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A Class of Nonlinear Generalized Riemann-Hilbert-Poincaré Problems for Holomorphic Functions

L. v. WOLFERSDORF

Mit Hilfe der Theorie pseudo-monotoner Operatoren wird die Existenz einer Lösung bei einer Klasse nichtlinearer verallgemeinerter Riemann-Hilbert-Poincaré-Probleme für eine holomorphe Funktion im Einheitskreis bewiesen.

С помощью теории, исевдо-монотонных операторов доказывается существование решения у одного класса нелинейных обобщенных задач Римана-Гильберта-Пуанкаре для аналитических функций в единичном круге.

By means of the theory of pseudo-monotone operators the existence of a solution of a class of nonlinear generalized Riemann-Hilbert-Poincaré problems for a holomorphic function in the unit disk is proved.

Introduction

In recent papers of the author [11, 14] existence theorems of the theory of maximal monotone operators and of Hammerstein equations in L_p spaces were applied to nonlinear Riemann-Hilbert, generalized Steklov, and generalized Poincaré problems for holomorphic functions in the unit disk. In the present paper the theory of pseudomonotone operators in the Sobolev space W_2 ¹ is utilized for proving corresponding existence theorems by a class of nonlinear generalized Riemann-Hilbert-Poincaré problems (nonlinear Vekua's problems) involving derivatives up to second order of the boundary values. Besides, by means of related regularizing approximations also some types of noncoercive problems of this kind are dealt with. In particular, some existence theorems of Landesman-Lazer's type are derived completing the theorems of such type obtained for the Riemann-Hilbert, the generalized Steklov, and the generalized Poincaré problem in [12, 14].

For classical work on nonlinear generalized Riemann-Hilbert-Poincaré problems we refer to Pogorzetski [8] and the papers quoted in the introduction of the monograph [4] by GUSEINOV and MUKHTAROV.

1. Statement of problem

Let $G: |z| < 1$ be the unit disk of the complex z plane with boundary $\Gamma: t = e^{is}$ $(-\pi \leq s \leq \pi)$. We deal with the following Problem A:.

To find a holomorphic function $w(z) = u(z) + iv(z)$ in G, which satisfies the boundary condition

$$
L[u, v] \equiv L_0[u, v] + L_1[v] + L_2[u] = f \quad \text{on} \quad \Gamma
$$

 (1)

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\n
$$
L_0[u, v] = -\varepsilon u_{ss} + \varkappa v_{ss}, \qquad L_1[v] = \alpha v + \beta v_s,
$$
\n
$$
L_2[u] = \varphi(u) u_s + \psi(u),
$$
\nadditional condition
\n
$$
v(0) = 0 \quad \text{in} \quad z^2 = 0.
$$
\n
$$
\geq 0, \times, \beta \geq 0, \text{ and } \alpha \text{ are given real constants, } \varphi
$$
\ns, and $f \in L_2(\Gamma)$ is the given right-hand side.
\nsee of (2) the boundary values $v = v(e^{is})$ of $v(z)$

and the additional condition

$$
v(0)=0 \quad \text{in} \quad z=0.
$$

 $L_0[u, v] = -\varepsilon u_{ss} + \varepsilon v_{ss},$ $L_1[v] = \alpha v + \beta v_s,$
 $L_2[u] = \varphi(u) u_s + \psi(u),$

and the additional condition
 $v(0) = 0$ in $z^2 = 0.$ (2)

Here $\varepsilon \ge 0$, $\varkappa, \beta \ge 0$, and α are given real constants, φ and ψ are given con functions, and $f \in L_2(\Gamma)$ is the given right-hand side.

Because of (2) the boundary values $v = v(e^{is})$ of $v(z)$ and $u = u(e^{is})$ of $u(z)$ on Γ are connected by the well-known relation $v = -Hu$ with the *Hilbert operator* where
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Here
funct
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$$
\begin{aligned}\n\mathbf{a} &= 0, \mathbf{x}, \beta \ge 0, \text{ and } \alpha \text{ are given real const.} \\
\mathbf{b} &= 0, \mathbf{x}, \beta \ge 0, \text{ and } \alpha \text{ are given real const.} \\
\mathbf{c} &= 0 \quad \text{and} \quad \mathbf{c} \quad \text{and} \\
\mathbf{d} &= 0 \quad \text{and} \quad \mathbf{c} \quad \text{and} \quad \mathbf{c} \quad \text{and} \quad \mathbf{c} \quad \text{and} \quad \mathbf{c} \quad \text{and} \\
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\mathbf{d} &= 0 \quad \text{and} \quad \mathbf{c} \quad \text{and} \quad \mathbf{c} \quad \
$$

We remark that an inhomogeneous additional condition $v(0) = \gamma$ can be reduced to the homogeneous condition (2) introducing the unknown function $w(z) - \gamma i$ instead *of w(z).* and the additional condition
 $v(0) = 0$ in $z^2 = 0$.

Here $\varepsilon \ge 0$, $x, \beta \ge 0$, and α are given real c

functions, and $f \in L_2(\Gamma)$ is the given right-ha

Because of (2) the boundary values $v = v$

are connected by the (*Hu*) (*s*) = $\frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\sigma}) \cot \frac{\sigma - s}{2} d\sigma$.

We remark that an inhomogeneous additional che homogeneous condition (2) introducing the

of $w(z)$.

In particular we are interested in the special
 $-v_{ss} + \lambda uu_s + \$ **'** *â(u,)'+ a1 (u,) +a2 (,* j) = *b()* for ,j E W2 ¹ (fl, '

In particular, we are interested in the special case

$$
-v_{ss} + \lambda u u_s + \mu u = f \tag{3}
$$

of (1) with constants $\lambda > 0$, $\mu \geq 0$, which may be regarded as a steady analogue of the well-known *Benjamin-Ono equation* of the theory of long internal gravity waves in a stratified fluid with infinite depth in the spatially periodic case.

A holomorphic function $w(z) = u(z) + iv(z)$ in G with boundary values $u(s) = u(e^{is})$ and $v(s) = -(Hu)$ *(s)* is said to be a *generalized solution* of Problem A if $u \in W_2^1(\Gamma)^1$ satisfies the integral relation the nomogeneous condition (2) introduction
of $w(z)$.
In particular we are interested in
 $-v_{ss} + \lambda u_x + \mu u = f$
of (1) with constants $\lambda > 0$, $\mu \ge 0$,
the well-known Benjamin-Ono equal
in a stratified fluid with infinite de
A $\begin{array}{l} \text{pecial case} \ \text{h\ may\ be\ ref} \ \text{f\ the\ theory} \ \text{the\ spatial} \ \text{in}\ (z) \ \text{in}\ G \ \text{with}\ \text{valized}\ \text{solution} \ \text{in}\ (\eta) \quad \text{for} \quad \eta \ \text{in}\ (\eta) \ \text{$ beneous condition (2) introducing the unknown function $\omega(x)$, γ , motion is
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 $\gamma_{\rm s} + \lambda u u_s + \mu u = \beta$ *-.*

$$
a_0(u,\eta) + a_1(u,\eta) + a_2(u,\eta) = b(\eta) \text{ for } \eta \in W_2^1(\Gamma), \tag{4}
$$

where, for $u, \eta \in W_2^1(\Gamma)$,

•

•

•

$$
b(\eta) = \int\limits_{\Gamma} f\eta \, ds, \qquad a_0(u,\eta) = \varepsilon \int\limits_{\Gamma} u'\eta' \, ds + \varepsilon \int\limits_{\Gamma} Hu' \cdot \eta' \, ds, \qquad (5)
$$

$$
a_1(u, \eta) = -\alpha \int_{\Gamma} Hu \cdot \eta \, ds - \beta \int_{\Gamma} Hu' \cdot \eta \, ds,
$$
\n
$$
a_2(u, \eta) = \int_{\Gamma} \varphi(u) u' \eta \, ds + \int_{\Gamma} \psi(u) \eta \, ds.
$$
\nHere the prime denotes derivatives with respect to s.
\nLemma: If $\varepsilon + x^2 > 0$, a generalized solution $w(z)$ of Problem A has boundary
\nvalues $u, v \in W_2^2(\Gamma)$ and the boundary condition (1) is fulfilled a.e. on Γ :
\nProof: Let $u \in W_2^1(\Gamma)$ be a solution of (4). We put
\n $U = f - \varphi(u) u' - \psi(u) + \alpha Hu + \beta Hu' \in L_2(\Gamma)$.
\nFrom (4) with $\eta = 1$ we have $\int_{\Gamma} U \, ds = 0$. Therefore, the boundary value problem

Here the prime denotes derivatives with respect to *s.*

Lemma: *If* $\varepsilon + x^2 > 0$, *a generalized solution,w(z) of Problem A has boundary values u, v* $\in W_2^2(\Gamma)$ and the boundary condition (1) is fulfilled a.e. on Γ : $e^x + x^2 > 0$, *u* generatized solution $f(x)$
 $f(x)$ and the boundary condition (1) is fu
 $f(x) \in W_2^{-1}(\Gamma)$ be a solution of (4). We put
 $-\varphi(u) u' - \psi(u) + \alpha Hu + \beta Hu' \in L_2(\Gamma)$
 $= 1$ we have $\int_U U ds = 0$. Therefore, th
 $\int_V + xv_{ss}$ Here the prime denotes derivatives with respect to *s*.

Lemma: If $\varepsilon + \varkappa^2 > 0$, a generalized solution, $w(z)$ of Problem values $u, v \in W_2^2(\Gamma)$ and the boundary condition (1) is fulfilled a.e.

Proof: Let $u \in W_2^1(\Gamma)$

$$
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$$

From (4) with $\eta = 1$ we have $\int U ds = 0$. Therefore, the boundary value problem

: Let
$$
u \in W_2^{-1}(I')
$$
 be a solution
\n
$$
U = f - \varphi(u) u' - \psi(u) + c
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\nwith $\eta = 1$ we have $\int_I U ds$
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$$
-\varepsilon u_{ss} + \kappa v_{ss} = U
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 on Γ
\n
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u(t)
$$
 includes that u is a (continu

1) $U = f - \varphi(u) u' - \psi(u) + \alpha Hu + \beta Hu' \in L_2(\Gamma)$.

From (4) with $\eta = 1$ we have $\int_{\Gamma} U ds = 0$. Therefore, the boundary value problem
 $\int_{\Gamma} u \in W_2^{-1}(\Gamma)$ includes that *u* is a (continuous) 2π -periodic function in *s*. Analo are continuous 2π -periodic functions in s if $u \in W_2^2(\Gamma)$.

(2)

(7)

has a unique solution $w_0(z) = u_0(z) + iv_0(z)$ with boundary values $u_0, v_0 \in W_2^2(\Gamma)$ satisfying the additional condition $w_0(0) = 0$. This solution may be easily constructed in closed form by trigonometric Fourier expansion, for instance. The function u_0 fulfils the identity (see (4)) Nonlinear Riemann-Hilb

has a unique solution $w_0(z) = u_0(z) + iv_0(z)$ with bou

satisfying the additional condition $w_0(0) = 0$. This soluti

in closed form by trigonometric Fourier expansion, for

fulfils the identity (see (*b*
 *a*₀ *a*₁ *a₁ a₂ a₂ <i>a*₂ *a***₂** *a***₁ ***a*₃ *a*₃ *a*₃

$$
a_0(u_0, \eta) = b(\eta) - a_1(u, \eta) - a_2(u, \eta) \quad \text{for} \quad \eta \in W_2^1(\Gamma). \tag{8}
$$

Putting $u_1 = u - u_0$, from (4) and (8) we have the integral relation

$$
a_0(u_1, \eta) = \int\limits_{\Gamma} \left[\varepsilon u_1' + \varkappa H u_1' \right] \eta' \, ds = 0 \quad \text{for} \quad \eta \in W_2^1(\Gamma).
$$

Hence it follows that the function $\epsilon u_1' + \varkappa H u_1'$ has the generalized derivative zero on Γ and therefore is a constant. But then \dot{u}_1 ' must be a constant and, due to its 2π periodicity, also u_1 itself. That is, $u = u_0 + u_1 \in W_2^2(\Gamma)$. Moreover, from (4) with $u \in W_2^2(\Gamma)$ by partial integration we obtain the integral relation *l* the additional condition $w_0(0) = 0$. This solution may be easily construct
form by trigonometric Fourier expansion, for instance. The function
identity (see (4))
 $a_0(u_0, \eta) = b(\eta) - a_1(u, \eta) - a_2(u, \eta)$ for $\eta \in W_2^1(\Gamma)$ Hence it follows that the function $\epsilon u_1' + \varkappa Hu_1'$ has the generalized d
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periodicity, also u_1 itself. That is, $u = u_0 + u_1 \in W_2^2(T)$. Moreove

$$
\int\limits_{\Gamma} \left\{L[u, -Hu] - \textstyle{\int}\right\} \eta \, ds = 0 \quad \text{for} \quad \eta \in W_2^1(\Gamma),
$$

which implies the validity of the boundary condition (1) *a.e.* on Γ

Remark: If $f, \varphi, \psi \in C^{\alpha}(\Gamma), 0 < \alpha < 1$, are Hölder' continuous functions with exponent α . then $\int_{\Gamma} \{L[u, -Hu] - f\} \eta ds = 0$ for $\eta \in W_2^1(I)$
which implies the validity of the boundary condition (
Remark: If $f, \varphi, \psi \in C^{\alpha}(\Gamma), 0 < \alpha < 1$, are Hölder continuous
then $u, v \in C^{2,\alpha}(\Gamma)$ have Hölder continuous derivatives o *u. i e. (P)* μ , ψ , ψ \in C²(*P*), $0 < \alpha < 1$, are Hölder continuous functions with exponent α , ν , $\nu \in C^{2,\alpha}(P)$ have Hölder continuous derivatives of second order with the same expo. Theorem 1: *U_{nder} U_l U_l u*_{*n*} *U*_{*n*} *u*_{*n*} *e C*^{*s*}(*T*), $0 < \alpha < 1$, are Hölder continuous functions with exponent α , α , α , α , β , α

We now prove the main theorem of this paper.

$$
u\psi(u) \ge \delta u^2 - D \qquad (\delta > 0, D \ge 0)
$$

Problem A possesses a generalized solution for any. $f \in L_2(\Gamma)$.

Proof: Problem A is equivalent to *the'operator equation*

$$
Au = b \quad \text{in} \quad X = W_2^1(\Gamma),
$$

Au = *b* in X = *W2 ¹ (P),* where $A = A_0 + A_1 + A_2$ and the operators A_k : $X = W_2^{-1}(F) \rightarrow X^* = W_2^{-1}(F)$ where $A = A_0 + A_1 + A_2$ and the operators $A_k: X = W_2^1(I) \rightarrow X^* = W_2^1(I)$
 $(k = 0, 1, 2)$ are defined by $a_k(u, \eta) = \langle A_k u, \eta \rangle_X$ for $u, \eta \in X$, and $b \in X^* = W_2^1(I)$ is 2. Basic existence theorem

We now prove the main theorem of this paper.

Theorem 1: *Under the additional assumptions* $\varepsilon > 0$ and
 $uv(u) \geq \delta u^2 - D$ ($\delta > 0, D \geq 0$)

Problem A possesses a generalized solution for any f *a(u, u)* = *ef U' ² ds* ^0, *a1(u, U) ⁼ ^flf - u ds* 0, (11)

The linear operators A_0 and A_1 are *continuous* since the Hilbert operator *H* is a continuous linear operator in $W_2^1(\Gamma)$. Besides, because of

$$
a_0(u, u) = \varepsilon \int_l u'^2 \, ds \geq 0, \qquad a_1(u, u) = \beta \int_l \frac{\partial u}{\partial r} u \, ds \geq 0,
$$
 (11)

where $\partial u/\partial r$ means the derivative of u in direction of the polar radius r , both operators A_0 and A_1 and therefore their sum $\overline{A}_1 = A_0 + A_1$ are *monotone*.

The operator A_2 is *completely continuous* in the sense that it maps weakly convergent sequences into strongly convergent ones. Assuming $u_n \rightharpoonup u$ in X, we have $||u_n||_X \leq K$ and, due to the compact embedding of $X = W_2^1(\Gamma)$ in $C(\Gamma)$, also $u_n \to u$ in $C(\Gamma)$. We have to show that $(k = 0, 1, 2)$ are defined by $a_k(u, \eta) = \langle A_1u, \eta \rangle_X$ for $u, \eta \in X$, and $b \in X^* = W_2^1(\Gamma)$ is

defined by (5).

The linear operators A_0 and A_1 are continuous since the Hilbert operator H is a

continuous linear oper and A_1 are continuous since then $W_2^{-1}(I)$. Besides, because of $l_s \geq 0$, $a_1(u, u) = \beta \int_{I'} \frac{\partial u}{\partial u}$
vative of u in direction of the rether sum $\overline{A}_1 = A_0 + A_1$ are their sum $\overline{A}_1 = A_0 + A_1$ are their sum $\overline{A}_$

$$
||A_2u_n - A_2u||_{X^*} = \sup_{||\eta||_X \le 1} |\langle A_2u_n - A_2u, \eta \rangle| \xrightarrow[n \to \infty]{} 0.
$$
 (12)

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 (10)

By partial integration from (6) it follows that

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al integration from (6) it follows that

$$
a_2(u, \eta) = \int\limits_{\Gamma} \psi(u) \eta \, ds - \int\limits_{\Gamma} \Phi(u) \eta' \, ds, \qquad \Phi \text{ a primitive of } \varphi.
$$

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\nBy partial integration from (6) it follows that
\n
$$
a_2(u, \eta) = \int_{\Gamma} \psi(u) \eta \, ds - \int_{\Gamma} \Phi(u) \eta' \, ds, \qquad \Phi \text{ a primitive of } \varphi.
$$
\nSince $\psi(u_n) \rightarrow \psi(u)$ and $\Phi(u_n) \rightarrow \Phi(u)$ in $C(\Gamma)$, we have
\n
$$
\sup_{\|\eta\|_X \leq 1} |\langle A_2 u_n - A_2 u, \eta \rangle|
$$
\n
$$
\leq \sup_{\|\eta\|_X \leq 1} |\varphi(u_n) - \psi(u)| |\eta(s)| \, ds + \sup_{\|\eta\|_X \leq 1} \int |\Phi(u_n) - \Phi(u)| |\eta'(s)| \, ds
$$
\n
$$
\leq \left(\int_{\Gamma} |\psi(u_n) - \psi(u)|^2 \, ds\right)^{1/2} + \left(\int_{\Gamma} |\Phi(u_n) - \Phi(u)|^2 \, ds\right)^{1/2} \rightarrow 0.
$$
\nThis proves (12).
\nAs the sum $A = \tilde{A}_1 + A_2$ of the continuous monotone operator \tilde{A}_1 and the com-
\npletely continuous one A_2 the operator A is pseudomontone. Since \tilde{A}_1 as a continuous
\nlinear operator and A_2 as a completely continuous one are bounded operators, so A is
\n*bounded*, too. Finally, owing to the assumption (9) there is
\n
$$
a_2(u, u) = \int_{\Gamma} u\psi(u) \, ds \geq \delta \int_{\Gamma} u^2 \, ds - 2\pi D
$$
\nand by (11)
\n
$$
a_0(u, u) + a_1(u, u) \geq \varepsilon \int_{\Gamma} u'^2 \, ds.
$$
\n(14)
\nTherefore, we have $\langle Au, u \rangle_x \geq \min(\varepsilon, \delta) ||u||_x^2 - 2\pi D$, and because of the assumption
\n $\varepsilon > 0$ (and $\delta > 0$) the operator A is coercitive.

This proves (12).

As the sum $A = \overline{A}_1 + A_2$ of the continuous monotone operator \overline{A}_1 and the completely continuous one A_2 the operator A is p seudomonotone. Since \bar{A}_1 as a continuous linear operator and $A_{\mathbf{2}}$ as a completely continuous one are bounded operators, so A is *bounded,* too. Finally, owing to the assumption (9) there is' **1** In sproves (12).

As the sum $A = \overline{A}_1 + A_2$ of the continuous

pletely continuous one A_2 the operator A is ps

linear operator and A_2 as a completely continuous

bounded, too. Finally, owing to the assumptic

$$
a_2(u, u) = \int\limits_{c} u\psi(u) \, ds \ge \delta \int\limits_{l} u^2 \, ds - 2\pi D \tag{13}
$$

 \bullet

and by

$$
a_0(u, u) + a_1(u, u) \ge \varepsilon \int_R u'^2 ds.
$$

Therefore, we have $\langle Au, u \rangle_X \ge \min(\varepsilon, \delta) ||u||_X^2 - 2\pi D$, and because of the assumption

 $\epsilon > 0$ (and $\delta > 0$) the operator A is *coercitive*.

The main theorem of the theory of pseudo-monotone operators by Brézis (cf. $[16: Theorem 27.2]$ now yields the existence of a solution u of the operator equation

Remark: Since the operator *A* also satisfies the condition $(S_+$ and hence) S_0 , the solution *u* of (10) is strong limit of a subsequence of solutions of the Galerkin equations of (10) with respect to an arbitrary basis in $\bar{X} = W_2^1(\Gamma)$ (cf. [16: Theorem 27.1]). Further, we remark that the question of uniqueness of the solution is an open problem. Of course, the solution is unique in the particular case: $\varphi = 0$, ψ a strictly increasing function. ence of a solution *u* of the operator equation
ies the condition $(S_+$ and hence) S_0 , the solution *u*
solutions of the Galerkin equations of (10) with
P) (cf. [16: Theorem 27.1]). Further, we remark
ion is an open 10. Incorem 27.2j) now yields the existence of a solution a of the

Remark: Since the operator A also satisfies the condition (S₊ and her

respect to an arbitrary basis in $X = W_2^1(\Gamma)$ (cf. [16: Theorem 27.1]),

that th **9 Fig. 10 C C Example 10 C C**

3. Non-coércive problems

We now deal with the *case* $\varepsilon = 0$, where the main term is $L_0[v] = \varepsilon v_{ss}$ in (1). The problem with the corresponding boundary condition *Leopropering the case* $\varepsilon = 0$ *, where

<i>L*₀*[v]* + *L*₁*(v]* + *L*₂*[u]* = *f*
 *L*₀*l U*] + *L*₁*(v)* + *L*₂*[u]* = *f*
 *d*dditional condition (2) is name ightharpoonup and with the case $\varepsilon = 0$, where the main term is $L_0[\text{oblem with the corresponding boundary condition}$
 $L_0[v] + L_1[v] + L_2[u] = f$ on Γ

d the additional condition (2) is named **Problem B.**

Theorem 2: *Under the additional assumptions* $\varepsilon \ge$

$$
L_0[v] + L_1[v] + L_2[u] = f \qquad \text{on } \Gamma \tag{15}
$$

$$
\varphi(u) \geq \nu > 0 \qquad (\varphi(u) \leq -\nu < 0), \qquad (16)
$$

and (9) Problem B possesses a generalized solution for any $f \in L_2(\Gamma)$.

Proof: We restrict ourselves to the case $\varkappa \geq 0$. In view of Theorem 1 the *perturbed* problem with the boundary condition $\varphi(u) \geq v > 0$ \vdots $(\varphi(u) \leq -v < 0),$ (16)
 and (9) *Problem B possesses a generalized solution for any* $f \in L_2(\Gamma)$.

Proof: We restrict ourselves to the case $x \geq 0$. In view of Theorem 1 the *perturbed*
 problem

$$
-\varepsilon u_{ss} + \varepsilon v_{ss} + \alpha v + \beta v_s + \varphi(u) u_s + \psi(u) = f \qquad \text{on } \Gamma \tag{17}
$$

(15)

(15)

(16)
 $L_2(\Gamma)$.

Theorem 1 the *perturbed*

on Γ (17)

n $w_{\epsilon}(z)$ with $u_{\epsilon} \in W_2^1(\Gamma)$

(4) and by the Lemma for any $\varepsilon > 0$. By definition u_{ε} satisfies the integral relation (4) and by the Lemma. u_{ϵ} , $v_{\epsilon} \in W_2^2(\Gamma)$ fulfil the boundary condition (17) a.e. on Γ .

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We show that the norms of u_{ϵ} in $X = W_2^{-1}(\Gamma)$ are bounded uniformly in ϵ . In view of (13), (14) from (4) for $\eta = u_t$, we obtain the inequality

$$
\varepsilon \int\limits_{\Gamma} u_{\epsilon}^{'2} ds + \delta \int\limits_{\Gamma} u_{\epsilon}^{2} ds \leq \int\limits_{\Gamma} f u_{\epsilon} ds + 2\pi D.
$$

Therefore we have $\delta ||u_{\ell}||_2^2 \leq ||f||_2 ||u_{\ell}||_2 + 2\pi D^2$ which implies the norm boundedness in $L_2(\Gamma)$.

$$
||u_{\epsilon}||_{2} \leq K_{0} \qquad \text{uniformly in} \quad \epsilon > 0. \tag{18}
$$

Further, multiplying (17) for $u = u_t$, $v = v_t$ by u_t and integrating over Γ yields the relation

$$
\underset{\Gamma}{\times}\int u_{\epsilon}v_{\epsilon}''\,ds+\alpha\int\limits_{\Gamma} u_{\epsilon}v_{\epsilon}\,ds+\int\limits_{\Gamma}\varphi(u_{\epsilon})\,u_{\epsilon}'^{2}\,ds=\int\limits_{\Gamma}fu_{\epsilon}'\,ds.
$$

Now there holds

$$
\int u'v''\,ds = \int u_s\,\frac{\partial u_s}{\partial r}\,ds \geq 0 \quad \text{for} \quad u \in W_2^2(\Gamma)
$$

so that $\star \int u_t' v_t'' ds \ge 0$ and, by (16), $\int \varphi(u_t) u_t'^2 ds \ge \star \int u_t'^2 ds$. Besides, $||v_t||_2$ $\leq ||u_{\epsilon}||_2 \leq K_0$ by (18). Hence we infer the inequality $\nu ||u_{\epsilon}||_2^2 \leq ||f||_2 + |\alpha| K_0 ||u_{\epsilon}||_2$ which implies the norm boundedness in $L_2(\Gamma)$

$$
||u_{\epsilon}'||_2 \leq K_1 \qquad \text{uniformly in} \quad \epsilon > 0. \tag{19}
$$

Finally, from (18) and (19) we have the estimate

$$
||u_{\epsilon}||_X \leq K \qquad \text{uniformly in } \epsilon > 0. \tag{20}
$$

Let $\varepsilon_n \to 0$ and put $u_n = u_{\varepsilon_n}$. Owing to (20) there exists a subsequence $\{u_{n_k}\}\$ of $\{u_n\}$ converging weakly in $X = W_2^{-1}(\Gamma)$ and therefore uniformly to a function $u \in W_2^{-1}(\Gamma)$. Then also the functions Hu_{n_k} converge weakly in $W_2^1(\Gamma)$ to $Hu \in W_2^1(\Gamma)$. Performing the limit $\varepsilon_{n_k} \to 0$ in (4), we obtain the identity

$$
\underset{\Gamma}{*} \int H u' \cdot \eta' ds + a_1(u, \eta) + a_2(u, \eta) = b(\eta) \qquad (\eta \in W_2^1(\Gamma))
$$

for u, i.e., $u \in W_2^1(\Gamma)$ is generalized solution of Problem B I

Remark: If $x \neq 0$, by the Lemma $u \in W_2^2(\Gamma)$.

Eurther we drop the assumption (16) on φ and prove

Theorem 3: Under the additional assumptions $x = 0$,

$$
|\varphi(u)| \le E_1 |u|^{\varrho} + D_1 \qquad (0 < \varrho < 2; E_1 \ge 0, D_1 \ge 0), \tag{21}
$$
\nassumption (9), and

the assumption (9),

$$
|\psi(u)| \le E_2 |u|^{\sigma} + D_2 \qquad (0 < \sigma < 5; E_2 \ge 0, D_2 \ge 0)
$$
 (22)

Problem B possesses a generalized solution for any $f \in L_2(\Gamma)$.

²) $\left\|\cdot\right\|_p$ denotes the norm in $L_p(\Gamma)$, $p > 1$.

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 $\begin{array}{r} \n\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot\n\end{array}$ Proof: We again consider the *perturbed problem* with the boundary condition (17) and (2). Due to the assumption (9) the estimate (18) holds again. Multiplying (17) for **•** *• •••*

Now we have
$$
||v_{\epsilon}||_2 \le ||u_{\epsilon}||_2 \le K_0
$$
 by (18) again,
\n
$$
||v_{\epsilon}||_2 \le ||u_{\epsilon}||_2 \le K_0
$$
 by (18) again,

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\nProof: We again consider the *perturbed problem* with the bou
\nand (2). Due to the assumption (9) the estimate (18) holds again
\n
$$
u = u_c
$$
, $v = v_c$ by v_c and integrating over Γ further yields the r
\n
$$
- \varkappa \int_{\Gamma} v_c^{\prime 2} ds + \alpha \int_{\Gamma} v_c^2 ds + \int_{\Gamma} \varphi(u_c) v_c u_c^{\prime \prime} ds + \int_{\Gamma} \psi(u_c) v_c
$$
\nNow we have $||v_c||_2 \leq ||u_c||_2 \leq K_0$ by (18) again,
\n
$$
\left| \int_{\Gamma} \psi(u_c) v_c ds \right| \leq E_2 \int_{\Gamma} |u_c|^{\sigma} |v_c| ds + D_2 \int_{\Gamma} |v_c| ds
$$
\n
$$
\leq E_2 K_0 ||u_c||_{2\sigma}^{\sigma} + D_2 \sqrt{2\pi} K_0
$$
\nby (22), and
\n
$$
\left| \int_{\Gamma} \varphi(u_c) v_c u_c^{\prime} ds \right| \leq E_1 \int |u_c|^{\rho} |v_c| |u_c^{\prime}| ds + D_1 \int |v_c| |u_c|
$$

 \cdot

-

0 -

$$
\begin{aligned}\n|\tilde{r} & \leq E_2 K_0 \|u_{\epsilon}\|_{2\sigma}^{\sigma} + D_2 \sqrt{2\pi} K_0 \\
\text{and} \\
\left| \int_{\Gamma} \varphi(u_{\epsilon}) v_{\epsilon} u_{\epsilon}' ds \right| & \leq E_1 \int_{\Gamma} |u_{\epsilon}|^{\rho} |v_{\epsilon}| |u_{\epsilon}'| ds + D_1 \int_{\Gamma} |v_{\epsilon}| |u_{\epsilon}'| ds \\
& \leq E_1 \| |u_{\epsilon}|^{\rho} |v_{\epsilon}|_{\rho} \|u_{\epsilon}'\|_{2} + D_1 K_0 \|u_{\epsilon}'\|_{2} \\
\text{Furthermore, by Hölder's inequality} \\
\| |u_{\epsilon}|^{\rho} |v_{\epsilon}|_{2} & \leq \|u_{\epsilon}\|_{2(\rho+1)} \|v_{\epsilon}\|_{2(\rho+1)} \leq A_{2(\rho+1)} \|u_{\epsilon}\|_{2(\rho+1)}^{\rho+1}, \\
\text{the norm of the Hilbert operator in } L_p(\Gamma), p > 1. \text{ Therefore,} \\
\text{on}\n\end{aligned}
$$

by (21). Furthermore, by Hölder's inequality

with A_p the norm of the Hilbert operator in $L_p(\Gamma),\, p>1.$ Therefore, there holds the estimation $\begin{array}{ccc} \n\vdots & \n\vdots & \n\end{array}$ by wired the set of the set

by (22), and
\n
$$
\left| \int_{\Gamma} \varphi(u_{\epsilon}) v_{\epsilon} u_{\epsilon}' ds \right| \leq E_{1} \int |u_{\epsilon}|^{p} |v_{\epsilon}| |u_{\epsilon}'| ds + D_{1} \int |v_{\epsilon}| |u_{\epsilon}'| ds
$$
\n
$$
\leq E_{1} \left| ||u_{\epsilon}|^{p} v_{\epsilon}||_{2} ||u_{\epsilon}'||_{2} + D_{1} K_{0} ||u_{\epsilon}'||_{2}
$$
\nby (21). Furthermore, by Hölder's inequality
\n
$$
\| |u_{\epsilon}|^{p} v_{\epsilon}||_{2} \leq \|u_{\epsilon}||_{2|_{\epsilon(\rho+1)}} ||v_{\epsilon}||_{2|_{\epsilon(\rho+1)}} \leq A_{2(\rho+1)} ||u_{\epsilon}||_{2(\rho+1)}^{2}
$$
\nwith A_{p} the norm of the Hilbert operator in $L_{p}(I)$, $p > 1$. Therefore, there holds the
\nestimation
\n
$$
|z| ||u_{\epsilon}'||_{2}^{2} = |z| ||v_{\epsilon}'||_{2}^{2}
$$
\n
$$
\leq K_{0} ||f||_{2} + |\alpha| K_{0}^{2} + D_{2} \sqrt{2\pi} K_{0} + E_{2} K_{0} ||u_{\epsilon}||_{2\sigma}^{2}
$$
\n
$$
+ \{E_{1} A_{2(\rho+1)} ||u_{\epsilon}||_{2(\rho+1)}^{2} + D_{1} K_{0} \} ||u_{\epsilon}'||_{2}.
$$
\n(23)
\nFinally, for any $U \in W_{2}^{-1}(I)$ satisfying the relation $\int_{I} U ds = 0$ the well-known
\ninterpolation inequality
\n
$$
||U||_{p} \leq C ||U'||_{2}^{2} ||U||_{2}^{1-\gamma}
$$
 (C > 0),
\nwhere $p \geq 2$ and $\gamma = 1/2 - 1/p$, is valid (cf. [7: Chap. II, Th. 2.2]). Taking $U = u - (1/2\pi) \int u ds$, from (24) for an arbitrary function $u \in W_{2}^{-1}(I)$ we obtain the inequality
\n
$$
||u||_{p} \leq c ||u'||_{2}^{1-\gamma} + d ||u||_{2}
$$
\n(25)
\nwith uniform positive constants c, d, depending

Finally, for any $U \in W_2^1(\Gamma)$ satisfying the relation $\int_U U ds = 0$ the well-known

$$
||U||_p \leq C ||U'||_2^{\gamma} ||U||_2^{1-\gamma} \qquad (C>0),
$$
\n(24)

where $p \ge 2$ and $\gamma = 1/2 - 1/p$, is valid (cf. [7: Chap. II, Th. 2.2]). Taking $U = u$ where $p \ge 2$ and $\gamma = 1/2 - 1/p$, is valid (cf. [7: Chap. 11, Th. 2.2]). Taking $U = u$
 $- (1/2\pi) \int u ds$, from (24) for an arbitrary function $u \in W_2^1(\Gamma)$ we obtain the ine-
quality Γ
 $\|u\|_p \le c \|u'\|_p^2 \|u\|_2^{1-\gamma} + d \|u\|$ quality *^r C* $||U'||_2$ ^{*'*} $||U||_2$ ^{1-*'*}

1 $\gamma = 1/2 - 1/p$,

from (24) for an
 c $||u'||_2$ ² $||u||_2$ ^{1-*'*} + sitive constants *c*, $+ |E_1 A_{2(q+1)}||u_t||_{2(q+1)} + D_1 K_0||u_t||_2.$
 con inequality
 $||U||_p \leq C ||U'||_2^*||U||_2^{1-\gamma}$ ($C > 0$),
 ≥ 2 and $\gamma = 1/2 - 1/p$, is valid (cf. [7: Chap. II, Th. 2.2]). Taking
 $\int u ds$, from (24) for an arbitrary function $u \in W$

$$
||u||_p \leq c ||u'||_2^{2} ||u||_2^{1-\gamma} + d ||u||_2
$$
\n(25)

with uniform positive constants c, d , depending only on p . Without loss of generality we suppose $\sigma \ge 1$ in (22) and choose $p = 2\sigma$ and $p = 2(\sigma + 1)$, respectively, in (25). Then we have we suppose $\sigma \ge 1$ in (22) and choose $p = 2\sigma$ and $p = 2(\rho + 1)$, respective

Then we have
 $||u_{\epsilon}||_{2\sigma} \le c_1 K_0^{1-\gamma_1} ||u_{\epsilon}'||_{2}^{\gamma_1} + d_1 K_0$

with $\gamma_1 = 1/2 - 1/\sigma$ so that $\sigma \gamma_1 = (\sigma - 1)/2 < 2$ since $\sigma < 5$, and
 $||u_{\$

$$
||u_{\epsilon}||_{2\sigma} \leqq c_1 K_0^{1-\gamma_1} ||u_{\epsilon}'||_{2^{\gamma_1}} + d_1 K_0 \tag{26}
$$

$$
\gamma_1 = 1/2 - 1/\sigma \text{ so that } \sigma \gamma_1 = (\sigma - 1)/2 < 2 \text{ since } \sigma < 5 \text{, and}
$$

$$
\|u_{\epsilon}\|_{2(\rho+1)} \leq c_2 K_0^{1-\gamma_1} \|u_{\epsilon}'\|_2^{\gamma_1} + d_2 K_0
$$
 (27)

with $\gamma_2 = 1/2 - 1/[2(\rho + 1)]$ so that $(\rho + 1)$ $\gamma_2 = \rho/2 < 1$ since $\rho < 2$.

 $||u_{\epsilon}||_{2\sigma} \leq c_1 K_0^{1-\gamma_1} ||u_{\epsilon}'||_{2}^{\gamma_1} + d_1 K_0$ (26)

th $\gamma_1 = 1/2 - 1/\sigma$ so that $\sigma \gamma_1 = (\sigma - 1)/2 < 2$ since $\sigma < 5$, and
 $||u_{\epsilon}||_{2(\rho+1)} \leq c_2 K_0^{1-\gamma_1} ||u_{\epsilon}'||_{2}^{\gamma_1} + d_2 K_0$ (27)

th $\gamma_2 = 1/2 - 1/[2(\rho + 1)]$ so th norms of u_{ϵ} in $L_2(\Gamma)$ follows, i.e., we again have the estimates (19) and (20). The rest with $\gamma_1 = 1/2 - 1/\sigma$ so that $\sigma \gamma_1 = (\sigma - 1)/2 < 2$ since $\sigma < 5$, and
 $||u_{\epsilon}||_{2(\rho+1)} \le c_2 K_0^{1-\gamma_1} ||u_{\epsilon}'||_2^{\gamma_1} + d_2 K_0$ (27)

with $\gamma_2 = 1/2 - 1/[2(\rho + 1)]$ so that $(\rho + 1) \gamma_2 = \rho/2 < 1$ since $\rho < 2$.

Hence, on account

Example 1: The problem (3) with constants $\lambda > 0$ and $\mu > 0$, i.e., $\kappa = -1$, $\alpha = \beta = 0$, $\varphi(u) = \lambda u$, $\psi(u) = \mu u$ fulfils the assumptions of Theorem 3.

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Sof Lande 4. Problems of Landesman-Lazer's type

Finally, we consider some problems without assuming the condition (9) for ψ . Firstly we make some brief remarks on the *case* $\psi = 0$ which will be also dealt with below -as a limit case of a Landesnian-Lazer's type problem. In this case the condition **f**
*f f f dandesman-L
<i>f f ds* = 0
f ds = 0
sly **necessary for the processary of Nonlinear Riemann-Hilbert-Poincaré Problems** 535
 azer's type
 blems without assuming the condition (9) for ψ . Firstly

on the case $\psi = 0$ which will be also dealt with below

n-Lazer's type problem. In this case **x** are consider some problems without assuming the condition (9) for ψ . Firstly
 xome brief remarks on the *case* $\psi = 0$ which will be also dealt, with below
 case of a Landesman-Lazer's type problem. In this cas 4. Problems of Landesman-Lazer's type

Finally, we consider some problems without assuming the conserve make some brief remarks on the *case* $\psi = 0$ which will be

as a limit case of a Landesman-Lazer's type problem. In

$$
\int\limits_{\Gamma} f\,ds=0
$$

is obviously necessary for the existence of a solution. If additionally $\alpha = 0$, Problem A can be reduced to the problem with the integrated boundary condition

$$
z\frac{\partial u}{\partial r} - \varepsilon \frac{\partial u}{\partial s} + \beta v + \varPhi(u) = F + C \tag{29}
$$

$$
\iint_{\Gamma} f ds = 0
$$

is obviously necessary for the existence of a so
blem A can be reduced to the problem with the i

$$
\alpha \frac{\partial u}{\partial r} - \varepsilon \frac{\partial u}{\partial s} + \beta v + \Phi(u) = F + C
$$

and (2), where C is a free constant,

$$
F(s) = \int_{0}^{s} f(\sigma) d\sigma, \qquad \Phi(u) = \int_{0}^{u} \varphi(\omega) d\omega.
$$

The problem described by the conditions (29) and

The problem described by the conditions (29) and (2) is a nonlinear generalized Poincaré problem and has been treated in the literature. We refer to the papers [14] for the general ease, [1, 7, 9, 10, 13, 15] for the case $\beta = 0$, [3, 5, 6] for the case $\varepsilon = \beta = 0$, [11; 12] for the case $x = 0$. See also [1, 10, 11, 13] for further references. Here we only consider two examples of this problem for illustration. $(u) = \int_{0}^{u} \varphi(\omega) d\omega.$

onditions (29) and (2) is a nonlinear generalize

literature. We refer to the papers [14] for the georgeneralize
 $(0, [3, 5, 6]$ for the case $\varepsilon = \beta = 0$, [11; 12] for

on the references. Here we

Example 2: The problem (3) with constants $\lambda>0$ and $\mu=0$ leads to the nonlinear Steklov problem

is problem for illustration.
\nExample 2: The problem (3) with constants
$$
\lambda > 0
$$
 and $\mu = 0$ leads to the nonlinear Steklov
\nvolume
\n
$$
\frac{\partial u}{\partial r} = \frac{\lambda}{2} u^2 - F - C
$$
 on Γ , (30)
\nwhich by [1: Example (2.6)] has a classical solution, $v \in C^2(\Omega)$, $C^2(\overline{\Omega})$ for $\mu = \frac{1}{2}$.

which by $[1:$ Example (2.6)] has a classical solution $u \in C^2(G) \cap C^1(\overline{G})$ for any Lipschitz continuous function *F* and constant $C > -$ min {*F*(s): $s \in \Gamma$ }. That means, the problem (3) with $\lambda > 0$ and $\mu = 0$ possesses a continuum of such solutions with bounded second derivative v'' of the boundary values *v* for any $f \in L_{\infty}(\Gamma)$ which fulfils (28). $\frac{\partial u}{\partial r} = \frac{\lambda}{2} u^2 - F - C$ on Γ , (30)

which by [1: Example (2.6)] has a classical solution $u \in C^2(G)$ o $C^1(\overline{G})$ for any Lipschitz con-

innuous function F and constant $C > -\min \{F(s) : s \in \Gamma\}$. That means, the prob

Example 3: By [12, 14] (cf. also the Remark to Theorem 4 below) the problem (29) with Example 3: By [12, 14] (cf. also the Remark to Theorem 4 below) the problem (29) with
(2), where $x > 0$ or $x = 0$ with $(\varepsilon \ge 0, \beta \ge 0$ and) $\varepsilon + \beta > 0$, respectively, has a (suitably
defined generalized) solution $\alpha \in I$ defined generalized) solution $u \in L_2(\Gamma)$ if *-*

$$
u\Phi(u) \ge -c |u| - d \qquad (c \ge 0, d \ge 0)
$$

and

$$
x > 0 \text{ or } x = 0 \text{ with } (s \ge 0, \beta \ge 0)
$$

generalized) solution $u \in L_2(\Gamma)$ if

$$
u \Phi(u) \ge -c |u| - d \qquad (c \ge 0, d \ge 0)
$$

$$
\Phi_{-} < \frac{1}{2\pi} \int_{\Gamma} [F(s) + C] ds < \Phi_{+},
$$

y constant C with

$$
\Phi_{-} < C + \frac{1}{2\pi} \int_{\Gamma} F(s) ds < \Phi_{+},
$$

i.e. for any coistant *C* with

$$
u\Phi(u) \ge -c |u| - d \qquad (c \ge 0, d \ge 0)
$$

\n
$$
\Phi_{-} < \frac{1}{2\pi} \int_{\Gamma} [F(s) + C] ds < \Phi_{+},
$$

\ny constant C with
\n
$$
\Phi_{-} < C + \frac{1}{2\pi} \int_{\Gamma} F(s) ds < \Phi_{+}, \quad \Phi_{+} = \lim_{u \to +\infty} \inf \Phi(u), \quad \Phi_{-} = \lim_{u \to -\infty} \sup \Phi(u),
$$

\nsmally,
\n
$$
|\Phi(u)| \le a |u| + b \qquad (a \ge 0, b \ge 0),
$$

\n
$$
u' \in L_{2}(\Gamma) \text{ and } u \in W_{2}^{-1}(\Gamma) \text{ is a generalized solution in the sense of point 1 above, under the assumptions (31) (and (32)) and } \Phi_{-} < \Phi_{+}
$$
 the Problem A with

If, additionally,

$$
\Phi(u) \leq a \left| u \right| + b \qquad (a \geq 0, b \geq 0),
$$

then also $u' \in L_2(\Gamma)$ and $u \in W_2^1(\Gamma)$ is a generalized solution in the sense of point 1 above.
Therefore, under the assumptions (31) (and (32)) and $\Phi \prec \Phi_+$ the Problem A with $\varkappa > 0$ If, additionally,
 $|\Phi(u)| \le a |u| + b$ $(a \ge 0, b \ge 0)$, (32)

then also $u' \in L_2(\Gamma)$ and $u \in W_2^{-1}(\Gamma)$ is a generalized solution in the sense of point 1 above.

Therefore, under the assumptions (31) (and (32)) and $\Phi_{-} < \Phi_{+$ for any $|\Phi(u)| \le a |u| + b$ $(a \ge 0, b \ge 0)$, $\Phi(u) = \Phi(u) + b$ $(a \ge 0, b \ge 0)$.

Therefore, under the assumptions (31) (and (32)) and $\Phi_{-} < \Phi_{+}$ the Problem A with $\varkappa > 0$ or $\varkappa = 0, \varepsilon + \beta > 0$ possesses a continuum of such gen

*:35**

(32)

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filled if Φ is a non-constant monotone increasing function. I.e., under the above-mentioned Filled if Φ is a non-constant monotone increasing function. I.e., under the above mentioned
filled if Φ is a non-constant monotone increasing function. I.e., under the above mentioned
restrictions on the parameters restrictions on the parameters α , ε , β , Problem A has solutions if the condition $\varphi(u) \ge 0$ is satisfied (for $\varphi(u) \equiv 0$, obviously, a solution u exists which is determined apart from an arbi-
trary additive c 536 L. v. WOLFERSDORF

L. v. WOLFERSDORF

filled if Φ is a non-constant monotone increasing function. I.e.,

restrictions on the parameters κ , ε , β , Problem A has solutions

satisfied (for $\varphi(u) \equiv 0$, obvio B. I. v. WOLFERSDORF

Let if Φ is a non-constant monotone increasing function. I.e., under the above-mentioned

trictions on the parameters x , ε , β , Problem A has solutions if the condition $\varphi(u) \ge 0$ is

sis ⁶
 \pm Theorem **4:** Under the additional monotone increasing function. I.e., under the above mentioned

trictions on the parameters z , ε , β , Problem A has solutions if the condition $\varphi(u) \ge 0$ is
 isting (f 536 L. v. WOLFERSDORF
 and a i a i a i and a i a <i>i a i a <i>i a i a i a i a i a i a i a i a i a i a i a i a i a i a i a i a i p(u) > —bo Jul - *^D0 (6o 0,- D >* 0) (33)

In case $\psi \neq 0$ there holds the following theorem of Landesman-Lazer's type for Problems A and B.

Theorem 4: Under the additional assumptions $z \geq 0$, $\alpha \leq 0$ ($z \leq 0$, $\alpha \geq 0$), (16),

$$
u\psi(u) \geq -\delta_0 |u| - D_0 \qquad (\delta_0 \geq 0, D_0 \geq 0)
$$
\n
$$
(33)
$$

Problem A possesses a generalized solution for each $f \in L_2(\Gamma)$ *satisfying the inequality*

In case
$$
\psi \neq 0
$$
 there holds the following theorem of Landesman-Lazer's type for
\noblems A and B.
\nTheorem 4: *Under the additional assumptions* $z \geq 0$, $\alpha \leq 0$ ($z \leq 0$, $\alpha \geq 0$), (16),
\n $u\psi(u) \geq -\delta_0 |u| - D_0$ ($\delta_0 \geq 0, D_0 \geq 0$) (33)
\n $oblem A possesses a generalized solution for each $f \in L_2(\Gamma)$ satisfying the inequality
\n $\psi_- < \frac{1}{2\pi} \int f ds < \psi_+$,
\n(34)
\nHere
\n $\psi_+ = \lim_{u \to +\infty} \inf \psi(u), \qquad \psi_- = \lim_{u \to -\infty} \sup \psi(u).$
\nWe remark that the condition (33) implies that $-\infty \leq \psi_- \leq \delta_0$, $-\delta_0 \leq \psi_+ \leq +\infty$. Of
\narse, for (34) to hold it is to assume that $\psi_- < \psi_+$.
\nProof: We consider the *perturbed problem* with the boundary condition$

where

$$
\psi_+ = \liminf_{u \to +\infty} \psi(u), \qquad \psi_- = \limsup_{u \to -\infty} \psi(u).
$$

We remark that the condition (33) implies that $-\infty \leq \psi_-\leq \delta_0$, $-\delta_0 \leq \psi_+\leq +\infty$. Of course, for (34) to hold it is to assume that $\psi_- < \psi_+$. $\Psi_+ = \lim_{u \to +\infty} \inf \psi(u),$ $\Psi_- = \limsup_{u \to -\infty} \psi(u).$

We remark that the condition (33) implies that $-\infty \le \Psi_- \le \delta_0$, $-\delta_0 \le \Psi_+ \le +\infty$. Of course, for (34) to hold it is to assume that $\Psi_- < \Psi_+$.

Proof: We consider the *p*

Proof: We consider the *perturbed problem* with the boundary condition

$$
-\varepsilon u_{ss} + \varkappa v_{ss} + \alpha v + \beta v_s + \varphi(u) u_s + \delta u + \psi(u) = f \qquad \text{on } \Gamma \tag{35}
$$

a generalized solution $w_0(z)$ with $u_0 \in W_2^2(\Gamma)$ for any $\delta > 0$. We again have to prove that the norms of u_{δ} in $X = W_2^{-1}(\Gamma)$ are uniformly bounded. $ev_{3s} + \alpha v_{3s} + \alpha v + \beta v_s + \varphi(u) u_s + \delta u + \psi(u) = f$ on Γ

dditional condition (2). By Theorems 1, 2 and the Lemma this problem

ized solution $w_3(z)$ with $u_3 \in W_2^2(\Gamma)$ for any $\delta > 0$. We again have to p

norms of u_s in

Multiplying (35) for $u = u_0$, $v = v_0$ by u_0' and integrating over Γ yields the relation

$$
\int u_{\delta}^{3} u_{\delta}^{3} u_{\delta}^{3} dx + \alpha \int u_{\delta}^{3} v_{\delta} ds + \int_{\Gamma} \varphi(u_{\delta}) u_{\delta}^{3} ds = \int_{\Gamma} f u_{\delta}^{3} ds.
$$

Now there hold the inequalities

re hold the inequalities
\n
$$
\int u'_0 v'_0 u'_0 ds \ge 0
$$
 and
$$
\int u'_0 v_0 ds \le 0.
$$
\n
$$
\int u'_0 v_0 ds \le 0.
$$
\n
$$
\int u'_0 u'_0 ds \le \int u'_0 ds
$$
\n
$$
\int u'_0 ds \le \int u'_0 ds
$$
\n
$$
\int u'_0 ds \le 0, \quad \text{where } \alpha \ge 0, \alpha \ge 0 \text{ and } \alpha \le 0, \alpha \ge 0, \text{ respectively.}
$$

On account of the assumption (16) we therefore have

$$
\int_{\Gamma} u_{\delta}^{\prime 2} ds \leqq \pm \int_{\Gamma} f u_{\delta}^{\prime} ds
$$

in cases $x \ge 0$, $\alpha \le 0$ and $x \le 0$, $\alpha \ge 0$, respectively. I.e., in both cases $\nu ||u_0||_2^2$
in cases $x \ge 0$, $\alpha \le 0$ and $x \le 0$, $\alpha \ge 0$, respectively. I.e., in both cases $\nu ||u_0||_2^2$
 $\le ||f||_2 ||u_0||_2$. This im $||f||_2 ||u_4||_2$. This implies the uniform boundedness of the norms of u_4 ' in $L_2(I)$. It remains to show that also the norms of u_0 themselves in $L_2(\Gamma)$ are uniformly bounded. We decompose $u_{\delta} = C_{\delta} + U_{\delta}$, where C_{δ} are constants and $\int U_{\delta} ds = 0$. Since in cases
 $\leq ||f||_2$ ||

remains

We de
 $U_0' = i$

selves a

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We have not the $\begin{aligned}\n\mathbf{v} &\int u_{\delta}^{(2)} ds \leq \pm \int f u_{\delta}^{(2)} ds \\
\mathbf{v} &\leq \int f u_{\delta}^{(2)} ds \\
\mathbf{v} &\leq \int f u_{\delta}^{(2)} ds\n\end{aligned}$
 $\begin{aligned}\n\mathbf{v} &\leq \int u_{\delta}^{(2)} ds \leq \int f u_{\delta}^{(2)} ds \\
\mathbf{v} &\leq \int f u_{\delta}^{(2)} ds \\
\mathbf{v} &\leq \int f u_{\delta}^{(2)} ds\n\end{aligned}$
 $\begin{aligned}\n\mathbf{v} &\leq \$

 $U_{\delta}^{\prime} = u_{\delta}^{\prime}$ and the norms of u_{δ}^{\prime} in $L_2(\Gamma)$ are uniformly bounded, the functions U_{δ} themselves are uniformly bounded':

$$
|U_{\delta}(s)|\leq L.
$$

We have to prove that also the constants C_{δ} are uniformly bounded. If this were not the case, there exists a sequence $\{C_{\delta_n}\}\$ going to $+\infty$ or $-\infty$ as $\delta_n \to 0$. From (35)

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we have the relation

$$
2\pi \delta_n C_{\delta_n} + \int\limits_{\Gamma} \psi(C_{\delta_n} + U_{\delta_n}) ds = \int\limits_{\Gamma} f ds. \tag{37}
$$

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the relation
 $2\pi\delta_nC_{\delta_n} + \int_p \psi(C_{\delta_n} + U_{\delta_n}) ds = \int_p f ds.$ (37)
 $\delta_n \to \infty$ as $\delta_n \to 0$, we apply Fatou's lemma to (37) taking into account that
($u \ge -\mu_0$ with a (positive) co If now $C_{\delta_n} \to \infty$ as $\delta_n \to 0$, we apply Fatou's lemma to (37) taking into account that
by (33) $\psi(u) \geq -\mu_0$ with a (positive) constant μ_0 for sufficiently large u, say $u \geq \lambda_0$,
and by (36) $C_{\delta_n} + U_{\delta_n} \geq \lambda$ *by* (33) $\psi(u) \ge -\mu_0$ with a (positive) constant μ_0 for sufficiently large u, say $u \ge \lambda_0$, and by (36) $C_{\delta_n} + U_{\delta_n} \ge \lambda_0$ for sufficiently large n. Hence we obtain the inequality

P1

$$
\int_{\Gamma} f ds \geq \liminf_{n \to \infty} \int_{\Gamma} \psi(C_{\delta_n} + U_{\delta_n}) ds \geq \int_{\Gamma} \liminf_{n \to \infty} \psi(C_{\delta_n} + U_{\delta_n}) ds \geq 2\pi\psi_+
$$

which is a contradiction to the right-hand side of (34). In the same way,the assumption $C_{\delta_n} \to -\infty$ as $\delta_n \to 0$ leads to a contradiction to the left-hand side of (34)

Coollary: *If* $\psi_-\leq \psi(u)\leq \psi_+$ *for all* $u \in \mathbb{R}$, *in particular, for a monotone nondecreasing function* ψ *, the condition (34) with* \leq *instead of* \lt *is obviously necessary for the solvability of Problem A because of the relation* contradiction to the right
 ∞ as $\delta_n \to 0$ leads to a c
ary: *If* $\psi_{-} \leq \psi(u) \leq \psi$
function ψ , the condition
lity of Problem A becaus
 $\int \psi(u) ds = \int f ds$

$$
\int \psi(u)\ ds = \int f\ ds
$$

following from (4) with $\eta = 1$ *. In the limit case* $\psi = 0$ *the above proof also goes through* with the assumption (34) replaced by (28) since for $\psi = 0$ from (37) and (28) it follows *that all constants* C_{δ_n} *vanish. The condition (28) is therefore necessary and sufficient in this case.*

Remark: In the particular case $\varepsilon = \varkappa = \beta = 0$ and $\varphi = 1$ ($\varphi = -1$) the existence assertion of Theorem 4 also holds true under the more general conditions that $\alpha \notin \{1, 2, ...\}$ ($\alpha \notin \{-1,$ $(-2,...)$) and the Carathéodory function $\psi = \psi(u, s)$ satisfies the assumption (33) with nonnegative functions $\delta_0 \in L_2(\Gamma)$; $D_0 \in L_1(\Gamma)$ and the assumption **s**: In the particular case $\varepsilon = \varkappa = \beta = 0$ and φ
14 also holds true under the more general cond the Carathéodory function $\psi = \psi(u, s)$ sat
netions $\delta_0 \in L_2(\Gamma)$; $D_0 \in L_1(\Gamma)$ and the assum
sup $|\psi(u, s)| \in L_1(\Gamma)$ for

This follows from Remark 111.3 to Theorem 111.6 of [2] like in the corresponding proof in [12], but there only the Theorem 111.6 of [2] itself has been used. The solution *u* lies in $L_2(\Gamma)$ with $\pm u' + \alpha v \in L_1(\Gamma)$. Finally, the same statement is true also in the case $\varepsilon = \varkappa = 0$, $\beta = 0$, *cotter interiors* $\psi = \psi(u, s)$ satisfies the assumption (33) with non-
negative functions $\delta_0 \in L_2(\Gamma)$; $D_0 \in L_1(\Gamma)$ and the assumption
sup $|\psi(u, s)| \in L_1(\Gamma)$ for any $R > 0$.
 $|u| \le R$
This follows from Remark III.3 to Th sup $|\psi(a, s)| \in L_1(1)$ for any $R > 0$
follows from Remark 111.3 to Theorem III.6 of
there only the Theorem III.6 of [2] itself has b
 $+\alpha v \in L_1(\Gamma)$. Finally, the same statement is
intrary and $\varphi = \pm 1$, where $u \in L_2(\Gamma)$ wi

REFERENCES

^I ...

- [1] AMANS, H.: Nonlinear elliptic equations with nonlinear boundary conditions. In: New. Developments in Differential Equations (Ed.: W. Ескнаus). Amsterdam: North-Holland there only the Theorem IFI.6 of [
 $+\alpha v \in L_1(\Gamma)$. Finally, the same
bitrary and $\varphi = \pm 1$, where $u \in I$
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FERENCES
AMANN, H.: Nonlinear elliptic en
Developments in Differential Equ
Publ. Comp. 1976, p. 43–63.
BRÉZIS Ditrary and $\varphi = \pm 1$, where $u \in L_2(\Gamma)$ with $\beta v' \pm 1$

FERENCES

AMANN, H.: Nonlinear elliptic equations with no

Developments in Differential Equations (Ed.: W.

Publ. Comp. 1976, p. 43–63.

BRÉZIS, H., and L. NIRENB
- [2] BRÉZIS, H., and L. NIRENBERG: Characterizations of the ranges of some nonlinear operators and applications to boundary value problems. Ann. Scuola Norm. Sup. Pisa 5 (1978), 225-326.
- [3] CUSHING, J. M.: Nonlinear Steklov problems on the unit circle. J. Math. Anal. Appl. 38 (1972), 766–783.
- [4] **FycEitlioB,** A. I'l., **ii** *X.* Ill. **MYXTAPOB:** BueReHile **B TeOpH}O lICJlIIIiClliiblX dnHryJlnpHbIx** интегральных уравнений. Москва: Изд-во Наука 1980.
- **[5] KLINGELHöFER,** K.: Nonlinear harmonic boundary value problems]. Arch. Rat. Mech. Anal. 31 (1968), 364-371. [4] Гусейнов, А. И., и Х. Ш. Мухтар интегральных уравнений. Москва

[5] Кымсецногев, К.: Nonlinear harm

Anal. 31 (1968), 364—371.

[6] Кымсецногев, К.: Nonlinear harm

stein integral equations). J. Math. A

[7] Ладыженск
	- **[6] KLINCELHOFER,** K.: Nonlinear harmonic boundary value problems IT (Modified Hammerstein integral equations). J. Math. Anal. Appl. 25 (1969); $592-606$.
	- [7] Ладыженская, О. А., Солонников, В. А., и Н. Н. Уральцева: Линейные и квазилинейные уравнения параболического типа. Москва: Изд-во Наука 1967.

- [8] Poconzelski, W.: Integral Equations and their Applications, Vol. I. Oxford: Pergamon Press, and Warszawa: PWN 1966.
- [9] CffLEIFF, M.: tYber ejnige nichtlineare Vera llgemeinerungen des Randwertproblems von Poincaré, 1. und 2. Teil. Wiss. Z. Univ. Halle 19 (1970), 87-93 und 95-100.
- [10] ScImIITT, K.: Boundary value problems for quzsilinear second order elliptic equations. Nonlinear Analysis 2 (1978), 263-309..
- [11] WOLFERSDORF, L. *V.:* Monotonicity methods for two classes of nonlinear boundary value (8) Poconzelski, W.: Integral Equations and their Applications, Vol. I. Oxford: Pergamon

Press, and Warszawa: PWN 1966.

(9) SCILETER, M.: Uber einige nichtlineare Verallgemeinerungen des Randwertproblems vor

Poincaré, problems with semilinear first order elliptic systems in the plane. Math. Nach. 109 (1982).
215-238. (8) POGORZELSKI, W.: Integral Equations and their A

Press, and Warszawa: PWN 1966.

[9] SCHLEIFF, M.: Uber einige nichtlineare Verallgeme

Poincaré, 1. und 2. Teil. Wiss. Z. Univ. Halle 19 (1

[10] SCHMITT, K.: Boundary
	- [12] WOLFERSDORF, L. v.: Landesman-Lazer's type boundary value problems for holomorphic functions. Math. Nachr. 114 (1983), 181-189.
	- [13] WOLFERSDORF, L. v.: On strongly nonlinear Poincaré boundary value problems for har-
	- [14] WOLFERSDORF, L. v.: A class of nonlinear generalized Poincaré problems for harmonic functions. Math. Nachr. 129 (1986), 103-108.
	- 1151 *VOLSRA-BOcUENEK,* J.: Problèine non . linéaire h derivee oblique. Ann. Polon. lat1i. **⁹** $(1961), 253 - 264.$
- $\begin{array}{l} \mathcal{L}_{\mathcal{A}}\left(\mathcal{L}_{\mathcal{A}}\right)=\mathcal{L}_{\mathcal{A}}\left(\mathcal{L}_{\mathcal{A}}\right)=\mathcal{L}_{\mathcal{A}}\left(\mathcal{L}_{\mathcal{A}}\right)=\mathcal{L}_{\mathcal{A}}\left(\mathcal{L}_{\mathcal{A}}\right)=\mathcal{L}_{\mathcal{A}}\left(\mathcal{L}_{\mathcal{A}}\right)=\mathcal{L}_{\mathcal{A}}\left(\mathcal{L}_{\mathcal{A}}\right)=\mathcal{L}_{\mathcal{A}}\left(\mathcal{L}_{\mathcal{A}}\right)=\mathcal{L}_{\mathcal{$ • [16] ZETDLER, E.: Vorlesungen fiber niclitlineare Funktionalanalysis II: Monotone Opeiatoren. Leipzig: BSB B. G. Teubner Verlagsgesellschaft 1977.

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