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On Commutators in Algebras of Unbounded Operators

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Für Diagonal-, Quasidiagonal- und endlichdimensionale Operatoren in topologischen Algebren unbeschränkter Operatoren werden verschiedene Kriterien angegeben, aus denen die Darstellbarkeit dieser Operatoren als Kommutatoren folgt. Es wird gezeigt, daß die Kommutatoren in der maximalen Op*-Algebra bezüglich der gleichmäßigen Topologie dicht sind. Außerdem werden einige einfache Eigenschaften von Selbstkommutatoren, mehrere Vermutungen und Probleme angeführt.

Для диагональных, квазидиагональных и конечномерных операторов в топологических алгебрах неограниченных операторов даются разные критерии, из которых следует представление этих операторов в виде коммутаторов. Показано, что коммутаторы плотны относительно равномерной топологии в максимальной Op*-алгебре. Приводятся также некоторые простые факты о самокоммутаторах, гипотезы и проблемы.

For diagonal, quasidiagonal and finite dimensional operators in topological algebras of unbounded operators there are given several criteria which imply their representation as commutators. It is proved that the commutators are dense in the maximal Op*-algebra with respect to the uniform topology. Simple facts about selfcommutators, some conjectures and problems are given.

1. Introduction

The structure of commutators in algebras of bounded operators (especially in $\mathscr{B}(\mathscr{H})$ and von Neumann algebras) was investigated by many authors. Let us only remember the paper of BROWN and PEARCY [2] which can be regarded as some final step in clarifying the situation for $\mathscr{B}(\mathscr{H})$: an operator $A \in \mathscr{B}(\mathscr{H})$ is a commutator if and only if it is not of the form $A = \lambda I + C$, $\lambda \neq 0$ real number and C a compact operator. Here, $\mathscr{H}$ is a separable, infinite dimensional Hilbert space. In finite dimensional spaces one has the classical result: a quadratic matrix is a commutator if and only if it has trace zero.

Unbounded operators enter if one considers the CCR (cf. [15]). To the author's knowledge up to now commutators in algebras of unbounded operators were not investigated. The present paper should be regarded as a first step toward a systematic study of commutators in the context of topological algebras of unbounded operators. The aim is first of all to stimulate such investigations by presenting some conjectures and problems on the basis of results obtained so far. We restrict ourselves to maximal Op*-algebras $\mathscr{L}^+(\mathscr{D})$ defined on domains of the form $\mathscr{D} = \mathscr{D}^\infty(T)$ (cf. Section 2). The paper is organized as follows. Section 2 contains the necessary notions, notations and preliminaries. Section 3 concerns diagonal and quasidiagonal operators. Here the possibility of representing an operator as commutator is related with the structure of the domain $\mathscr{D}$ of the algebra. Several criteria are given which imply that such operators are commutators. In Section 4 we consider finite dimensional operators. If $\mathscr{D}$ is not of type (I), any finite dimensional operator is a commutator. If $\mathscr{D}$ is of type (I), then it is proved that "enough" finite dimensional operators are commutators. Combining considerations of Sections 3 and 4 one gets as a main result that the commutators are $\tau_{\mathcal{D}}$-dense in $\mathcal{L}^+(\mathcal{D})$. This is the analogous result to the bounded case. In Section 5 there are collected some facts about selfcommutators. Finally, in Section 6 we formulate some conjectures as well as problems.

2. Preliminaries

For a dense linear manifold $\mathcal{D}$ in a separable Hilbert space $\mathcal{H}$ the set $\mathcal{L}^+(\mathcal{D}) = \{A : A\mathcal{D} \subset \mathcal{D}, A^*\mathcal{D} \subset \mathcal{D}\}$ forms a $*$-algebra with respect to the usual operations and the involution $A \to A^+ = A^* \mid \mathcal{D}$. An Op*-*algebra* $\mathcal{A}(\mathcal{D})$ is a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ containing the identity operator I. The *graph topology* t on $\mathcal{D}$ induced by $\mathcal{L}^+(\mathcal{D})$ is given by the family of seminorms

$$\varphi \to \|A\varphi\| \quad \left(A \in \mathcal{L}^+(\mathcal{D})\right).$$

Among the many possible topologies on $\mathcal{L}^+(\mathcal{D})$ we mention only the *uniform topology* $\tau_{\mathcal{D}}$ [8] given by the seminorms

$$A \to \|A\|_{\mathcal{M}} = \sup_{\varphi,\psi \in \mathcal{M}} |\langle\varphi, A\psi\rangle| \qquad (\mathcal{M} \subset \mathcal{D}\ t\text{-bounded}).$$

The set

$$\mathcal{C}(\mathcal{D}) = \{C \in \mathcal{L}^+(\mathcal{D}) : C\mathcal{M} \text{ is relatively } t\text{-compact for all } t\text{-bounded } \mathcal{M} \subset \mathcal{D}\}$$

is a two-sided $*$-ideal in $\mathcal{L}^+(\mathcal{D})$. It appears that this set is a very appropriate generalization of the ideal of compact operators in $\mathcal{B}(\mathcal{H})$ [7, 12]: If $\mathcal{D}[t]$ is an (F)-space, then the $\tau_{\mathcal{D}}$-closure of the set of finite dimensional operators of $\mathcal{L}^+(\mathcal{D})$ coincides with $\mathcal{C}(\mathcal{D})$.

In this paper we consider only (F)-domains of the form

$$\mathcal{D} = \mathcal{D}^\infty(T) = \bigcap_{n \geq 0} \mathcal{D}(T^n)$$

where $T = T^* \geqq I$ is a selfadjoint operator which can be supposed to have the structure

$$T\varphi_n = t_n\varphi_n \quad (n \in \mathbf{N}), \qquad (\varphi_n) \text{ an orthonormal basis in } \mathcal{H}.$$

If necessary, one can suppose $t_n \in \mathbf{N}$. Write shortly $T \sim (t_n)$ or more precisely, $T \sim (t_n), (\varphi_n)$. Further we use the following notations. Let $T = \int_1^\infty \lambda\, dE_\lambda$ be the spectral resolution of T. Then the operators $P_\mu = \int_1^\mu dE_\lambda$ belong to $\mathcal{L}^+(\mathcal{D})$ for all $1 \leqq \mu < \infty$ and $\mathcal{H}^\mu = P_\mu\mathcal{H} \subset \mathcal{D}$. It will be frequently used that, for all $A \in \mathcal{L}^+(\mathcal{D})$, $A = \tau_{\mathcal{D}} - \lim_{\mu\to\infty} P_\mu A P_\mu$. We will make use of the classification of domains as provided in [10, 11] (cf. also [3, 4]). To fix the notations we repeat some facts:

$\mathcal{D}$ *is of type* (I) $\left(\mathcal{D} \in (\mathrm{I})\right)$ if there is no infinite dimensional Hilbert space $\mathcal{H}_0 \subset \mathcal{D}$. This is equivalent to $\lim t_n = \infty$.

$\mathcal{D}$ *is of type* (II) $\left(\mathcal{D} \in (\mathrm{II})\right)$ if there is a splitting $\mathcal{D} = \mathcal{H}_0 \oplus \mathcal{D}_0$, $\mathcal{H}_0$ an infinite dimensional Hilbert space, $\mathcal{D}_0 \in (\mathrm{I})$. This is equivalent to a decomposition $(t_n) = (t_n^0) \cup (t_n^1)$, with a bounded sequence (t_n^0) and $\lim t_n^1 = \infty$, i.e. $\mathcal{D} = \mathcal{H}_0 \oplus \mathcal{D}^\infty(T_1)$ where $T_1 \sim (t_n^1), (\psi_n)$ for some orthonormal basis (ψ_n) in $\mathcal{H}_0^\perp$.

$\mathcal{D}$ *is of type* (III) $(\mathcal{D} \in (\mathrm{III}))$ if T has infinite many eigenvalues t_n' with infinite multiplicity and $\lim t_n' = \infty$. Therefore if $\mathcal{H}_n$ denotes the eigenspace to t_n', one has

$$\mathcal{D} = \Big(\sum_{(t_n')} \oplus \mathcal{H}_n\Big) \oplus \mathcal{D}_0 \quad \text{with} \quad \mathcal{D}_0 \in (\mathrm{I}) \quad \text{or} \quad \mathcal{D}_0 = \{0\}.$$

The sum means the following:

$$\sum_{(t_n')} \oplus \mathcal{H}_n = \Big\{\psi = \sum \oplus \psi_n : \psi_n \in \mathcal{H}_n, \sum (t_n')^{2k} \|\psi_n\|^2 < \infty \quad \forall k \in \mathrm{N}\Big\}.$$

Especially, $\mathcal{D} \in (\mathrm{III_A})$ if T can be chosen such that $\mathcal{D}_0 = \{0\}$.

In Section 3 we will use the fact that all $\mathcal{H}_n$ can be identified with some Hilbert space $\mathcal{H}$ (e.g. via some fixed isometric isomorphism). Thus it makes sense to consider any element $\psi_n \in \mathcal{H}_n$ as an element of $\mathcal{H}_j$ for any $j \neq n$. In what follows we will fix the orthonormal basis (φ_n), and if it is not indicated otherwise, all constructions will be done with respect to this basis. In general, it is even necessary to fix also the ordering $\varphi_1, \varphi_2, \ldots$ (i.e. (φ_n) and $(\varphi_{\pi(n)})$ for some permutation π of N are in general not equivalent with respect to the constructions).

An operator $A \in \mathcal{L}^+(\mathcal{D})$ is called *commutator* or *representable as commutator* if $A = [B, C] \equiv BC - CB$ for some $B, C \in \mathcal{L}^+(\mathcal{D})$. The following operators are frequently used:

diagonal operator D_a:	$D_a\varphi_n = a_n\varphi_n$,
right shift R:	$R\varphi_n = \varphi_{n+1}$,
weighted right shift R_a:	$R_a\varphi_n = a_n\varphi_{n+1}$,
left shift L:	$L\varphi_n = \varphi_{n-1}$, $\varphi_0 = 0$,
weighted left shift L_a:	$L_a\varphi_n = a_n\varphi_{n-1}$, $\varphi_0 = 0$.

As above we often write $D_a \equiv D \sim (a_n)$. Here $a = (a_n)$ is a sequence of complex numbers. To use such operators it is necessary to decide whether or not they belong to $\mathcal{L}^+(\mathcal{D})$. It is easy to write down some formal conditions, namely:

$$D_a \in \mathcal{L}^+(\mathcal{D}) \quad \text{iff} \quad |a_n| \leqq C t_n^r \quad \text{for some} \quad C, r > 0; \tag{1}$$

$R_a \in \mathcal{L}(\mathcal{D})$ $(L_a \in \mathcal{L}(\mathcal{D}))$ iff for all $l \in \mathrm{N}$ there exist $C(l), r(l) > 0$, so that for all $n \in \mathrm{N}$

$$|a_n|\, t_{n+1}^l \leqq C(l)\, t_n^{r(l)} \qquad (|a_n|\, t_{n-1}^l \leqq C(l)\, t_n^{r(l)}). \tag{2}$$

For reasons which will become clear a little bit later it is useful to introduce some more general notions.

Definition 2.1: a) A sequence (s_n), $s_n > 0$, is said to be *shift-admissible* if $s_{n+1} \leqq C s_n^r$ for some $C, r > 0$ and all n.

b) Let (t_n) be a sequence, $t_n > 0$. A sequence (a_n) of complex numbers is said to be (t_n)*-addable* (or T*-addable* if $T \sim (t_n)$) if, for some $C, r > 0$,

$$|b_n| \leqq C t_n^r \quad \text{with} \quad b_n = \sum_{j=1}^{n} a_j \quad \text{for all} \quad n. \tag{3}$$

Since most of the representations of operators as commutators use (explicitly or implicitly) shift operators, it seems worthwhile to add some remarks. In general the estimations (2) are not very helpful to decide whether or not $R, L \in \mathcal{L}^+(\mathcal{D})$. The reason is, roughly speaking, that in the sequence (t_n) there appear eigenvalues with infinite multiplicity arranged in a complicated manner. So, it may happen that (t_n) is not shift-admissible, i.e. $R, L \notin \mathcal{L}^+(\mathcal{D})$, but

$(t_{\pi(n)})$ is shift-admissible for some permutation π of $\mathbf{N}$. Without proof we state some observations:

1. $\mathcal{D} \in$ (I): $R, L \in \mathcal{L}^+(\mathcal{D})$ means that (t_n) does not increase too fast. Note that $t_n \sim n^n$ or even $t_n \sim (n!)^n$ is not yet too fast.
2. $\mathcal{D} \in$ (II): R, L are never in $\mathcal{L}^+(\mathcal{D})$. More exactly, there is no permutation π of $\mathbf{N}$ such that $(t_{\pi(n)})$ is shift-admissible.
3. $\mathcal{D} \in$ (III): If (t_n') is shift-admissible, then automatically $\mathcal{D} \in (\text{III}_\text{A})$ and there is a permutation π of $\mathbf{N}$ such that $(t_{\pi(n)})$ is also shift-admissible.

The next lemma states that in case $\mathcal{D} \in$ (I) the notions in Definition 2.1 are independent of the representing operator T.

Lemma 2.1: *Let $\mathcal{D} = \mathcal{D}^\infty(T) = \mathcal{D}^\infty(S) \in$ (I), $T\varphi_n = t_n\varphi_n$, $S\psi_n = s_n\psi_n$ and let R_a, L_a (R_a', L_a') be the weighted shift operators corresponding to (φ_n), $\big((\psi_n)\big)$. Then*

(i) $R_a, L_a \in \mathcal{L}^+(\mathcal{D})$ iff $R_a', L_a' \in \mathcal{L}^+(\mathcal{D})$;
(ii) (a_n) is T-addable iff (a_n) is S-addable.

Proof: In [13] there was proved that $Cs_n^{1/r} \leq t_n \leq Ds_n^r$ $(n \in \mathbf{N};\ C, D, r > 0)$. Therefore, if an estimation of type (2) or (3) is valid for (t_n), it is also valid for (s_n) and vice versa (of course with other constants) ∎

Most of the further considerations are based on matrix representations of operators. If not stated otherwise, we always use the representation with respect to the canonical basis (φ_n) and write $A \sim (A_{mn})$ with $A_{mn} = \langle \varphi_m, A\varphi_n \rangle$.

3. Diagonal and quasidiagonal operators

In this section we demonstrate some typical features for commutators in $\mathcal{L}^+(\mathcal{D})$. Therefore we do not start with the most general result in this context but prefer a more inductive representation of the results.

A) *We start with* $\mathcal{D} \in$ (I). Let $D_a \in \mathcal{L}^+(\mathcal{D})$, $a = (a_n)$. Then one has the formal relation

$$D_a = L_bR - RL_b \quad \text{with} \quad b = (b_n) \quad \text{and} \quad b_n = \sum_{j=1}^{n} a_j. \tag{4}$$

If $R, L \in \mathcal{L}^+(\mathcal{D})$, then, due to $L_b = D_bL$, $L_b \in \mathcal{L}^+(\mathcal{D})$ iff $D_b \in \mathcal{L}^+(\mathcal{D})$.

Lemma 3.1: *Let $\mathcal{D} \in$ (I), $R, L \in \mathcal{L}^+(\mathcal{D})$ and D_a a diagonal operator with T-addable sequence $a = (a_n)$. Then D_a is a commutator given by* (4).

If one uses representation (4), one has, so to say, two contrary restrictions. The first one is $R, L \in \mathcal{L}^+(\mathcal{D})$, i.e. (t_n) should not increase too wild. On the other hand, if (t_n) increases "slowly", then it will happen that some (or even many) diagonal operators D_a will not have T-addable diagonal sequences a. For example, if $T \sim (\log(n+1))$, then this sequence itself is not T-addable. Thus, it is not surprising, that there are optimal cases, i.e. (t_n) which increase "optimal".

Lemma 3.2: *Let s be the Schwartz space of rapidly decreasing sequences. Each of the following conditions implies that any diagonal operator $D_a \in \mathcal{L}^+(\mathcal{D})$ is a commutator:*

(i) $\mathcal{D}$ is isomorphic to s.
(ii) $\mathcal{D}$ is isomorphic to a sequence space contained in s and $R, L \in \mathcal{L}^+(\mathcal{D})$.

Proof: First let us remark that the isomorphisms mentioned above are understood with respect to the basis (φ_n).

(i) Without loss of generality we may suppose that $t_n = n$. Then $D_a \in \mathcal{L}^+(\mathcal{D})$ means $|a_n| \leqq Cn^r$ for some $C, r > 0$. Since

$$|b_n| = \left|\sum_{j=1}^{n} a_j\right| \leqq \sum_{j=1}^{n} |a_j| \leqq C \sum_{j=1}^{n} j^r \leqq Cn^{r+1},$$

the sequence (a_n) is T-addable. Clearly, $R, L \in \mathcal{L}^+(\mathcal{D})$, so the proof is complete.

(ii) Here one combines an analogous estimation as in (i) with the estimation $n \leqq Ct_n^s$ for some $C, s > 0$, where $T \sim (t_n)$ and $\mathcal{D}^\infty(T) \subseteqq s$ (cf. [10, 11]) ∎

Remark that in the proof above it is implicitly used that both, $R, L \in \mathcal{L}^+(\mathcal{D})$ and the T-addability of (a_n) does not depend on the concrete representation $\mathcal{D} = \mathcal{D}^\infty(T)$. As mentioned in Section 2 this is only true for $\mathcal{D} \in$ (I).

Now we turn to quasidiagonal operators. We use again some ideas from the bounded case. Let (S_{ij}) be a matrix the entries of which can be numbers as well as operators. Then define another matrix (W_{ij}) associated with (S_{ij}) by the following rule:

$$W_{1j} = 0 \quad \text{for} \quad j \geqq 1, \qquad W_{ij} = \begin{cases} 0 & \text{for} \quad i, j \leqq 0 \\ W_{i-1,j-1} + S_{i-1,j} & \text{for} \quad i \geqq 2,\ j \geqq 1. \end{cases} \tag{5}$$

In [1] the following was proved: Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \cdots$, $S = (S_{ij})$ an operator on $\mathcal{H}$ with $S_{ij} \in \mathcal{B}(\mathcal{H}_0)$, $\sum \|S_{ij}\| < \infty$. Then $S = LW - WL$ where $W = (W_{ij})$ is defined by (5) and gives an operator on $\mathcal{H}$ and $L = (L_{ij})$, $L_{ij} = \delta_{i+1,j} I_{\mathcal{H}_0}$, $I_{\mathcal{H}_0}$ the identity on $\mathcal{H}_0$.

Let us return to $\mathcal{L}^+(\mathcal{D})$ with $\mathcal{D} \in$ (I). For $S \in \mathcal{L}^+(\mathcal{D})$, $S \sim (S_{ij})$ one defines the corresponding $(W_{ij}) = W$ and has formally

$$S = LW - WL. \tag{6}$$

Here L is the left shift. In case $S = D_a$ (6) is equivalent to (4). If we again suppose $R, L \in \mathcal{L}^+(\mathcal{D})$, the only question to decide is whether or not (W_{ij}) defines an operator $W \in \mathcal{L}^+(\mathcal{D})$, i.e. to give (6) sense as a relation in $\mathcal{L}^+(\mathcal{D})$. In general it is a difficult task to prove that a given matrix defines an operator of a given class (say bounded or belonging to $\mathcal{L}^+(\mathcal{D})$ and so on). Most easily one can handle this for quasidiagonal operators. An operator $Q \in \mathcal{L}^+(\mathcal{D})$ is said to be *quasidiagonal* (with respect to (φ_n), as usual) if its matrix representation (Q_{ij}) has only finite many lower and upper subdiagonals different from zero, i.e. schematically:

$$Q \sim \begin{pmatrix} & & 0 \\ Q_{-m} & \cdots Q_0 \cdots & Q_n \\ 0 & & \end{pmatrix}.$$

Denote these subdiagonals (from left to right) by $Q_{-m}, \ldots, Q_0, \ldots, Q_n$ and write $Q = [Q_{-m}, \ldots, Q_n]$. Moreover denote the sequences of the matrix elements corresponding to Q_j by

$$q^{(j)} = (q_k^{(j)}), \quad \text{i.e.} \quad q_k^{(j)} = \begin{cases} Q_{k,k+1} & (k \in \mathbf{N};\ 0 \leqq j \leqq n), \\ Q_{k-j,k} & (k \in \mathbf{N};\ -m \leqq j < 0). \end{cases}$$

Proposition 3.3: *Let* $\mathcal{D} \in$ (I), $R, L \in \mathcal{L}^+(\mathcal{D})$. *A quasidiagonal operator* $Q = [Q_{-m}, \ldots, Q_n] \in \mathcal{L}^+(\mathcal{D})$ *is a commutator if all sequences* $q^{(j)}$ $(-m \leqq j \leqq n)$ *are* T*-addable. In this case one possible representation is given by* $Q = LW - WL$, *where* L *is the left shift and* W *is the operator corresponding to* (W_{ij}) *given by* (5).

Proof: That (W_{ij}) defines an operator $W \in \mathcal{L}^+(\mathcal{D})$ can be seen by the following considerations. The definition of W implies that this is a quasidiagonal operator of the same type as Q:

$$W = [W_{-m}, \ldots, W_n] = [W_{-m}, 0, \ldots, 0] + \cdots + [0, \ldots, 0, W_n]. \tag{7}$$

The corresponding sequences $w^{(j)}$ are formed by the partial sums $w_n^{(j)} = q_1^{(j)} + \cdots + q_n^{(j)}$. Observe that any term in (7) is obtained from a diagonal operator $D_{w^{(j)}}$ via application of some power of R or L. Therefore the assumption implies that any $D_{w^{(j)}} \in \mathcal{L}^+(\mathcal{D})$ by (1) and any term in (7) also defines an operator belonging to $\mathcal{L}^+(\mathcal{D})$. Thus, $W \in \mathcal{L}^+(\mathcal{D})$ and the proof is complete ∎

Corollary: *Let* $\mathcal{D} \in$ (I), $R, L \in \mathcal{L}^+(\mathcal{D})$. *Then* R_a, L_a *are commutators if* $a = (a_n)$ *is* T*-addable.*

Remark that for R_a, L_a the T-addability of $a = (a_n)$ automatically implies $R_a, L_a \in \mathcal{L}^+(\mathcal{D})$.

B) For $\mathcal{D} \in$ (II) we remark only the following. Since in this case $\mathcal{D} = \mathcal{H}_0 \oplus \mathcal{D}^\infty(T_1)$, $\mathcal{D}^\infty(T_1) \in$ (I) (cf. Section 2) the results for the bounded case and for type (I) can be combined to identify a lot of operators of the form

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \quad B \in \mathcal{B}(\mathcal{H}_0), \quad C \in \mathcal{L}^+\big(\mathcal{D}^\infty(T_1)\big),$$

as commutators. Moreover, operators of the form

$$A = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \quad \text{and hence also} \quad A = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$$

are commutators as can be seen e.g. from

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

C) *Next let* $\mathcal{D} \in$ (III). In view of the considerations of Section 2 we consider first $\mathcal{D} \in (\mathrm{III_A})$, i.e.

$$\mathcal{D} = \sum_{(t_n')} \oplus \mathcal{H}_n. \tag{8}$$

If $A = \sum \oplus A_n$, $A_n \in \mathcal{B}(\mathcal{H}_n)$ and $\|A_n\| \leqq C(t_n')^s$ for all n and appropriate $C, s > 0$, then $A \in \mathcal{L}^+(\mathcal{D})$. In the case that the sum representing A contains only finite many terms different from zero, the results of the bounded case can be applied (cf. also Prop. 3.4). In the general case one could proceed as follows. Suppose $A_n = [B_n, C_n]$. If $B = \sum \oplus B_n$ and $C = \sum \oplus C_n$ belong to $\mathcal{L}^+(\mathcal{D})$, then $A = [B, C]$. But this procedure seems not to be very useful for concrete applications. Let us therefore describe another possibility to construct commutators. To do so we introduce a generalized shift operator which does not correspond to the basis (φ_n) but to the representation (8). Let $\psi = (\psi_1, \psi_2, \ldots) \in \mathcal{D}$ (cf. Section 2). Then define

$$R\psi = (0, \psi_1, \psi_2, \ldots), \qquad L\psi = (\psi_2, \psi_3, \ldots).$$

Analogously to $\mathcal{D} \in$ (I) one has: $R, L \in \mathcal{L}^+(\mathcal{D})$ if and only if (t_n') is shift-admissible. Now we use the matrix representation of $A \in \mathcal{L}^+(\mathcal{D})$ with respect to (8), i.e. $A \sim (A_{ij})$, $A_{ij} \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_j) \cong \mathcal{B}(\mathcal{H})$. It is natural to call an operator of the form

$$A = \begin{pmatrix} A_1 & 0 & 0 & \dots \\ 0 & A_2 & 0 & \dots \\ 0 & 0 & A_3 & \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

generalized diagonal operator or *diagonal operator with respect* to (8). The notion of T-addability is generalized as follows. Let $A = \sum \oplus A_n \in \mathcal{L}^+(\mathcal{D})$ a generalized diagonal operator and $\|A_n\| = a_n$. Then A is called *T-addable* if (a_n) is (t_n')-addable.

Proposition 3.4: *Suppose* $\mathcal{D} = \mathcal{D}^\infty(T) \in (\mathrm{III_A})$, (t_n') *shift-admissible and* $A \in \mathcal{L}^+(\mathcal{D})$ *a T-addable generalized diagonal operator. Then A is a commutator.*

Proof: As in the proof of Lemma 3.1 it is seen that formally

$$A = W\tilde{R} - \tilde{R}W \tag{9}$$

with

$$\tilde{R} \sim \begin{pmatrix} 0 & 0 & 0 & \dots \\ I & 0 & 0 & \\ 0 & I & 0 & \\ \vdots & & \ddots & \end{pmatrix},$$

$$W \sim \begin{pmatrix} 0 & A_1 & 0 & 0 & \dots \\ 0 & 0 & A_1 + A_2 & 0 & \dots \\ 0 & 0 & 0 & A_1 + A_2 + A_3 & \\ \vdots & \vdots & & & \ddots \end{pmatrix}.$$

This representation is also understood with respect to (8). Clearly, $\tilde{R} \in \mathcal{L}^+(\mathcal{D})$. To see that $W \in \mathcal{L}^+(\mathcal{D})$ one estimates as follows. Let $\psi = \sum \oplus \psi_n \in \mathcal{D}$. Then

$$\|T^j W\psi\|^2 = \left\| T^j \sum_{j=1}^{\infty} \oplus \left(\sum_{k=1}^{n} A_k \right) \psi_{n+1} \right\|^2$$

$$\leqq \sum_{n=1}^{\infty} (t_{n+1}')^{2j} \left(\sum_{k=1}^{n} \|A_k\| \right)^2 \|\psi_{n+1}\|^2 = \sum_{n=1}^{\infty} (t_{n+1}')^{2j} \left(\sum_{k=1}^{n} a_k \right)^2 \|\psi_{n+1}\|^2.$$

Since (a_n) was (t_n')-addable, we have $a_1 + \dots + a_n \leqq C(t_n')^r$ $(n \in \mathbb{N};\ C, r > 0)$ and the estimation can be continued:

$$\|T^j W\psi\|^2 \leqq C^2 \sum_{n=1}^{\infty} (t_{n+1}')^{2(j+r)} \|\psi_{n+1}\|^2 < \infty.$$

The estimation of $\|T^j W^*\psi\|^2$ is almost the same and therefore omitted. Thus $W \in \mathcal{L}^+(\mathcal{D})$ ∎

Now we could prove several variants of results analogous to Lemma 3.2 but we mention only one of them.

Corollary: *If* $t_n' \sim n^\beta$ *for some* $\beta > 0$, *then any bounded generalized diagonal operator is a commutator.*

In a similar way as for $\mathcal{D} \in$ (I) one can also handle generalized quasidiagonal operators.

Now let us give a result which is valid in the general case $\mathcal{D} \in$ (III). First remark that in a natural manner operators $A \in \mathcal{B}(\mathcal{H}^\mu)$, $\mu > 1$ can be viewed as elements of $\mathcal{L}^+(\mathcal{D})$ via identifying A and $A \oplus 0$, 0 the zero operator on $(\mathcal{H}^\mu)^\perp$.

Proposition 3.5: *Let $\mathcal{D} \in$ (III). Then any operator $A \in \mathcal{B}(\mathcal{H}^\mu)$ $(1 \leqq \mu < \infty)$ is a commutator.*

Proof: It is reduced to the bounded case. Let $\nu > \mu$ be such that $\dim (\mathcal{H}^\nu \ominus \mathcal{H}^\mu) = \infty$. Then A has a large kernel if it is considered as operator in $\mathcal{B}(\mathcal{H}^\nu)$. Hence it is a commutator [6]. Consequently, A is also a commutator in $\mathcal{L}^+(\mathcal{D})$ ∎

Remember that for any $\mathcal{D}^\infty(T)$ on $\mathcal{B}(\mathcal{H}^\mu)$ the topology induced by $\tau_\mathcal{D}$ coincides with the usual operator norm topology. Moreover, the commutators are norm dense in $\mathcal{B}(\mathcal{H})$ for any infinite dimensional separable Hilbert space [2]. Combining these observations with the fact that the set of all $A \in \mathcal{B}(\mathcal{H}^\mu)$ (for all $1 \leqq \mu < \infty$) is $\tau_\mathcal{D}$-dense in $\mathcal{L}^+(\mathcal{D})$, one gets

Proposition 3.6: *Let $\mathcal{D} = \mathcal{D}^\infty(T) \in$ (II) or $\mathcal{D} = \mathcal{D}^\infty(T) \in$ (III). Then the commutators are $\tau_\mathcal{D}$-dense in $\mathcal{L}^+(\mathcal{D})$.*

In Section 4 this result will be generalized to include also the case $\mathcal{D} \in$ (I). Because this will be use finite dimensional operators we did not include it in this section.

4. Finite dimensional operators

The aim of this section is to show that most of the finite dimensional operators are commutators. For $\mathcal{D} \in$ (II) or $\mathcal{D} \in$ (III) the problem can be reduced to the bounded case:

Proposition 4.1: *Let $\mathcal{D} \notin$ (I). Then any finite dimensional operator $F \in \mathcal{L}^+(\mathcal{D})$ is a commutator.*

Proof: *Let* $F = \sum_{i=1}^{n} \langle \psi_i, \cdot \rangle \chi_i$. Since $\mathcal{D} \notin$ (I), there is an infinite dimensional Hilbert space $\mathcal{H}_0 \subset \mathcal{D}$. Put $\mathcal{H}_1 = \text{lin}\,\{\psi_i, \chi_i, \mathcal{H}_0\}$. Then $\mathcal{H}_1 \subset \mathcal{D}$ and $F_1 = F \mid \mathcal{H}_1 \in \mathcal{B}(\mathcal{H}_1)$ has a large kernel. Consequently $F_1 = [A_1, B_1]$, $A_1, B_1 \in \mathcal{B}(\mathcal{H}_1)$. Thus $F = F_1 \oplus 0 = [A, B]$ with $A = A_1 \oplus 0$ and $B = B_1 \oplus 0$. Here 0 denotes the zero operator on $\mathcal{H}_1^\perp$. Clearly $A, B \in \mathcal{L}^+(\mathcal{D})$ ∎

Now we consider domains $\mathcal{D} \in$ (I). The first result says that there are "enough" commutators in $\mathcal{L}^+(\mathcal{D})$, cf. also Proposition 3.6.

Proposition 4.2: *Let $\mathcal{D} \in$ (I). Then the commutators are $\tau_\mathcal{D}$-dense in $\mathcal{L}^+(\mathcal{D})$.*

Proof: Let $\mathcal{H}_n = \text{lin}\,\{\varphi_1, \ldots, \varphi_n\}$ and Q_n the projection onto $\mathcal{H}_n$. Then

$$A = \tau_\mathcal{D} - \lim Q_n A Q_n = \tau_\mathcal{D} - \lim A_n \quad \text{for all } A \in \mathcal{L}^+(\mathcal{D}). \tag{10}$$

The finite dimensional $A_n = Q_n A Q_n$ has matrix representation $A_n \sim \begin{pmatrix} A_n & 0 \\ 0 & 0 \end{pmatrix}$. Let $a_n = \operatorname{Tr} A_n$. Then

$$A_n = \tau_\mathcal{D} - \lim_j B_j^{(n)} \quad \text{with} \quad B_j^{(n)} = \begin{pmatrix} A_n & & & 0 \\ & \ddots & & \\ & & -a_n & \\ & & & \ddots \\ 0 & & & \end{pmatrix} \tag{11}$$

where $-a_n$ stands at the diagonal place with number $(n + j + 1)$. Since $\operatorname{Tr} B_j^{(n)} = 0$ for all j, n, the $B_j^{(n)}$ are commutators in $\mathcal{L}^+(\mathcal{D})$. Relations (10) and (11) together give the desired result ∎

Combining Propositions 3.6 and 4.2 we get a main result of the paper.

Theorem 4.3: *Let $\mathcal{D} = \mathcal{D}^\infty(T)$. Then the commutators are $\tau_{\mathcal{D}}$-dense in $\mathcal{L}^+(\mathcal{D})$.*

This theorem has a nice corollary which is worthwhile being mentioned.

Corollary: *Let $\mathcal{D} = \mathcal{D}^\infty(T)$. Then on $\mathcal{L}^+(\mathcal{D})$ there are no non-zero $\tau_{\mathcal{D}}$-continuous complex homomorphisms (i.e. multiplicative linear functionals).*

Our final aim in this section is to show that if $R, L \in \mathcal{L}^+(\mathcal{D})$, any finite dimensional operator is a commutator.

Proposition 4.4: *Let $\mathcal{D} = \mathcal{D}^\infty(T)$ and $R, L \in \mathcal{L}^+(\mathcal{D})$. Then any finite dimensional operator $F \in \mathcal{L}^+(\mathcal{D})$ is a commutator.*

To separate the technical details from the main idea of the proof we start with two lemmata.

Lemma 4.5: *Let $F \in \mathcal{L}^+(\mathcal{D})$ be finite dimensional, $F = \sum_{j=1}^{n} \langle \chi_j, \cdot \rangle \psi_j$, (ψ_j) an orthonormal set. Then there is an operator S such that $S = S^* \geqq I$, $S\psi_j = s_j\psi_j$, $(\psi_j) \subset \mathcal{D}$ an orthonormal basis and $\mathcal{D} = \mathcal{D}^\infty(S)$.*

Proof: We give only a sketch of the proof. First, we use a fact which seems to be well known, but for which we cannot give a reference. Our domain $\mathcal{D}[t] = \mathcal{D}^\infty(T)\,[t]$ is an (F)-space with unconditional basis (φ_n), $\mathcal{D}_1 = \operatorname{lin}\{\psi_1, \ldots, \psi_n\}$ is a topologically complemented subspace. Let $P_1 \in \mathcal{L}^+(\mathcal{D})$ be the orthoprojection onto $\mathcal{D}_1$. The above mentioned fact consists in $\mathcal{D}_2 = (I - P_1)\,\mathcal{D}$ having also an unconditional basis.
Next we apply a result of Mitjagin [14]. Let $E = \mathcal{D}^\infty(T)\,[t]$ and $X \subset E$ a complemented subspace with unconditional basis. Then X is topologically isomorphic to a coordinate subspace of E. Especially, X is isomorphic to some $\mathcal{D}^\infty(B)$ where $B = B^* \geqq I$ is a selfadjoint operator in $\mathcal{H}_2 = \overline{\mathcal{D}}_2$, $B\varrho_n = b_n\varrho_n$ for some orthonormal basis (ϱ_n) in $\mathcal{H}_2$. Then one can put $S = I_k \oplus B$, I_k the identity on $\mathcal{D}_1$ and $\psi_{n+k} = \varrho_n$ for all n ∎

Let us remark that in what follows we will apply Lemma 2.1 several times without explicitly mentioning it. The advantage of the representation $\mathcal{D} = \mathcal{D}^\infty(S)$ described in Lemma 4.5 is a simple matrix representation of F (now with respect to (ψ_n)):

$$F \sim (f_{ij}) = \begin{pmatrix} f_{11} & f_{12} & f_{13} & \cdots \\ \vdots & \vdots & \vdots & \\ f_{k1} & f_{k2} & f_{k3} & \cdots \\ 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \end{pmatrix} = \begin{pmatrix} F_1 & F_2 & F_3 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

where the right-hand matrix has $(k \times k)$-matrices as entries: $F_1 = (f_{ij})$, $1 \leqq i$, $j \leqq k$; ..., $F_n = (f_{ij})$, $1 \leqq i \leqq k$, $(n-1)\,k + 1 \leqq j \leqq nk$, ... Clearly, $F = \sum F_n$ in an obvious manner. This matrix representation suggests a splitting of $\mathcal{H}$: $\mathcal{H} = \sum \oplus \mathcal{H}_n$, $\dim \mathcal{H}_n = k$. Put P_n to be the projection onto $\mathcal{H}_n$. Without loss of generality we may suppose that $1 \leqq s_j$. Now put $a_n = s_{nk}$ and form a new operator A by setting $A \mid \mathcal{H}_n = a_n I_k$. Using the assumption $R, L \in \mathcal{L}^+(\mathcal{D})$ we get

$\mathcal{D} = \mathcal{D}^\infty(A)$, more explicitly

$$\begin{aligned}\mathcal{D} &= \sum_{(a_n)} \oplus \mathcal{H}_n \\ &= \left\{\varphi = \sum \oplus \overline{\varphi}_n : \overline{\varphi}_n \in \mathcal{H}_n,\ \sum a_n^{2m} \|\overline{\varphi}_n\|^2 < \infty \text{ for all } m \in \mathbf{N}\right\} \\ &= \left\{\varphi : \sum \|P_n\varphi\|^2 a_n^{2m} < \infty \quad \text{for all} \quad m \in \mathbf{N}\right\}. \end{aligned} \tag{12}$$

Lemma 4.6: *With the notations above one has*

$$\sum a_n^{2m} \|F_{n+1}\|^2 < \infty \quad \textit{for all } m \in \mathbf{N}. \tag{13}$$

Proof: Using $F_n = P_1 F P_n$ one immediately gets $\|F_n\|^2 \leqq \sum_{j=1}^{k} \|P_n \chi_j\|^2$. This together with (12) applied to $\varphi = \chi_1, \ldots, \chi_n$ successively gives

$$\sum a_n^{2m} \|F_n\|^2 \leqq \sum_n \left(\sum_{j=1}^{k} \|P_n \chi_j\|^2\right) a_n^{2m} < \infty.$$

Since $a_n \uparrow$, the assertion follows ∎

Proof of Proposition 4.4: Refering to Lemma 4.5 and the considerations before Lemma 4.6 we use the representation $\mathcal{D} = \mathcal{D}^\infty(A)$ described there. If we put

$$C = \begin{pmatrix} -F_2 & -F_3 & -F_4 & \cdots \\ F_1 & 0 & 0 & \cdots \\ 0 & F_1 & 0 & \cdots \\ 0 & 0 & F_1 & \\ \vdots & \vdots & & \ddots \end{pmatrix}, \quad D = \begin{pmatrix} 0 & I_k & 0 & 0 & \cdots \\ 0 & 0 & I_k & 0 & \cdots \\ 0 & 0 & 0 & I_k & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix},$$

one has formally $F = DC - CD$. Again, $D \in \mathcal{L}^+(\mathcal{D})$ by the assumptions. To check $C \in \mathcal{L}^+(\mathcal{D})$ we first estimate

$$\begin{aligned}\|A^m C\varphi\|^2 &= a_1^{2m} \left\|\sum_{n=1}^{\infty} F_{n+1}\overline{\varphi}_n\right\|^2 + \sum_{n=2}^{\infty} a_n^{2m} \|F_1 \overline{\varphi}_{n-1}\|^2 \\ &\leqq C_1 a_1^{2m} \|\varphi\|^2 + \sum_{n=2}^{\infty} a_n^{2m} \|\overline{\varphi}_{n-1}\|^2 < \infty.\end{aligned}$$

Here (12) and $R, L \in \mathcal{L}^+(\mathcal{D})$ are used. It remains to estimate

$$\begin{aligned}\|A^m C^*\varphi\|^2 &= \sum_{n=1}^{\infty} a_n^{2m} \|F_{n+1}^* \overline{\varphi}_1 + F_1^* \overline{\varphi}_{n+1}\|^2 \\ &\leqq 2\left\{\sum_{n=1}^{\infty} a_n^{2m} \|F_{n+1}^*\|^2 \|\overline{\varphi}_1\|^2 + \sum_{n=1}^{\infty} a_n^{2m} \|F_1^*\|^2 \|\overline{\varphi}_{n+1}\|^2\right\} < \infty.\end{aligned}$$

The first sum is finite due to Lemma 4.6; the second sum is finite because of (12) and $a_n \uparrow$ ∎

5. Selfcommutators

A special kind of commutators are so-called selfcommutators. Some informations about the situation in the bounded case can be taken from [6]. Concerning the unbounded case we will give only some preliminary results.

Definition 5.1: An operator $S \in \mathcal{L}^+(\mathcal{D})$ is said to be a *self-commutator* (or *representable as a selfcommutator*) if $S = [A, A^+]$ for some $A \in \mathcal{L}^+(\mathcal{D})$.

Clearly, selfcommutators are symmetric ($S = S^+$). The following results are well known [6, 15]:

1. *If $S \in \mathcal{B}(\mathcal{H})$ is a selfcommutator, then 0 belongs to the spectrum of S and so S is not invertible in $\mathcal{B}(\mathcal{H})$.*

2. *If A is a closed operator and on $\mathcal{D}(A^*A) = \mathcal{D}(AA^*)$ one has $AA^* - A^*A = I$, then A^*A has eigenvalues, 0, 1, 2, ... all with the same multiplicity.*

Turning to $\mathcal{L}^+(\mathcal{D})$ let us remark that Property 1 above is not valid for $\mathcal{L}^+(\mathcal{D})$. Further, we consider here only selfcommutators $S = [A, A^+]$ for such A that AA^+ and A^+A are essentially selfadjoint. We have $\sigma(AA^+) \cup \{0\} = \sigma(A^+A) \cup \{0\}$. Moreover, if A is a closed operator, then A^*A and AA^* have the same non-zero eigenvalues with the same multiplicity [5].

In the next lemma we collect some further properties related with selfcommutators.

Lemma 5.1: *Let $\mathcal{D} = \mathcal{D}^\infty(T) \in$ (I), $0 \leqq S \sim (s_n)$, (φ_n) a diagonal operator in $\mathcal{L}^+(\mathcal{D})$ with $S \in$ (I) (cf. [10]). If $S = AA^+ - A^+A$, the following statements are true:*

(i) $\overline{A^+A}, \overline{AA^+}, \bar{A}, \overline{A^+} \in$ (I).

(ii) *Let (a_n) be the eigenvalues of $\overline{A^+A}$. Then $s_n \leqq a_n$. Moreover, $(a_n) \subset \sigma(AA^+)$ and if $0 \in \sigma(AA^+)$, then $0 \notin \sigma_{\mathrm{ess}}(AA^+)$, i.e. 0 can be only an eigenvalue with finite multiplicity.*

Proof: (i) $AA^+ = S + A^+A$ leads immediately to

$$\langle S\varphi, \varphi\rangle \leqq \langle S\varphi, \varphi\rangle + \langle A^+A\varphi, \varphi\rangle = \langle AA^+\varphi, \varphi\rangle = \|A^+\varphi\|^2. \tag{14}$$

Therefore, $\mathcal{D}(S^{1/2}) \supset \mathcal{D}(\overline{A^+}) \supset \mathcal{D}(\overline{AA^+})$ and consequently $\mathcal{D}(\overline{A^+}), \mathcal{D}(\overline{AA^+}) \in$ (I). Here we used that $\mathcal{D}(S) \in$ (I) implies $\mathcal{D}(S^{1/2}) \in$ (I). To see $\mathcal{D}(\bar{A}) \in$ (I), suppose that there is an infinite dimensional Hilbert space $\mathcal{H}_0 \subset \mathcal{D}(\bar{A})$. Then $\mathcal{H}_0 \cap \mathcal{D}$ is infinite dimensional and for $\varphi, \psi \in \mathcal{K}_0$, $\mathcal{K}_0$ the unit ball in $\mathcal{H}_0$, one has $\sup |\langle\varphi, A^+\psi\rangle| = \sup |\langle A\varphi, \psi\rangle| < \infty$. Thus A^+ is bounded on $\mathcal{H}_0 \cap \mathcal{D}$, i.e. $\overline{\mathcal{H}_0 \cap \mathcal{D}} \subset \mathcal{D}(A^+)$ in contradiction to $\mathcal{D}(A^+) \in$ (I). Hence $\mathcal{D}(\bar{A}) \in$ (I) and so $\mathcal{D}(\overline{A^+A}) \in$ (I), too.

(ii) By (14) and the minimax principle one gets $s_n \leqq a_n$. The property stated before the Lemma gives $(a_n) \subset \sigma(AA^+)$ and (i) means especially that $0 \notin \sigma_{\mathrm{ess}}(AA^+)$, because this would imply that $\mathcal{D}(\overline{AA^+}) \notin$ (I) ∎

Now we indicate some simple conditions which imply that diagonal operators (with respect to (φ_n)) are selfcommutators. This should be also compared with Lemma 3.1.

Lemma 5.2: *Let $\mathcal{D} = \mathcal{D}^\infty(T) \in$ (I), $R, L \in \mathcal{L}^+(\mathcal{D})$, $0 \leqq D \sim (d_n)$, $D \in \mathcal{L}^+(\mathcal{D})$ so that $d = (d_n)$ is T-addable. Then D is a selfcommutator.*

Proof: We give at once infinite many such representations, namely: let A_k be the operator which corresponds to the matrix

$$\begin{pmatrix} 0 & \cdots & 0 & a_1 & 0 & \cdots & \\ \vdots & & & & a_2 & & \\ & & & & & \ddots & \\ 0 & \cdots & & & & & a_n \\ \vdots & & & & & & & \ddots \end{pmatrix}$$

i.e. in the k^{th} upper subdiagonal there stands the sequence (a_n). With other words $A_k \sim (a_{ij}^{(k)})$, $a_{ij}^{(k)} = \delta_{i,j-k} a_i$. Then the assumptions guarantee that $A_k \in \mathcal{L}^+(\mathcal{D})$ and it is easy to see that $D = [A_k, A_k^+]$ with appropriately chosen (a_n) (cf. also the next remark) ∎

Remark: In the classical case $D = I$, of course D and AA^+ commute and this is an essential point in the proof of Result 2 above. The representations of D used in the proof of Lemma 5.2 also lead to operators $A_k A_k^+$ and $A_k^+ A_k$ commuting with D. Furthermore, the spectra of $A_k A_k^+$ and $A_k^+ A_k$ are easily determined. Clearly, $A_k A_k^+ = D_c$ with $c = (a_1^2, a_2^2, a_3^2, \ldots)$, $A_k^+ A_k = D_{c'}$ with $c' = (0, \ldots, 0, a_1^2, a_2^2, \ldots)$ where stand k zeros. Therefore one has:

$$a_1^2 = d_1, \quad a_2^2 = d_2, \ldots, \quad a_k^2 = d_k;$$

$$a_{k+1}^2 = d_{k+1} + d_1, \quad a_{k+2}^2 = d_{k+2} + d_2, \ldots, \quad a_{2k}^2 = d_{2k} + d_k;$$

$$a_{nk+1}^2 = d_{nk+1} + \cdots + d_{2k+1} + d_{k+1} + d_1, \ldots, a_{(n+1)k}^2 = d_{(n+1)k} + \cdots + d_{2k} + d_k.$$

So we see a clear structure of the spectrum of $A_k A_k^+$ for any k. A little bit more formally it can be described as follows. Consider a decomposition of (d_n) into k-blocks:

$$(d_n) = \bigcup_{j=0}^{\infty} D_j, \qquad D_j = (d_{jk+1}, \ldots, d_{(j+1)k}).$$

For these ordered k-tuples define $D_i + D_j$ as an elementwise addition and for $i < j$ put $(D_i, D_j) = (d_{ik+1}, \ldots, d_{(i+1)k}, d_{jk+1}, \ldots, d_{(j+1)k})$. Then $(a_n^2) = (D_0, D_0 + D_1, D_0 + D_1 + D_2, \ldots)$ is the sequence of eigenvalues of AA^+. The sequence c' above gives the eigenvalues of A^+A. Furthermore, if each eigenvalue of D has equal multiplicity $d \geqq 1$, then by appropriate choice of k ($1 \leqq k \leqq d$) one can generate a spectrum of A^+A with (homogeneous) multiplicity k. Thus, one sees also in which manner the eigenvalues $(0, 1, 2, \ldots)$ with equal multiplicity arise in the case $D = I$ (cf. Result 2 above).

Let us remark that the case $\mathcal{D} \in (\mathrm{III_A})$ can also be handled if one modifies the notions of diagonal and shift operators as in the second part of Section 3.

6. Concluding remarks

We conclude with a small section which contains some conjectures and problems. But first let us remark the following. If one considers algebras of unbounded operators, then operators which are not commutators in algebras of bounded operators now can become commutators. The most famous example is the identity operator which is a commutator only if one includes also unbounded operators. A next step would be to leave even algebras and to go over to topological quasi-*-algebras as introduced by LASSNER ([9] and the references therein). In the context of quasi-*-algebras again a lot of operators become commutators. The detail will be published.

Let us now collect some conjectures and problems. They are partially modified by the constructions done so far; some of them may appear to be trivial.

Although the representation of an operator as commutator is not unique, it seems — at least for diagonal operators — that the growth of the diagonal sequence (a_n) determines the "degree of unboundedness" of A or/and B in $D_a = AB - BA$ (cf. Section 3). Thus, the following conjecture may be true.

Conjecture 1: *If $D_a \in \mathcal{L}^+(\mathcal{D})$ has a representation $D_a = AB - BA$, $A, B \in \mathcal{L}^+(\mathcal{D})$, then D_a has also the representation (4). More specifically, if $\mathcal{D} \in$ (I), then T is a commutator if and only if (t_n) is shift-admissible and T-addable.*

The next conjecture is partially modified by the first one.

Conjecture 2: a) *Let* $\mathcal{D} \in$ (I). *If the identity* I *is a commutator, then* $\mathcal{D}[t]$ *is a nuclear space.*

b) *If* $\mathcal{D} \in$ (II), *the identity* I *is not a commutator.*

c) *If* $\mathcal{D} \in$ (I); $R, L \in \mathcal{L}^+(\mathcal{D})$ *and* (t_n) *is* T*-addable, then any* $A \in \mathcal{L}^+(\mathcal{D})$ *is a commutator.*

The following conjecture contains a guess under which conditions any "compact" operator, i.e. any operator from $\mathcal{C}(\mathcal{D})$ is a commutator.

Conjecture 3: *If* $\mathcal{D} \in$ (II) $(\mathcal{D} \in (\mathrm{III_A}))$ *and* (t_n^1) $((t_n'))$ *are shift-admissible and* (t_n^1)*-addable* ((t_n')*-addable*, resp.), *then any* $A \in \mathcal{C}(\mathcal{D})$ *is a commutator. In case that* $A \in \mathcal{C}(\mathcal{D})$ *is a commutator,* $A = [B, C]$, *is it possible to take* $B, C \in \mathcal{C}(\mathcal{D})$?

We conclude with some

Problems: a) Extend the results in an appropriate way to general Op*-algebras and to the case where $\mathcal{D}[t]$ is a general (F)-space) (or at least more general then $\mathcal{D}^\infty(T)$).

b) Let $A \in \mathcal{L}^+(\mathcal{D})$ be an operator with $\mathrm{Tr}\, A = 0$. Is then A a commutator?

c) Under which general conditions on D and A one can prove that $\sigma(AA^+)$ has a structure similar to that described in the Remark following Lemma 5.2? Especially: Let $D \sim (d_n)$, $D = AA^+ - A^+A$, D and AA^+ commute. Describe $\sigma(AA^+)$.

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REFERENCES

[1] Brown, A., Halmos, P., and C. Pearcy: Commutators of operators on Hilbert space. Can. J. Math. 17 (1965), 695–708.
[2] Brown, A., and C. Pearcy: Structure of commutators of operators. Ann. Math. 82 (1965), 112–127.
[3] Dixmier, J.: Sur les variétés J d'un espace de Hilbert. J. Math. Pures Appl. (9) 28 (1949), 321–358.
[4] Dixmier, J.: Etude sur les variétés et les opérateurs de Julia avec quelques applications. Bull. soc. math. de France 77 (1949), 11–101.
[5] Dunford, N., and J. T. Schwartz: Linear operators. New York–London: Intersci. Publ. 1963.
[6] Halmos, P.: A Hilbert space problem book. Toronto–London: van Nostrand 1967.
[7] Kürsten, K.-D.: Two-sided ideals of certain classes of unbounded operators. Math. Nachr. (to appear).
[8] Lassner, G.: Topological algebras of operators. Rep. Math. Phys. 3 (1972), 279–293.
[9] Lassner, G.: Algebras of unbounded operators and quantum dynamics. Physica A 124 (1984), 471–479.
[10] Lassner, G., and W. Timmermann: Classification of domains of closed operators. Rep. Math. Phys. 9 (1976), 157–170.
[11] Lassner, G., and W. Timmermann: Classification of domains of operator algebras. Rep. Math. Phys. 9 (1976), 205–217.
[12] Löffler, F., and W. Timmermann: The Calkin representation for a certain class of algebras of unbounded operators. Rev. Roum. Math. Pures Appl. 31 (1986), 891–903.

[13] Митягин, Б. С.: Эквивалентность базисов в гильбертовых шкалах. Studia Math. **37** (1971), 111–137.
[14] Митягин, Б. С.: Структура подпространств бесконечной гильбертовой шкалы. В книге: Теория операторов в линейных пространствах (Дрогобыч 1974). Москва: Изд-во „Наука" 1976.
[15] Putnam, C. R.: Commutation properties of Hilbert space operators and related topics. New York: Springer-Verlag 1967.

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