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The Convergence of Rothe's Method for Parabolic Differential Equations

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Es wird die Konvergenz des Rothe-Verfahrens für quasilineare parabolische Differentialgleichungen unter für die Existenzaussage typischen Voraussetzungen bewiesen. Grundlage hierfür sind geeignete Approximationsschemata für Evolutionstripel und die Beobachtung, daß das Rothe-Verfahren den Forderungen der Stabilität und der Konsistenz genügt.

Доказывается сходимость метода Роте для квазилинейных параболических дифференциальных уравнений при типических предположениях о существовании. Основание этого — подходящая аппроксимация для триплета пространств и наблюдение, что метод Роте выполняет условия устойчивости и состоятельности.

The convergence of Rothe's method for quasilinear parabolic differential equations is proved under typical assumptions which guarantee existence. The investigations are based on appropriate approximation schemes for evolution tripels and on the observation that Rothe's method satisfies the requirements of stability and consistency.

1. Introduction

Consider the mixed problem for a quasilinear parabolic differential equation

$$\begin{aligned} & u_t(x,t) + \sum_{|\alpha|\leqq m} (-1)^{|\alpha|} D^\alpha A_\alpha\big(t,x,Du(x,t)\big) = f(x,t) \\ & \text{in } Q_T = \Omega \times [0,T], \\ & u(x,0) = u_0(x) \qquad \text{in } \Omega, \\ & D^\beta u(x,t) = 0 \qquad \text{on } \partial\Omega \times [0,T] \quad \text{for} \quad 0 \leqq |\beta| \leqq m-1 \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^N$, T is some fixed positive number and $Du = (D^\beta u)_{|\beta|\leqq m}$ ($m \geqq 1$). In abstract form (1) may be written as

$$\begin{aligned} & u'(t) + A(t)\,u(t) = f(t) \qquad \text{for all} \quad t \in (0,T), \\ & u(0) = u_0. \end{aligned} \tag{2}$$

The Rothe method introduced by Rothe [12] more than fifty years ago consists in replacing (2) by a family of elliptic equations using discretization in time and difference quotients:

$$\left.\begin{aligned} h^{-1}\big(u_h(t_1) - u_h(t_0)\big) &+ A(t_1)\,u_h(t_1) = f_h(t_1) \\ h^{-1}\big(u_h(t_2) - u_h(t_1)\big) &+ A(t_2)\,u_h(t_2) = f_h(t_2) \\ &\vdots \\ h^{-1}\big(u_h(t_n) - u_h(t_{n-1})\big) &+ A(t_n)\,u_h(t_n) = f_h(t_n) \end{aligned}\right\} \tag{3}$$

with $u_h(t_0) = u_0$. Here $h = T/n$, $n \in \mathbb{N}$, denotes the parameter of discretization and $t_k = kh$ $(k = 0, 1, \ldots, n)$. The system (3) may be regarded as an implicite scheme for the successive determination of $u_h(t_1), \ldots, u_h(t_n)$ from the given datum $u_h(t_0) = u_0$.

Since there exists a plenty of methods for the (approximate) solution of elliptic equations Rothe's method has gained significance for numerical mathematics (cf. Rektorys [11], where many theoretical and practical questions concerning Rothe's method are enlighted). Nečas [8] applied Rothe's method to abstract parabolic equations in a Hilbert space and Kačur [4, 5] to equations of the general form (1). One of the typical results in these papers is, roughly speaking, the following: The piecewise constant and the piecewise linear functions constructed from the solutions $u_h(t_i)$ of (3) converge to the solution of (1) in the norm of $L_\infty(0, T, L_2(\Omega))$ as $h \to 0$, i.e. $n \to \infty$, under a hypothesis like e.g. $u_0 \in \mathring{W}_2^m(\Omega)$, $A(0)\, u_0 \in L_2(\Omega)$, $f \in L_\infty(0, T, L_2(\Omega))$. If $A(0)$ satisfies certain regularity assumptions, this means that u_0 must belong to the Sobolev space $W_2^{2m}(\Omega) \cap \mathring{W}_2^m(\Omega)$. Under these assumptions existence and uniqueness results are slightly better obtained by nonlinear semigroup theory and the solution is more regular. Here we shall use the weaker assumptions from the existence theory established by Browder and Lions — Strauss (cf. Lions [7]). Gröger [3] proves the convergence of an iteration method which in the case of equation (2) with $f = 0$ reads as

$$\frac{1}{h}\left(u_h(t_i) - u_h(t_{i-1})\right) + \int\limits_{t_{i-1}}^{t_i} A(t)\, u_h(t_i)\, dt = 0.$$

He reduces it to Rothe's equation (3) if $t \mapsto A(t)$ is Lipschitz continuous and also gives stronger convergence results under additional regularity assumptions on A, u_0.

It is the aim of this paper to prove the convergence of Rothe's method under those typical hypotheses on the operator A, the initial datum u_0 and the right-hand side f of (1) which guarantee existence. We would like to mention that Rothe's method sometimes is called (transversal) method of lines in distinction from the longitudinal method of lines investigated in detail by Walter (cf. [14]).

2. Abstract parabolic equations and their approximation

2.1 Existence theorem

It is the purpose of this section to state the general conditions under which problem (2) has a solution. Simultaneously we want to present the minor changes necessary for the application of a difference method, like Rothe's method. Suppose $(V, \|\cdot\|_V)$ is a reflexive and separable Banach space densely embedded in a separable Hilbert space $(H, |\cdot|)$ with scalar product $(\cdot, \cdot)$, V^* denotes the dual space to V and $\langle \cdot, \cdot \rangle$ is the pairing between V^* and V. By identification of H with its dual we get the triple of spaces $V \subset H \subset V^*$. The basis of our subsequent considerations are the spaces

$$X = L_p(0, T, V), \qquad X^* = L_q(0, T, V^*),$$
$$W = W_p^1(0, T, V, H) = \{u \in L_p(0, T, V) \mid u' \in L_q(0, T, V^*)\}$$

where $p \geqq 2$, $p^{-1} + q^{-1} = 1$. The norms are defined by

$$\|u\|_X = \left(\int\limits_0^T \|u(t)\|_V^p\, dt\right)^{1/p}, \qquad \|u^*\|_{X^*} = \left(\int\limits_0^T \|u^*(t)\|_{V^*}^q\, dt\right)^{1/q}$$

and $\|u\|_W = \|u\|_X + \|u'\|_{X^*}$ for $u \in X$, $u^* \in X^*$, and $u \in W$, respectively. Remember that $W \hookrightarrow C([0, T], H)$. For details cf. Gajewski, Gröger and Zacharias [2: Ch. 4], Lions [7: Ch. 2], Wloka [15: Ch. 4], Zeidler [16: Ch. 23].

Theorem 1: *Let $A(t)$ $(0 \leqq t \leqq T)$ be a family of monotone mappings from V into V^* with the following properties:*

(H1) *Coerciveness: $\langle A(t)\,\bar{u}, \bar{u}\rangle \geqq \gamma[\bar{u}]^p - \gamma_1$ for all $\bar{u} \in V$, $t \in [0, T]$ where $\gamma, \gamma_1 > 0$ are constants and $[\cdot]: V \to \mathbf{R}$ denotes a seminorm satisfying $[\cdot] + \lambda\,|\cdot| \geqq c_0\,\|\cdot\|_V$ for some constants $\lambda, c_0 > 0$.*

(H2) *Growth condition: $\|A(t)\,\bar{u}\|_{V^*} \leqq c_1(1 + \|\bar{u}\|_V^{p/q})$ for all $\bar{u} \in V$, $t \in [0, T]$ where $c_1 > 0$ is a constant.*

(H3) *Continuity: $A(t_l)\,\bar{w}_l \to A(t)\,\bar{w}$ in V^* for all sequences $(t_l) \subset [0, T]$, $(\bar{w}_l) \subset V$ with $t_l \to t$, $\bar{w}_l \to \bar{w}$ in V as $l \to \infty$.*

For the right-hand side and the initial value we assume

(H4) $f \in L_q(0, T, V^*)$,

(H5) $u_0 \in H$.

Define $(Au)(t) = A(t)\,u(t)$. Then the problem

$$u' + Au = f, \qquad u(0) = u_0 \tag{4}$$

has exactly one solution $u \in W = W_p^1(0, T, V, H)$.

Proof: Cf. Gajewski, Gröger and Zacharias [2: Ch. 6], Lions [7: Ch. 2], Zeidler [16: Ch. 30].

Remarks: 1. If $A(t)$ is hemicontinuous and (H2), (H3) above are replaced by

(H2′) $\|A(t)\,\bar{u}\|_V \leqq g(t) + c_1\,\|u\|_V^{p/q}$ with $g \in L_q(0, T)$,

(H3′) the function $t \mapsto \langle A(t)\,\bar{u}, \bar{v}\rangle$ is measurable on $[0, T]$ for all $\bar{u}, \bar{v} \in V$

the conclusion of Theorem 1 is still valid. 2. The case $1 < p < 2$ may be treated, too (cf. Gajewski, Gröger and Zacharias [2: p. 142]).

2.2 Approximation scheme

In order to prove the convergence of Rothe's method in the next section we construct approximation schemes adapted for the application to equations (2), (3). We use methods of Aubin [1], Raviart [10], and Temam [13], but we must pay attention to the fact that our aim is to approximate mappings from $[0, T] \subset \mathbf{R}$ into Banach spaces instead of real-valued functions in the cited references.

A. Grid functions: Set $h = T/n$ $(n \in \mathbf{N})$, $t_{k,h} = kh$ $(k = 0, 1, \ldots, n)$. If it is clear to which subdivision (characterized by h) $t_{k,h}$ belongs we also write t_k instead of $t_{k,h}$. Define the grids

$$\mathcal{G}_h = \{t_k \mid k = 1, \ldots, n\} \quad \text{and} \quad \overline{\mathcal{G}}_h = \{t_k \mid k = 0, 1, \ldots, n\},$$

and the following spaces of grid functions:

$$X_h = L_p(\mathcal{G}_h, V) = \left\{u_h : \mathcal{G}_h \to V \mid \|u_h\|_{X_h} = \left(h \sum_{i=1}^{n} \|u_h(t_i)\|_V^p\right)^{1/p}\right\},$$

$$X_h^* = L_q(\mathcal{G}_h, V^*) = \left\{f_h : \mathcal{G}_h \to V^* \mid \|f_h\|_{X_h^*} = \left(h \sum_{i=1}^{n} \|f_h(t_i)\|_{V^*}^q\right)^{1/q}\right\},$$

$$W_h = W^1(\bar{\mathcal{G}}_h) = \{u_h : \bar{\mathcal{G}}_h \to H \mid u_h(t_i) \in V \quad \text{for} \quad i = 1, \dots, n;$$
$$\|u_h\|_{W_h} = \|u\|_{X_h} + \|\nabla u_h\|_{X_h^*}\}$$

where $\nabla u_h(t_i) = h^{-1}\big(u_h(t_i) - u_h(t_{i-1})\big)$ is the backward difference quotient.

B. The prolongation p_h^0: For any grid function defined on $\mathcal{G}_h$ with values in V (H, V^*, respectively) we set

$$(p_h^0 u_h)(t) = \sum_{i=1}^{n} u_h(t_i)\, \chi_i(t)$$

where χ_i is the characteristic function of $(t_{i-1}, t_i]$.

C. The prolongation operators p_h and p_h^1: First, we define the mapping $p_h : W^1(\bar{\mathcal{G}}_h) \to L_\infty(0, T, H)$ by

$$(p_h u_h)(t) = (p_h^0 u_h)(t) \qquad \text{for} \qquad t \in (0, T], \quad u_h \in W^1(\bar{\mathcal{G}}_h).$$

Furthermore we set

$$(p_h^1 u_h)(t) = u_h(t_{i-1}) + \frac{1}{h}\big(u_h(t_i) - u_h(t_{i-1})\big)(t - t_{i-1})$$

if $t_{i-1} \leqq t \leqq t_i$ $(i = 1, \dots, n)$, $u_h \in W^1(\bar{\mathcal{G}}_h)$. The functions $p_h^1 u_h$ (often called *Rothe functions*) are piecewise linear mappings from $[0, T]$ into H, thus $p_h^1 : W^1(\bar{\mathcal{G}}_h) \to L_\infty(0, T, H)$.

D. The restriction operators r_h and $\tilde{r}_h$: There exists a linear continuous mapping $P : W_p^1(0, T, V, H) \to W_p^1(\mathbf{R}, V, H)$ satisfying $(Pu)(t) = u(t)$ for $t \in [0, T]$ and $(Pu)(t) = 0$ in the exterior of some fixed compact set K containing $[0, T]$ (it is easy to see that the proof of NEČAS [9: Theorem 2.3.9/pp. 75–76] concerning the extension of functions from usual Sobolev spaces carries over to our situation with obvious changes (Bochner's integral, triple of spaces $V \subset H \subset V^*$)). With this preparation we may define $r_h : W_p^1(0, T, V, H) \to W^1(\bar{\mathcal{G}}_h)$ by

$$(r_h u)(t_i) = \frac{1}{h} \int_{t_{i-1}}^{t_i} (Pu)(t)\, dt, \qquad i = 0, 1, \dots, n.$$

Thus r_h is an averaging operator. Further, for any $f \in L_q(0, T, V^*)$ we define $\tilde{r}_h : L_q(0, T, V^*) \to L_q(\mathcal{G}_h, V^*)$ by

$$(\tilde{r}_h f)(t_i) = \frac{1}{h} \int_{t_{i-1}}^{t_i} f(t)\, dt, \qquad i = 1, \dots, n.$$

E. Properties of the approximation scheme: Let us consider the approximation scheme

$$\begin{array}{ccc} L = L_\infty(0, T, H) & \xleftarrow{\;\;\omega\;\;} & W = W_p^1(0, T, V, H) \\ \uparrow {\scriptstyle p_h,\, p_h^1} & & \downarrow {\scriptstyle r_h} \\ & \longleftarrow & W_h = W^1(\bar{\mathcal{G}}_h) \end{array} \qquad (\Gamma)$$

Here ω denotes the continuous embedding $W \hookrightarrow L$. We show (for proofs cf. Appendix 1) that the approximation is *stable* (cf. RAVIART [10], TEMAM [13]).

Lemma 1: *We have*

(i) $\|p_h^0 u_h\|_{L_p(0,T,V)} = \|u_h\|_{L_p(\mathcal{S}_h,V)}$ *for all* $u_h \in L_p(\mathcal{S}_h, V)$,

(ii) $\|p_h u_h\|_{L_\infty(0,T,H)} \leqq C \|u_h\|_{W^1(\overline{\mathcal{S}}_h)}$ *for all* $u_h \in W^1(\overline{\mathcal{S}}_h)$

where $C > 0$ *is a constant independent of* h.

Lemma 2: *We have*

(i) $\|r_h u\|_{W^1(\overline{\mathcal{S}}_h)} \leqq C \|u\|_W$ *for all* $u \in W_p{}^1(0, T, V, H)$,

(ii) $\|\tilde{r}_h f\|_{L_q(\mathcal{S}_h,V^*)} \leqq C \|f\|_{L_q(0,T,V^*)}$ *for all* $f \in L_q(0, T, V^*)$ *where* $C > 0$ *is a constant independent of* h.

Lemma 3 (Convergence of the approximation scheme): *We have*

(i) $p_h^0 r_h u \to u$ *in* $L_p(0, T, V)$,

(ii) $p_h^0 \nabla r_h u \to u'$ *in* $L_q(0, T, V^*)$,

(iii) $p_h r_h u \to \omega u$ *in* $L_\infty(0, T, H)$

for all $u \in W_p{}^1(0, T, V, H)$ *as* $h \to 0$.

2.3 Convergence result

We now investigate the convergence of Rothe's method. For a given $f \in L_q(0, T, V^*)$ we define the right-hand side of (3) as $\tilde{r}_h f$ and get the following system of equations for the determination of the grid function u_h:

$$\left.\begin{aligned} u_h(t_0) &= u_{h0}, \\ h^{-1}u_h(t_1) + A(t_1)\, u_h(t_1) &= (\tilde{r}_h f)\,(t_1) + h^{-1}u_h(t_0), \\ &\vdots \\ h^{-1}u_h(t_n) + A(t_n)\, u_h(t_n) &= (\tilde{r}_h f)\,(t_n) + h^{-1}u_h(t_{n-1}). \end{aligned}\right\} \tag{5}$$

Theorem 2: *Suppose hypotheses* **(H1)–(H5)** *of Theorem* 1 *are satisfied. Assume*

(H6) $(u_{h0}) \subset H$ *is a sequence with* $u_{h0} \to u_0$ *in* H *as* $h \to 0$ *(e.g.* $u_{h0} = u_0$ *for all* h *is a possible choice).*

Then

(i) *for any* $h = T/n$, $n \in \mathbf{N}$, (5) *possesses a unique solution* $u_h \in W^1(\overline{\mathcal{S}}_h)$,

(ii) $\sup_h \|u_h\|_{W_1(\overline{\mathcal{S}}_h)} < \infty$,

(iii) $p_h u_h \to \omega u$ *in* $L_\infty(0, T, H)$ *as* $h \to 0$.

Corollary: *For the sequence of the Rothe functions* (cf. C) *we have the convergence* $p_h{}^1 u_h \to \omega u$ *in* $C([0, T], H)$ *as* $h \to 0$.

Proof of Theorem 2: (i): Let us define a mapping $A_{i,h}: V \to V^*$ $(i = 1, \ldots, n)$ by

$$A_{i,h}\overline{u} = \frac{1}{h}\,\overline{u} + A(t_i)\,\overline{u} \quad \text{for} \quad \overline{u} \in V.$$

We claim that $A_{i,h}$ is monotone, continuous and coercive. Clearly

$$\begin{aligned} \langle A_{i,h}\overline{u} - A_{i,h}\overline{v}, \overline{u} - \overline{v}\rangle &= \frac{1}{h}\,|\overline{u} - \overline{v}|^2 + \langle A(t_i)\,\overline{u} - A(t_i)\,\overline{v}, \overline{u} - \overline{v}\rangle \\ &\geqq \frac{1}{h}\,|\overline{u} - \overline{v}|^2 \geqq 0 \end{aligned} \tag{6}$$

for all $\bar{u}, \bar{v} \in V$ since $A(t)$ is monotone for all $t \in [0, T]$. The continuity of $A_{i,h}$ follows immediately from (H3) and the continuity of the embedding $V \hookrightarrow H$. To verify that $A_{i,h}$ is coercive we have

$$(\|u\|_V)^{-1} \langle A_{i,h}\bar{u}, \bar{u}\rangle = (\|\bar{u}\|_V)^{-1} (h^{-1} |\bar{u}|^2 + \langle A(t_i)\bar{u}, \bar{u}\rangle)$$

$$\geqq -\frac{\gamma_1}{\|\bar{u}\|_V} + c_0 \frac{h^{-1} |\bar{u}|^2 + \gamma[\bar{u}]^p}{\lambda |\bar{u}| + [\bar{u}]}$$

$$\geqq -\frac{\gamma_1}{\|\bar{u}\|_V} + \begin{cases} c_0 \min (h^{-1}\lambda^{-1} |\bar{u}|, \gamma[\bar{u}]^{p-1}) & \text{if} \quad |\bar{u}|, [\bar{u}] > 0, \\ c_0 h^{-1}\lambda^{-1} |\bar{u}| & \text{if} \quad [\bar{u}] = 0, |\bar{u}| > 0, \\ c_0\gamma[u]^{p-1} & \text{if} \quad |\bar{u}| = 0, [\bar{u}] > 0, \end{cases}$$

$$\to \infty \quad \text{if} \quad \|\bar{u}\|_V \to \infty$$

because of hypothesis (H1) of Theorem 1. We now consider the system (5). Since $u_0 \in H \subset V^*$, $(\tilde{r}_h f)(t_1) \in V^*$, the well-known existence theorem for monotone, hemicontinuous, and coercive operators (cf. LIONS [7: Theorem 2.1] gives the existence of a solution $u_h(t_1) \in V$ of the first equation of (5). Uniqueness follows from (6). The remaining equations of (5) are treated analogously.

(ii): a) We multiply the first equation of (5) by $u_h(t_1)$, the second by $u_h(t_2)$, etc. This gives

$$\frac{1}{h} |u_h(t_i)|^2 + \langle A(t_i)\, u_h(t_i), u_h(t_i)\rangle$$

$$= \langle (\tilde{r}_h f)(t_i), u_h(t_i)\rangle + \frac{1}{h} \big(u_h(t_{i-1}), u_h(t_i)\big),$$

$i = 1, \ldots, n$. Hypothesis (H1) implies

$$\frac{1}{h} |u_h(t_i)|^2 + \gamma[u_h(t_i)]^p$$

$$\leqq \gamma_1 + \|(\tilde{r}_h f)(t_i)\|_{V^*} \|u_h(t_i)\|_V + \frac{1}{h} |u_h(t_{i-1})|\, |u_h(t_i)|,$$

$$\frac{1}{2} |u_h(t_i)|^2 + h\gamma[u_h(t_i)]^p$$

$$\leqq h\gamma_1 + h \|(\tilde{r}_h f)(t_i)\|_{V^*} \|u_h(t_i)\|_V + \frac{1}{2} |u_h(t_{i-1})|^2.$$

Summing up the first k of these equations we get by virtue of the inequalities of Hölder and Young

$$\frac{1}{2} |u_h(t_k)|^2 + \gamma h \sum_{i=1}^{k} [u_h(t_i)]^p$$

$$\leqq kh\gamma_1 + \left(h \sum_{i=1}^{k} \|(\tilde{r}_h f)(t_i)\|_{V^*}^q\right)^{1/q} \left(h \sum_{i=1}^{k} \|u_h(t_i)\|_V^p\right)^{1/p} + \frac{1}{2} |u_h(t_0)|^2$$

$$\leqq kh\gamma_1 + C \|\tilde{r}_h f\|_{L_q(\mathcal{S}, V^*)} \left[\left(h \sum_{i=1}^{k} |u_h(t_i)|^p\right)^{1/p}\right.$$

$$\left. + \left(h \sum_{i=1}^{k} [u_h(t_i)]^p\right)^{1/p}\right] + \frac{1}{2} |u_h(t_0)|^2$$

$$\leqq kh\gamma_1 + C_\gamma \|\tilde{r}_h f\|^q_{L_q(\mathcal{S}_h, V^*)} + \frac{\gamma}{2}\left(h \sum_{i=1}^{k} [u_h(t_i)]^p\right)$$

$$+ C \|\tilde{r}_h f\|_{L_q(\mathcal{S}_h, V^*)} \left(h \sum_{r=1}^{k} |u_h(t_i)|^p\right)^{1/p} + \frac{1}{2} |u_h(t_0)|^2$$

and

$$\frac{1}{2} |u_h(t_k)|^2 + \frac{\gamma}{2} h \sum_{i=1}^{k} [u_h(t_i)]^p$$

$$\leqq kh\gamma_1 + C_\gamma \|\tilde{r}_h f\|^q_{L_q}$$

$$+ \frac{1}{2} C^2 \|\tilde{r}_h f\|^2_{L_q} + \frac{1}{2}\left(h \sum_{i=1}^{k} |u_h(t_i)|^p\right)^{2/p} + \frac{1}{2} |u_h(t_0)|^2.$$

Thus by Lemma 2/(ii)

$$|u_h(t_k)|^2 + \gamma h \sum_{i=1}^{k} [u_h(t_i)]^p \leqq C_f + \left(h \sum_{i=1}^{k} |u_h(t_i)|^p\right)^{2/p} + |u_h(t_0)|^2 \tag{7}$$

where $C_f > 0$ is a constant depending on $f \in L_q(0, T, V^*)$ only. Especially we have

$$|u_h(t_k)|^p \leqq C_{f,u_0}\left(1 + h \sum_{i=1}^{k} |u_h(t_i)|^p\right), \qquad k = 1, \ldots, n,$$

where C_{f,u_0} depends on f and $u_0 \in H$ only. From this and a discrete variant of Gronwall's Lemma (cf. Lemma 4/Appendix 2) we get $|u_h(t_k)| \leqq C$ for all $h = T/n$ $(n \in \mathbf{N})$ and $k = 1, \ldots, n$ where C is a constant independent from h and k. Now (7) with $k = n$ gives $h \sum_{i=1}^{n} [u_h(t_i)]^p \leqq \tilde{C}_{f,u_0}$. Therefore

$$\|u_h\|_{L_p(\mathcal{S}_h, V)} \leqq \tilde{\tilde{C}}_{f,u_0} \qquad \text{for all } h.$$

Here again $\tilde{C}_{f,u_0}$ and $\tilde{\tilde{C}}_{f,u_0}$ are constants depending on f und u_0 only.

b) From the Rothe equations (5) we get

$$h^{-1}\big(u_h(t_i) - u_h(t_{i-1})\big) = (\tilde{r}_h f)(t_i) - A(t_i)\, u_h(t_i)$$

for $i = 1, \ldots, n$, i.e.

$$\|\nabla u_h(t_i)\|^q_{V^*} \leqq C\big(\|(\tilde{r}_h f)(t_i)\|^q_{V^*} + \|A(t_i)\, u_h(t_i)\|^q_{V^*}\big)$$

$$\leqq C\big(1 + \|(\tilde{r}_h f)(t_i)\|^q_{V^*} + \|u_h(t_i)\|^p_V\big)$$

by virtue of hypothesis (H2). Therefore it follows from part a) that

$$h \sum_{i=1}^{n} \|\nabla u_h(t_i)\|^q_{V^*} \leqq C\big(1 + \|\tilde{r}_h f\|^q_{L_q(\mathcal{S}_h, V^*)} + \|u_h\|^p_{L_p(\mathcal{S}_h, V)}\big) \leqq C$$

i.e. $\sup_h \|u_h\|_{W^1(\bar{\mathcal{S}}_h)} < \infty$.

(iii): a) First we try to give an estimate for the difference of two solutions u_h, v_h of Rothe's equations (3) corresponding to different right-hand sides f_h, g_h. The resulting implication "(9), (10) $\Rightarrow$ (11)" is a *stability property* of our numerical procedure. Suppose

$$h^{-1}\big(u_h(t_i) - u_h(t_{i-1})\big) + A(t_i)\, u_h(t_i) = f_h(t_i),$$

$$h^{-1}\big(v_h(t_i) - v_h(t_{i-1})\big) + A(t_i)\, v_h(t_i) = g_h(t_i)$$

for $i = 1, \ldots, n$. Scalar multiplication of these equations by $u_h(t_i) - v_h(t_i)$ and subtraction gives

$$\frac{1}{h} |u_h(t_i) - v_h(t_i)|^2 + \langle A(t_i)\, u_h(t_i) - A_h(t_i)\, v_h(t_i), u_h(t_i) - v_h(t_i)\rangle$$
$$= \langle f_h(t_i) - g_h(t_i), u_h(t_i) - v_h(t_i)\rangle + \frac{1}{h} \left(u_h(t_{i-1}) - v_h(t_{i-1}), u_h(t_i) - v_h(t_i)\right).$$

Summing up the first k of these equations we get by virtue of the monotonicity of $A(t_i)$ and Hölder's inequality (cf. proof of (ii)/a))

$$\frac{1}{2} |u_h(t_k) - v_h(t_k)|^2 \leqq \left(h \sum_{i=1}^{k} \|f_h(t_i) - g_h(t_i)\|_{V^*}^q\right)^{1/q}$$
$$\times \left(h \sum_{i=1}^{k} \|u_h(t_i) - v_h(t_i)\|_V^p\right)^{1/p} + \frac{1}{2} |u_h(t_0) - v_h(t_0)|^2.$$

Thus we can give the estimate

$$|u_h(t_k) - v_h(t_k)|^2$$
$$\leqq 2 \|f_h - g_h\|_{L_q(\mathcal{S}_h, V^*)} \|u_h - v_h\|_{L_p(\mathcal{S}_h, V)} + |u_h(0) - v_h(0)|^2 \tag{8}$$

for $k = 1, \ldots, n$.

b) Let us suppose now we are given two sequences (f_h), $(g_h) \subset L_p(\mathcal{S}_h, V^*)$ ($h = T/n$, $n \in \mathbb{N}$) bounded in the sense

$$\sup_h \|f_h\|_{L_q(\mathcal{S}_h, V^*)}, \qquad \sup_h \|g_h\|_{L_q(\mathcal{S}_h, V^*)} < \infty$$

and satisfying

$$\|f_h - g_h\|_{L_q(\mathcal{S}_h, V^*)} \to 0 \qquad \text{as} \qquad h \to 0. \tag{9}$$

Furthermore we assume that

$$u_h(0) \to u_0 \quad \text{and} \quad v_h(0) \to u_0 \quad \text{in} \quad H \text{ as } h \to 0. \tag{10}$$

From the proof of (ii) we conclude that $\sup_h \|u_h\|_{W_h}$, $\sup_h \|v_h\|_{W_h} < \infty$. Then it follows from (8)—(10) that $\sup_{1 \leqq k \leqq n} |u_h(t_k) - v_h(t_k)| \to 0$ as $h \to 0$, i.e.

$$\|p_h u_h - p_h v_h\|_{L_\infty(0,T,H)} \to 0 \qquad \text{as} \qquad h \to 0. \tag{11}$$

c) In this point we want to show that our discretization method is *consistent* with the original problem (4) (cf. (12), (15)). Suppose $u \in W$ is the unique solution of (4) (cf. Theorem 1). Set $u_h(t_i) = (r_h u)\,(t_i)$ in the left-hand side of Rothe's equations (3) ($i = 0, \ldots, n$; $h = T/n$) and consider the resulting right-hand side

$$\frac{1}{h} \left((r_h u)\,(t_i) - (r_h u)\,(t_{i-1})\right) + A(t_i)\,(r_h u)\,(t_i) =: g_h(t_i),$$

$i = 1, \ldots, n$. We intend to use part b) to show that

$$\|p_h u_h - p_h r_h u\|_{L_\infty(0,T,H)} \to 0 \qquad \text{as} \qquad h \to 0.$$

In view of Lemma 3/(iii) this implies assertion (iii). That (10) is valid follows from hypothesis (H6) of the theorem and Lemma 3/(iii). Now we consider the grid function

$$\psi_h : \mathcal{S}_h \to V^*, \qquad \psi_h(t_i) = A(t_i)\,(r_h u)\,(t_i) \qquad (i = 1, \ldots, n).$$

If follows from hypothesis (H2) of Theorem 1 that

$$\|\psi_h(t_i)\|_{V^*}^q \leqq \text{const}\,(1 + \|(r_h u)(t_i)\|_V^p).$$

Thus by Lemma 2/(i) $\sup_h \|\psi_h\|_{L_q(\mathcal{T}_h, V^*)} < \infty$. We claim that

$$p_h^0 \psi_h \to Au \text{ in } L_q(0, T, V^*) \quad \text{as} \quad h \to 0. \tag{12}$$

From Lemma 3/(i) we get $p_h^0 r_h u(t) \to u(t)$ in V a.e. on $[0, T]$ as $h \to 0$. A reasoning similar to the proof of the completeness of L_p-spaces (cf. KUFNER, JOHN and FUCIK [6: p. 74] gives the existence of a function $w \in L_p(0, T)$ such that $\|p_h^0 r_h u(t)\|_V \leqq w(t)$ a.e. on $[0, T]$. Since $(p_h^0 \psi_h)(t) = A(t_i)(r_h u)(t_i)$ if $t_{i-1,h} < t \leqq t_{i,h}$ it follows that

$$\|p_h^0 \psi_h(t)\|_{V^*}^q \leqq C(1 + \|p_h^0 r_h u(t)\|_V^p) \leqq C\big(1 + w(t)^p\big) \tag{13}$$

a.e. on $[0, T]$. For $t_{i-1,h} < t \leqq t_{i,h}$ we have

$$\begin{aligned}\|(p_h^0 \psi_h)(t) - A(t)\,u(t)\|_{V^*} &= \|A(t_{i,h})(r_h u)(t_{i,h}) - A(t)\,u(t)\|_{V^*} \\ &= \|A(t_{i,h})(p_h^0 r_h u)(t) - A(t)\,u(t)\|_{V^*}.\end{aligned}$$

Thus it remains to use (H3) of Theorem 1 to see that

$$\|(p_h^0 \psi_h)(t) - A(t)\,u(t)\|_{V^*} \to 0 \quad \text{a.e. on } [0, T] \quad \text{as} \quad h \to +0. \tag{14}$$

Now we get (12) from (13), (14) and Lebesgue's Theorem on Dominated Convergence. Together with Lemma 3/(ii) (12) implies

$$p_h^0 g_h \to u' + Au = f \quad \text{in} \quad L_q(0, T, V^*) \quad \text{as} \quad h \to 0. \tag{15}$$

Equivalently $\|g_h - \tilde{r}_h f\|_{L_q(\mathcal{T}_h, V^*)} \to 0$ as $h \to 0$ and the conclusion $\|p_h u_h - \omega u\|_{L_\infty(0,T,H)} \to 0$ follows from part b) and Lemma 3/(iii) ∎

Proof of the Corollary: We have for any $h > 0$

$$\begin{aligned}\sup_{0 \leqq t \leqq T} |p_h^1 u_h(t) - p_h u_h(t)| &\leqq \sup_{1 \leqq k \leqq n} |u_h(t_{k,h}) - u_h(t_{k-1,h})| \\ &\leqq \sup_k |u_h(t_{k,h}) - \omega u(t_{k,h})| + \sup_k |\omega u(t_{k,h}) - \omega u(t_{k-1,h})| \\ &\quad + \sup_k |\omega u(t_{k-1,h}) - u_h(t_{k-1,h})| \\ &\leqq 2 \sup_s |p_h u_h(s) - \omega u(s)| + \sup_{|\tau - s| \leqq h} |\omega u(\tau) - \omega u(s)|.\end{aligned}$$

The result follows since $p_h u_h \to \omega u$ in $L_\infty(0, T, H)$ as $h \to 0$ and $W_p^1(0, T, V, H) \hookrightarrow C([0, T], H)$ ∎

3. Application to parabolic differential equations

Suppose Ω is a bounded domain in $\mathbf{R}^N$, $\Omega \in \mathcal{C}^{0,1}$ (cf. KUFNER, JOHN and FUCIK [6: p. 305], T is some fixed positive number, $Q_T = \Omega \times [0, T]$ and $Du = (D^\beta u)_{|\beta| \leqq m}$ is the tupel of all spatial derivatives up to the order m ($m \geqq 1$) of a function u $Q_T \to \mathbf{R}$. We are going to apply Bothe's method to the parabolic equation

$$u_t(x, t) + \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^\alpha A_\alpha\big(t, x, Du(x, t)\big) = f(x, t) \qquad \text{in } Q_T$$

with prescribed initial and boundary data

$$u(x, 0) = u_0(x) \quad \text{for} \quad x \in \Omega,$$

$$D^\beta u(x, t) = 0 \quad \text{on} \quad \partial\Omega \times [0, T] \quad \text{for} \quad 0 \leqq |\beta| \leqq m - 1.$$

We prepare the use of the results of Chapter 2 by the following definitions.

Spaces:

$$V = \mathring{W}_p^m(\Omega), \qquad H = L_2(\Omega), \qquad X = L_p(0, T, V), \qquad X^* = L_q(0, T, V^*),$$

$$W = W_p^1(0, T, V, H) = \{u \in X \mid u' \in X^*\} \qquad (p \geqq 2, p^{-1} + q^{-1} = 1).$$

Here $\mathring{W}_p^m$ is the usual Sobolev space defined e.g. in NEČAS [9: p. 64].

Operator A:

$$(Au)(t) := A(t)\, u(t) \quad \text{for} \quad t \in [0, T]$$

where $A(t)\,\bar{u} \in V^*$ is defined for $\bar{u}, \bar{v} \in V$ by

$$\langle A(t)\, \bar{u}, \bar{v} \rangle = \int_\Omega \Big(\sum_{|\alpha| \leqq m} A_\alpha\big(t, x, D\bar{u}(x, t)\big)\, D^\alpha \bar{v}(x) \Big)\, dx. \tag{16}$$

Right-hand side f:

Suppose $f \in L_q(Q_T)$. Then an element $b \in X^*$ is given by

$$\langle b(t), v \rangle = \int_\Omega f(x, t)\, \bar{v}(x)\, dx \quad \text{for} \quad \text{all } \bar{v} \in V.$$

With these preliminaries we can formulate the convergence theorem.

Theorem 3: *Assume that the following hypotheses are satisfied.*

a) *Carathéodory condition: For all α, $|\alpha| \leqq m$, let $A_\alpha : [0, T] \times \Omega \times \mathbf{R}^\mu \to \mathbf{R}$ be a function such that $x \mapsto A_\alpha(t, x, D)$ is measurable on Ω for all $t \in [0, T]$, $D = (D^\beta) \in \mathbf{R}^\mu$, $(t, D) \mapsto A_\alpha(t, x, D)$ is continuous on $[0, T] \times \mathbf{R}^\mu$ for almost all $x \in \Omega$ (μ is the cardinal number of the set $\{\alpha : |\alpha| \leqq m\}$).*

b) *Growth condition:*

$$|A_\alpha(t, x, D)| \leqq d(x) + C \sum_{|\beta| \leqq m} |D^\beta|^{p-1}$$

for all $t \in [0, T]$, $x \in \Omega$, $D = (D^\beta) \in \mathbf{R}^\mu$ where $d \in L_q(\Omega)$ and $C > 0$ is constant

c) *Monotonicity:*

$$\sum_{|\alpha| \leqq m} \big(A_\alpha(t, x, D) - A_\alpha(t, x, D')\big)\,(D^\alpha - D'^\alpha) \geqq 0$$

for all $t \in [0, T]$, $x \in \Omega$, $D = (D^\beta)$, $D' = (D'^\beta) \in \mathbf{R}^\mu$.

d) *Coerciveness:*

$$\sum_{|\alpha| \leqq m} A_\alpha(t, x, D)\, D^\alpha \geqq \gamma_0 \sum_{|\beta| \leqq m} |D^\beta|^p - K(x)$$

for all $t \in [0, T]$, $x \in \Omega$, $D \in \mathbf{R}^\mu$ where $K \in L_1(\Omega)$, $\gamma_0 > 0$ constant.

e) *Initial datum: $u_0 \in H$. Further, we suppose that $(u_{h0}) \subset H$ is a sequence with $u_{h0} \to u_0$ in H as $h \to 0$.*

f) *Right-hand side: $f \in L_q(Q_T)$.*

Then all the hypotheses of Theorem 1 *and Theorem* 2 *are satisfied and Rothe's method converges.*

Proof: That $A(t)$ ($t \in [0, T]$) is a monotone mapping follows immediately from hypothesis c). The growth condition b) and (16) give $A(t): V \to V^*$. It is obvious that the family $A(t)$ satisfies the growth condition (H2) of Theorem 1.

Let $(t_l) \subset [0, T]$, $(\overline{w}_l) \subset V$ be sequences such that $t_l \to t$ and $\overline{w}_l \to \overline{w}$ in V. We intend to show that for all α, $|\alpha| \leqq m$,

$$I_l := \int_\Omega |A_\alpha(t_l, x, D\overline{w}_l(x)) - A_\alpha(t, x, D\overline{w}(x))|^q \, dx \xrightarrow[l\to\infty]{} 0.$$

First we observe that passing to a subsequence if necessary

$$D^\beta \overline{w}_l(x) \to D^\beta \overline{w}(x) \quad \text{and} \quad |D^\beta \overline{w}_l(x)| \leqq g(x) \quad \text{a.e. on } \Omega$$

for some function $g \in L_p(\Omega)$ ($|\beta| \leqq m$). Again, cf. Kufner, John and Fucik [6: p. 74]. Thus

$$A_\alpha(t_l, x, D^\beta \overline{w}_l(x)) \xrightarrow[l\to\infty]{} A_\alpha(t, x, D^\beta \overline{w}(x)) \qquad \text{a.e. on } \Omega.$$

This and the growth condition b) permit to apply Lebesgue's Theorem on Dominated Convergence. Therefore $I_l \to 0$ as $l \to \infty$ by virtue of an argument concerning subsequences (cf. Zeidler [16: Ch. 10].

The coerciveness hypothesis of Theorem 1 clearly follows from condition d) above ∎

Appendix 1

We are going to prove the stability and convergence properties of the approximation scheme Γ which were stated in Section 2.2.

Proof of Lemma 1: (i): The definition of p_h^0 gives

$$\|p_h^0 u_h\|^p_{L_p(0,T,V)} = h \sum_{i=1}^n \|u_h(t_i)\|_V^p = \|u_h\|^p_{L_p(\mathcal{S}_h,V)}$$

(ii): This proof is more difficult since it constitutes the discrete version of the embedding result $W_p^1(0, T, V, H) \hookrightarrow C([0, T], H)$, i.e. after a possible change on a set of measure zero any function $u \in W = W_p^1(0, T, V, H)$ is continuous and

$$\|u\|_{L_\infty(0,T,H)} \leqq c \, \|u\|_W, \qquad c \quad \text{a constant}. \tag{A.1}$$

We use the idea for the demonstration of (A.1) in Gajewski, Gröger and Zacharias [2: pp. 144, 148].

a) First we want to show that for some constant $C_1 > 0$, independent of h,

$$\|p_h^0 u_h\|_{L_\infty(0,T,V^*)} \leqq C_1 \, \|u_h\|_{W^1(\overline{\mathcal{S}}_h)} \quad \text{for} \quad u_h \in W^1(\overline{\mathcal{S}}_h). \tag{A.2}$$

Clearly for any t_k we have $u_h(t_k) = u_h(t_0) + h \sum_{i=1}^k \nabla u_h(t_i)$; define $v_h \in W^1(\overline{\mathcal{S}}_h)$ by $v_h(0) = 0$, $v_h(t_k) = h \sum_{i=1}^k \nabla u_h(t_i)$. It follows that for all k

$$\|v_h(t_k)\|_{V^*} \leqq h \sum_{i=1}^n \|\nabla u_h(t_i)\|_{V^*} \leqq C \left(h \sum_{i=1}^n \|\nabla u_h(t_i)\|_{V^*}^q\right)^{1/q}. \tag{A.3}$$

Since $u_h(t_k) - v_h(t_k) = u_h(t_0)$ for all k we get

$$T^{1/p} \|u_h(t_0)\|_{V^*} = \left(h \sum_{i=1}^{n} \|u_h(t_i) - v_h(t_i)\|_{V^*}^p\right)^{1/p}$$

$$\leqq \|u_h\|_{L_p(\mathcal{S}_h,V^*)} + \left(h \sum_{i=1}^{n} \|v_h(t_i)\|_{V^*}^p\right)^{1/p}$$

$$\leqq C(\|u_h\|_{L_p(\mathcal{S}_h,V)} + \sup_{1\leqq i\leqq n} \|v_h(t_i)\|_{V^*})$$

$$\leqq C(\|u_h\|_{L_p(\mathcal{S}_h,V)} + \|\nabla u_h\|_{L_q(\mathcal{S}_h,V^*)}) \leqq C \|u_h\|_{W^1(\mathcal{S}_h)} \tag{A.4}$$

by virtue of (A.3) and the continuous embedding $V \subset H \subset V^*$. Using (A.3) again (A.4) implies (A.2).

b) We note the formula of discrete integration by parts

$$\big(u_h(t_k), v_h(t_k)\big) - \big(u_h(t_{l-1}), v_h(t_l)\big)$$
$$= h \sum_{i=l}^{k} \langle \nabla u_h(t_i), v_h(t_i)\rangle + h \sum_{i=l+1}^{k} \langle \nabla v_h(t_i), u_h(t_{i-1})\rangle \tag{A.5}$$

if $k = 2, 3, \ldots, n$; $l = 1, \ldots, k-1$;

$$\big(u_h(t_1), v_h(t_1)\big) - \big(u_h(t_0), v_h(t_1)\big) = h \langle \nabla u_h(t_1), v_h(t_1)\rangle$$

if $k = 1$. Suppose $\varphi_h: \mathcal{S}_h \to \mathbf{R}$. Then the discrete version of the formula for differentiating a product reads, for $i = 1, \ldots, n$, as

$$\nabla(\varphi_h u_h)\,(t_i) = \big(\nabla\varphi_h(t_i)\big)\, u_h(t_i) + \varphi_h(t_{i-1})\, \nabla u_h(t_i). \tag{A.6}$$

c) We choose an arbitrary C^1-function $\varphi: [0, T] \to \mathbf{R}$ with $\varphi(0) = 0$, $\varphi(T) = 1$ (e.g. $T^{-1}t$). For any $h > 0$, φ defines a grid function φ_h via $\varphi_h(t_k) = \varphi(t_k)$ for $k = 0, 1, \ldots, n$. Set

$$v_h(t_k) = \varphi_h(t_k)\, u_h(t_k) \qquad \text{for} \quad k = 0, 1, \ldots, n,$$
$$w_h(t_k) = u_h(t_{k+1}) - \varphi_h(t_{k+1})\, u_h(t_{k+1}) \quad \text{for} \quad k = 0, 1, \ldots, n-1.$$

We use formula (A.5) for v_h, u_h with $l = 1$ and w_h, u_h with $l = k$, $k = n-1$, respectively. This gives

$$\big(v_h(t_k), u_h(t_k)\big) = h \sum_{i=1}^{k} \langle \nabla(\varphi_h u_h)\,(t_i), u_h(t_i)\rangle$$
$$+ h \sum_{i=2}^{k} \langle \nabla u_h(t_i), \varphi_h(t_{i-1})\, u_h(t_{i-1})\rangle \tag{A.7}$$

for $k = 1, \ldots, n$ defining the second term on the right-hand side of (A.7) to be zero if $k = 1$. Further

$$-\big(w_h(t_{k-1}), u_h(t_k)\big) = h \sum_{i=k}^{n-1} \langle \nabla w_h(t_i), u_h(t_i)\rangle$$
$$+ h \sum_{i=k+1}^{n-1} \langle \nabla u_h(t_i), w_h(t_{i-1})\rangle \tag{A.8}$$

for $k = 1, \ldots, n - 1$ with the convention that the second term on the right-hand side is zero if $k = n - 1$. Subtracting (A.8) from (A.7) yields

$$|u_h(t_k)|^2 = h \sum_{i=1}^{k} \{\nabla\varphi_h(t_i)\,\langle u_h(t_i), u_h(t_i)\rangle + \varphi_h(t_{i-1})\,\langle \nabla u_h(t_i), u_h(t_i)\rangle\}$$
$$+ h \sum_{i=2}^{k} \varphi_h(t_{i-1})\,\langle \nabla u_h(t_i), u_h(t_{i-1})\rangle + h \sum_{i=k}^{n-1} \langle \nabla u_h(t_{i+1}), u_h(t_i)\rangle$$
$$- h \sum_{i=k}^{n-1} \nabla\varphi_h(t_{i+1})\,\langle u_h(t_{i+1}), u_h(t_i)\rangle - \sum_{i=k}^{n-1} \varphi_h(t_i)\,\langle \nabla u_h(t_{i+1}), u_h(t_i)\rangle$$
$$+ h \sum_{i=k+1}^{n-1} \{\langle \nabla u_h(t_i), u_h(t_i)\rangle - \varphi_h(t_i)\,\langle \nabla u_h(t_i), u_h(t_i)\rangle\}$$

for $k = 1, \ldots, n - 1$. It follows that for these k

$$|u_h(t_k)|^2 \leqq C \left(h \sum_{i=1}^{n} \|\nabla u_h(t_i)\|_{V^*}^q\right)^{1/q} \left(h \sum_{i=1}^{n} \|u_h(t_i)\|_V^p\right)^{1/p}$$
$$+ C \sup_{1\leqq i\leqq n} \|u_h(t_i)\|_{V^*} \left(h \sum_{i=1}^{n} \|u_h(t_i)\|_V^p\right)^{1/p} \qquad \text{(A.9)}$$

by virtue of (A.2) and Hölder's inequality where the constant C depends on φ and p, only. From (A.7) with $k = n$ and $\varphi(T) = 1$ we immediately see that (A.9) is valid for $k = n$, too. Thus a) gives the norm estimate $|u_h(t_k)|^2 \leqq C\,\|u_h\|_{W^1(\bar{\mathfrak{Z}}_h)}^2$ for $k = 1, \ldots, n$, with C independent of h, i.e. $\|p_h u_h\|_{L_\infty(0,T,H)} \leqq C\,\|u_h\|_{W^1(\bar{\mathfrak{Z}}_h)}$ for all $u_h \in W^1(\bar{\mathfrak{Z}}_h)$ ∎

Proof of Lemma 2: a) Consider

$$\|r_h u\|_{L_p(\mathfrak{Z}_h,V)}^p = h \sum_{i=1}^{n} \left(\left\|\frac{1}{h}\int_{t_{i-1}}^{t_i} (Pu)\,(t)\,dt\right\|_V\right)^p \leqq h \sum_{i=1}^{n} \left(\frac{1}{h}\int_{t_{i-1}}^{t_i} \|(Pu)\,(t)\|_V\,dt\right)^p$$
$$\leqq \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|(Pu)\,(t)\|_V^p\,dt = \int_0^T \|(Pu)'(t)\|_V^p\,dt.$$

b) Further

$$\|\nabla r_h u\|_{L_q(G_h,V^*)}^q = h \sum_{i=1}^{n} \left\|\frac{1}{h}\left(\frac{1}{h}\int_{t_{i-1}}^{t_i} (Pu)\,(t)\,dt - \frac{1}{h}\int_{t_{i-2}}^{t_{i-1}} (Pu)\,(t)\,dt\right)\right\|_{V^*}^q$$
$$= h \sum_{i=1}^{n} \left\|\frac{1}{h}\int_{t_{i-1}}^{t_i} \frac{1}{h}\big((Pu)\,(t) - (Pu)\,(t-h)\big)\,dt\right\|_{V^*}^q$$
$$= h \sum_{i=1}^{n} \left\|\frac{1}{h}\int_{t_{i-1}}^{t_i} \left(\frac{1}{h}\int_{t-h}^{t} (Pu)'\,(s)\,ds\right) dt\right\|_{V^*}^q$$
$$\leqq \frac{1}{h} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\int_{t-h}^{t} \|(Pu)'\,(\bar{s})\|_{V^*}^q\,ds\right) dt$$

$$= \frac{1}{h} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\int_0^h \|(Pu)'(\eta + t - h)\|_{V^*}^q \, d\eta \right) dt$$

$$= \frac{1}{h} \int_0^h \sum_{i=1}^{n} \left(\int_{t_{i-1}}^{t_i} \|(Pu)'(\eta + t - h)\|_{V^*}^q \, dt \right) d\eta$$

$$\leqq \frac{1}{h} \int_0^h \left(\int_{\mathbf{R}} \|(Pu)'(\eta + t - h)\|_{V^*}^q \, dt \right) d\eta = \int_{\mathbf{R}} \|(Pu)'(t)\|_{V^*}^q \, dt$$

where Jensen's inequality and the theorem of Tonelli were used. Observe that supp Pu is a compact subset of $\mathbf{R}$. Assertion (i) follows from a), b) and the continuity of P ∎

Proof of Lemma 3: To begin with we remember the fact that the set $C^1([0, T], V)$ is dense in W (cf. GAJEWSKI, GRÖGER and ZACHARIAS [2: p. 144]. Since the linear mappings

$$p_h^0 r_h: \quad W \to L_p(0, T, V),$$
$$p_h^0 \nabla r_h: W \to L_q(0, T, V^*),$$
$$p_h r_h: \quad W \to L_\infty(0, T, H)$$

have operator norms bounded independently of $h > 0$ by virtue of Lemma 1 and Lemma 2 it suffices to prove the assertions (i)—(iii) for $u \in C^1([0, T], V)$ (cf. RAVIART [10], TEMAM [13]).

(i): Suppose $t_{i-1} < t \leqq t_i$. We have

$$\|u(t) - (p_h^0 r_h u)(t)\|_V = \left\| u(t) - \frac{1}{h} \int_{t_{i-1}}^{t_i} (Pu)(s) \, ds \right\|_V$$

$$= \left\| \frac{1}{h} \int_{t_{i-1}}^{t_i} \big(u(t) - u(s)\big) \, ds \right\|_V \leqq \frac{1}{h} \int_{t_{i-1}}^{t_i} \|u(t) - u(s)\|_V \, ds.$$

For any $\varepsilon > 0$ there exists $h_0(\varepsilon) > 0$ such that $|t - s| < h_0$ implies $\|u(t) - u(s)\|_V < \varepsilon$ because u is uniformly continuous on $[0, T]$. Thus

$$\|u(t) - (p_h^0 r_h u)(t)\|_V < \varepsilon \quad \text{if} \quad h < h_0(\varepsilon), \tag{A.10}$$

i.e. $\|u(t) - (p_h^0 r_h u)(t)\|_V^p \to 0$ as $h \to 0$.

(ii): Let us consider an approximation scheme for the space $W_p^1(\mathbf{R}, V, H)$ for a moment. For any $h > 0$ we define the grid $\mathcal{R}_h = \{t_k = hk \mid k = 0, \pm 1, \pm 2, \ldots\}$ and the spaces of grid functions $L_p(\mathcal{R}_h, V)$, $L_q(\mathcal{R}_h, V^*)$, and $W^1(\mathcal{R}_h)$ analogously to Section 2.2/A. Set

$$(\hat{r}_h w)(t_k) = \frac{1}{h} \int_{t_{k-1}}^{t_k} w(t) \, dt \qquad \big(w \in W_p^1(\mathbf{R}, V, H); k = \pm 1, \pm 2, \ldots\big).$$

Then the proof of Lemma 2 shows that $\hat{r}_h: W_p^1(\mathbf{R}, V, H) \to W^1(\mathcal{R}_h)$ $(h > 0)$ is a family of linear mappings bounded independently of h:

$$\|\hat{r}_h w\|_{W^1(\mathcal{R}_h)} \leqq c \, \|w\|_{W_p^1(\mathbf{R}, V, H)} \qquad \text{for all } w \in W_p^1(\mathbf{R}, V, H).$$

Suppose $t_{i-1} < t \leqq t_i$. Then it is easy to see that

$$\|w'(t) - (p_h^0 \nabla \hat{r}_h w)(t)\|_{V^*} \leqq \frac{1}{h^2} \int_{t_{i-1}}^{t_i} \left(\int_{s-h}^{s} \|w'(t) - w'(\tau)\|_{V^*} \, d\tau \right) ds \qquad \text{(A.11)}$$

for $w \in W_p^1(\mathbf{R}, V, H)$. Since $C_0^1(\mathbf{R}, V)$ is dense in $W_p^1(\mathbf{R}, V, H)$ (Gajewski, Gröger and Zacharias [2: p. 144] and Wloka [15: p. 70]) we may derive from (A.11) that $p_h^0 \nabla \hat{r}_h w \to w'$ in $L_q(\mathbf{R}, V^*)$ as $h \to 0$ in a fashion similar to part a) of this proof. As $(p_h^0 \nabla \hat{r}_h Pu)(t) = (p_h^0 \nabla r_h u)(t)$ for $t \in [0, T]$ we get

$$p_h^0 \nabla r_h u \underset{h \to 0}{\longrightarrow} u' \quad \text{in} \quad L_q(0, T, V^*) \quad \text{for} \quad u \in W_p^1(0, T, V, H).$$

(iii): This follows from inequality (A.10) and Lemma 1/(ii) by virtue of the embedding $V \hookrightarrow H$ ∎

Appendix 2

We prove a discrete variant of *Gronwall's Lemma*.

Lemma 4: *Suppose a family of grid functions $u_h: \mathcal{S}_h \to \mathbf{R}$ $(h = T/n, n \in \mathbf{N})$ satisfies the growth condition*

$$u_h(t_{k,h}) \leqq a + bh \sum_{i=1}^{k} u_h(t_{i,h}) \qquad \text{(A.12)}$$

for all $k = 1, \ldots, n$ and $h > 0$ where $a, b > 0$ are constant. Then, with $C > 0$ being independent of $h > 0$, $\sup_{h>0} \left(\max_{1 \leqq k \leqq n} u_h(t_{k,h}) \right) \leqq C$.

Proof: We may assume that $u_h(t_k) \leqq M_h$ for all $k = 1, \ldots, n$ where M_h depends upon h, in general. Therefore in a first step $u_h(t_k) \leqq a + bhkM_h$ by virtue of (A.12). Substituting this into (A.12) again we get in the second step

$$u_h(t_k) \leqq a + abhk + b^2h^2M_h \sum_{i=1}^{k} i = a + abhk + \frac{1}{2} b^2h^2M_hk(k+1).$$

One may prove by induction on m that after m steps the iteration of this procedure gives

$$\begin{aligned} u_h(t_k) \leqq a \Bigg\{ &\binom{k-1}{0} + \binom{k}{1} bh \\ &+ \binom{k+1}{2} (bh)^2 + \cdots + \binom{k+m-2}{m-1} (bh)^{m-1} \Bigg\} \\ &+ M_h \binom{k+m-1}{m} (bh)^m \end{aligned}$$

for all $k = 1, \ldots, n$; $m \geqq 1$. Thus, if $bh < 1$, Cauchy's theorem on the product of series applied to the geometric series $(1 + bh + (bh)^2 + \cdots)$ implies that $(m \to \infty;$ $k = 1, \ldots, n)$

$$u_h(t_k) \leqq a(1 + bh + (bh)^2 + \cdots)^k = a \left(\frac{1}{1 - bh} \right)^k.$$

Thus

$$\max_{t_k \in \mathcal{T}_h} u_h(t_k) \leqq a \left(\frac{1}{1-bh}\right)^n = a \left(1 + \frac{bT}{n-bT}\right)^n \to a\,e^{bT}$$

as $n \to \infty$, i.e. $h \to 0$. Therefore we may conclude that

$$\sup_h \left(\max_{t_{k,h} \in \mathcal{T}_h} u_h(t_{k,h})\right) \leqq C, \qquad C > 0 \text{ constant } \blacksquare$$

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