_4(L)$ is an equilibrium solution of (5), $Eu_0 + F(\alpha, u_0) = 0$, then

(i) u_0 is asymptotically stable if $\operatorname{Re} \lambda < 0$ for all $\lambda \in \sigma(E + d_uF(\alpha, u_0))$,
(ii) u_0 is unstable if $\operatorname{Re} \lambda > 0$ for some $\lambda \in \sigma(E + d_uF(\alpha, u_0))$.

Remark: If we consider E as an operator acting on $H_0(L)$, with Dom $E = H_2(L)$, then E still has the properties stated by Lemmata 1 and 2. But F is not a smooth mapping from $H_0(L)$ to $H_0(L)$. The principle of linearized stability then only holds under more complicated circumstances, involving fractional operators (see e.g. [8] for detail). One might expect some equilibrium solutions $u \in H_2(L) - H_4(L)$ to get lost. However, slight extensions of the regularity results in [5: Lecture 5] exclude this possibility as can be shown.

3. Bifurcating branches

3.1 The bifurcation problem

Below the operators $E_2 = D\Delta$ and ∂_j ($j = 1, 2$) are the same as in Section 2.2. For $a = (a_1, a_2) \in \mathbf{R}^2$ and $f \in H_3^1(L)$ we set

$$(a\nabla) f = a_1 \partial_2 f + a_2 \partial_2 f.$$

The matrices $B_j(x)$ are now taken to be constant. The operator $E_1 = \sum B_j(x) \partial_j$ can then be cast into the form

$$B = ((b^{ij}\nabla)), \qquad b^{ij} \in \mathbf{R}^2 \qquad \text{for} \qquad i, j = 1, 2,$$

transforming $u = (u_1, u_2) \in H_3(L)$ into $\left(\sum (b^{1j}\nabla) u_j, \sum (b^{2j}\nabla) u_j\right)$. The operator E_0 will always be a Fréchet derivative

$$E_0 = d_uF(\alpha, v), \qquad v \in H_4(L),$$

where $F(\alpha, u) = (F_1(\alpha, u_1, u_2), F_2(\alpha, u_1, u_2))$ is polynomial in α, u_1, u_2, of degree $k \geqq 2$ in u_1, u_2. The period L is kept fixed in this chapter.

We now introduce the first of a series of assumptions enabling us to apply the apparatus of bifurcation theory.

(A 1) There is a number $\alpha_0 \in \mathbf{R}$, an interval $I = (-\varepsilon, \varepsilon)$ and an analytic mapping $u: I \to \mathbf{R}^2$ such that $F(\alpha_0 + \delta, u(\delta)) = 0$ for all $\delta \in I$.

Based on this assumption we define the operators

$$T(\delta) = D\Delta + B + d_u F(\alpha_0 + \delta, u(\delta)), \qquad \delta \in I;$$

acting on $H_2(L)$ with Dom $T(\delta) = H_4(L)$. They have the properties listed in Lemmata 1 and 2. Set

$$E(k) = \{\zeta \exp(i2\pi L^{-1}k \cdot x) \mid \zeta \in \mathbf{C}^2\}.$$

Since $H_2(L) = \sum \oplus E(k)$, the operators $T(\delta)$ are determined by their action on the invariant subspaces $E(k)$, which is described in terms of matrices $B(k)$, $M_L(k, \delta)$, defined as follows:

1. $B(k) = (b^{ij}k)$, where $b^{ij} \cdot k = b_1{}^{ij}k_1 + b_2{}^{ij}k_2$,
2. $M_L(k, \delta) = -(2\pi L^{-1})^2 \, |k|^2 \, D + i2\pi L^{-1}B(k) + d_u F(\alpha_0 + \delta, u(\delta))$;
3. $T(\delta)\left(\sum \zeta_k \exp(i2\pi L^{-1}k \cdot x)\right) = \sum \left(M_L(k, \delta)\, \zeta_k\right) \exp(i2\pi L^{-1}k \cdot x)$.

After these preparations we state the following

Lemma 3: *The spectrum $\sigma_L(T(\delta))$ can be described in terms of the spectra $\sigma(M_L(k, \delta))$ as $\sigma_L(T(\delta)) = \cup\, \sigma(M_L(k, \delta))$.*

(A 2): Among the matrices $M_L(k, 0)$, $k \in \mathbf{Z}^2$, there is exactly one, say $M_L(k^0, 0)$, such that $0 \in \sigma(M_L(k^0, 0))$. Moreover we assume
(i) 0 is a simple eigenvalue of $M_L(k^0, 0)$,
(ii) if $\eta_0{}^t M_L(k^0, 0) = 0$, $M_L(k^0, 0)\, \zeta_0 = 0$ and $\eta_0, \zeta_0 \neq 0$, then $(\eta_0, B_1\zeta_0) \neq 0$ where B_1 is the linear term in the expansion

$$d_u F(\alpha_0 + \delta, u(\delta)) = d_u F(\alpha_0, u(0)) + \delta B_1 + O(\delta^2).$$

The "wave" vector k^0 and the eigenvectors η_0, ζ_0 in this assumption are kept fixed henceforth. By assumption (A 2) there is a real analytic function λ, $\lambda(\delta) = \lambda_1\delta + O(\delta^2)$, such that

(i) $\lambda(\delta)$ is a simple eigenvalue of $M_L(k^0, \delta)$,
(ii) $\lambda_1 = (\eta_0, \zeta_0)^{-1}\, (\eta_0, B_1\zeta_0) \neq 0$.

Since $M_L(k, 0) \neq M_L(k^0, 0)$ if $|k|^2 \neq |k^0|^2$, $T(\delta)$ has a real eigenvalue $\lambda(\delta)$, crossing $\lambda = 0$ at non-zero speed, while $\sigma_L(T(\delta)) - \{\lambda(\delta)\}$ is bounded away from $\lambda = 0$ for small δ. By assumption (A1) on the other hand $u(\delta)$, $\delta \in I$, is a trivial solution branch of the equilibrium equation $D\Delta u + Bu + F(\alpha_0 + \delta; u) = 0$. This suggests to look for non-trivial bifurcating solution branches and to investigate their stability. Two cases will be discussed:

(A) $F(\alpha, u)$ is a gradient, $k^0 = 0$ and the operator B is symmetric, i.e. $b^{12} = b^{21}$;
(B) $F(\alpha, u)$ and $k^0 \neq 0$ are arbitrary but $B = 0$.

Case (A) is related to the Landau-Ginzburg equations treated in [3, 4]. Case (B) has the most interesting features and is discussed at length. The general case $B \neq 0$ is not considered since it is "generically" simpler than case (B) and does not give new insights.

3.2 Liapunov-Schmidt equations

Lemmata 1 and 2, and (A1), (A2) enable us to introduce Liapunov-Schmidt equations. Let $k^0, \ldots, k^s \in \mathbf{Z}^2$ be the list of all different wave vectors such that $M_L(k^0, 0) = \cdots = M_L(k^s, 0)$. The eigenspace of the Fréchet derivative $T(0)$ at the eigenvalue 0 is spanned by the elements $f_j = \zeta_0 \exp(i2\pi L^{-1}k^j \cdot x)$, $j = 0, \ldots, s$.

Lemma 4: *$T(0)$ has index 0 at the eigenvalue 0.*

Proof: Assume $-\lambda \in \varrho(T(0))$ (with $\varrho(T(0))$ the resolvent set of $T(0)$); define $G_\lambda = (T(0) + \lambda)^{-1}$. The lemma states, if $(\lambda G_\lambda - 1) f = g$ and $(\lambda G_\lambda - 1) g = 0$, then $(\lambda G_\lambda - 1) f = 0$. Thus let f, g satisfy the first two equations. Then $f, g \in H_4(L)$ and satisfy

$$T(0)\, g = 0, \tag{6}$$

$$T(0)\, f + \lambda g = 0. \tag{7}$$

By (6),

$$g = \sum_0^s \alpha_j \zeta_0 \exp(i2\pi L^{-1}k^j \cdot x) \qquad \text{for suitable } \alpha_j \in \mathbf{C}.$$

With the Fourier expansion $f = \sum \xi_k \exp(i2\pi L^{-1}k \cdot x)$ we can express (7) as

$$0 = \sum{}' M_L(k, 0)\, \xi_k \exp(i2\pi L^{-1}k \cdot x) + \sum_{j=0}^{s} \big(M_L(k^0, 0)\, \xi_{k^j} + \lambda\alpha_j\zeta_0\big) \exp(i2\pi L^{-1}k^j \cdot x) \tag{8}$$

where $\sum'$ means the summation over vectors $k \notin \{k^0, \ldots, k^s\}$. From (8) we get

$$\xi_k = 0 \quad \text{for} \quad k \notin \{k^0, \ldots, k^s\}, \qquad M_L(k^0, 0)\, \xi_{k^j} + \lambda\alpha_j\zeta_0 = 0. \tag{9}$$

Now $\lambda \neq 0$, $\eta_0{}^t M_L(k^0, 0) = 0$ and $(\eta_0, \zeta_0) \neq 0$ by (A2). This together with (9) implies $\alpha_j = 0$ and $\xi_{k^j} = \beta_j \xi_0$ for suitable $\beta_j \in \mathbf{C}$. Thus $f = \sum \beta_j \zeta_0 \exp(i2\pi L^{-1}k^j \cdot x)$ is itself an eigenfunction of $T(0)$ at 0, proving the lemma ∎

Next we need some notation. Set $f_j^* = \eta_0 \exp(i2\pi L^{-1}k^j \cdot x)$, $j = 0, \ldots, s$. The equations

$$Pf = \sum \langle f_j^*, f_j\rangle^{-1} \langle f_j^*, f\rangle f_j \qquad \text{and} \qquad K = I - P$$

define bounded projection operators P, K onto the eigenspace $\{g \mid T(0)\, g = 0\}$ and range Ran $T(0)$, respectively. Moreover $T(0)$ has a bounded inverse $G = \big(T(0)\, K\big)^{-1}$ with Dom G = Ran G = Ran $T(0)$. Our aim is to find small pairs δ, w solving the equation

$$D\Delta w + Bw + F\big(\alpha_0 + \delta, u(\delta) + w\big) = 0. \tag{10}$$

By using the decomposition $Pw = \sum \alpha_j f_j$, $Kw = g$, we can replace (10) by the equivalent set of $s + 2$ equations

$$g + GK \left\{B(\delta) \left(\sum \alpha_j f_j + g\right) + \sum_{p=2}^{m} R_p(\delta) \left(\sum \alpha_j f_j + g\right)^p\right\} = 0,$$

$$\left\langle f_j^*, \left\{B(\delta) \left(\sum \alpha_j f_j + g\right) + \sum_{p=2}^{m} R_p(\delta) \left(\sum \alpha_j f_j + g\right)^p\right\}\right\rangle = 0; \tag{11}$$

$j = 0, \ldots, s.$

Thereby,

$$F(\alpha_0 + \delta, u(\delta) + w) = F(\alpha_0 + \delta, u(\delta)) + d_u F(\alpha_0 + \delta, u(\delta)) w + \sum_{p=2}^{m} R_p(\delta) w^p$$

and

$$d_u F(\alpha_0 + \delta, u(\delta)) = d_u F(\alpha_0, u(0)) + B(\delta).$$

By well-known theorems a small solution pair δ, w of (11) is a small solution pair of (10) and conversely (see e.g. [11, 12] for detail). If we look for solutions lying in a suitable invariant subspace H, we get Liapunov-Schmidt equations having exactly the same form as (11), except that the set $\{f_0, \ldots, f_s\}$ of eigenfunctions is replaced by another set which spans $\{f \mid T(0) f = 0\} \cap H$.

3.3 Standing waves

Prior to coming to the topic of this paragraph, we quickly get rid of case (A) in Section 3.1. A glance at the assumptions (A 1), (A 2) and case (A) shows that we are led to seek bifurcating solutions of the equation $F(\alpha_0 + \delta, u(\delta) + w) = 0$, $w \in \mathbf{R}^2$, a situation known as "bifurcation at a simple eigenvalue". We content ourselves summarizing the facts and omitting the evident proof.

Lemma 5: *Under the assumption of case* (A) *the Liapunov-Schmidt equations* (11) *have a unique bifurcating solution branch, namely a real analytic pair* $\delta(r) \in \mathbf{R}$, $w(r) \in \mathbf{R}^2$, *with* r *from a neighbourhood of* $r = 0$, *which satisfies* $F(\alpha_0 + \delta(r), u(\delta(r)) + w(r)) = 0$, *for every* r, *and* $\delta(0) = 0$, $w(0) = 0$.

In order to lessen the high degeneracy prevailing under the assumptions of case (B) we shall restrict ourselves from now on to the invariant subspace $H_2^e(L)$ of even functions. An element $u \in H_2^e(L)$ is given by its "even" Fourier series

$$u = \sum \zeta_k \cos(2\pi L^{-1} k \cdot x)$$

where

$$\zeta_k = \zeta_{-k} \quad \text{and} \quad \|u\|_2^2 = \frac{1}{2} \sum |\zeta_k|^2 \left(1 + (2\pi L^{-1})^4 |k|^4\right).$$

The action of $T(\delta)$ on $u \in H_4(L) \cap H_2^e(L)$ is described by

$$T(\delta)\, u = \sum M(k, \delta)\, \zeta_k \cos(2\pi L^{-1} k \cdot x)$$

where now

$$M(k, \delta) = -(2\pi L^{-1})^2 |k|^2 D + d_u F(\alpha_0 + \delta, u(\delta)).$$

Next, let $k^0 \neq 0$ be as postulated by (A 2) and case (B). A set $\{k^0, \ldots, k^N\} \subset \mathbf{Z}^2$ of "wave" vectors is called *complete* if

1. $|k^j|^2 = |k^0|^2$, $j = 1, \ldots, N$,
2. $|k|^2 = |k^0|^2$, $k \in \mathbf{Z}^2$, implies $k = \pm k^j$ for some index j,
3. there are no $i \neq j$ with $k^i = \pm k^j$.

We assume a complete set to be given in a fixed way and define, for $j = 0, \ldots, N$,

$$\varphi_j(x) = \zeta_0 \cos(2\pi L^{-1} k^j \cdot x) \quad \text{and} \quad \varphi_j^*(x) = \eta_0 \cos(2\pi L^{-1} k^j \cdot x).$$

The set $\{\varphi_0, \ldots, \varphi_N\}$ spans the eigenspace $H_2^e(L) \cap \{f \mid T(0) f = 0\}$, while the set $\{\varphi_0^*, \ldots, \varphi_N^*\}$ forms a dual basis in $H_2^e(L)$. Moreover

$$\langle \varphi_i^*, \varphi_j \rangle = \frac{1}{2} (\eta_0, \zeta_0) \left(1 + (2\pi L^{-1})^4 |k^0|^4\right) \delta_{ij}.$$

By proper scaling we achieve $\langle \varphi_i^*, \varphi_j \rangle = \delta_{ij}$, what will be assumed henceforth.

For bifurcational purposes, $H_2^e(L)$ is still too large. A tool which provides suitable smaller subspaces is given by the lemma below. For a set $\chi \subseteq \mathbf{Z}^2$, let $\chi^* \subseteq \mathbf{Z}^2$ be the smallest set containing χ, closed against integer linear combinations. Let $H_2^e(L/\chi)$ be the closed subspace of $H_2^e(L)$ comprehending precisely those elements $u \in H_2^e(L)$ with a Fourier series of the form $\sum \zeta_k \cos (2\pi L^{-1} k \cdot x)$, $k \in \chi^*$.

Lemma 6: *With the help of the previous definitions we can state:*

(i) *If $u \in H_2^e(L/\chi) \cap H_4(L)$, then $D\Delta u \in H_2^e(L/\chi)$.*
(ii) *If $u \in H_2^e(L/\chi)$, then $Au \in H_2^e(L/\chi)$ for any constant 2×2 matrix A.*
(iii) *If $u \in H_2^e(L/\chi)$, then $F(u) \in H_2^e(L/\chi)$.*

Proof: Clauses (i) and (ii) are evident. Since F is polynomial, (iii) holds for finite sums $\sum \zeta_k \cos (2\pi L^{-1} k \cdot x)$, $k \in \chi^*$. By the Banach-algebra property of the scalar space $H_2^1(L)$ there is for every c_0 a c_1 such that $\|F(u) - F(v)\|_2 \leqq c_1 \|u - v\|_2$, provided $\|u\|_2, \|v\|_2 \leqq c_0$. Then clause (iii) can be proved by approximation for arbitrary $u \in H_2^e(L/\chi)$ ∎

Standing waves are obtained by taking for χ any of the sets $\chi_i = \{k^i\}$, $i = 0, \ldots, N$. Evidently, $\{f \mid T(0) = 0\} \cap H_2^e(L/\chi_i) = \{a\varphi_i \mid a \in \mathbf{C}\}$. In order to find bifurcating branches of our basic equation (10) in Section 3.2 which lie in $H_2^e(L/\chi_i)$ we formulate the Liapunov-Schmidt equations in $H_2^e(L/\chi_i)$

$$0 = g + GK \left\{ B(\delta)\,(r\varphi_i + g) + \sum_{p=2}^{m} R_p(\delta)\,(r\varphi_i + g)^p \right\}, \tag{12}$$

$$0 = \left\langle \varphi_i^*, B(\delta)\,(r\varphi_i + g) + \sum_{p=2}^{m} R_p(\delta)\,(r\varphi_i + g)^p \right\rangle. \tag{13}$$

Here, K and $G = \big(T(0)\,K\big)^{-1}$ are tacitly restricted to $H_2^e(L/\chi_i)$ and $g \in H_2^e(L/\chi_i)$ satisfies $\langle \varphi_i^*, g \rangle = 0$.

Theorem 1: *In $H_2^e(L/\chi_i)$ the equation $D\Delta w + F\big(\alpha_0 + \delta, u(\delta) + w\big) = 0$ has a unique, real analytic solution branch $\delta(r) = r^2\bar{\tau}(r)$, $w(r) = r\varphi_i + r^2\bar{h}(r)$, with $\langle \varphi_i^*, \bar{h}(r) \rangle = 0$ and small r. The values $\bar{\tau}(0)$, $\bar{h}(0)$ are determined by the equations*

$$\begin{aligned} &\bar{h}(0) + GKR_2(0)\,\varphi_i^2 = 0, \\ &\bar{\tau}(0)\,\langle \varphi_i^*, B_1\varphi_i \rangle + 2\langle \varphi_i^*, R_2(0)\,\varphi_i \bar{h}(0) \rangle + \langle \varphi_i^*, R_3(0)\,\varphi_i^3 \rangle = 0. \end{aligned} \tag{14}$$

Proof: We show that the Liapunov-Schmidt equations (12), (13) have a unique solution branch in $H_2^e(L/\chi_i)$ with the required properties. By the Implicit Function Theorem, (12) admits a unique, real analytic solution $g(r, \delta)$. Inspection shows that $g(0, \delta) = 0$. By setting $g = rh$ we obtain from (12) the following equation for h

$$0 = h + GK \left\{ B(\delta)\,(\varphi_i + h) + \sum_{p=2}^{m} R_p(\delta)\, r^{p-1} (\varphi_i + h)^p \right\}. \tag{15}$$

In order to evaluate (13) we recall that $B(\delta) = B_1\delta + B_2\delta^2 + \cdots = B_1\delta + B_2(\delta)$. By inserting $g = rh$ into (13) we get after some rearrangements

$$0 = \delta\langle \varphi_i^*, B_1, \varphi_i \rangle + \left\langle \varphi_i^*, B_2(\delta)\,\varphi_i + B(\delta)\,h + \sum_{p=2}^{m} R_p(\delta)\, r^{p-1} (\varphi_i + h)^p \right\rangle. \tag{16}$$

Now $(\eta_0, B_1\zeta_0) \neq 0$ and hence $\langle \varphi_i^*, B_1\varphi_i \rangle \neq 0$ by assumption (A2). A glance at (15) on the other hand shows that $h(0, 0) = 0$, i.e. $h = rh_1 + \delta h_2$. From this and the Implicit Function Theorem it follows that (16) admits a unique real analytic solution $\delta = \delta(r)$ for small r, which satisfies $\delta(0) = 0$. Thus $\delta(r) = r\tau(r)$ for real analytic τ, and $h(r, \delta(r)) = r(h_1 + \tau(r)\, h_2) = r\tilde{h}(r)$. By inserting $r\tau(r)$ into (16) and by comparing powers in r we obtain $0 = \tau(0)\, \langle \varphi_i^*, B_1\varphi_i \rangle + \langle \varphi_i^*, R_2(0)\, \varphi_i^2 \rangle$. Since $\langle \varphi_i^*, R_2(0)\, \varphi_i^2 \rangle$ is easily seen to vanish, we obtain $\tau(r) = r\bar{\tau}(r)$ for real analytic $\bar{\tau}$. Equations (14) now follow if we substitute $r\tilde{h}$ for h in (15) and $r^2\bar{\tau}$ for δ in (16) by comparison of powers in r ∎

Remark: If we set $u = u(\delta(r)) + w(r)$, then we obtain the standing wave solutions mentioned in the introduction. To make it evident, we observe that u has a Fourier series of the form

$$\sum \zeta_p \cos(2\pi L^{-1} p k^i \cdot x), \qquad p \in \mathbf{Z}.$$

Thus if we set $\Phi(z) = \sum \zeta_p \cos(pz)$, $p \in \mathbf{Z}$, then $u = \Phi(2\pi L^{-1} k^i \cdot x)$. Since $\sum |\zeta_p|^8 < \infty$, Φ is in $C^2[0, 2\pi]$ and, being a solution of a suitable ordinary differential equation, it is also analytic. The stability of these standing wave solutions will be our main concern. For simplicity we refer to the pair $\delta(r)$, $w(r) \in H_2^e(L/\chi_i)$ as the *i-th standing wave branch* or the *standing wave branch associated with* k^i.

4. Stability properties

As pointed out in the introduction and elaborated in the next chapter, there is a tendency toward instability. Nevertheless positive results exist on standing waves, which hold if the nonlinearity $F(\alpha, u)$ satisfies an additional assumption. We recall that by assumptions (A1) and (A2) there is an analytic function λ, $\lambda(\delta) = \lambda_1\delta + O(\delta^2)$, such that $\lambda(\delta)$ is a simple eigenvalue of $M_L(k^0, \delta)$. In order to establish the familiar context of stability we need a further assumption

(A3) $\qquad \lambda_1 > 0 \quad \text{and} \quad \sigma_L(T(0)) - \{0\} \subseteqq \{z \mid \operatorname{Re} z < 0\}.$

This assumption may hold for some period L but not for an other $L' \neq L$. We are allowing for this fact, saying that $T(\delta)$ satisfies (A3) in $H_2^e(L)$. For notational simplicity we assume in this paragraph $\alpha_0 = 0$ and $u(0) = 0$.

Let $\delta_i(r)$, $w_i(r)$ be the standing wave branch associated with the wave vector k^i according to Theorem 1. A stability analysis amounts to a study of the spectrum of the Fréchet derivative $D\Delta + d_uF(\delta_i(r), u(\delta_i(r)) + w_i(r))$ in a neighbourhood of $r = 0$. The first step is provided by the theory of analytic perturbations of an isolated eigenvalue of finite multiplicity due to RELLICH [7, 10]. To start with, we set $d_uF(\delta_i(r), u(\delta_i(r)) + w_i(r)) = G^i(r)$ and observe that $G^i(r)$ is an analytic family of bounded linear operators mapping $H_2^e(L)$ into itself. Thus $G^i(r)$ admits an expansion in powers of r

$$G^i(r) = d_uF(0, 0) + D^i(r), \qquad D^i(r) = rG_r^i(0) + \frac{1}{2} r^2 G_{rr}^i(0) + \cdots.$$

The perturbed eigenvalue equation to study then is

$$\left(T(0) + D^i(r)\right)\left(\sum_0^N \zeta_j\varphi_j + g\right) = \lambda\left(\sum_0^N \zeta_j\varphi_j + g\right),$$
$$\langle \varphi_j^*, g \rangle = 0 \quad \text{for} \quad j = 0, \ldots, N.$$

By a straigtforward computation omitted for reasons of space we obtain the equation for the perturbed eigenvalue λ

$$0 = \operatorname{Det}\left(\lambda E + \left(\langle \varphi_j^*, -D^i(r)\,\varphi_k + D^i(r)\left(1 - GK(\lambda - D^i(r))\right)^{-1} GKD^i(r)\,\varphi_k \rangle\right)\right).$$

In a next step we will show that the matrix whose determinant is taken in this equation is in fact diagonal.

Lemma 7: *Let $k^0, k^1, k^2 \in \mathbf{Z}^2$ be wave vectors such that*

(i) $k^1 \neq \pm k^2$,
(ii) $|k^0|^2 = |k^1|^2 = |k^2|^2$,
(iii) $k^0 = \alpha k^1 + \varepsilon k^2$ *for some* $\alpha \in \mathbf{Z}$ *and* $\varepsilon \in \{-1, 1\}$.

Then $\alpha = 0$ *and* $k^0 = \pm k^2$.

Proof: Let α, ε satisfy (iii) and assume $\alpha \neq 0$. From (ii), (iii) and Schwarz's inequality we infer

$$|\alpha| \, |k^1|^2 = 2 \, |(k^1, k^2)| \leqq 2 \, |k^1| \, |k^2| = 2 \, |k^1|^2 \tag{17}$$

and thus $|\alpha| \leqq 2$.

Case $|\alpha| = 2$: According to (17) we have $|(k^1, k^2)| = |k^1| \, |k^2|$ and thus by Schwarz's equality case $k^2 = \lambda k^1$ with $\lambda \in \{-1, 1\}$ by assumption (ii) contradicting (i).

Case $|\alpha| = 1$: Since $|\varepsilon| = 1$ we immediately obtain from (ii), (iii)

$$2 \, |(k^1, k^2)| = |k^1|^2 = |k^1| \, |k^2| . \tag{18}$$

Now let $k^1 = (a, b)$, $k^2 = (c, d)$. Without loss of generality we may assume $b \geqq 0$, $d \geqq 0$. In addition let n, μ and n, ϱ be the polar coordinates of k^1 and k^2, respectively (i.e. $n^2 = |k^1|^2 = |k^2|^2$); without loss of generality we may also assume $\mu < \varrho$. Then we infer from (18) $\tan (\varrho - \mu) = \sqrt{3}$.

Subcase $a \neq 0$ *and* $c \neq 0$: Since $ad - bc = 0$ we obtain by the Addition Theorem of Tangents $dc^{-1} = \left(\sqrt{3} + ba^{-1}\right) \left(1 - \sqrt{3}\, ba^{-1}\right)^{-1}$ and thus $\sqrt{3} = (ad - bc)(ac + bd)^{-1}$, contradicting the irrationality of $\sqrt{3}$.

Subcase $a = 0$: Here we have $dc^{-1} = \tan \varrho = -\left(\sqrt{3}\right)^{-1}$, in contradiction with the irrationality of $\sqrt{3}$.

Subcase $c = 0$: Then $\mu = \pi/6$ and thus $ba^{-1} = \tan (\pi/6) = 1/\sqrt{3}$, giving a contradiction again ∎

For future use we introduce the following convention: An even element f from any of our Sobolev spaces is said to *have its wave vectors in the set* $S \subseteqq \mathbf{Z}^2$ if its Fourier series has the form $f = \sum \xi_k \cos (2\pi L^{-1} k \cdot x)$, $k \in S$.

Lemma 8: *Let k^i, k^j $(i \neq j)$ be two wave vectors from the complete set and $f \in H_2^c(L)$ having its wave vectors in the set $\{\alpha k^i + k^j \mid \alpha \in \mathbf{Z}\}$. Let $B = \left(b_{pq}(x)\right)$ be a 2×2 matrix whose even entries belong to $H_2^1(L)$ and have their wave vectors in $\{\alpha k^i \mid \alpha \in \mathbf{Z}\}$. Then Kf, $(GK)^{-1} Kf$ and Bf will have their wave vectors in $\{\alpha k^i + k^j \mid \alpha \in \mathbf{Z}\}$, too.*

Proof: That Kf and $(GK)^{-1} Kf$ have their wave vectors in the set mentioned above, follows from the fact that both K and $(GK)^{-1} K$ leave the subspaces $\{\xi \cos (2\pi L^{-1} \times k \cdot x) \mid \xi \in \mathbf{C}^2\}$ invariant for any $k \in \mathbf{Z}^2$. In order to prove the lemma for Bf it suffices to demonstrate that if $a(x), b(x)$ are even members of the scalar space $H_2^1(L)$ with $a(x)$ having its wave vectors in $\{\alpha k^i \mid \alpha \in \mathbf{Z}\}$ and $b(x)$ having its wave vectors in $\{\alpha k^i + k^j \mid \alpha \in \mathbf{Z}\}$, then $a(x)\, b(x)$ will have its wave vectors in $\{\alpha k^i + k^j \mid \alpha \in \mathbf{Z}\}$. Now this evidently holds if $a(x), b(x)$ are even trigonometric polynomials. The general case then follows from the Banach-algebra property of $H_2^1(L)$ by an approximation argument ∎

Lemma 9: *The spectrum of the Fréchet derivative $T(0) + D^i(r)$ in a neighbourhood of zero is decribed by a set $\{\lambda^{ij} \mid j = 0, \ldots, N\}$ of functions, real analytic in a neighbourhood of $r = 0$, such that*

(i) $\lambda^{ij}(0) = 0$,
(ii) $\lambda^{ij}(r)$ *is the unique local solution of*

$$0 = \lambda + \langle \varphi_j^*, -D^i(r) \, \varphi_j + D^i(r) \left(1 - GK(\lambda - D^i(r))\right)^{-1} GKD^i(r) \, \varphi_j \rangle .$$

Proof: Set

$$f_p^i = -D^i(r)\,\varphi_p + D^i(r)\left(1 - GK(\lambda - D^i(r))\right)^{-1} GKD^i(r)\,\varphi_p.$$

As pointed out above, the spectrum of the Fréchet derivative $T(0) + D^i(r)$ in a neighbourhood of zero is given by the solutions of the determinant equation

$$0 = \operatorname{Det}\left(\lambda E + (\langle \varphi_j^*, f_p^i \rangle)\right),$$

with E being the identity. The lemma is proved if we can show that the matrix $(a_{jp}) = (\langle \varphi_j^*, f_p^i \rangle)$ is in fact diagonal. Thus assume $p \neq j$. For λ, r small,

$$\left(1 - GK(\lambda - D^i(r))\right)^{-1} = \sum \left(GK(\lambda - D^i(r))\right)^n.$$

Now $\left(GK(\lambda - D^i(r))\right)^n GKD^i(r)\,\varphi_p$ and $D^i(r)\,\varphi_p$ have their wave vectors in the set $\{\alpha k^i + k^p \mid \alpha \in \mathbf{Z}\}$ according to Lemma 8. Since the space of elements in $H_2^e(L)$ having their wave vectors in the mentioned set is closed, f_p^i has its wave vectors in this set, too. Since $j \neq p$, $k^p \neq \pm k^j$. Thus by Lemma 7, $\alpha k^i + k^p \neq \pm k^j$ and hence $\langle \varphi_j^*, f_p^i \rangle = 0$. From this, existence and uniqueness of the functions λ^{ij} follow immediately from the Implicit Function Theorem ∎

As a preparation to the main theorem we note

Lemma 10: *For the functions λ^{ij} the equations $(d\lambda^{ij}/dr)\,(0) = 0$ hold.*

Proof: By Lemma 9, $\lambda^{ij}(r) = \lambda_1^{ij} r + O(r^2)$. From assertion (ii) in Lemma 9 we infer $\lambda_1^{ij} = \langle \varphi_j^*, D_r^i(0)\,\varphi_j \rangle$ where

$$D_r^i = \frac{dD^i}{dr} = \frac{dG^i}{dr} = d_{u\delta}F\left(\delta_i(r), u(\delta_i(r)) + w_i(r)\right)\frac{d\delta_i}{dr}$$

$$+ d_{uu}F\left(\delta_i(r), u(\delta_i(r)) + w_i(r)\right)\left(\frac{du}{d\delta}\frac{d\delta_i}{dr} + \frac{dw_i}{dr}\right).$$

Now $(d\delta_i/dr)\,(0) = 0$ and $(dw_i/dr)\,(0) = \varphi_i$ according to Theorem 1. Thus $\lambda_1^{ij} = \langle \varphi_j^*, d_{uu}F(0,0)\,\varphi_i\varphi_j \rangle$, which is easily seen to vanish ∎

Theorem 2: *Assume that besides assumptions* (A 1)–(A 3)

$$d_{uu}F(0,0) = 0 \quad \text{and} \quad \left(\eta_0, d_{uuu}F(0,0)\,\zeta_0^3\right) \neq 0.$$

(i) *If the i-th branch of standing waves $\delta_i(r)$, $w_i(r)$ is stable in $H_2^e(L/\{k^i\})$ for small $r \neq 0$ (i.e. against small perturbations belonging to $H_2(L/\{k^i\})$, then it will be stable in $H_2(L)$ and every other branch of standing waves $\delta_m(r)$, $w_m(r)$ will be stable in $H_2^e(L)$.*

(ii) *If $\sigma_{nL}(T(0)) - \{0\} \subseteq \{z \mid \operatorname{Re} z < 0\}$ for a particular integer $n > 1$ and the i-th standing wave branch $\delta_i(r)$, $w_i(r)$ is stable in $H_2^e(L/\{k^i\})$, then every branch of standing waves $\delta_m(r)$, $w_m(r)$ is stable in $H_2^e(nL)$ (i.e. against nL-periodic perturbations).*

Proof: To start with, we compute the coefficient λ_2^{ij} in the expansion $\lambda^{ij}(r) = \lambda_2^{ij} r^2 + \cdots$ (see Lemma 10), ignoring for the moment whether $i = j$ or $i \neq j$. From Lemma 9/(ii) we infer that λ_2^{ij} has the form

$$\lambda_2^{ij} = \frac{1}{2}\langle \varphi_j^*, D_{rr}^i(0)\,\varphi_j \rangle - \langle \varphi_j^*, D_r^i(0)\,GKD_r^i(0)\,\varphi_j \rangle.$$

Here is, as before

$$D^i(r) = G^i(r) - G^i(0) \quad \text{and} \quad G^i(r) = d_uF\left(\delta_i(r), u(\delta_i(r)) + w_i(r)\right).$$

From the expression for $D_r^i (= dD^i/dr)$ given in the proof of Lemma 10 it follows that $D_{rr}^i (= d^2D^i/dr^2)$ is a sum $t_1 + \cdots + t_6$ where

$$t_1(r) = d_{u\delta}F(\delta_i(r), f_i(r)) \frac{d^2\delta_i}{dr^2},$$

$$t_2(r) = d_{\delta\delta u}F(\delta_i(r), f_i(r)) \left(\frac{d\delta_i}{dr}\right)^2,$$

$$t_3(r) = d_{\delta uu}F(\delta_i(r), f_i(r)) \frac{d\delta_i}{dr} \left(\frac{du}{d\delta}\frac{d\delta_i}{dr} + \frac{dw_i}{dr}\right),$$

$$t_4(r) = d_{uu}F(\delta_i(r), f_i(r)) \left(\frac{du^2}{d\delta^2}\left(\frac{d\delta_i}{dr}\right)^2 + \frac{du}{d\delta}\left(\frac{d^2\delta_i}{dr^2}\right) + \frac{d^2w_i}{dr^2}\right),$$

$$t_5(r) = d_{\delta uu}F(\delta_i(r), f_i(r)) \frac{d\delta_i}{dr} \left(\frac{du}{d\delta}\frac{d\delta_i}{dr} + \frac{dw_i}{d\delta}\right),$$

$$t_6(r) = d_{uuu}F(\delta_i(r), f_i(r)) \left(\frac{du}{d\delta}\frac{d\delta_i}{dr} + \frac{dw_i}{dr}\right)^2$$

and where $f_i(r) = u(\delta_i(r)) + w_i(r)$. Since $d_{uu}F(0, 0) = 0$ by assumption, it follows from Theorem 1 that $t_p(0) = 0$ for $p = 2, 3, 4, 5$. It remains

$$D_{rr}^i(0) = 2d_{u\delta}F(0, 0)\, \bar{\tau}(0) + d_{uuu}F(0, 0)\, \varphi_i^2.$$

Now, as already shown, $D_r^i(0) = d_{uu}F(0, 0)\, \varphi_i$, i.e. $D_r^i(0) = 0$ by our assumption. Thus we get

$$\lambda_2^{ij} = \langle \varphi_j^*, d_{u\delta}F(0, 0)\, \varphi_j \rangle\, \bar{\tau}(0) + \frac{1}{2} \langle \varphi_j^*, d_{uuu}F(0, 0)\, \varphi_i^2 \varphi_j \rangle. \tag{19}$$

In order to evaluate this we recall the second equation in (14) defining $\bar{\tau}(0)$, and observe that

$$R_2(0) = d_{uu}F(0, 0), \qquad R_3(0) = d_{uuu}F(0, 0),$$

$$B_1 = d_{u\delta}F(0, 0) + d_{uu}F(0, 0) \frac{du}{d\delta}(0).$$

Since $d_{uu}F(0, 0) = 0$ we find

$$\bar{\tau}(0) = -\langle \varphi_i^*, d_{u\delta}F(0, 0)\, \varphi_i \rangle^{-1} \langle \varphi_i^*, d_{uuu}F(0, 0)\, \varphi_i^3 \rangle.$$

Inserting this into (19) and observing that $\langle \varphi_i^*, d_{u\delta}F(0, 0)\, \varphi_i \rangle = \langle \varphi_j^*, d_{u\delta}F(0, 0)\, \varphi_j \rangle$ yields

$$\lambda_2^{ij} = \frac{1}{2} \langle \varphi_j^*, d_{uuu}F(0, 0)\, \varphi_i^2 \varphi_j \rangle - \langle \varphi_i^*, d_{uuu}F(0, 0)\, \varphi_i^3 \rangle.$$

By a further evaluation based on our definitions of the scalar product we obtain

$$\lambda_2^{ii} = -\frac{3}{16} \left(\eta_0, d_{uuu}F(0, 0)\, \zeta_0^3\right) \left(1 + (2\pi L^{-1})^4 |k^i|^4\right). \tag{20}$$

If $i \neq j$, then

$$\lambda_2^{ij} = \frac{1}{4} \left(\eta_0, d_{uuu}F(0, 0)\, \zeta_0^3\right) \left(1 + (2\pi L^{-1})^4 |k^j|^4\right). \tag{21}$$

Since $|k^0|^2 = \dots = |k^N|^2$ we have

$$\lambda_2^{ii} = \lambda_2^{mm} \quad \text{for} \quad m = 0, \dots, N,$$

$$\text{if} \quad p \neq q, i \neq j, \quad \text{then} \quad \lambda_2^{pq} = \lambda_2^{ij}. \tag{22}$$

(i): Since $(\eta_0, d_{uuu}F(0, 0)\,\zeta_0^3) \neq 0$, we have $\lambda_2^{pq} \neq 0$ for all $p, q = 0, \dots, N$, by (20), (21). If the standing wave branch $\delta_i(r)$, $w_i(r)$ is stable in $H_2^e(L/\{k^i\})$ for small $r \neq 0$, then necessarily $\lambda_2^{ii} < 0$ and thus $(\eta_0, d_{uuu}F(0, 0)\,\zeta_0^3) > 0$ by (20). But (21), (22) then imply $\lambda_2^{pq} < 0$ for all $p, q = 0, \dots, N$ what entails the stability of all standing waves $\delta_m(r)$, $w_m(r)$ in $H_2^e(L)$, for $m = 0, \dots, N$.

(ii): We first observe the facts

$$|nk^0|^2 = \dots = |nk^N|^2, \qquad M_{nL}(nk, \delta) = M_L(k, \delta), \qquad k \in \mathbf{Z}^2,$$

$$H_2^e(L/\{k\}) = H_2^e(nL/\{nk\}) \quad \text{for any} \quad k \in \mathbf{Z}^2.$$

This, together with the assumptions of part (ii) implies that (A1)—(A3) apply to the wave vectors $nk^0, \dots, nk^N$. Thus every standing wave branch $\delta_p(r)$, $w_p(r)$ $(p = 0, \dots, N)$ is also a standing wave branch in $H_2^e(nL/\{nk^p\})$, now belonging to the wave vector nk^p. Eventually the set $\{nk^0, \dots, nk^N\}$ has to be extended into a complete set $\{\tilde{k}^0, \dots, \tilde{k}^M\}$ where $N \leqq M$ and $\tilde{k}^j = nk^j$ for $j \leqq N$. The eigenfunctions associated with, are

$$\tilde{\varphi}_j = \zeta_0 \cos\left(2\pi(nL)^{-1}\, \tilde{k}^j \cdot x\right), \qquad \tilde{\varphi}_j^* = \eta_0 \cos\left(2\pi(nL)^{-1}\, \tilde{k}^j \cdot x\right)$$

with $\tilde{\varphi}_j = \varphi_j$, $\tilde{\varphi}_j^* = \varphi_j^*$ for $j \leqq N$. Finally we have standing wave branches $\tilde{\delta}_p(r)$, $\tilde{w}_p(r)$, $p \leqq M$, where $\tilde{\delta}_p(r) = \delta_p(r)$, $\tilde{w}_p(r) = w_p(r)$ for $p \leqq N$. In order to determine the stability of the branches $\tilde{\delta}_p(r)$, $\tilde{w}_p(r)$ we compute again the coefficients $\tilde{\lambda}_2^{ij}$, which, as before, are determined by the equations (20)—(22), except that the scalar product in $H_2^e(nL)$ has to taken now. But since $(2\pi L^{-1})^4\, |k^i|^4 = (2\pi(nL)^{-1})^4\, |nk^i|^4$ we infer

$$\tilde{\lambda}_2^{pq} = \begin{cases} \lambda_2^{ii} & \text{for all } p = 0, \dots, M,\ p = q, \\ \lambda_2^{ij} & \text{if } p \neq q,\ p, q = 0, \dots, M,\ i \neq j \ \text{ and } \ j = 0, \dots, N. \end{cases}$$

Since $\lambda_2^{ii} < 0$ and $\lambda_2^{ij} < 0$ according to the assumed stability of $\delta_i(r)$, $w_i(r)$ in $H_2^e(L)$ we infer the stability of all branches $\tilde{\delta}_p(r)$, $\tilde{w}_p(r)$ $(p = 0, \dots, M)$ in $H_2^e(nL)$ ∎

Remarks: 1. A particular case with $d_{uu}F(0, 0) = 0$ arises if $F(0, -u) = -F(0, u)$. 2. Whether Theorem 2 applies to solution branches other than standing waves is not known to us. A verification seems to depend on non-trivial number-theoretic properties of the sets $\{(x, y) \mid x^2 + y^2 = n, x, y \in \mathbf{Z}\}$. 3. Theorem 2 can be cast into a thumb rule as follows:

(i) If $d_{uu}F(0, 0) = 0$ and if the standing wave branches $\delta_i(r)$, $w_i(r)$ are stable in $H_2^e(L)$, then they will remain stable in $H_2^e(nL)$, provided the trivial branch $u(\delta)$ remains stable in $H_2^e(nL)$ for small $\delta < 0$.

Put it the other way round:

(ii) If $d_{uu}F(0, 0) = 0$, then $\delta_i(r), w_i(r)$ can only lose stability in $H_2^e(nL)$ if the trivial branch $u(\delta)$ loses stability in $H_2^e(nL)$ for small $\delta < 0$.

5. Instability results

5.1 The gradient case

The first situation giving rise to instabilities is provided by case (A) in Section 3.1. Here we are back in the full space $H_2(L)$, the system to investigate is

$$D\Delta u + Bu + F(\alpha, u) = 0$$

with $B = ((b^{ij}\nabla))$ symmetric, i.e. $b^{ij} = b^{ji}$, and where F is a gradient, i.e. $F(\alpha, u) = \nabla V(\alpha, u)$ for some potential V. Now we have as in Section 3

$$M_L(k, \delta) = -(2\pi L^{-1})^2 |k|^2 D + i2\pi L^{-1}B(k) + d_uF(\alpha_0 + \delta, u(\delta))$$

with $B(k) = ((b^{ij}k))$, and the spectrum of $T(\delta) = D\Delta + B + d_uF(\alpha_0 + \delta, u(\delta))$ is $\sigma_L(T(\delta)) = \bigcup_k (M_L(k, \delta))$.

Lemma 11: *Let A, B be hermitean $n \times n$ matrices, λ^* the largest eigenvalue of A and ζ an eigenvalue of $A + iB$. Then* $\operatorname{Re} \zeta \leqq \lambda^*$.

Proof: Assume $(A + iB)x = \zeta x$, $(x; x) = 1$, where $(\cdot, \cdot)$ is the usual hermitean scalar product. Then $\operatorname{Re} \zeta = (x, Ax) \leqq \lambda^*$ ∎

Lemma 12: *Let $k^0 \neq 0$ in assumption* (A2). *Then there is a $\bar\lambda \in \sigma_L(T(0))$ with $\bar\lambda > 0$.*

Proof: Since F is a gradient, $C = d_uF(\alpha_0, u(0))$ is real and symmetric. Let λ^* and $\bar\lambda$ be the largest eigenvalues of $A = -(2\pi L^{-1})^2 |k^0|^2 D + C$ and C, respectively. Since D is positive, $\lambda^* < \bar\lambda$. By assumption (A2), $0 \in \sigma(M_L(k^0, 0))$ where $M_L(k^0, 0) = A + i2\pi L^{-1}B(k^0)$. By Lemma 11, $0 \leqq \lambda^* < \bar\lambda$. Since $C = M_L(0, 0)$, $\bar\lambda \in \delta_L(T(0))$ ∎

Theorem 3: *If $\delta(r)$, $w(r)$ is any stable solution branch bifurcating from the trivial branch $u(\delta)$, then $w(r)$ must necessarily be spatially constant and satisfy $F(\alpha_0 + \delta(r), u(\delta(r)) + w(r)) = 0$ identically in r, i.e. $\delta(r)$, $w(r)$ is itself trivial.*

Proof: Let k^0 be the wave vector in assumption (A2). If $k^0 \neq 0$, then there is a $\bar\lambda \in \sigma_L(T(0))$ with $\bar\lambda > 0$ by Lemma 12. From this and a perturbation argument it follows that for any bifurcating branch $\delta(r)$, $w(r)$ the Fréchet derivative

$$D\Delta + B + d_uF(\alpha_0 + \delta(r), u(\delta(r)) + w(r))$$

has an eigenvalue in the vicinity of $\bar\lambda$, provided r is small. But this would imply instability. The theorem then follows from Lemma 5 ∎

It was somewhat disappointing that the two-component Landau-Ginzburg equation investigated in [3] falls under the scope of Theorem 3.

5.2 Generic instability for large periods

Our next aim is to show that although $T(\delta)$ may satisfy assumption (A3) in $H_2^e(L)$ for some L, it will necessarily violate (A3) in $H_2^e(nL)$ for large integers n, provided some generic assumption holds. This means that the trivial branch $u(\delta)$, $\delta \in I$, becomes unstable in $H_2^e(nL)$ as $n \uparrow \infty$, and as a consequence of this, that all bifurcating branches become unstable in $H_2^e(nL)$ as $n \uparrow \infty$. To start with, we take it for granted that the assumptions (A1) and (A2) hold as before, but not necessarily (A3). Instead of it, we suppose another "generic" condition, namely assumption

(A4) $(\eta_0, D\zeta_0) \neq 0$, with η_0, ζ_0 as in (A2).

As a preparation consider the matrix

$$A(\delta, \varepsilon) = d_uF(\delta, u(\delta)) - ((2\pi L^{-1})^2 |k^0|^2 + \varepsilon) D$$

with k^0 as in assumption (A2). From (A2) and perturbational arguments it follows that there is an analytical function $\lambda = \lambda(\delta, \varepsilon)$ defined in a neighbourhood of (0, 0) such that

1. $\lambda(0, 0) = 0$,
2. $\lambda(\delta, \varepsilon)$ is a simple eigenvalue of $A(\delta, \varepsilon)$,
3. $\lambda(\delta, \varepsilon) = \lambda_1\delta + \lambda_2\varepsilon + \cdots$, where $(\eta_0, \zeta_0)\, \lambda_1 = (\eta_0, B_1\zeta_0)$ and $(\eta_0, \zeta_0)\, \lambda_2 = -(\eta_0, D\zeta_0)$.

From assumptions (A2) and (A4) we infer

4. $\lambda_1 \neq 0$, $\lambda_2 \neq 0$.

Lemma 13: *For every sufficiently small $\varepsilon_0 \neq 0$ having the same sign as λ_2 there is a $\mu_1 > 0$ and a μ_2 with $0 < \mu_2 < |\varepsilon_0|$ such that $\lambda(\delta, \varepsilon) \geqq \varepsilon_0\lambda_2/4$ whenever $|\delta| \leqq \mu_1$ and $|\varepsilon - \varepsilon_0| \leqq \mu_2$ hold.*

Proof: Assume e.g. $\lambda_2 > 0$. Since $\lambda(\delta, \varepsilon) = \lambda_1\delta + \varepsilon\big(\lambda_2 + H(\delta, \varepsilon)\big)$ with $H(0, 0) = 0$ there is a $\mu_0 > 0$ such that $\lambda_2 + H(\delta, \varepsilon) \geqq 3\lambda_2/4$ if $|\delta|, |\varepsilon| \leqq \mu_0$. Then for a fixed ε_0 with $0 < \varepsilon_0 \leqq \mu_0/2$ there is a μ_1 with $0 < \mu_1 \leqq \mu_0$ such that $\lambda(\delta, \varepsilon_0) \geqq \varepsilon_0\lambda_2/2$ if $|\delta| < \mu_1$. The existence of μ_2 finally follows from the continuity of $\lambda(\delta, \varepsilon)$ in the rectangle $[-\mu_1, \mu_2] \times [0, \mu_0]$ ∎

Lemma 14: *Let the assumptions* (A1), (A2) *and* (A4) *hold for L. There is an $\varepsilon_0 \neq 0$ and an $n_0 > 0$ such that for every integer $n \geqq n_0$ there is a wave vector $k \in \mathbf{Z}^2$ with the property that $M_{nL}(k, 0)$ has an eigenvalue $\lambda \geqq |\varepsilon_0|\, |\lambda_2|/4$.*

Proof: Assume e.g. $\lambda_2 > 0$. We can chose $\varepsilon_0 > 0$ in the last lemma so as to satisfy $0 < \varepsilon_0 - \mu_2 < \varepsilon_0 + \mu_2 < L^{-2}$. For a given integer $n > 0$ let $q > 0$ be the integer determined by $(q-1)^2 < n^2\, |k^0|^2 \leqq q^2$. From this choice of q, ε_0, μ_2 we infer

$$(2\pi)^2\, |k^0|^2\, L^{-2} + \varepsilon_0 + \mu_2 < (2\pi)^2\, (q+n)^2\, (nL)^{-2}. \tag{23}$$

Now set $m_p = (2\pi)^2\, (q+p)^2\, (nL)^{-2}$ for $p = -1, 0, \ldots, n$. Then we find $m_{p+1} - m_p \leqq (2\pi)^2\, (nL)^{-2}\big(n(|k^0| + 1) + 3\big)$ for $p < n$. We thus may take n so large that $m_{p+1} - m_p < 2\mu_2$ holds. Now consider the wave vectors $d_p = (q+p, 0)$ for $p = -1, 0, \ldots, n$. From (23) and our choice of n it follows that for at least one $p \in \{1, \ldots, n-1\}$ the inequalities

$$(2\pi)^2\, |k^0|^2/L^2 + \varepsilon_0 - \mu_2 < (2\pi)^2\, d_p^{\,2}/(nL)^2 < (2\pi)^2\, |k^0|^2/L^2 + \varepsilon_0 + \mu_2$$

hold. By our choice of ε_0, μ_2 and by the last lemma it follows that $M_{nL}(d_p, 0)$ has an eigenvalue $\lambda \geqq \varepsilon_0\lambda_2/4$ ∎

The main instability result is given by

Theorem 4: *Suppose that assumptions* (A1), (A2) *and* (A4) *hold for the period L. Then there is an integer $n_0 > 0$ such that every solution branch $\delta(r)$, $w(r)$ which eventually bifurcates from the trivial branch $u(\delta)$, $\delta \in I$, in $H_2^{\mathrm{e}}(L)$ is unstable in $H_2^{\mathrm{e}}(nL)$, provided $n \geqq n_0$.*

Proof: Let n_0, ε_0, λ_2 be as in the previous lemma and assume $n \geqq n_0$. Set as before $T(0) = D\Delta + d_uF(0, 0)$. Since the spectrum $\sigma_{nL}\big(T(0)\big)$ of $T(0)$ in $H_2^{\mathrm{e}}(nL)$ is given by $\bigcup \sigma\big(M_{nL}(k, 0)\big)$, it follows from Lemma 14 that $\sigma_{nL}\big(T(0)\big)$ contains a real eigenvalue $\lambda \geqq |\varepsilon_0|\, |\lambda_2|/4$. By a perturbation argument we derive that every solution branch $\delta(r)$, $w(r)$ which eventually bifurcates in $H_2^{\mathrm{e}}(nL)$ from the trivial branch $u(\delta)$, $\delta \in I$, has an eigenvalue in the set $\{z \mid \operatorname{Re} z \geqq |\varepsilon_0|\, |\lambda_2|/8\}$, provided r is sufficiently small. Since every solution branch $\delta(r)$, $w(r)$ in $H_2^{\mathrm{e}}(L)$ also belongs to $H_2^{\mathrm{e}}(nL)$, the theorem is proved ∎

5.3 Instability against other perturbations

Let $\delta_0(r)$, $w_0(r) \in H_2^e(L)$ be the standing wave branch associated with k^0, given by Theorem 1, let $L' \neq L$ be any other period and consider the initial value problem

$$\begin{aligned} \dot{u}_\varepsilon &= D\Delta u_\varepsilon + F\big(\delta_0(r), u_\varepsilon\big), \\ u_\varepsilon(0) &= u\big(\delta_0(r)\big) + w_0(r) + \varepsilon \cos\big(2\pi(L')^{-1} k^0 \cdot x\big). \end{aligned} \tag{24}$$

The problem arises if for small ε the solution u_ε remains close to the equilibrium solution $u\big(\delta_0(r)\big) + w_0(r)$. One of the smallest spaces within which we could try to solve (24) is the Hilbert space H_a of almost periodic even functions having Fourier series of the form

$$u = \sum \xi_{pq} \cos\big(2\pi(pL^{-1} + qL'^{-1})\, k^0 \cdot x\big), \qquad p, q \in \mathbf{Z},$$

provided with the norm

$$\|u\|^2 = \sum |\xi_{pq}|^2 \big(1 + (2\pi)^4 (pL^{-1} + qL'^{-1})^4 |k^0|^4\big) < \infty.$$

One should then extend Kielhöfer's approach to H_a and to operators with continuous spectrum. The elaboration of this is not within the scope of this paper. We have to content ourselves with a few indications suggesting that in spaces like H_a instability prevails. At the root of this is again Lemma 13. Let, for $k \in \mathbf{Z}^2$, $M'(k, \delta)$ be the matrix $-(2\pi)^2 |k|^2 D + d_uF\big(\delta, u(\delta)\big)$; thus $M'(kL^{-1}, \delta) = M_L(k, \delta)$. Suppose that assumptions (A1), (A2) and (A4) are holding for the period L, with $k^0 \in \mathbf{Z}^2$ as in (A2). Thus prepared we state

Lemma 15: *Let L, L' be rationally independent. Then there are $a, b \in \mathbf{Z}$ such that $M'\big((aL^{-1} + bL'^{-1})\, k^0, 0\big)$ has an eigenvalue $\lambda \geqq |\varepsilon_0|\, |\lambda_2|/4$, with ε_0, λ_2 as in Lemma 13.*

Proof: Assume e.g. $\lambda_2 > 0$. By Lemma 13 there is $\mu_2 > 0$ with $\mu_2 < \varepsilon_0$ such that $-\big((2\pi L^{-1})^2 |k^0|^2 + \varepsilon\big) D + d_uF(0, 0)$ has an eigenvalue $\lambda \geqq \varepsilon_0\lambda_2/4$, provided $|\varepsilon - \varepsilon_0| \leqq \mu_2$. Now define the real number $\eta > 0$ by $(2\pi)^2 |k^0|^2 \eta^2 = (2\pi L^{-1})^2 |k^0|^2 + \varepsilon_0$. Since L^{-1}, L'^{-1} are rationally independent, we can make $|aL^{-1} + bL'^{-1} - \eta|$ as small as we like it by a suitable choice of $a, b \in \mathbf{Z}$. From these remarks the lemma follows immediately ∎

Remark: $aL^{-1} + bL'^{-1}$ in the lemma may be taken arbitrarily close to L^{-1}.

Next, let a, b be two integers as provided by Lemma 15. Consider the closed subspace $H^* \subseteqq H_a$ of the elements with Fourier series

$$\sum \xi_p \cos\big(2\pi(pL^{-1} + bL'^{-1})\, k^0 \cdot x\big), \qquad p \in \mathbf{Z}.$$

Finally, let $\tilde{H} \subseteqq H_a$ be the subspace of elements u whose Fourier series satisfy

$$\sum |\xi_{pq}|^2 \big(1 + (2\pi)^8 (pL^{-1} + qL'^{-1})^8 |k^0|^8\big) < \infty, \qquad p, q \in \mathbf{Z}.$$

On $\tilde{H}$ we define $T(0) = D\Delta + d_uF(0, 0)$ by its componentwise action, i.e.

$$T(0)\, u = \sum \big(M'((pL^{-1} + qL'^{-1})\, k^0, 0)\, \xi_{pq}\big) \cos\big(2\pi(pL^{-1} + qL'^{-1})\, k^0 \cdot x\big).$$

It will be routine to show that $T(0)$ leaves H^* invariant, that it has compact resolvent on H^*, and that its spectrum in H^* is given by

$$\sigma^*\big(T(0)\big) = \bigcup_p \sigma\big(M'((pL^{-1} + bL'^{-1})\, k^0, 0)\big).$$

From the last lemma we infer that $T(0)$ has a real eigenvalue $\lambda > 0$ in H^*. Now each member $w_0(r)$ of the standing wave branch $\delta_0(r)$, $w_0(r)$ is also a member of H_a. Only routine work is required in order to define the action of the Fréchet derivative $T(r) = D\Delta + d_u F(\delta_0(r), u(\delta_0(r)) + w_0(r))$ on $\tilde{H}$ properly. In particular it turns out that $T(r)$ leaves H^* invariant and that the resolvent of $T(r)$ on H^* is compact. From this, from the property of $T(0)$ mentioned above, and from the usual perturbation argument we derive that $T^*(r)$ has an eigenvalue λ_r with $\operatorname{Re} \lambda_r > 0$, provided r is sufficiently small. This means, if the principle of linearized stability carries over to the present situation, then the standing wave branch $\delta_0(r)$, $w_0(r)$ is necessarily instable in H_a. Thus instability seems to be rather the rule than the exception. This does of course not exclude the possibility that $u(\delta_0(r)) + w_0(r)$ is stable against the particular perturbations of the form $\varepsilon \cos(2\pi L'^{-1} k^0 \cdot x)$. Whether this is the case or not is unknown to us.

6. Conclusion

We conclude with some remarks on the chosen frame. The restriction to two-component vectors $u(u_1, u_2)$ is for simplicity only: more components would not exhibit new phenomena, one component narrows the range of possibilities considerably. The restriction to space dimension $n = 2$ (i.e. to two space variables (x_1, x_2)) is more serious. The choice $n \leq 3$ is justified by the physical background mentioned in the introduction and also by the Banach-algebra property of the space $H_2^1(L)$ which simplifies the theory considerably. The results of Chapter 5 carry over literally to $n = 1$ and $n = 3$, while Theorem 2 loses its interest in case of $n = 1$, in which bifurcation from a single eigenvalue prevails. Whether Theorem 2 holds for dimension $n = 3$ is not known to us; we have not been able to prove the crucial Lemma 7 for dimension $n = 3$. The case of functions $u(x_1, x_2)$, L_1-periodic in x_1 and L_2-periodic in x_2, with L_1, L_2 rationally independent behaves rather like the case $n = 1$ and has been omitted therefore. Finally we have chosen a polynomial nonlinearity F partly because of the physical background and partly for simplicity, but it is clear that weaker assumptions on F would suffice (e.g. analycity). We have treated standing wave branches exclusively, although it is easy to see that there are many more bifurcating solution branches. It is a difficult task to describe and to classify all of them under appropriate generic conditions. We have only obtained some partial results in this direction. E.g. it can be shown that each subspace $H_2^c(L/\{k^i, k^j\})$ for $i \neq j$ contains exactly four bifurcation branches, two of which are of course the standing wave branches associated with k^i and k^j, respectively. But even this restricted result is not quite simple to prove and has been omitted for lack of space. Questions about the stability behaviour of bifurcating branches other than standing waves (such as whether Theorem 2 applies to them or not) immediately lead to elementary but tricky questions about the number-theoretic properties of the sets $\{(x, y) \mid x^2 + y^2 = n, x, y \in \mathbf{Z}\}$. Finally we note that there is an abundance of examples which fall under the scope of Theorem 2, but since their discussion would require some place we have renounced to describe them.

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