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A Class of Nonlinear Generalized Riemann-Hilbert-Poincaré Problems for Holomorphic Functions

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Mit Hilfe der Theorie pseudo-monotoner Operatoren wird die Existenz einer Lösung bei einer Klasse nichtlinearer verallgemeinerter Riemann-Hilbert-Poincaré-Probleme für eine holomorphe Funktion im Einheitskreis bewiesen.

С помощью теории псевдо-монотонных операторов доказывается существование решения у одного класса нелинейных обобщенных задач Римана-Гильберта-Пуанкаре для аналитических функций в единичном круге.

By means of the theory of pseudo-monotone operators the existence of a solution of a class of nonlinear generalized Riemann-Hilbert-Poincaré problems for a holomorphic function in the unit disk is proved.

Introduction

In recent papers of the author [11, 14] existence theorems of the theory of maximal monotone operators and of Hammerstein equations in L_p spaces were applied to nonlinear Riemann-Hilbert, generalized Steklov, and generalized Poincaré problems for holomorphic functions in the unit disk. In the present paper the theory of pseudo-monotone operators in the Sobolev space W_2^1 is utilized for proving corresponding existence theorems by a class of nonlinear generalized Riemann-Hilbert-Poincaré problems (nonlinear Vekua's problems) involving derivatives up to second order of the boundary values. Besides, by means of related regularizing approximations also some types of noncoercive problems of this kind are dealt with. In particular, some existence theorems of Landesman-Lazer's type are derived completing the theorems of such type obtained for the Riemann-Hilbert, the generalized Steklov, and the generalized Poincaré problem in [12, 14].

For classical work on nonlinear generalized Riemann-Hilbert-Poincaré problems we refer to Pogorzelski [8] and the papers quoted in the introduction of the monograph [4] by Guseinov and Mukhtarov.

1. Statement of problem

Let $G: |z| < 1$ be the unit disk of the complex z plane with boundary $\Gamma: t = e^{is}$ $(-\pi \leqq s \leqq \pi)$. We deal with the following **Problem A**:

To find a holomorphic function $w(z) = u(z) + iv(z)$ in G, which satisfies the boundary condition

$$L[u, v] \equiv L_0[u, v] + L_1[v] + L_2[u] = f \quad \text{on} \quad \Gamma, \tag{1}$$

where

$$L_0[u, v] = -\varepsilon u_{ss} + \varkappa v_{ss}, \qquad L_1[v] = \alpha v + \beta v_s,$$

$$L_2[u] = \varphi(u)\, u_s + \psi(u),$$

and the additional condition

$$v(0) = 0 \quad \text{in} \quad z = 0. \tag{2}$$

Here $\varepsilon \geqq 0$, $\varkappa$, $\beta \geqq 0$, and α are given real constants, φ and ψ are given continuous functions, and $f \in L_2(\Gamma)$ is the given right-hand side.

Because of (2) the boundary values $v = v(e^{is})$ of $v(z)$ and $u = u(e^{is})$ of $u(z)$ on Γ are connected by the well-known relation $v = -Hu$ with the *Hilbert operator*

$$(Hu)(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\sigma}) \cot \frac{\sigma - s}{2}\, d\sigma.$$

We remark that an inhomogeneous additional condition $v(0) = \gamma$ can be reduced to the homogeneous condition (2) introducing the unknown function $w(z) - \gamma i$ instead of $w(z)$.

In particular we are interested in the special case

$$-v_{ss} + \lambda u u_s + \mu u = f \tag{3}$$

of (1) with constants $\lambda > 0$, $\mu \geqq 0$, which may be regarded as a steady analogue of the well-known *Benjamin-Ono equation* of the theory of long internal gravity waves in a stratified fluid with infinite depth in the spatially periodic case.

A holomorphic function $w(z) = u(z) + iv(z)$ in G with boundary values $u(s) = u(e^{is})$ and $v(s) = -(Hu)(s)$ is said to be a *generalized solution* of Problem A if $u \in W_2^1(\Gamma)$[1]) satisfies the integral relation

$$a_0(u, \eta) + a_1(u, \eta) + a_2(u, \eta) = b(\eta) \quad \text{for} \quad \eta \in W_2^1(\Gamma), \tag{4}$$

where, for $u, \eta \in W_2^1(\Gamma)$,

$$b(\eta) = \int_\Gamma f\eta\, ds, \qquad a_0(u, \eta) = \varepsilon \int_\Gamma u'\eta'\, ds + \varkappa \int_\Gamma Hu' \cdot \eta'\, ds, \tag{5}$$

$$\left.\begin{aligned} a_1(u, \eta) &= -\alpha \int_\Gamma Hu \cdot \eta\, ds - \beta \int_\Gamma Hu' \cdot \eta\, ds, \\ a_2(u, \eta) &= \int_\Gamma \varphi(u)\, u'\eta\, ds + \int_\Gamma \psi(u)\, \eta\, ds. \end{aligned}\right\} \tag{6}$$

Here the prime denotes derivatives with respect to s.

Lemma: *If $\varepsilon + \varkappa^2 > 0$, a generalized solution $w(z)$ of Problem A has boundary values $u, v \in W_2^2(\Gamma)$ and the boundary condition (1) is fulfilled a.e. on Γ.*

Proof: Let $u \in W_2^1(\Gamma)$ be a solution of (4). We put

$$U = f - \varphi(u)\, u' - \psi(u) + \alpha Hu + \beta Hu' \in L_2(\Gamma).$$

From (4) with $\eta = 1$ we have $\int_\Gamma U\, ds = 0$. Therefore, the boundary value problem

$$-\varepsilon u_{ss} + \varkappa v_{ss} = U \quad \text{on} \quad \Gamma \tag{7}$$

[1]) $u \in W_2^1(\Gamma)$ includes that u is a (continuous) 2π-periodic function in s. Analogously, u and u_s are continuous 2π-periodic functions in s if $u \in W_2^2(\Gamma)$.

has a unique solution $w_0(z) = u_0(z) + iv_0(z)$ with boundary values $u_0, v_0 \in W_2^2(\Gamma)$ satisfying the additional condition $w_0(0) = 0$. This solution may be easily constructed in closed form by trigonometric Fourier expansion, for instance. The function u_0 fulfils the identity (see (4))

$$a_0(u_0, \eta) = b(\eta) - a_1(u, \eta) - a_2(u, \eta) \quad \text{for} \quad \eta \in W_2^1(\Gamma). \tag{8}$$

Putting $u_1 = u - u_0$, from (4) and (8) we have the integral relation

$$a_0(u_1, \eta) = \int_\Gamma [\varepsilon u_1' + \varkappa H u_1'] \, \eta' \, ds = 0 \quad \text{for} \quad \eta \in W_2^1(\Gamma).$$

Hence it follows that the function $\varepsilon u_1' + \varkappa H u_1'$ has the generalized derivative zero on Γ and therefore is a constant. But then u_1' must be a constant and, due to its 2π-periodicity, also u_1 itself. That is, $u = u_0 + u_1 \in W_2^2(\Gamma)$. Moreover, from (4) with $u \in W_2^2(\Gamma)$ by partial integration we obtain the integral relation

$$\int_\Gamma \{L[u, -Hu] - f\} \, \eta \, ds = 0 \quad \text{for} \quad \eta \in W_2^1(\Gamma),$$

which implies the validity of the boundary condition (1) a.e. on Γ ∎

Remark: If $f, \varphi, \psi \in C^\alpha(\Gamma)$, $0 < \alpha < 1$, are Hölder continuous functions with exponent α, then $u, v \in C^{2,\alpha}(\Gamma)$ have Hölder continuous derivatives of second order with the same exponent α.

2. Basic existence theorem

We now prove the main theorem of this paper.

Theorem 1: *Under the additional assumptions $\varepsilon > 0$ and*

$$u\psi(u) \geqq \delta u^2 - D \qquad (\delta > 0, D \geqq 0) \tag{9}$$

Problem A possesses a generalized solution for any $f \in L_2(\Gamma)$.

Proof: Problem A is equivalent to the *operator equation*

$$Au = b \quad \text{in} \quad X = W_2^1(\Gamma), \tag{10}$$

where $A = A_0 + A_1 + A_2$ and the operators $A_k: X = W_2^1(\Gamma) \to X^* = W_2^1(\Gamma)$ $(k = 0, 1, 2)$ are defined by $a_k(u, \eta) = \langle A_k u, \eta \rangle_X$ for $u, \eta \in X$, and $b \in X^* = W_2^1(\Gamma)$ is defined by (5).

The linear operators A_0 and A_1 are *continuous* since the Hilbert operator H is a continuous linear operator in $W_2^1(\Gamma)$. Besides, because of

$$a_0(u, u) = \varepsilon \int_\Gamma u'^2 \, ds \geqq 0, \qquad a_1(u, u) = \beta \int_\Gamma \frac{\partial u}{\partial r} u \, ds \geqq 0, \tag{11}$$

where $\partial u/\partial r$ means the derivative of u in direction of the polar radius r, both operators A_0 and A_1 and therefore their sum $\bar{A}_1 = A_0 + A_1$ are *monotone*.

The operator A_2 is *completely continuous* in the sense that it maps weakly convergent sequences into strongly convergent ones. Assuming $u_n \rightharpoonup u$ in X, we have $\|u_n\|_X \leqq K$ and, due to the compact embedding of $X = W_2^1(\Gamma)$ in $C(\Gamma)$, also $u_n \to u$ in $C(\Gamma)$. We have to show that

$$\|A_2 u_n - A_2 u\|_{X^*} = \sup_{\|\eta\|_X \leqq 1} |\langle A_2 u_n - A_2 u, \eta \rangle| \xrightarrow[n \to \infty]{} 0. \tag{12}$$

By partial integration from (6) it follows that

$$a_2(u, \eta) = \int_\Gamma \psi(u)\, \eta\, ds - \int_\Gamma \Phi(u)\, \eta'\, ds, \qquad \Phi \text{ a primitive of } \varphi.$$

Since $\psi(u_n) \to \psi(u)$ and $\Phi(u_n) \to \Phi(u)$ in $C(\Gamma)$, we have

$$\begin{aligned}
&\sup_{\|\eta\|_X \leqq 1} |\langle A_2 u_n - A_2 u, \eta\rangle| \\
&\leqq \sup_{\|\eta\|_X \leqq 1} \int_\Gamma |\psi(u_n) - \psi(u)|\, |\eta(s)|\, ds + \sup_{\|\eta\|_X \leqq 1} \int_\Gamma |\Phi(u_n) - \Phi(u)|\, |\eta'(s)|\, ds \\
&\leqq \left(\int_\Gamma |\psi(u_n) - \psi(u)|^2\, ds\right)^{1/2} + \left(\int_\Gamma |\Phi(u_n) - \Phi(u)|^2\, ds\right)^{1/2} \to 0.
\end{aligned}$$

This proves (12).

As the sum $A = \bar{A}_1 + A_2$ of the continuous monotone operator $\bar{A}_1$ and the completely continuous one A_2 the operator A is *pseudomonotone*. Since $\bar{A}_1$ as a continuous linear operator and A_2 as a completely continuous one are bounded operators, so A is *bounded*, too. Finally, owing to the assumption (9) there is

$$a_2(u, u) = \int_\Gamma u\psi(u)\, ds \geqq \delta \int_\Gamma u^2\, ds - 2\pi D \tag{13}$$

and by (11)

$$a_0(u, u) + a_1(u, u) \geqq \varepsilon \int_\Gamma u'^2\, ds. \tag{14}$$

Therefore, we have $\langle Au, u\rangle_X \geqq \min(\varepsilon, \delta)\, \|u\|_X^2 - 2\pi D$, and because of the assumption $\varepsilon > 0$ (and $\delta > 0$) the operator A is *coercitive*.

The main theorem of the theory of pseudo-monotone operators by Brézis (cf. [16: Theorem 27.2]) now yields the existence of a solution u of the operator equation (10) ∎

Remark: Since the operator A also satisfies the condition (S_+ and hence) S_0', the solution u of (10) is strong limit of a subsequence of solutions of the Galerkin equations of (10) with respect to an arbitrary basis in $X = W_2^1(\Gamma)$ (cf. [16: Theorem 27.1]). Further, we remark that the question of uniqueness of the solution is an open problem. Of course, the solution is unique in the particular case: $\varphi = 0$, ψ a strictly increasing function.

3. Non-coercive problems

We now deal with the *case* $\varepsilon = 0$, where the main term is $L_0[v] = \varkappa v_{ss}$ in (1). The problem with the corresponding boundary condition

$$L_0[v] + L_1[v] + L_2[u] = f \qquad \text{on } \Gamma \tag{15}$$

and the additional condition (2) is named **Problem B.**

Theorem 2: *Under the additional assumptions* $\varkappa \geqq 0$ ($\varkappa \leqq 0$),

$$\varphi(u) \geqq \nu > 0 \qquad \left(\varphi(u) \leqq -\nu < 0\right), \tag{16}$$

and (9) *Problem B possesses a generalized solution for any* $f \in L_2(\Gamma)$.

Proof: We restrict ourselves to the case $\varkappa \geqq 0$. In view of Theorem 1 the *perturbed problem* with the boundary condition

$$-\varepsilon u_{ss} + \varkappa v_{ss} + \alpha v + \beta v_s + \varphi(u)\, u_s + \psi(u) = f \qquad \text{on } \Gamma \tag{17}$$

and the additional condition (2) possesses a generalized solution $w_\varepsilon(z)$ with $u_\varepsilon \in W_2^1(\Gamma)$ for any $\varepsilon > 0$. By definition u_ε satisfies the integral relation (4) and by the Lemma $u_\varepsilon, v_\varepsilon \in W_2^2(\Gamma)$ fulfil the boundary condition (17) a.e. on Γ.

We show that the norms of u_ε in $X = W_2^1(\Gamma)$ are bounded uniformly in ε. In view of (13), (14) from (4) for $\eta = u_\varepsilon$ we obtain the inequality

$$\varepsilon \int_\Gamma u_\varepsilon'^2 \, ds + \delta \int_\Gamma u_\varepsilon^2 \, ds \leqq \int_\Gamma f u_\varepsilon \, ds + 2\pi D.$$

Therefore we have $\delta \, \|u_\varepsilon\|_2^2 \leqq \|f\|_2 \, \|u_\varepsilon\|_2 + 2\pi D$[2]) which implies the norm boundedness in $L_2(\Gamma)$

$$\|u_\varepsilon\|_2 \leqq K_0 \qquad \text{uniformly in} \quad \varepsilon > 0. \tag{18}$$

Further, multiplying (17) for $u = u_\varepsilon$, $v = v_\varepsilon$ by u_ε' and integrating over Γ yields the relation

$$\varkappa \int_\Gamma u_\varepsilon' v_\varepsilon'' \, ds + \alpha \int_\Gamma u_\varepsilon' v_\varepsilon \, ds + \int_\Gamma \varphi(u_\varepsilon) \, u_\varepsilon'^2 \, ds = \int_\Gamma f u_\varepsilon' \, ds.$$

Now there holds

$$\int_\Gamma u' v'' \, ds = \int_\Gamma u_s \frac{\partial u_s}{\partial r} \, ds \geqq 0 \quad \text{for} \quad u \in W_2^2(\Gamma)$$

so that $\varkappa \int_\Gamma u_\varepsilon' v_\varepsilon'' \, ds \geqq 0$ and, by (16), $\int_\Gamma \varphi(u_\varepsilon) \, u_\varepsilon'^2 \, ds \geqq \nu \int_\Gamma u_\varepsilon'^2 \, ds$. Besides, $\|v_\varepsilon\|_2 \leqq \|u_\varepsilon\|_2 \leqq K_0$ by (18). Hence we infer the inequality $\nu \, \|u_\varepsilon'\|_2^2 \leqq \{\|f\|_2 + |\alpha| \, K_0\} \, \|u_\varepsilon'\|_2$ which implies the norm boundedness in $L_2(\Gamma)$

$$\|u_\varepsilon'\|_2 \leqq K_1 \qquad \text{uniformly in} \quad \varepsilon > 0. \tag{19}$$

Finally, from (18) and (19) we have the estimate

$$\|u_\varepsilon\|_X \leqq K \qquad \text{uniformly in} \quad \varepsilon > 0. \tag{20}$$

Let $\varepsilon_n \to 0$ and put $u_n = u_{\varepsilon_n}$. Owing to (20) there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging weakly in $X = W_2^1(\Gamma)$ and therefore uniformly to a function $u \in W_2^1(\Gamma)$. Then also the functions Hu_{n_k} converge weakly in $W_2^1(\Gamma)$ to $Hu \in W_2^1(\Gamma)$. Performing the limit $\varepsilon_{n_k} \to 0$ in (4), we obtain the identity

$$\varkappa \int_\Gamma Hu' \cdot \eta' \, ds + a_1(u, \eta) + a_2(u, \eta) = b(\eta) \qquad \big(\eta \in W_2^1(\Gamma)\big)$$

for u, i.e., $u \in W_2^1(\Gamma)$ is generalized solution of Problem B ∎

Remark: If $\varkappa \neq 0$, by the Lemma $u \in W_2^2(\Gamma)$.

Further we drop the assumption (16) on φ and prove

Theorem 3: *Under the additional assumptions $\varkappa \neq 0$,*

$$|\varphi(u)| \leqq E_1 \, |u|^\varrho + D_1 \qquad (0 < \varrho < 2;\, E_1 \geqq 0,\, D_1 \geqq 0), \tag{21}$$

the assumption (9), *and*

$$|\psi(u)| \leqq E_2 \, |u|^\sigma + D_2 \qquad (0 < \sigma < 5;\, E_2 \geqq 0,\, D_2 \geqq 0) \tag{22}$$

Problem B possesses a generalized solution for any $f \in L_2(\Gamma)$.

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[2]) $\|\cdot\|_p$ denotes the norm in $L_p(\Gamma)$, $p > 1$.

Proof: We again consider the *perturbed problem* with the boundary condition (17) and (2). Due to the assumption (9) the estimate (18) holds again. Multiplying (17) for $u = u_\varepsilon$, $v = v_\varepsilon$ by v_ε and integrating over Γ further yields the relation

$$-\varkappa \int_\Gamma v_\varepsilon'^2\, ds + \alpha \int_\Gamma v_\varepsilon^2\, ds + \int_\Gamma \varphi(u_\varepsilon)\, v_\varepsilon u_\varepsilon'\, ds + \int_\Gamma \psi(u_\varepsilon)\, v_\varepsilon\, ds = \int_\Gamma f v_\varepsilon\, ds.$$

Now we have $\|v_\varepsilon\|_2 \leqq \|u_\varepsilon\|_2 \leqq K_0$ by (18) again,

$$\left| \int_\Gamma \psi(u_\varepsilon)\, v_\varepsilon\, ds \right| \leqq E_2 \int_\Gamma |u_\varepsilon|^\sigma\, |v_\varepsilon|\, ds + D_2 \int_\Gamma |v_\varepsilon|\, ds$$
$$\leqq E_2 K_0\, \|u_\varepsilon\|_{2\sigma}^\sigma + D_2 \sqrt{2\pi}\, K_0$$

by (22), and

$$\left| \int_\Gamma \varphi(u_\varepsilon)\, v_\varepsilon u_\varepsilon'\, ds \right| \leqq E_1 \int_\Gamma |u_\varepsilon|^\varrho\, |v_\varepsilon|\, |u_\varepsilon'|\, ds + D_1 \int_\Gamma |v_\varepsilon|\, |u_\varepsilon'|\, ds$$
$$\leqq E_1\, \||u_\varepsilon|^\varrho\, v_\varepsilon\|_2\, \|u_\varepsilon'\|_2 + D_1 K_0\, \|u_\varepsilon'\|_2$$

by (21). Furthermore, by Hölder's inequality

$$\||u_\varepsilon|^\varrho\, v_\varepsilon\|_2 \leqq \|u_\varepsilon\|_{2(\varrho+1)}^\varrho\, \|v_\varepsilon\|_{2(\varrho+1)} \leqq A_{2(\varrho+1)}\, \|u_\varepsilon\|_{2(\varrho+1)}^{\varrho+1}$$

with A_p the norm of the Hilbert operator in $L_p(\Gamma)$, $p > 1$. Therefore, there holds the estimation

$$|\varkappa|\, \|u_\varepsilon'\|_2^2 = |\varkappa|\, \|v_\varepsilon'\|_2^2$$
$$\leqq K_0\, \|f\|_2 + |\alpha|\, K_0^2 + D_2 \sqrt{2\pi}\, K_0 + E_2 K_0\, \|u_\varepsilon\|_{2\sigma}^\sigma$$
$$+ \{E_1 A_{2(\varrho+1)}\, \|u_\varepsilon\|_{2(\varrho+1)}^{\varrho+1} + D_1 K_0\}\, \|u_\varepsilon'\|_2. \tag{23}$$

Finally, for any $U \in W_2^1(\Gamma)$ satisfying the relation $\int_\Gamma U\, ds = 0$ the well-known *interpolation inequality*

$$\|U\|_p \leqq C\, \|U'\|_2^\gamma\, \|U\|_2^{1-\gamma} \qquad (C > 0), \tag{24}$$

where $p \geqq 2$ and $\gamma = 1/2 - 1/p$, is valid (cf. [7: Chap. II, Th. 2.2]). Taking $U = u - (1/2\pi) \int_\Gamma u\, ds$, from (24) for an arbitrary function $u \in W_2^1(\Gamma)$ we obtain the inequality

$$\|u\|_p \leqq c\, \|u'\|_\gamma^2\, \|u\|_2^{1-\gamma} + d\, \|u\|_2 \tag{25}$$

with uniform positive constants c, d, depending only on p. Without loss of generality we suppose $\sigma \geqq 1$ in (22) and choose $p = 2\sigma$ and $p = 2(\varrho + 1)$, respectively, in (25). Then we have

$$\|u_\varepsilon\|_{2\sigma} \leqq c_1 K_0^{1-\gamma_1}\, \|u_\varepsilon'\|_2^{\gamma_1} + d_1 K_0 \tag{26}$$

with $\gamma_1 = 1/2 - 1/\sigma$ so that $\sigma\gamma_1 = (\sigma - 1)/2 < 2$ since $\sigma < 5$, and

$$\|u_\varepsilon\|_{2(\varrho+1)} \leqq c_2 K_0^{1-\gamma_2}\, \|u_\varepsilon'\|_2^{\gamma_2} + d_2 K_0 \tag{27}$$

with $\gamma_2 = 1/2 - 1/[2(\varrho + 1)]$ so that $(\varrho + 1)\, \gamma_2 = \varrho/2 < 1$ since $\varrho < 2$.

Hence, on account of $\varkappa \neq 0$, from (23), (26), (27) the uniform boundedness of the norms of u_ε' in $L_2(\Gamma)$ follows, i.e., we again have the estimates (19) and (20). The rest of the proof is the same as in the proof of Theorem 2 ∎

Example 1: The problem (3) with constants $\lambda > 0$ and $\mu > 0$, i.e., $\varkappa = -1$, $\alpha = \beta = 0$, $\varphi(u) = \lambda u$, $\psi(u) = \mu u$ fulfils the assumptions of Theorem 3.

4. Problems of Landesman-Lazer's type

Finally, we consider some problems without assuming the condition (9) for ψ. Firstly we make some brief remarks on the *case* $\psi = 0$ which will be also dealt with below as a limit case of a Landesman-Lazer's type problem. In this case the condition

$$\int_\Gamma f\,ds = 0 \tag{28}$$

is obviously necessary for the existence of a solution. If additionally $\alpha = 0$, Problem A can be reduced to the problem with the integrated boundary condition

$$\varkappa \frac{\partial u}{\partial r} - \varepsilon \frac{\partial u}{\partial s} + \beta v + \Phi(u) = F + C \tag{29}$$

and (2), where C is a free constant,

$$F(s) = \int_0^s f(\sigma)\,d\sigma, \qquad \Phi(u) = \int_0^u \varphi(\omega)\,d\omega.$$

The problem described by the conditions (29) and (2) is a nonlinear generalized Poincaré problem and has been treated in the literature. We refer to the papers [14] for the general case, [1, 7, 9, 10, 13, 15] for the case $\beta = 0$, [3, 5, 6] for the case $\varepsilon = \beta = 0$, [11; 12] for the case $\varkappa = 0$. See also [1, 10, 11, 13] for further references. Here we only consider two examples of this problem for illustration.

Example 2: The problem (3) with constants $\lambda > 0$ and $\mu = 0$ leads to the nonlinear Steklov problem

$$\frac{\partial u}{\partial r} = \frac{\lambda}{2} u^2 - F - C \qquad \text{on } \Gamma, \tag{30}$$

which by [1: Example (2.6)] has a classical solution $u \in C^2(G) \cap C^1(\overline{G})$ for any Lipschitz continuous function F and constant $C > -\min\{F(s) : s \in \Gamma\}$. That means, the problem (3) with $\lambda > 0$ and $\mu = 0$ possesses a continuum of such solutions with bounded second derivative v'' of the boundary values v for any $f \in L_\infty(\Gamma)$ which fulfils (28).

Example 3: By [12, 14] (cf. also the Remark to Theorem 4 below) the problem (29) with (2), where $\varkappa > 0$ or $\varkappa = 0$ with ($\varepsilon \geqq 0, \beta \geqq 0$ and) $\varepsilon + \beta > 0$, respectively, has a (suitably defined generalized) solution $u \in L_2(\Gamma)$ if

$$u\Phi(u) \geqq -c\,|u| - d \qquad (c \geqq 0, d \geqq 0) \tag{31}$$

and

$$\Phi_- < \frac{1}{2\pi} \int_\Gamma [F(s) + C]\,ds < \Phi_+,$$

i.e. for any constant C with

$$\Phi_- < C + \frac{1}{2\pi} \int_\Gamma F(s)\,ds < \Phi_+, \quad \Phi_+ = \liminf_{u\to+\infty} \Phi(u), \quad \Phi_- = \limsup_{u\to-\infty} \Phi(u).$$

If, additionally,

$$|\Phi(u)| \leqq a\,|u| + b \qquad (a \geqq 0, b \geqq 0), \tag{32}$$

then also $u' \in L_2(\Gamma)$ and $u \in W_2{}^1(\Gamma)$ is a generalized solution in the sense of point 1 above.

Therefore, under the assumptions (31) (and (32)) and $\Phi_- < \Phi_+$ the Problem A with $\varkappa > 0$ or $\varkappa = 0$, $\varepsilon + \beta > 0$ possesses a continuum of such generalized solutions $u \in L_2(\Gamma)$ ($u \in W_2{}^1(\Gamma)$) for any $f \in L_2(\Gamma)$ which satisfies (28). The assumptions (31) and $\Phi_- < \Phi_+$ are especially fulfilled if Φ is a non-constant monotone increasing function. I.e., under the above-mentioned restrictions on the parameters $\varkappa$, ε, β, Problem A has solutions if the condition $\varphi(u) \geqq 0$ is satisfied (for $\varphi(u) \equiv 0$, obviously, a solution u exists which is determined apart from an arbitrary additive constant).

In case $\psi \neq 0$ there holds the following theorem of Landesman-Lazer's type for Problems A and B.

Theorem 4: *Under the additional assumptions $\varkappa \geqq 0$, $\alpha \leqq 0$ ($\varkappa \leqq 0$, $\alpha \geqq 0$), (16), and*

$$u\psi(u) \geqq -\delta_0 |u| - D_0 \qquad (\delta_0 \geqq 0, D_0 \geqq 0) \tag{33}$$

Problem A possesses a generalized solution for each $f \in L_2(\Gamma)$ satisfying the inequality

$$\psi_- < \frac{1}{2\pi} \int_\Gamma f\, ds < \psi_+, \tag{34}$$

where

$$\psi_+ = \liminf_{u \to +\infty} \psi(u), \qquad \psi_- = \limsup_{u \to -\infty} \psi(u).$$

We remark that the condition (33) implies that $-\infty \leqq \psi_- \leqq \delta_0$, $-\delta_0 \leqq \psi_+ \leqq +\infty$. Of course, for (34) to hold it is to assume that $\psi_- < \psi_+$.

Proof: We consider the *perturbed problem* with the boundary condition

$$-\varepsilon u_{ss} + \varkappa v_{ss} + \alpha v + \beta v_s + \varphi(u)\, u_s + \delta u + \psi(u) = f \qquad \text{on } \Gamma \tag{35}$$

and the additional condition (2). By Theorems 1, 2 and the Lemma this problem has a generalized solution $w_\delta(z)$ with $u_\delta \in W_2{}^2(\Gamma)$ for any $\delta > 0$. We again have to prove that the norms of u_δ in $X = W_2{}^1(\Gamma)$ are uniformly bounded.

Multiplying (35) for $u = u_\delta$, $v = v_\delta$ by u_δ' and integrating over Γ yields the relation

$$\varkappa \int_\Gamma u_\delta' v_\delta''\, ds + \alpha \int_\Gamma u_\delta' v_\delta\, ds + \int_\Gamma \varphi(u_\delta)\, u_\delta'^2\, ds = \int_\Gamma f u_\delta'\, ds.$$

Now there hold the inequalities

$$\int_\Gamma u_\delta' v_\delta''\, ds \geqq 0 \quad \text{and} \quad \int_\Gamma u_\delta' v_\delta\, ds \leqq 0.$$

On account of the assumption (16) we therefore have

$$\nu \int_\Gamma u_\delta'^2\, ds \leqq \pm \int_\Gamma f u_\delta'\, ds$$

in cases $\varkappa \geqq 0$, $\alpha \leqq 0$ and $\varkappa \leqq 0$, $\alpha \geqq 0$, respectively. I.e., in both cases $\nu \|u_\delta'\|_2{}^2 \leqq \|f\|_2 \|u_\delta'\|_2$. This implies the uniform boundedness of the norms of u_δ' in $L_2(\Gamma)$. It remains to show that also the norms of u_δ themselves in $L_2(\Gamma)$ are uniformly bounded. We decompose $u_\delta = C_\delta + U_\delta$, where C_δ are constants and $\int_\Gamma U_\delta\, ds = 0$. Since $U_\delta' = u_\delta'$ and the norms of u_δ' in $L_2(\Gamma)$ are uniformly bounded, the functions U_δ themselves are uniformly bounded:

$$|U_\delta(s)| \leqq L. \tag{36}$$

We have to prove that also the constants C_δ are uniformly bounded. If this were not the case, there exists a sequence $\{C_{\delta_n}\}$ going to $+\infty$ or $-\infty$ as $\delta_n \to 0$. From (35) we have the relation

$$2\pi\delta_n C_{\delta_n} + \int_\Gamma \psi(C_{\delta_n} + U_{\delta_n})\, ds = \int_\Gamma f\, ds. \tag{37}$$

If now $C_{\delta_n} \to \infty$ as $\delta_n \to 0$, we apply Fatou's lemma to (37) taking into account that by (33) $\psi(u) \geqq -\mu_0$ with a (positive) constant μ_0 for sufficiently large u, say $u \geqq \lambda_0$, and by (36) $C_{\delta_n} + U_{\delta_n} \geqq \lambda_0$ for sufficiently large n. Hence we obtain the inequality

$$\int_\Gamma f\, ds \geqq \liminf_{n\to\infty} \int_\Gamma \psi(C_{\delta_n} + U_{\delta_n})\, ds \geqq \int_\Gamma \liminf_{n\to\infty} \psi(C_{\delta_n} + U_{\delta_n})\, ds \geqq 2\pi\psi_+$$

which is a contradiction to the right-hand side of (34). In the same way the assumption $C_{\delta_n} \to -\infty$ as $\delta_n \to 0$ leads to a contradiction to the left-hand side of (34) ∎

Corollary: *If $\psi_- \leqq \psi(u) \leqq \psi_+$ for all $u \in \mathbf{R}$, in particular, for a monotone non-decreasing function ψ, the condition (34) with $\leqq$ instead of $<$ is obviously necessary for the solvability of Problem A because of the relation*

$$\int_\Gamma \psi(u)\, ds = \int_\Gamma f\, ds$$

following from (4) with $\eta = 1$. In the limit case $\psi = 0$ the above proof also goes through with the assumption (34) replaced by (28) since for $\psi = 0$ from (37) and (28) it follows that all constants C_{δ_n} vanish. The condition (28) is therefore necessary and sufficient in this case.

Remark: In the particular case $\varepsilon = \varkappa = \beta = 0$ and $\varphi = 1$ ($\varphi = -1$) the existence assertion of Theorem 4 also holds true under the more general conditions that $\alpha \notin \{1, 2, \ldots\}$ ($\alpha \notin \{-1, -2, \ldots\}$) and the Carathéodory function $\psi = \psi(u, s)$ satisfies the assumption (33) with non-negative functions $\delta_0 \in L_2(\Gamma)$, $D_0 \in L_1(\Gamma)$ and the assumption

$$\sup_{|u| \leqq R} |\psi(u, s)| \in L_1(\Gamma) \qquad \text{for any} \quad R > 0.$$

This follows from Remark III.3 to Theorem III.6 of [2] like in the corresponding proof in [12], but there only the Theorem III.6 of [2] itself has been used. The solution u lies in $L_2(\Gamma)$ with $\pm u' + \alpha v \in L_1(\Gamma)$. Finally, the same statement is true also in the case $\varepsilon = \varkappa = 0$, $\beta \neq 0$, α arbitrary and $\varphi = \pm 1$, where $u \in L_2(\Gamma)$ with $\beta v' \pm u' + \alpha v \in L_1(\Gamma)$ (cf. [14]).

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