Zeitschrift für Analysis
und ihre Anwendungen
Bd. 7 (1) 1988, S. 19–39

The Conditional Lindeberg-Trotter Operator in the Resolution of Limit Theorems with Rates for Dependent Random Variables. Applications to Markovian Processes

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Unter Ausnutzung der Erkenntnisse über bedingte Erwartungen wird ein bedingter Lindeberg-Trotter-Operator definiert, der die Eigenschaften des klassischen Lindeberg-Trotter-Operators auf den Fall abhängiger Zufallsvariabler erweitert. Somit werden allgemeine Grenzwertsätze für Summen abhängiger Zufallsvariabler mit o- und O-Ordnung bewiesen, die auf den Zentralen Grenzwertsatz, das schwache Gesetz großer Zahlen und vor allem auf Markov-Prozesse, auch im Falle von starker Konvergenz, angewendet werden.

Используя свойства условного математического ожидания определяется условный оператор Линдеберга-Троттера, который расширяет свойства классического оператора Линдеберга-Троттера на случай зависящих случайных величин. Этим доказываются общие предельные теоремы для сумм зависящих случайных величин с o- и O-порядками, которые применяются к центральной предельной теореме, к ослабленному закону больших чисел и особенно (также в случае сильной сходимости) к марковским процессам.

Making use of the properties of conditional expectations, a conditional Lindeberg-Trotter operator is defined which extends the properties of the classical Lindeberg-Trotter operator to the case of dependent random variables. This approach enables one to establish general limit theorems equipped with little-o and large-O rates for sums of dependent random variables; these are applied to several versions of the central limit theorem, the weak law of large numbers, and especially to Markovian processes, not only in the case of weak convergence in distribution but partially also for strong convergence.

1. Introduction

The limit theorems of probability theory have been presented at various levels of generality and application, both with respect to the structure of the random variables or processes considered and to types of limit laws considered. The main results of this paper refer to the weak convergence with large-O as well as little-o rates of the normalized sums $\varphi(n) S_n = \varphi(n) (X_1 + \cdots + X_n)$ (where $\varphi: \mathbf{N} \to \mathbf{R}^+$, $\varphi(n) \to 0$ as $n \to \infty$) of possibly dependent random variables to suitable limiting random variables Z. Here Z is always assumed to be φ-decomposable into independent components Z_i $(= Z_{i,n})$, $1 \leq i \leq n$ (i.e. for the distribution P_Z of Z one has $P_Z = P_{\varphi(n)(Z_1 + \cdots + Z_n)}$ for each $n \in \mathbf{N}$). In order to be able to apply (elegant) operator-theoretical methods in the proofs the convergence of the sequence $(\varphi(n) S_n)$ will be expressed in terms of a generalization of the Lindeberg-Trotter operator. For independent random variables the analysis carries through if the operator $V_X: C \to C$ is defined in its classical form by

$$(V_X f)(y) = \int_{\mathbf{R}} f(u + y)\, dF_X(u) = \mathrm{E}[f(X + y)] \qquad (y \in \mathbf{R}). \tag{1.1}$$

However, in the instance of not necessarily independent nor identically distributed random variables, thus for arbitrarily dependent random variables, one does not have the basic property that $(V_{X_1+\cdots+X_n})f = (V_{X_1}V_{X_2}\ldots V_{X_n})f$. For this purpose one turns to the concept of conditional expectations and defines the conditional Trotter operator $V_{X|\mathfrak{G}}$ of X relative to a sub-σ-algebra $\mathfrak{G}$ of $\mathfrak{A}$ in a probability space $(\Omega, \mathfrak{A}, P)$ by

$$(V_{X|\mathfrak{G}}f)(y) = \inf_{x\in A_\alpha(y,f)} \mathrm{E}[f(X+x) \mid \mathfrak{G}],$$

$A_\alpha(y;f) = \{x \in \mathbf{Q}: f(x) > f(y)\}$ with $y \in B_{\alpha x} = \{y \in \mathbf{R}: |x-y| < \alpha\}$, $\alpha, x \in \mathbf{Q}$ ($=$ set of rationals). Note that the family

$$\bar{\mathscr{B}} = \{B_{\alpha x}: x, \alpha \in \mathbf{Q}, \alpha > 0\} \tag{1.2}$$

is a base of the topological space $(\mathbf{R}, \mathscr{T})$, where $\mathscr{T}$ is the family of all open subsets of $\mathbf{R}$. This space is in particular a *Polish space*; it guarantees the existence of a regular conditional probability distribution of X relative to $\mathfrak{G}$. The formation of the infimum over all $x \in A_\alpha(y, f)$ is necessary to ensure the appropriate properties of the conditional operator needed for the proofs.

If $(X_k, \mathfrak{G}_k)$ is a sequence[1]) of couples, where the X_k are possibly dependent, real, P-integrable random variables on Ω, and the $\mathfrak{G}_k$ form a non-decreasing sequence of sub-σ-algebras of $\mathfrak{A}$, then the general limit theorem of Section 5, Theorem 2, on the weak convergence of $\varphi(n)\, S_n$ to Z, yields the estimate

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1}f - V_Zf\| = \mathcal{O}\left(\omega_r\left(\left[\frac{\varphi(n)^r}{(r-1)!}M(n;\mathfrak{G}_k)\right]^{1/r}; f; C\right)\right) \tag{1.3}$$

for any $f \in C$. Here ω_r is the rth modulus of continuity defined in (2.1), and $M(n; \mathfrak{G}_k)$ in (2.12). The basic assumption that $(X_k, \mathfrak{G}_k)$ has to fulfill is a suitable (conditional) pseudo-moment condition, namely (5.1). It is the only assumption which restricts the dependence structure of the random variables X_i in question. It regulates the dependence of the X_i amongst themselves, with the associated sub-σ-algebras $\mathfrak{G}_i$, together with the decomposition-components Z_i of Z. Such conditions are discussed in [8].

Although no directly comparable results for dependent random variables may be found in the literature, GUDYNAS [18] does correlate the rate of convergence of metrics comparable to the left side of (1.3) with metrics expressed in terms of conditional pseudo-moments. Pseudo-moments themselves have recently been also employed by ZOLOTAREV [32], PADITZ [27] and SAZONOV and ULYANOV [28] in work on the central limit theorem for independent random variables.

Theorem 1 of Section 4 provides a little-o counterpart of Theorem 2; it is a generalization of the corresponding result in [8]. This time the assumptions include in addition some generalized Lindeberg-type conditions (see (2.11)). Whereas the foregoing two theorems involve weak convergence, Theorem 7 of Section 6 deals with the strong convergence of $\varphi(n)\, S_n$ towards Z, equipped with $\mathcal{O}$-rates. The result is reduced to Theorem 2 by applying a lemma, found implicitly in ZOLOTAREV [30].

Applications of the results presented are to be found in the wide area of stochastic processes. Of great importance, particularly in renewal and queuing theory (see e.g. [20]), are Markov processes to which Section 7 is dedicated. The basic limit theorems for such processes, namely rather general versions of the weak law of

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[1]) We will write a sequence briefly as (a_k) instead of $(a_k)_{k\in\mathbf{N}}$, $\mathbf{N}$ the set of naturals.

large numbers (see Theorem 12), and especially of the central limit theorem expressed in terms of both weak (Theorem 10) and strong convergence (Theorem 11), all equipped with rates, are of as great significance as is an examination of the behaviour of the increments (Theorem 13) and transition functions (see e.g. [13]). If such topics are not treated explicitly in Section 7, they may nevertheless be followed up from the results presented. Thus Theorem 11 is a central limit theorem for sums of Markovian dependent random variables with respect to strong convergence. Under a suitable pseudo-moment condition it yields

$$\|F_{n^{-1/2}S_n} - F_{X^*}\| = \mathcal{O}(n^{(2-r)/(2r+2)}) \qquad (n \to \infty).$$

In the case $r = 4$ this means a rate of $\mathcal{O}(n^{-1/5})$. In the analogous situation for weak convergence the order is even $\mathcal{O}(n^{-1})$.

Many authors have investigated this matter. (In the case of o-rates one may check the discussion in [8].) The majority of them, instead of employing pseudo-moment conditions, used Doeblin's condition respectively conditions on the coefficient of ergodicity (see e.g. [14]). Connections between these two as well as with the coefficient of correlation or with mixing conditions are pointed out in LIFSHITS [24] and BRADLEY [3]. In particular, O'BRIEN [26] employed a strong mixing hypothesis for a proof of a central limit theorem for chain-dependent processes. HEINRICH [19] and NAGAEV [25] used Doeblin's hypothesis in their examination of the rate of convergence in a central limit theorem for Markov chains. LIFSHITS [23] computed the order $\mathcal{O}(n^{-1/2})$ for a central limit theorem for Markov chains in the case of strong convergence under conditions on the maximum coefficient of correlation. This result was generalized by GUDYNAS [17]. Further papers in the matter are due to LANDERS and ROGGE [22], BOLTHAUSEN [2], GORDIN and LIFSHITS [16], SIRAZHDINOV and FORMANOV [29], and BRADLEY [4]. All in all, most of these articles use conditions which imply that the random variables are in some sense "asymptotically independent". The question in regard to these conditions as well as to our pseudo-moment condition is in how far they actually restrict Markovian dependence and so the applicabilities.

It should also be mentioned that our main Theorems 9—13 in the particular case of identically distributed random variables may be applied to give assertions concerning stationary processes. In fact, the results and methods of this paper could be applied to many other related problems. Theorem 9 is the most general limit theorem with rates for Markov processes of this paper. A main problem in applying it is the determination of the suitable limiting random variable Z and its possible decomposition components. In the instance of convergence in distribution for independent random variables there exists a theorem to the effect that the limiting random variable of $S_n = X_1 + \cdots + X_n$ has an infinitely divisible distribution (see e.g. [5: p. 196]). Further, possible connections between infinite divisibility and φ-decomposability have been touched upon (see [10]). This may be of help in determining Z in the dependent case. Finally, Sections 5/B and C are not to be forgotten. They deal with a rather general central limit theorem for dependent random variables equipped with $\mathcal{O}$-rates (Theorems 3, 4) as well as with a generalization of the weak law of large numbers (Theorem 5). The counterpart for the central limit theorem in the case of strong convergence is formulated and established in Section 6.

2. Notations and preliminaries

In the following, $C = C(\mathbf{R})$ will denote the vector space of all real-valued, bounded, uniformly continuous functions defined on the reals $\mathbf{R}$, endowed with the supremum norm $\|\cdot\|$. We set

$$C^0 = C, \qquad C^r = \{g \in C : g^{(j)} \in C, 1 \leq j \leq r\} \quad \text{for } r \in \mathbf{N},$$

the seminorm on C^r being given by $|g| = \|g^{(r)}\|$. For any $f \in C$ and $t \geqq 0$ the K-functional is defined by

$$K(t; f; C, C^r) = \inf \{\|f - g\| + t\,|g| : g \in C^r\}.$$

It is equivalent to the rth modulus of continuity, defined for $f \in C$ by

$$\omega_r(t; f; C) = \sup \left\{\left\| \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(u + kh)\right\| : |h| \leq t\right\},$$

in the sense that there are constants $c_{1,r}, c_{2,r} > 0$, independent of f and $t \geqq 0$, such that (see [6: pp. 192, 258])

$$c_{1,r}\omega_r(t^{1/r}; f; C) \leqq K(t; f; C, C^r) \leqq c_{2,r}\omega_r(t^{1/r}; f; C). \tag{2.1}$$

Lipschitz classes of index $r \in \mathbf{N}$ and order $\alpha \in (0, r]$ will be needed. They are defined by $\mathrm{Lip}\,(\alpha; r; C) = \{f \in C : \omega_r(t; f; C) \leqq L_f t^\alpha\}$, L_f being the so-called *Lipschitz constant.*

Several preliminaries from probability theory will be noted. Let $(\Omega, \mathfrak{A}, P)$ denote a probability space with set Ω, σ-algebra $\mathfrak{A}$ and probability measure P, $\mathfrak{B}$ the σ-algebra of Borel sets in $\mathbf{R}$, $\mathfrak{Z}(\Omega, \mathfrak{A}) = \{X : \Omega \to \mathbf{R} : X \text{ is } \mathfrak{A}, \mathfrak{B}\text{-measurable}\}$ the set of all real random variables on Ω, and $\mathfrak{L}(\Omega, \mathfrak{A}, P) = \{X \in \mathfrak{Z}(\Omega, \mathfrak{A}) : X \text{ is } P\text{-integrable}\}$ the set of all real P-integrable random variables on Ω. An important concept needed for the proofs will be the *conditional expectation* (see e.g. [1: p. 292]), to be denoted for $X \in \mathfrak{L}(\Omega, \mathfrak{A}, P)$ and each sub-σ-algebra $\mathfrak{G} \subset \mathfrak{A}$ by $\mathrm{E}[X \mid \mathfrak{G}]$. If Y also belongs to $\mathfrak{L}(\Omega, \mathfrak{A}, P)$, and $\mathfrak{G}'$ is a further sub-σ-algebra of $\mathfrak{A}$, then there hold the properties (see e.g. [1: p. 293f.], [15: p. 188f.])

$$\mathrm{E}\big[\mathrm{E}[X \mid \mathfrak{G}]\big] = \mathrm{E}[X]; \qquad \mathrm{E}[X \mid \mathfrak{G}_0] = \mathrm{E}[X] \quad \text{a.s. for } \mathfrak{G}_0 = \{\Omega, \emptyset\}; \tag{2.2}$$

$$X \leqq Y \quad \text{a.s. implies} \quad \mathrm{E}[X \mid \mathfrak{G}] \leqq \mathrm{E}[Y \mid \mathfrak{G}] \quad \text{a.s.}; \tag{2.3}$$

$$X = c \quad \text{a.s., some} \quad c \in \mathbf{R}, \quad \text{implies} \quad \mathrm{E}[X \mid \mathfrak{G}] = c \quad \text{a.s.}; \tag{2.4}$$

$$\mathrm{E}[(\alpha X + \beta Y) \mid \mathfrak{G}] = \alpha \mathrm{E}[X \mid \mathfrak{G}] + \beta \mathrm{E}[Y \mid \mathfrak{G}] \quad \text{a.s.} \qquad (\alpha, \beta \in \mathbf{R}); \tag{2.5}$$

$$\mathrm{E}[X \mid \mathfrak{G}] = \mathrm{E}[X] \quad \text{a.s. provided the } \sigma\text{-algebra } \mathfrak{A}(X), \text{ generated by } X, \text{ is independent of } \mathfrak{G}; \tag{2.6}$$

$$\mathrm{E}\big[\mathrm{E}[X \mid \mathfrak{G}] \mid \mathfrak{G}'\big] = \mathrm{E}\big[\mathrm{E}[X \mid \mathfrak{G}'] \mid \mathfrak{G}\big] = \mathrm{E}[X \mid \mathfrak{G}] \quad \text{a.s. provided} \quad \mathfrak{G} \subset \mathfrak{G}'. \tag{2.7}$$

Results on topology and regular conditional distributions will also be needed. Let $\mathscr{T}$ be the family of all open subsets (in classical sense) of $\mathbf{R}$; then the space $(\mathbf{R}, \mathscr{T})$ is a topological space having a countable base. One base is the family of sets $\mathscr{B}$ of (1.2). A topological space with this base is a complete, separable metric space. In fact, a topological space possessing a countable base and defined via a complete metric space is said to be *Polish* (according to Bourbaki). A Polish space is known to be a Borel space, $(\mathbf{R}, \mathfrak{B})$ here.

The aim now is to represent the conditional expectation as an integral. For this purpose two concepts need be recapitulated. If $\mathfrak{G} \subset \mathfrak{A}$ is a σ-algebra and $X \in \mathfrak{Z}(\Omega, \mathfrak{A})$, a function $P_X{}^\wedge : \Omega \times \mathfrak{A} \to \mathbf{R}$ is said to be a *regular conditional probability distribution* of X relative to $\mathfrak{G}$, if it satisfies the conditions (see e.g. [21: p. 372ff.]):

(i) $P_X{}^\wedge(\omega, \cdot)$ is a probability measure for every $\omega \in \Omega$;

(ii) $P_X{}^\wedge(\cdot, A) \in \mathfrak{Z}(\Omega, \mathfrak{G})$ for every $A \in \mathfrak{A}$;

(iii) $\int_G P_X{}^\wedge\big(\omega, X^{-1}(A)\big)\, dP = P\big(G \cap X^{-1}(A)\big)$ for every $A \in \mathfrak{A}$, $G \in \mathfrak{G}$.

The function $F_X: \mathbf{R} \times \Omega \to \mathbf{R}$, $F_X(x \mid \mathfrak{G}) = F_X(x \mid \mathfrak{G})(\omega) = P_X^\wedge(\omega, (-\infty, x])$ a.s. $(x \in \mathbf{R})$, is called a *conditional distribution function* of X with respect to $\mathfrak{G}$. Note that if $(\Omega, \mathfrak{A}, P)$ is an arbitrary probability space, and $\mathfrak{G}$ an arbitrary sub-σ-algebra of $\mathfrak{A}$, then for each $X \in \mathfrak{Z}(\Omega, \mathfrak{A})$ there always exists a regular conditional distribution (and so also a conditional distribution function) of X with respect to $\mathfrak{G}$ (see e.g. [21: p. 373]). This is due to the fact that $(\mathbf{R}, \mathfrak{B})$ is a Borel space. Now to the integral representation. Let $X \in \mathfrak{L}(\Omega, \mathfrak{A}, P)$, $\mathfrak{G}$ be a sub-σ-algebra of $\mathfrak{A}$, $g: \mathbf{R} \to \mathbf{R}$ a Borel-measurable function with $\mathrm{E}[g(X)] < \infty$, and $F_X(x \mid \mathfrak{G})$ be a conditional distribution function of X relative to $\mathfrak{G}$. Then there exists a $G \in \mathfrak{G}$ with $P(G) = 0$ such that for all $\omega \in \Omega \setminus G$ (see [21: p. 375])

$$\mathrm{E}[g(X) \mid \mathfrak{G}](\omega) = \int g(x)\, d\big(F_X(x \mid \mathfrak{G})(\omega)\big). \tag{2.8}$$

For the proofs Lindeberg-type conditions will be needed. They will all be formulated for $X_k, Z_k \in \mathfrak{Z}(\Omega, \mathfrak{A})$, all $k \in \mathbf{N}$. If $X_k^s \in \mathfrak{L}(\Omega, \mathfrak{A}, P)$ for some $s \in (0, \infty)$ and all $k \in \mathbf{N}$, then the sequence (X_k) is said to satisfy a *generalized Lindeberg condition of order* s (see e.g. [12]), if for every $\delta > 0$

$$\left(\sum_{k=1}^{n} \int_{|x| \geq \delta/\varphi(n)} |x|^s\, dF_{X_k}(x)\right) \Big/ \left(\sum_{k=1}^{n} \mathrm{E}[|X_k|^s]\right) \to 0 \qquad (n \to \infty). \tag{2.9}$$

If $X_k^s, Z_k^s \in \mathfrak{L}(\Omega, \mathfrak{A}, P)$ or $|X_k - Z_k|^s \in \mathfrak{L}(\Omega, \mathfrak{A}, P)$ for all $k \in \mathbf{N}$, then the sequences (X_k) and (Z_k) satisfy a *generalized pseudo-Lindeberg condition of order* s if for every $\delta > 0$

$$\sum_{k=1}^{n} \int_{|x| \geq \delta/\varphi(n)} |x|^s\, d\big(F_{X_k}(x) - F_{Z_k}(x)\big) = \begin{cases} o_\delta\big(M(n)\big) & \text{or} \\ o_\delta\big(V(n)\big) \end{cases} \qquad (n \to \infty) \tag{2.10}$$

where

$$M(n) = \sum_{k=1}^{n} (\mathrm{E}[|X_k|^s] + \mathrm{E}[|Z_k|^s]), \qquad V(n) = \sum_{k=1}^{n} (\mathrm{E}[|X_k - Z_k|^s]).$$

In regard to this paper, a further generalization of this condition is basic. If for the sequences $(X_k, \mathfrak{G}_k)$ and (Z_k) there holds $(E[|X_k|^s \mid \mathfrak{G}_k] - E[|Z_k|^s]) < \infty$ for some $s \in (0, \infty)$ and all $k \in \mathbf{N}$, then they are said to satisfy a *conditional pseudo-Lindeberg condition of order* s if for every $\delta > 0$

$$\sum_{k=1}^{n} \int_{|x| \geq \delta/\varphi(n)} |x|^s\, d\big(F_{X_k}(x \mid \mathfrak{G}_k) - F_{Z_k}(x)\big) = o_\delta\big(M(n; \mathfrak{G}_k)\big) \qquad (n \to \infty), \tag{2.11}$$

where

$$M(n; \mathfrak{G}_k) = \sum_{k=1}^{n} \int_{\mathbf{R}} |x|^s\, d\big(F_{X_k}(x \mid \mathfrak{G}_k) - F_{Z_k}(x)\big). \tag{2.12}$$

It should be remarked that (2.11) coincides with the second possibility of (2.10) in the case that $\mathfrak{A}(X_k)$ and $\mathfrak{G}_k$ are independent, since then $M(n; \mathfrak{G}_k) = M(n)$. Further, condition (2.10) is automatically fulfilled (compare Lemma 1 in [8]) if (2.9) is satisfied for both (X_k) and (Z_k).

As already mentioned in the introduction, the Trotter operator plays an important role in establishing rates of convergence for independent random variables. For the development of corresponding assertions in the instance of dependent random variables a new operator concept — closely related to the usual Trotter operator — will be used in this paper. To elucidate the connections, let us first recall the most important properties for the Trotter operator defined in (1.1).

Lemma 1: *Let $X, Y \in \mathfrak{Z}(\Omega, \mathfrak{A})$. Let $X_1, \ldots, X_n$ and $Z_1, \ldots, Z_n$ be independent random variables belonging to $\mathfrak{Z}(\Omega, \mathfrak{A})$. Then*

a) *V_X is a positive, linear operator satisfying $\|V_X f\| \leqq \|f\|$ $(f \in C)$;*

b) *$V_X = V_Y$ provided X and Y are identically distributed;*

c) *V_X and V_Y are commutative provided X and Y are independent;*

d) *$V_{S_n} f = V_{X_1} V_{X_2} \ldots V_{X_n} f$ $(f \in C)$;*

e) $\left\| V_{S_n} f - V_{\sum_{k=1}^{n} Z_k} f \right\| \leqq \sum_{k=1}^{n} \|V_{X_k} f - V_{Z_k} f\|$ $(f \in C)$.

The following lemma, a slight generalization of Lemma 1e), will play a decisive role in the proof of Lemma 5.

Lemma 2: *Let $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$ be contraction endomorphisms of C such that $U_i U_j$ is only defined for $i \leqq j$ but the V_i may commute amongst themselves and $U_i V_j = V_j U_i$ for any i, j. Then $\|U_1 \ldots U_n f - V_1 \ldots V_n f\| \leqq \|U_1 - V_1\| + \cdots + \|U_n - V_n\|$.*

Proof: Set $U_0 = I$. For $f \in C$,

$$\sum_{i=1}^{n} \left(U_1 \ldots U_{i-1}(U_i - V_i) V_{i+1} \ldots V_n\right) f$$

$$= \sum_{i=1}^{n} (U_1 \ldots U_{i-1} U_i V_{i+1} \ldots V_n) f - \sum_{i=1}^{n} (U_1 \ldots U_{i-1} V_i V_{i+1} \ldots V_n) f$$

$$= \sum_{j=1}^{n} (U_1 \ldots U_j V_{j+1} \ldots V_n) f - \sum_{j=1}^{n} (U_1 \ldots U_{j-1} V_j \ldots V_n) f$$

$$= U_1 \ldots U_n f - V_1 \ldots V_n f.$$

Now the restricted commutativity is brought into play. Indeed,

$$\|\left(U_1 \ldots U_{i-1}(U_i - V_i) V_{i+1} \ldots V_n\right) f\| \leqq \|(U_i - V_i) f\|$$

since

$$\begin{aligned} \|\left(U_1 \ldots U_{i-1}(U_i - V_i) V_{i+1} \ldots V_n\right) f\| &\leqq \|\left((U_i - V_i) V_{i+1} \ldots V_n\right) f\| \\ &\leqq \|V_{i+1} \ldots V_n (U_i - V_i) f\| \\ &\leqq \|(U_i - V_i) f\| \quad \blacksquare \end{aligned}$$

3. The conditional Trotter operator

The idea behind the conditional Trotter operator is a proper exploitation of the properties of conditional expectations. Assertions concerning them are generally valid only almost surely, thus for each individual $y \in \mathbf{R}$ but not uniformly for all $y \in \mathbf{R}$. (See (3.1).) In order to achieve the latter fact, which is especially important in an operator theoretical approach, one makes use of the concept of Polish spaces introduced in Section 2.

Definition 1: Let $(X, \mathfrak{G})$ be a couple, where $X \in \mathfrak{L}(\Omega, \mathfrak{A}, P)$ and $\mathfrak{G}$ is an arbitrary sub-σ-algebra of $\mathfrak{A}$. The *conditional Trotter operator* $V_{X|\mathfrak{G}}: C \to C \times \mathfrak{Z}(\Omega, \mathfrak{G})$ of $(X, \mathfrak{G})$ is defined for $f \in C$ by

$$V_{X|\mathfrak{G}} f(y) = \inf_{x \in A_x(y,f)} \mathrm{E}[f(X + x) \mid \mathfrak{G}] \qquad (y \in \mathbf{R}). \tag{3.1}$$

The fact that a Polish space like $(\mathbf{R}, \mathfrak{B})$ has a countable base assures that the infimum is taken only countably often. This means that operations dealing with the conditional Trotter operator are valid a.s. for all $y \in \mathbf{R}$. The condition "$f(x) > f(y)$" is necessary to ensure that the infimum is taken at $x = y$ in case $\mathfrak{A}(x)$ and $\mathfrak{G}$ are independent, so that the conditional Trotter operator coincides with the classical one.

The most important properties of this operator which is uniquely determined up to a set of measure zero by definition are collected in the following lemmas; below one has set $\left(V_{X|\mathfrak{G}}f(y)\right)^{\sim}(\omega) = (V_{X|\mathfrak{G}}f)(y, \omega)$.

Lemma 3: *Let $(X, \mathfrak{G})$ be a couple with $X \in \mathfrak{L}(\Omega, \mathfrak{A}, P)$ and $\mathfrak{G}$ an arbitrary sub-σ-algebra of $\mathfrak{A}$, and let $f, g \in C$. Then*

a) $V_{X|\mathfrak{G}}f(y, \cdot) \in \mathfrak{Z}(\Omega, \mathfrak{G})$ $(y \in \mathbf{R})$;

b) $\sup_{y\in\mathbf{R}} |(V_{X|\mathfrak{G}}f)(y, \omega)| \leq \|f\|$ $(\omega \in \bar{G}_1, f \in C)$ *for some* $G_1 \in \mathfrak{G}$ *with* $P(G_1) = 0$;

c) $(V_{X|\mathfrak{G}}f)(\cdot, \omega) \in C$ $(\omega \in \bar{G}_2)$ *for some* $G_2 \in \mathfrak{G}$ *with* $P(G_2) = 0$;

d) $\left(V_{X|\mathfrak{G}}(\gamma f + \beta g)\right)(\cdot, \omega) = \gamma(V_{X|\mathfrak{G}}f)(\cdot, \omega) + \beta(V_{X|\mathfrak{G}}g)(\cdot, \omega)$ $(\omega \in \bar{G}_3; \beta, \gamma \in \mathbf{R})$ *for some* $G_3 \in \mathfrak{G}$ *with* $P(G_3) = 0$;

e) $V_{X|\mathfrak{G}}f(y, \omega) = V_X f(y)$ $(\omega \in \bar{G}_4)$ *for some* $G_4 \in \mathfrak{G}$ *with* $P(G_4) = 0$, *provided* $\mathfrak{A}(X)$ *is independent of* $\mathfrak{G}$;

f) $V_{X|\mathfrak{G}}f(y, \omega) = \inf_{x\in A_\alpha(y,f)} \int_{\mathbf{R}} f(u + x)\, dF_X(u \mid \mathfrak{G})(\omega)$ $(\omega \in \bar{G}_5)$ *for some* $G_5 \in \mathfrak{G}$ *with* $P(G_5) = 0$.

Proof: a) By definition of conditional expectations, $\mathrm{E}[Z \mid \mathfrak{G}] \in \mathfrak{Z}(\Omega, \mathfrak{G})$ for each $Z \in \mathfrak{Z}(\Omega, \mathfrak{G})$. So part a) follows by Definition 1 with $Z = f(X + x)$.

b) In view of (2.3) and (2.4) there exists a set $G = G(x)$ with $P(G) = 0$ such that

$$\mathrm{E}[f(X + x) \mid \mathfrak{G}](\omega) \leq \mathrm{E}[\|f\| \mid \mathfrak{G}](\omega) = \|f\| \quad \text{a.s.} \tag{3.2}$$

for each fixed $x \in \mathbf{Q}$ and $\omega \in \bar{G}$. Setting $G_1 = \bigcup_{x\in\mathbf{Q}} G(x)$, then $P(G_1) = 0$, and so (3.2) holds for all $x \in \mathbf{Q}$. The fact that there is only a countable number of infima yields part b).

c) Since $V_{X|\mathfrak{G}}f$ is bounded a.s. by part b), it remains to show that it is uniformly continuous a.s. Because $f \in C(\mathbf{R})$, $|f(y_1) - f(y_2)| < \varepsilon$ for all $y_1, y_2 \in \mathbf{R}$ with $|y_1 - y_2| < \delta$, so that $\sup_{u\in\mathbf{R}} |f(u + y_1) - f(u + y_2)| < \varepsilon$. By (2.3), (2.4) and the special structure of $A_\alpha(y, f)$, it follows that

$$\begin{aligned} &|V_{X|\mathfrak{G}}f(y_1, \omega) - V_{X|\mathfrak{G}}f(y_2, \omega)| \\ &= \left| \inf_{x\in A_\alpha(y_1,f)} \mathrm{E}[f(X + x) \mid \mathfrak{G}] - \inf_{x\in A_\alpha(y_2,f)} \mathrm{E}[f(X + x) \mid \mathfrak{G}] \right| \\ &\leq \sup_{u\in\mathbf{R}} \left| \inf_{x\in A_\alpha(y_1,f)} f(u + x) - \inf_{x\in A_\alpha(y_2,f)} f(u + x) \right| \\ &= \sup_{u\in\mathbf{R}} |f(u + y_1) - f(u + y_2)| < \varepsilon \quad \text{a.s.}, \end{aligned}$$

noting that the infimum is taken on the closure of the range of $A_\alpha(y_1, f)$ (or $A_\alpha(y_2, f)$); since $f(x) > f(y_1)$, the minimal value can only coincide with the value at y_1 (or y_2). This establishes part c).

d) By (2.5) it follows that

$$\inf_{x\in A_\alpha(y,\gamma f+\beta g)} \mathrm{E}[(\gamma f+\beta g)(X+x)\mid \mathfrak{G}](\omega)$$
$$=\inf_{x\in A_\alpha(y,\gamma f+\beta g)} \{\gamma \mathrm{E}[f(X+x)\mid\mathfrak{G}]+\beta\mathrm{E}[f(X+x)\mid\mathfrak{G}]\}$$
$$=\gamma\inf_{x\in A_\alpha(y,f)}\mathrm{E}[f(X+x)\mid\mathfrak{G}]+\beta\inf_{x\in A_\alpha(y,f)}\mathrm{E}[f(X+x)\mid\mathfrak{G}]$$

This gives part d).

e) Since $\mathfrak{A}(x)$ and $\mathfrak{G}$ are independent, one has by (2.6),

$$V_{X\mid\mathfrak{G}}f(y,\omega)=\inf_{x\in A_\alpha(y,f)}\mathrm{E}[f(X+x)]=\mathrm{E}[f(X+y)]=V_Xf(y)\quad\text{a.s.},$$

noting that the infimum is taken on the closure of the range of $A_\alpha(y,f)$.

f) The fact that Polish spaces are Borel spaces ensures the regularity of $F_X(u\mid\mathfrak{G})(\omega)$ which is in particular $\mathfrak{G}$-measurable for each fixed $B\in\mathfrak{B}$ as well as a measure for each fixed ω. This together with (2.8) gives part f) ∎

Corollary: *Let $(\Omega,\mathfrak{A},P)$, $\mathfrak{G}$, X and f be given as in Lemma 3. There exists a set $G\in\mathfrak{G}$ with $P(G)=0$ such that $(V_{X\mid\mathfrak{G}}f)(\cdot,\omega)$ is a linear operator of C into itself for each $\omega\in\overline{G}$, satisfying $\|V_{X\mid\mathfrak{G}}f(\cdot,\omega)\|\leqq\|f\|$.*

Indeed, with G_1, G_2, G_3 given as in Lemma 3b)—d), $(V_{X\mid\mathfrak{G}}f)(\cdot,\omega)$ is a contraction endomorphism on C for each $\omega\in\overline{G}$.

Lemma 4: *Let $(X_n)\subset\mathfrak{L}(\Omega,\mathfrak{A},P)$ be a sequence of random variables, and $(\mathfrak{G}_n)$ a non-decreasing sequence of sub-σ-algebras of $\mathfrak{A}$. Then for each $f\in C$,*

$$V_{X_1\mid\mathfrak{G}_1}\big(V_{X_2\mid\mathfrak{G}_2}(\dots V_{X_n\mid\mathfrak{G}_n}f(\cdot)\dots)\big)(y,\omega)=(V_{X_1\mid\mathfrak{G}_1}V_{X_2\mid\mathfrak{G}_2}\dots V_{X_n\mid\mathfrak{G}_n}f)(y,\omega)$$
$$=(V_{S_n\mid\mathfrak{G}_1}f)(y,\omega)\quad a.s.\quad(y\in\mathbf{R};\ n\in\mathbf{N}).$$

If, in particular, $\mathfrak{G}_1=\{\Omega,\emptyset\}$, then

$$(V_{X_1\mid\mathfrak{G}_1}V_{X_2\mid\mathfrak{G}_2}\dots V_{X_n\mid\mathfrak{G}_n}f)(y,\omega)=V_{S_n}f(y)\quad a.s.\quad(y\in\mathbf{R};\ n\in\mathbf{N}).$$

Proof: First take $n=2$. By (2.2) and (2.7),

$$(V_{X_1\mid\mathfrak{G}_1}V_{X_2\mid\mathfrak{G}_2}f)(y,\omega)=\Big(V_{X_1\mid\mathfrak{G}_1}\Big\{\inf_{\bar{x}\in A_\alpha(\cdot,f)}\mathrm{E}\big[f\big(X_2+\bar{x}(\cdot)\big)\mid\mathfrak{G}_2\big]\Big\}\Big)(y,\omega)$$
$$=\inf_{x\in A_\alpha(y,V_{X_2\mid\mathfrak{G}_2}f)}\mathrm{E}\Big[\Big\{\inf_{\bar{x}\in A_\alpha(\cdot,f)}\mathrm{E}\big[f\big(X_2+x(\cdot)\big)\mid\mathfrak{G}_2\big]\Big\}(X_1+x)\mid\mathfrak{G}_1\Big](\omega)$$
$$=\inf_{x\in A_\alpha(y,V_{X_2\mid\mathfrak{G}_2}f)}\ \inf_{\bar{x}\in A_\alpha(\cdot,f)}\mathrm{E}\Big[\mathrm{E}\big[f\big(X_2+\bar{x}(\cdot)\big)\mid\mathfrak{G}_2\big](X_1+x)\mid\mathfrak{G}_1\Big](\omega)$$
$$=\inf_{x\in A_\alpha(y,V_{X_2\mid\mathfrak{G}_2}f)}\mathrm{E}[f(X_2+X_1+x)\mid\mathfrak{G}_1](\omega),$$

noting that $\mathrm{E}\Big[\mathrm{E}\big[f\big(X_2+\bar{x}(\cdot)\big)\mid\mathfrak{G}_2\big](X_1+x)\mid\mathfrak{G}_1\Big](\omega)=\mathrm{E}\big[f\big(X_2+\bar{x}(X_1+x)\big)\mid\mathfrak{G}_1\big](\omega)$, implying that the inner infimum is taken over the closure of the range of $A_\alpha(X_1+x,f)$. Since the latter infimum is equal to $\mathrm{E}[f(X_1+X_2+x)\mid\mathfrak{G}_1](\omega)$, the proof is complete since

$$\inf_{x\in A_\alpha(y,f)}\mathrm{E}[f(X_2+X_1+x)\mid\mathfrak{G}_1](\omega)=V_{X_1+X_2\mid\mathfrak{G}_1}f(y,\omega).$$

The general result now follows by induction, and the particular case by Lemma 3e) ∎

Lemma 5: *Let* (X_n) *and* $(\mathfrak{G}_n)$ *be given as in Lemma 4. If* $(Z_n) \subset \mathfrak{L}(\Omega; \mathfrak{A}, P)$ *is a further sequence, it being assumed that the* Z_n *are independent themselves as well as of the* X_n*, then for each* $f \in C$,

$$\left\| V_{S_n|\mathfrak{G}_1} f(y, \omega) - V_{\sum_{k=1}^n Z_k} f(y) \right\| \leqq \sum_{k=1}^n \| V_{X_k|\mathfrak{G}_k} f(y, \omega) - V_{Z_k} f(y) \|.$$

If in particular $\mathfrak{G}_k = \{\Omega, \emptyset\}$*, all* $k \in \mathbf{N}$*, then*

$$\left\| V_{S_n} f(y) - V_{\sum_{k=1}^n Z_k} f(y) \right\| \leqq \sum_{k=1}^n \| V_{X_k} f(y) - V_{Z_k} f(y) \| \qquad (n \in \mathbf{N}).$$

The proof follows by the corollary of Lemma 3 and Lemmas 3e), 2 and 1 ∎

4. General limit theorems for dependent random variables with o-rates

In our following main approximation theorem for sums of possibly dependent random variables X_i and their corresponding sub-σ-algebras $\mathfrak{G}_i$, endowed with o-rates, the conditional Trotter operator, introduced in Section 3, and the conditional pseudo-Lindeberg condition (2.11) are of great importance.

Theorem 1: *Let* $(X_k, \mathfrak{G}_k)$ *be a sequence of couples, the* X_k *being real-valued random variables from* $\mathfrak{L}(\Omega, \mathfrak{A}, P)$ *and the* $\mathfrak{G}_k$ *a non-decreasing sequence of sub-σ-algebras of* $\mathfrak{A}$. *Let* Z *be a* φ*-decomposable random variable with decomposition components* Z_k, $k \in \mathbf{N}$. *Assume that* $\mathrm{E}[|X_k|^r \mid \mathfrak{G}_k] < \infty$ *a.s. and* $\mathrm{E}[|Z_k|^r] < \infty$ *for* $k \in \mathbf{N}$ *and an* $r \in \mathbf{N} \setminus \{1\}$. *If, furthermore, the sequences* $(X_k, \mathfrak{G}_k)$ *and* (Z_k) *satisfy a conditional pseudo-Lindeberg condition* (2.11) *of order* r*, and*

$$\sum_{k=1}^n \{\mathrm{E}[X_k^j \mid \mathfrak{G}_k] - \mathrm{E}[Z_k^j]\} = o\big(\varphi(n^r)\, M(n; \mathfrak{G}_k)\big) \qquad (1 \leqq j \leqq r-1;\ n \to \infty) \tag{4.1}$$

with $M(n; \mathfrak{G}_k)$ *of* (2.12), *then there holds for* $f \in C^r$

$$\| V_{\varphi(n) S_n|\mathfrak{G}_1} f - V_Z f \| = o\big(\varphi(n)^r\, M(n; \mathfrak{G}_k)\big). \tag{4.2}$$

If, in particular, $\mathfrak{G}_1 = \{\Omega, \emptyset\}$*, then*

$$\| V_{\varphi(n) S_n} f - V_Z f \| = o\big(\varphi(n)^r\, M(n; \mathfrak{G}_k)\big). \tag{4.3}$$

Proof: In view of Lemma 5 there holds

$$\left\| V_{\varphi(n) S_n|\mathfrak{G}_1} f - V_{\varphi(n) \sum_{k=1}^n Z_k} f \right\| \leqq \sum_{k=1}^n \| V_{\varphi(n) X_k|\mathfrak{G}_k} f - V_{\varphi(n) Z_k} f \|.$$

Furthermore, one has on account of set-function-theoretical aspects,

$$\left| \inf_{x \in A_\alpha(y; f)} \left\{ \mathrm{E}\big[f\big(\varphi(n)\, X_k + x\big) \,\big|\, \mathfrak{G}_k\big]\right\} - \mathrm{E}\big[f\big(\varphi(n)\, Z_k + y\big)\big] \right|$$

$$\leqq \sup_{x \in A_\alpha(y; f)} \left\{ \big| \mathrm{E}\big[f\big(\varphi(n)\, X_k + x\big) \mid \mathfrak{G}_k\big] - \mathrm{E}\big[f\big(\varphi(n)\, Z_k + x\big)\big] \big| \right\}.$$

So it suffices to estimate the following difference. By the integral representation (2.8), and Taylor's formula applied twice to $f(u + x)$, one has

$$\begin{aligned}
&\left|\mathrm{E}\left[f(\varphi(n)\, X_k + x) \mid \mathfrak{G}_k\right] - \mathrm{E}\left[f(\varphi(n)\, Z_k + x)\right]\right| \\
&= \left| \int_{\mathbf{R}} f(u + x)\, d\left(F_{\varphi(n) X_k}(u \mid \mathfrak{G}_k)(\omega) - F_{\varphi(n) Z_k}(u)\right)\right| \\
&\leq \left| \int_{\mathbf{R}} \left\{\sum_{j=0}^{r} \frac{\varphi(n)^j\, u^j}{j!}\, f^{(j)}(x)\right\} d\left(F_{X_k}(u \mid \mathfrak{G}_k) - F_{Z_k}(u)\right)\right. \\
&\quad \left. + \int_{\mathbf{R}} \left\{\frac{1}{r!}\, \varphi(n)^r\, u^r \left(f^{(r)}(\eta) - f^{(r)}(x)\right)\right\} d\left(F_{X_k}(u \mid \mathfrak{G}_k) - F_{Z_k}(u)\right)\right|,
\end{aligned} \tag{4.4}$$

where $|\eta - x| \leq \varphi(n)\, |u|$. Since $f^{(r)} \in C$, to any $\varepsilon > 0$ there is a $\delta(\varepsilon)$ such that $|f^{(r)}(\eta) - f^{(r)}(x)| < \varepsilon$ for $|\eta - x| < \delta$. But since $\varphi(n) = o(1)$, to $\delta > 0$ and $u \in \mathbf{R}$ there is an $n \in \mathbf{N}$ with $|\eta - x| \leq \varphi(n)\, |u| < \delta$. So, splitting up the range $\mathbf{R}$ in (4.4) into $\{u \in \mathbf{R}\colon |u| < \delta/\varphi(n)\}$ and its complementary set, yields for the remainder

$$\left|\left(\int_{|u| < \delta/\varphi(n)} + \int_{|u| \geq \delta/\varphi(n)}\right) \frac{1}{r!}\, \varphi(n)^r\, u^r \left(f^{(r)}(\eta) - f^{(r)}(x)\right) d\left(F_{X_k}(u \mid \mathfrak{G}_k) - F_{Z_k}(u)\right)\right|$$

the estimate

$$\begin{aligned}
&\left|\frac{\varphi(n)^r}{r!}\, \varepsilon\left(\mathrm{E}[|X_k|^r \mid \mathfrak{G}_k] - \mathrm{E}[|Z_k|^r]\right)\right| \\
&+ \left|\frac{\varphi(n)^r}{r!}\, 2\, \|f^{(r)}\| \int_{|u| \geq \delta/\varphi(n)} u^r d\left(F_{X_k}(u \mid \mathfrak{G}_k) - F_{Z_k}(u)\right)\right|.
\end{aligned}$$

Combining these estimates, one has

$$\begin{aligned}
&\left|\mathrm{E}\left[f(\varphi(n)\, X_k + x) \mid \mathfrak{G}_k\right] - \mathrm{E}[f(\varphi(n)\, Z_k + x)]\right| \\
&\leq \left|\sum_{j=0}^{r} \frac{\varphi(n)^j}{j!}\, f^{(j)}(x) \int_{\mathbf{R}} u^j d\left(F_{X_k}(u \mid \mathfrak{G}_k) - F_{Z_k}(u)\right)\right| \\
&\quad + \left|\frac{\varphi(n)^r}{r!}\, \varepsilon\left(\mathrm{E}[|X_k|^r \mid \mathfrak{G}_k] - \mathrm{E}[|Z_k|^r]\right)\right| \\
&\quad + \left|\frac{\varphi(n)^r}{r!}\, 2\, \|f^{(r)}\| \int_{|u| \geq \delta/\varphi(n)} u^r d\left(F_{X_k}(u \mid \mathfrak{G}_k) - F_{Z_k}(u)\right)\right|.
\end{aligned}$$

Summing up this inequality over k from 1 to n, the first term has the order $\sum_{j=0}^{r} \left(\varphi(n)^j/j!\right) \|f^{(j)}\|\, o\left(\varphi(n)^r\, M(n; \mathfrak{G}_k)\right)$, the sum over j being bounded. The second also has the desired order by choosing a suitable $\varepsilon > 0$. Concerning the third term, one has by (2.11)

$$\sum_{k=1}^{n} \left| \frac{\varphi(n)^r}{r!} 2 \|f^{(r)}\| \int\limits_{|u| \geq \delta/\varphi(n)} u^r d\big(F_{X_k}(u \mid \mathfrak{G}_k) - F_{Z_k}(u)\big) \right|$$

$$= \frac{2 \|f^{(r)}\|}{r!} \varphi(n)^r \left| \sum_{k=1}^{n} \int\limits_{u \geq \delta/\varphi(n)} u^r d\big(F_{X_k}(u \mid \mathfrak{G}_k) - F_{Z_k}(u)\big) \right|$$

$$= (2 \|f^{(r)}\|/r!)\, o\big(\varphi(n)^r M(n; \mathfrak{G}_k)\big).$$

All in all, one has the estimate

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1} f - V_Z f\|$$

$$\leq \sup_{y \in \mathbf{R}} \sup_{x \in A_\alpha(y,f)} \left\{ \sum_{j=0}^{r} \frac{\varphi(n)^j}{j!} 2 \|f^{(j)}\| + \frac{\varepsilon}{r!} + \frac{2 \|f^{(r)}\|}{r!} \right\} o\big(\varphi(n)^r M(n; \mathfrak{G}_k)\big)$$

$$= o\big(\varphi(n)^r M(n; \mathfrak{G}_k)\big). \tag{4.5}$$

This yields (4.2). The estimate (4.3) follows with Lemma 3e) ∎

Corollary: *If the random variables X_i as well as the decomposition components Z_i, $i \in \mathbf{N}$, are additionally identically distributed, as well as all $\mathfrak{G}_i$ are equal to another, then assumption* (4.1) *implies for* $f \in C$

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1} f - V_Z f\| = o\big(\varphi(n)^r n\, |\mathrm{E}[X_1 \mid \mathfrak{G}_1] + \mathrm{E}[Z_1]|^r\big) \quad (n \to \infty).$$

The result will follow from Theorem 1 if the conditional pseudo-Lindeberg condition (2.11) for the $(X_i, \mathfrak{G}_i)$ and Z_i can now be shown to follow for $\varphi(n) = o(1)$. But for identically distributed random variables with $\mathfrak{G}_i = \mathfrak{G}_j$, $i \neq j$, this condition reduces to

$$\int\limits_{|x| \geq \delta/\varphi(n)} |x|^r d\big(F_{X_k}(x \mid \mathfrak{G}_k) - F_{Z_k}(x)\big) = o_\delta(1) \quad \text{for each } \delta > 0,$$

which is automatically satisfied since $\delta/\varphi(n) \to \infty$, $n \to \infty$ ∎

Remarks: 1. The term $\|V_{\varphi(n)S_n|\mathfrak{G}_1} f - V_Z f\|$ in (4.2) tends to zero for $n \to \infty$ if $\varphi(n)^r M(n; \mathfrak{G}_k)$ is bounded. In the case of the corollary this is fulfilled for $\varphi(n) = n^{-1/r}$. The constant in the convergence estimate is, according to (4.5),

$$\sum_{j=0}^{r} \frac{\varphi(n)^j}{j!} \|f^{(j)}\| + \frac{\varepsilon}{r!} + \frac{2 \|f^{(r)}\|}{r!}.$$

2. It should be mentioned that the conditional Trotter operator method used in this paper permits a generalization of the theorems and results obtained in an earlier paper [8] by means of the modified Dvoretzky extension of the classical Trotter operator approach. In this sense the results of [8] would all follow by Theorem 1.

5. A General limit theorem with $\mathcal{O}$-rates. Applications to the central limit theorem and weak law of large numbers

A. General results

The following general limit theorem with $\mathcal{O}$-rates for arbitrary random variables is a generalization of the comparable Theorem 1 in [9].

Theorem 2: *Let $(X_k, \mathfrak{G}_k)$ be a sequence of couples, where (X_k) is a sequence of possibly dependent random variables from $\mathfrak{L}(\Omega, \mathfrak{A}, P)$, and $(\mathfrak{G}_k)$ a non-decreasing se-*

quence of sub-σ-algebras of $\mathfrak{A}$. *Let* Z *be a* φ*-decomposable random variable with decomposition components* Z_k, $k \in \mathbf{N}$. *Assume that* $\mathrm{E}[|X_k|^r \mid \mathfrak{G}_k] < \infty$ *a.s. as well as* $\mathrm{E}[|Z_k|^r] < \infty$ *for* $k \in \mathbf{N}$ *and an* $r \in \mathbf{N} \setminus \{1\}$. *Let furthermore*

$$\sum_{k=1}^{n} \{\mathrm{E}[X_k{}^j \mid \mathfrak{G}_k] - \mathrm{E}[Z_k{}^j]\} = \mathcal{O}\left(\frac{\varphi(n)^r}{(r-1)!} M(n; \mathfrak{G}_k)\right)$$
$$(1 \leqq j \leqq r-1;\ n \to \infty). \tag{5.1}$$

Under these hypotheses one has for any $f \in C$

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1}f - V_Z f\| \leqq 2c_{2,r}N_1\omega_r\left(\left[\frac{\varphi(n)^r}{(r-1)!} M(n; \mathfrak{G}_k)\right]^{1/r}; f; C\right),$$

$c_{2,r}$ *being the constant of* (2.1) *and* N_1 *the constant of the "*$\mathcal{O}$*"-order of* (5.1). *If in particular* $\mathfrak{G}_1 = \{\Omega, \emptyset\}$, *then*

$$\|V_{\varphi(n)S_n}f - V_Z f\| \leqq 2c_{2,r}N_1\omega_r\left(\left[\frac{\varphi(n)^r}{(r-1)!} M(n; \mathfrak{G}_k)\right]^{1/r}; f; C\right).$$

Proof: In view of (2.3) and (2.4) one has for $f \in C$ and any $g \in C^r$,

$$\begin{aligned}
&\Big|\inf_{x\in A_\alpha(y,f)}\big\{\mathrm{E}\big[f(\varphi(n)S_n + x) \mid \mathfrak{G}_1\big]\big\} - \mathrm{E}[f(Z+y)]\Big| \\
&\leqq \Big|\inf_{x\in A_\alpha(y,f)}\big\{\mathrm{E}\big[f(\varphi(n)S_n + x) \mid \mathfrak{G}_1\big]\big\} - \inf_{x\in A_\alpha(y,g)}\big\{\mathrm{E}\big[g(\varphi(n)S_n + x) \mid \mathfrak{G}_1\big]\big\}\Big| \\
&\quad + \Big|\inf_{x\in A_\alpha(y,g)}\big\{\mathrm{E}\big[g(\varphi(n)S_n + x) \mid \mathfrak{G}_1\big]\big\} - \mathrm{E}[g(Z+y)]\Big| \\
&\quad + |\mathrm{E}[g(Z+y)] - \mathrm{E}[f(Z+y)]| \\
&\leqq 2\,\|f-g\| + \Big|\inf_{x\in A_\alpha(y,g)}\big\{\mathrm{E}\big[g(\varphi(n)S_n + x) \mid \mathfrak{G}_1\big] - \mathrm{E}[g(Z+x)]\big\}\Big|.
\end{aligned} \tag{5.2}$$

Further, on account of Lemma 5,

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1}g - V_Z g\| \leqq \sum_{k=1}^{n} \|V_{\varphi(n)X_k|\mathfrak{G}_k}g - V_{\varphi(n)Z_k}g\|. \tag{5.3}$$

Thirdly, there holds the estimate

$$\begin{aligned}
&\Big|\inf_{x\in A_\alpha(y,g)}\big\{\mathrm{E}\big[g(\varphi(n)X_k + x) \mid \mathfrak{G}_k\big]\big\} - \mathrm{E}\big[g(\varphi(n)Z_k + x)\big]\Big| \\
&\leqq \sup_{x\in A_\alpha(y,g)}\Big\{\Big|\mathrm{E}\big[g(\varphi(n)X_k + x) \mid \mathfrak{G}_k\big] - \mathrm{E}\big[g(\varphi(n)Z_k + x)\big]\Big|\Big\}.
\end{aligned} \tag{5.4}$$

Fourthly, on account of the integral representation (2.8), and Taylor's formula of order $r-1$ applied to both $g(u+x)$,

$$\begin{aligned}
&\Big|\mathrm{E}\big[g(\varphi(n)X_k + x) \mid \mathfrak{G}_k\big] - \mathrm{E}\big[g(\varphi(n)Z_k + x)\big]\Big| \\
&\leqq \left|\left[\int_{\mathbf{R}} \left\{\sum_{j=0}^{r-1} \frac{u^j}{j!} g^{(j)}(x)\right\} dF_{\varphi(n)X_k}(u \mid \mathfrak{G}_k)\right] - \left[\int_{\mathbf{R}} \sum_{j=0}^{r-1} \frac{u^j}{j!} g^{(j)}(x)\, dF_{\varphi(n)Z_k}(u)\right]\right| \\
&\quad + \left|\int_{\mathbf{R}} \frac{1}{(r-2)!}\left[\int_0^1 (1-t)^{r-2}\,\{g^{(r-1)}(x+tu) - g^{(r-1)}(x)\}\,u^{r-1}\,dt\right] dF_{\varphi(n)X_k}(u \mid \mathfrak{G}_k)\right. \\
&\quad \left. - \int_{\mathbf{R}} \frac{1}{(r-2)!}\left[\int_0^1 (1-t)^{r-2}\,\{g^{(r-1)}(x+tu) - g^{(r-1)}(x)\}\,u^{r-1}\,dt\right] dF_{\varphi(n)Z_k}(u)\right|.
\end{aligned}$$

Since $g \in C^r$, $g^{(r-1)} \in \mathrm{Lip}\,(1; 1; C)$ with Lipschitz constant $L_g = \|g^{(r)}\|$. So fifthly, for $0 < t \leqq 1$, $|\{g^{(r-1)}(x + tu) - g^{(r-1)}(x)\}\, u^{r-1}| \leqq \|g^{(r)}\|\, |u|^r$, thus

$$\sum_{k=1}^{n} \sum_{j=0}^{r-1} \left|\mathrm{E}\big[g\big(\varphi(n) X_k + x\big) \mid \mathfrak{G}_k\big] - \mathrm{E}\big[g\big(\varphi(n)\, Z_k + x\big)\big]\right|$$

$$\leqq \sum_{k=1}^{n} \sum_{j=0}^{r-1} \left| \frac{1}{j!}\, g^{(j)}(x) \left\{ \int_{\mathbf{R}} u^j d[F_{\varphi(n)X_k}(u \mid \mathfrak{G}_k) - F_{\varphi(n)Z_k}(u)] \right\} \right|$$

$$+ \frac{\|g^{(r)}\|}{(r-1)!} \left| \sum_{k=1}^{n} \int_{\mathbf{R}} |u|^r\, d[F_{\varphi(n)X_k}(u \mid \mathfrak{G}_k) - F_{\varphi(n)Z_k}(u)] \right|.$$

But by (2.12) this whole expression is of order $\mathcal{O}\big(\varphi(n)^r/(r-1)!\, M(n; \mathfrak{G}_k)\big)$. All in all, by (5.2)–(5.4),

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1} f(y) - V_Z f(y)\|$$

$$\leqq 2\, \|f - g\| + \sum_{i=1}^{n} \sup_{y \in \mathbf{R}} \left| \sup_{x \in A_\alpha(y,f)} \big\{\mathrm{E}\big[g\big(\varphi(n)\, X_i\big) \mid \mathfrak{G}_i\big] - \mathrm{E}\big[g\big(\varphi(n)\, Z_i\big)\big]\big\} \right|$$

$$\leqq 2K \left(N_2\, \frac{\varphi(n)^r}{(r-1)!}\, M(n, \mathfrak{G}_k); f; C; C^r \right).$$

This establishes the general result. The particular case follows noting Lemma 3e) ∎

Corollary: *Let the assumptions of Theorem 2 be satisfied.*

a) *If further* $f \in \mathrm{Lip}\,(\alpha; r; C)$, $\alpha \in (0, r]$, *then*

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1} f - V_Z f\| \leqq 2c_{2,r} N_1 L_f \left(\frac{\varphi(n)^r}{(r-1)!}\, M(n; \mathfrak{G}_k) \right)^{\alpha/r}.$$

b) *If the* $X_1, X_2, \ldots$ *are in addition identically distributed, where* $\mathfrak{G}_k = \mathfrak{G}_1$, $k \in \mathbf{N}$, *and the* $Z_1, Z_2, \ldots$ *are also identically distributed, then*

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1} f - V_Z f\| \leqq 2c_{2,r} N_1 L_f\, \frac{\varphi(n)^\alpha}{(r-1)!}\, n^{\alpha/r} (\mathrm{E}[|X_1|^r \mid \mathfrak{G}_1] - \mathrm{E}[|Z_1|^r])^{\alpha/r}.$$

c) *In case* $\varphi(n) = o(n^{-1/r})$ *one has* $\|V_{\varphi(n)S_n|\mathfrak{G}_1} f - V_Z f\| = o_f(1)$, *the constant being given by*

$$2c_{2,r} N_1 L_f\, \frac{1}{(r-1)!}\, (\mathrm{E}[|X_1|^r \mid \mathfrak{G}_1] - \mathrm{E}[|Z_1|^r])^{\alpha/r}. \tag{5.5}$$

d) *In the classical case* $\varphi(n) = n^{-1/2}$ *one has the order* $O_f(n^{(2-r)/2})$, *where the constant is given in* (5.5).

Remark: As already mentioned in the introduction, Theorem 2 and the Corollary are the most general theorems known to us in the matter. They are generalizations of the comparable results for independent random variables [7] and those for Martingale difference arrays [11]. Possible applications are indicated in the introduction. A comparable result of other authors is e.g. [18].

B. The central limit theorem with $\mathcal{O}$-rates

As an application of the general Theorem 2, a central limit theorem for dependent random variables, endowed with $\mathcal{O}$-rates, will be formulated with the help of the conditional Trotter operator.

Theorem 3: *Let $(X_k, \mathfrak{G}_k)$ be a sequence of couples as in Theorem 2, and let X^* be a standard normally distributed random variable. Assume that $E[|X_k|^r \mid \mathfrak{G}_k] < \infty$ a.s. for $k \in \mathbf{N}$ and an $r \in \mathbf{N} \setminus \{1\}$. If*

$$\sum_{k=1}^{n} \{E[X_k^j \mid \mathfrak{G}_k] - a_k^j E[X^{*j}]\} = \mathcal{O}\left(\frac{\varphi(n)^r}{(r-1)!} M(n;\mathfrak{G}_k)\right)$$

where $(a_k) \subset \mathbf{R}$ and $M(n;\mathfrak{G}_k) = \sum_{k=1}^{n} (E[|X_k|^r \mid \mathfrak{G}_k] - E[|a_k X^*|^r])$, then one has for $f \in C$

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1} f - V_{X^*} f\| \leqq 2c_{2,r} N_1 \omega_r \left(\left[\frac{\varphi(n)^r}{(r-1)!} M(n;\mathfrak{G}_k)\right]^{1/r}; f; C\right).$$

Proof: The theorem follows by Theorem 2, noting that X^* is φ-decomposable (see e.g. [11]) with $P_{X^*} = P_{\varphi(n)\sum_{k=1}^{n} Z_k}$, where the decomposition components Z_k are normally distributed random variables with mean zero and variance a_k^2; they may, without loss of generality (see [1]), be chosen to be independent amongst themselves as well as of the random variables X_k. ∎

Let us now formulate some handy versions of the central limit theorem for dependent random variables.

Theorem 4: *Let $(X_k, \mathfrak{G}_k)$, (a_k) and X^* be given as in Theorem 3.*

a) *If especially $\mathfrak{G}_1 = \{\Omega, \emptyset\}$, then, for $f \in C$,*

$$\|V_{\varphi(n)S_n} f - V_{X^*} f\| \leqq 2c_{2,r} N_1 \omega_r \left(\left[\frac{\varphi(n)^r}{(r-1)!} M(n;\mathfrak{G}_k)\right]^{1/r}; f; C\right).$$

b) *If, additionally, $f \in \mathrm{Lip}(\alpha; r; C)$, $\alpha \in (0, r]$, then*

$$\|V_{\varphi(n)S_n} f - V_{X^*} f\| = \mathcal{O}(\varphi(n)^\alpha M(n;\mathfrak{G}_k)^{\alpha/r}), \tag{5.6}$$

where the constant is given by $2c_{2,r} N_1 L_f/(r-1)!$.

c) *In the special case that the $X_1, X_2, \ldots$ are identically distributed as well as $a_i = a_j$, $i \neq j$, and $\mathfrak{G}_k = \{\Omega, \emptyset\}$, $k \in \mathbf{N}$, the order in (5.6) is $\mathcal{O}(\varphi(n)^\alpha n^{\alpha/r})$, with constant*

$$2c_{2,r} N_1 L_f (E[|X_1|^r] - E[|a_k X^*|^r])^{\alpha/r}/(r-1)!. \tag{5.7}$$

d) *In the classical case, where $\varphi(n) = A_n^{-1} := (a_1^2 + \cdots + a_n^2)^{-1/2}$, the order in (5.6) is $\mathcal{O}(A_n^{-\alpha} n^{\alpha/r})$ with constant (5.7).*

e) *If $a_i = a_j$, $i \neq j$, then $A_n = n^{1/2} a_1$. If $a_1 = 1$, so that the Z_i are standard normally distributed, one has for $f \in \mathrm{Lip}(\alpha; r, C)$ the estimate*

$$\|V_{n^{-1/2}S_n} f - V_{X^*} f\| = \mathcal{O}(n^{\alpha(2-r)/2r}).$$

Observe that the latter estimate yields convergence provided $r > 2$, the constant being (5.7) with $a_1 = 1$.

C. The weak law of large numbers with $\mathcal{O}$-rates

In the following two versions of the weak law of large numbers are formulated. The first, a rather general version, will follow from Theorem 2.

Theorem 5: *Let $(X_k, \mathfrak{G}_k)$ be a sequence as in Theorem 2. Let $Z = Z_0$ be a trivial random variable, i.e., $P(Z_0 = 0) = 1$. Assume that $\mathrm{E}[|X_k|^r \mid \mathfrak{G}_k] := u_{rk} < \infty$ a.s. for $k \in \mathbf{N}$ and an $r \in \mathbf{N} \setminus \{1\}$. Let furthermore*

$$\sum_{k=1}^{n} \mathrm{E}[X_k^j \mid \mathfrak{G}_k] = \mathcal{O}\left(\frac{\varphi(n)^r}{(r-1)!} U_{rn}\right) \quad (1 \leqq j \leqq r,\ n \to \infty), \tag{5.8}$$

where $U_{rn} = u_{r1} + \cdots + u_{rn}$. Then one has for $f \in C$

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1} f - f(0)\| \leqq 2c_{2,r}N_1\omega_r\left(\left[\frac{\varphi(n)^r}{(r-1)!} U_{rn}\right]^{1/r}; f; C\right).$$

Proof: Noting that Z_0 is φ-decomposable, and transforming the conditions and results of Theorem 2 to the situation of $Z = Z_0$, the theorem follows directly by Theorem 2 ∎

Corollary: *Under the assumptions of Theorem 5 there holds for $f \in \mathrm{Lip}\,(\alpha; r; C)$, $\alpha \in (0, r]$,*

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1} f - f(0)\| \leqq 2c_{2,r}N_1L_f\left(\frac{\varphi(n)^r}{(r-1)!} U_{rn}\right)^{1/r}.$$

Noting the equivalence (see [1: p. 220] of $\lim_{n\to\infty} P(\{\varphi(n) S_n \geqq \varepsilon\}) = 0$, $\varepsilon > 0$, with $\lim_{n\to\infty} |\mathrm{E}[f(\varphi(n) S_n)] - f(0)| = 0$ for $f \in C^r$, any $r > 0$, one has the following

Theorem 6: *Let $(X_k, \mathfrak{G}_k)$ be given as in Theorem 2, where $u_{rk} < \infty$ for an $r \in \mathbf{N} \setminus \{1\}$ and $\mathfrak{G}_1 = \{\Omega, \emptyset\}$. Let $Z = Z_0$, and let (5.8) hold. If furthermore $\varphi(n)^r U_{rn} = o(1)$, then $\lim_{n\to\infty} P(\{\varphi(n) S_n \geqq \varepsilon\}) = 0$ $(\varepsilon > 0)$.*

In the case that the $X_1, X_2, \ldots$ are identically distributed, and $\mathfrak{G}_k = \{\Omega, \emptyset\}$, $k \in \mathbf{N}$, condition $\varphi(n)^r U_{rn} = o(1)$ is equivalent to $\varphi(n)^r n = o(1)$.

6. Strong convergence in distribution

In this section we will carry over our results for the weak convergence with large $\mathcal{O}$-orders of Section 5 to the case of strong convergence of the distribution function of the normed sum $\varphi(n) S_n$ to an arbitrary, φ-decomposable random variable Z. In order to achieve this aim we need

Lemma 6: *Let Y be a real-valued random variable with distribution function F_Y such that a constant $M_Y > 0$ exists with*

$$|F_Y(t) - F_Y(s)| \leqq M_Y |t - s| \qquad (s, t \in \mathbf{R},\ s < t). \tag{6.1}$$

Then for each random variable X and each constant $M_2 > 0$ there exists a constant $M = M(M_Y, M_2)$ such that for the so-called Kolmogorov metric between the distribution functions F_Y and F_X, there holds for an arbitrary, fixed $r \in \mathbf{N}$,

$$\sup_{t\in\mathbf{R}} |F_X(t) - F_Y(t)| \leqq M \left\{\sup_{f\in D} |\mathrm{E}[f(X)] - \mathrm{E}[f(Y)]|\right\}^{1/r+1}.$$

Here $D = \{f \in C^{r-1}; f^{(r-1)} \in \mathrm{Lip}_{M_2}(1; 1; C)\}$, *with uniformly bounded Lipschitz constant* $L_f(r-1) \leq M_2$.

This Lemma is to be found implicitly in ZOLOTAREV [30] (see also [31]), and formulated explicitly in [11]. Let us now consider the general convergence theorem for the strong convergence, as mentioned above.

Theorem 7: *Let* $(X_k, \mathfrak{G}_k)$ *be a sequence of couples, where* (X_k) *is a sequence from* $\mathfrak{L}(\Omega, \mathfrak{A}, P)$ *and* $(\mathfrak{G}_k)$ *is a non-decreasing sequence of sub-σ-algebras of* $\mathfrak{A}$ *with* $\mathfrak{G}_1 = \{\Omega, \emptyset\}$. *Let Z be a φ-decomposable random variable for which condition* (6.1) *holds. Assume that* $\mathrm{E}[|X_k|^r \mid \mathfrak{G}_k] < \infty$ *a. s. as well as* $\mathrm{E}[|Z_k|^r] < \infty$ *for* $k \in \mathbf{N}$ *and some* $r \in \mathbf{N} \setminus \{1\}$. *Let furthermore*

$$\sum_{k=1}^{n} \{\mathrm{E}[X_k^j \mid \mathfrak{G}_k] - \mathrm{E}[Z_k^j]\} = \mathcal{O}\left(\frac{\varphi(n)^r}{(r-1)!} M(n; \mathfrak{G}_k)\right)$$

$$(1 \leq j \leq r-1; n \to \infty).$$

Under these hypotheses

$$\|F_{\varphi(n)S_n} - F_Z\| = \mathcal{O}\left(\varphi(n)^{r/(r+1)} M(n; \mathfrak{G}_k)^{1/(r+1)}\right) \tag{6.2}$$

with constant

$$\frac{M}{(r-1)!}\left(\|f^{(r)}\| + \sum_{j=1}^{r-1} \frac{1}{j!} \|f^{(j)}\|\right)^{1/r+1} \qquad (f \in D), \tag{6.3}$$

where M is given in Lemma 6 *and the other factors of the constant come from the proof of Theorem* 2.

Proof: The term which has to be estimated is divided into two parts as at the end of the proof of Theorem 2. The part with $g \in C^r$ is estimated as in the fifth step of this proof, and has the bound

$$\left\{\frac{1}{(r-1)!}\left(\varphi(n)^r M(n; \mathfrak{G}_k)\right)\left(\|g^{(r)}\| + \sum_{j=1}^{r-1} \frac{1}{j!} \|g^{(j)}\|\right)\right\}.$$

This bound holds for all $g \in C^r$, where $g^{(r-1)} \in \mathrm{Lip}\,(1; 1; C)$. The set of these g is an upper-set of D. This means that the estimate (6.2) follows by applying Lemma 6 ∎

Corollary: a) *If in particular* $\mathrm{E}[|X_k|^r \mid \mathfrak{G}_k] \leq M_r$ *a.s. and* $\mathrm{E}[|Z_k|^r] \leq M_r^*$ *uniformly for all* $k \in \mathbf{N}$, *then*

$$\|F_{\varphi(n)S_n} - F_Z\| = \mathcal{O}\left(\varphi(n)^{r/(r+1)} n^{1/(r+1)} (M_r + M_r^*)^{1/(r+1)}\right).$$

b) *If* $X_1, X_2, \ldots$ *are identically distributed with* $\mathfrak{G}_k = \mathfrak{G}_1$, $k \in \mathbf{N}$, *and* $Z_1, Z_2, \ldots$ *are also identically distributed, then*

$$\|F_{\varphi(n)S_n} - F_Z\| = \mathcal{O}\left(\varphi(n)^{r/(r+1)} n^{1/(r+1)} (\mathrm{E}[|X_1|^{r/(r+1)}] - \mathrm{E}[|Z_1|^{r/(r+1)}])\right).$$

c) *If furthermore* $\varphi(n) = o(n^{-r})$, *then the estimate from part* a) *gives convergence.*
d) *In case* $\varphi(n) = n^{-1/2}$ *one has rhe order* $\mathcal{O}(n^{(2-r)/2(r+1)})$.

Proof: Part a) follows by Theorem 6, using the estimate

$$M(n; \mathfrak{G}_k) := \sum_{k=1}^{n} \{\mathrm{E}[|X_k|^r \mid \mathfrak{G}_k] - \mathrm{E}[|Z_k|^r]\} \leq n\{M_r + M_r^*\}.$$

The other parts follow immediately ∎

Remark: Part a) of the corollary coincides exactly with Theorem 8 in [9]. This indeed shows that Theorem 6 in this paper is a deep generalization of Theorem 8 there. The exact constants in the different cases of the corollary follow always by (6.3).

Let us now apply Theorem 7 to a version of the central limit theorem.

Theorem 8: *Let $(X_k, \mathfrak{G}_k)$ be as in Theorem 7. Let X^* be a standard normally distributed random variable satisfying condition* (6.1). *Assume that* $\mathrm{E}[|X_k|^r \mid \mathfrak{G}_k] =: M_{kr} < \infty$ *a.s. for $k \in \mathbf{N}$ and an $r \in \mathbf{N} \setminus \{1\}$. If*

$$\sum_{k=1}^{n} \{\mathrm{E}[X_k^j \mid \mathfrak{G}_k] - a_k^j \mathrm{E}[X^{*j}]\} = \mathcal{O}\left(\frac{\varphi(n)^r}{(r-1)!} M(n; \mathfrak{G}_k)\right)$$

$$(1 \leqq j \leqq r-1,\ n \to \infty),$$

where $(a_k) \subset \mathbf{R}$, then

$$\|F_{\varphi(n)S_n} - F_{X^*}\| = \mathcal{O}\left(\varphi(n)^{r/(r+1)} M(n; \mathfrak{G}_k)^{1/(r+1)}\right). \tag{6.4}$$

Here $M(n; \mathfrak{G}_k) := \sum_{k=1}^{n} \{\mathrm{E}[|X_k|^r \mid \mathfrak{G}_k] - |a_k|^r \mathrm{E}[|X^*|^r]\}$, *and the constant in* (6.4) *is given for $f \in D$ by* (6.3).

This theorem follows immediately by Theorems 7 and 3 ∎

Naturally it would be possible to formulate further different versions of Theorem 8 as applications of the corollary of Theorem 7.

7. Applications to Markovian processes

A. General assumptions

Let us first formulate some preparatory lemmas and definitions.

Definition 2: A sequence (X_i) of real random variables on some probability space $(\Omega, \mathfrak{A}, P)$ is said to be

a) *dependent from below* if, for each $1 \leqq i \leqq n$, $n \in \mathbf{N}$,

$$P(X_i \in B \mid X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) = P(X_i \in B \mid X_{i-1}) \quad \text{a.s.} \quad (B \in \mathfrak{B});$$

b) *expectationally dependent from below* if, for each $1 \leqq i \leqq n$, $n \in \mathbf{N}$,

$$\mathrm{E}[X_i \mid X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n] = \mathrm{E}[X_i \mid X_1, \ldots, X_{i-1}] \quad \text{a.s.}$$

Lemma 7: a) *If X is any random variable, $\mathfrak{E}$, $\mathfrak{F}$ are two sub-σ-algebras of $\mathfrak{A}$, then $P(X \in B \mid \mathfrak{E}) = P(X \in B \mid \mathfrak{F})$ for all $B \in \mathfrak{B}$ implies $E[X \mid \mathfrak{E}] = \mathrm{E}[X \mid \mathfrak{F}]$ a.s.*

b) *If (X_i) is a sequence of random variables that is dependent from below, then it is expectationally dependent from below.*

Definition 3: A *Markovian process with discrete time parameter* is a sequence of random variables (X_i) on some probability space $(\Omega, \mathfrak{A}, P)$ possessing the Markov property

$$P(X_i \in B \mid X_1, \ldots, X_{i-1}) = P(X_i \in B \mid X_{i-1}) \quad (B \in \mathfrak{B};\ i \geqq 2). \tag{7.1}$$

If (X_i) is a Markovian process, then the random variables $Y_i := X_i - X_{i-1}$, $X_0 := 0$ a.s., are called the *increments*.

Definition 4: The Markovian process (X_i) is called a *Markovian process with dependent increments* if the Y_i are dependent. Otherwise the process is called a *process with independent increments.* In both cases $X_n := Y_1 + \cdots + Y_n$.

Lemma 8: *If (X_i) is a Markovian processes, then the sequence of increments (Y_i) is expectationally dependent from below.*

Remark: Definitions 2—4 and Lemmas 7, 8 as well as their proofs are explicitly given in [8]. In this paper results for general limit theorems for Markovian processes with o-rates are formulated and proved.

In order to apply the results of Sections 4—6 to Markovian processes one has to give explicitly the sequence $(X_k, \mathfrak{G}_k)$. If (X_k) is such a process, then the appropriate sequence of sub-σ-algebras $(\mathfrak{G}_k)$ is the sequence with $\mathfrak{G}_k = \mathfrak{A}(X_1, \ldots, X_{k-1})$, $\mathfrak{G}_1 = \{\Omega, \emptyset\}$. If one regards the sequence of increments (Y_k), the appropriate sub-σ-algebras are given by $\mathfrak{G}_k = \mathfrak{A}(Y_1, \ldots, Y_{k-1})$; $\mathfrak{G}_1 = \{\Omega, \emptyset\}$ according to Lemma 8.

Let us now formulate another important lemma, needed in the following theorems. It states that the expectation of a Markovian process depends only upon the expectation of its immediate predecessor.

Lemma 9: *Let $(X_k, \mathfrak{G}_k)$ be defined as above, with $\mathfrak{G}_k = \mathfrak{A}(X_1, \ldots, X_{k-1})$. Then for each function h, it holds* $\mathrm{E}[h(X_k) \mid \mathfrak{G}_k] = \mathrm{E}[h(X_k) \mid X_{k-1}]$.

Proof: *Noting* (7.1) *with* $B = (-\infty, u]$,

$$\mathrm{E}[h(X_k) \mid \mathfrak{G}_k] := \int_{\mathbf{R}} h(u)\, dF_{X_k}(u \mid \mathfrak{G}_k)$$

$$= \int_{\mathbf{R}} h(u)\, dF_{X_k}(u \mid X_{k-1}) =: \mathrm{E}[h(X_k) \mid X_{k-1}] \quad \blacksquare$$

B. General limit theorem. A central limit theorem and weak law of large numbers

Let us first formulate the general result.

Theorem 9: *Let $(X_k, \mathfrak{G}_k)$ be a sequence of couples, (X_k) being a Markovian process and $\mathfrak{G}_k = \mathfrak{A}(X_1, \ldots, X_{k-1})$ sub-σ-algebras of $\mathfrak{A}$. Let Z be a φ-decomposable random variable with* $\mathrm{E}[Z] = 0$. *Assume that*

$$\mathrm{E}[|X_k|^r \mid X_{k-1}] < \infty \quad a.s. \tag{7.2}$$

as well as $\mathrm{E}[|Z_k|^r] < \infty$ *for* $k \in \mathbf{N}$ *and some* $r \in \mathbf{N} \setminus \{1\}$. *Let, furthermore,*

$$\sum_{k=1}^{n} \{\mathrm{E}[X_k^j \mid X_{k-1}] - \mathrm{E}[Z_k^j]\} = \mathcal{O}\left(\frac{\varphi(n)^r}{(r-1)!}\, \overline{M}(n; X_k)\right) \quad (1 \leqq j \leqq r-1;\ n \to \infty), \tag{7.3}$$

where $\overline{M}(n; X_k) = \sum_{k=1}^{n} \{\mathrm{E}[|X_k|^r \mid X_{k-1}] - \mathrm{E}[|Z_k|^r]\}$. *Then one has for any* $f \in C$

$$\|V_{\varphi(n)S_n} f - V_Z f\| \leqq 2c_{2,r} N_1 \omega_r \left(\left[\frac{\varphi(n)^r}{(r-1)!}\, \overline{M}(n; X_k)\right]^{1/r}; f; C\right).$$

Proof: In order to apply Theorem 2, its assumption needs to be checked. In fact, the condition $\mathrm{E}[|X_k|^r \mid \mathfrak{G}_k] < \infty$ follows by (7.2) and Lemma 9 with $h(u) = |u|^r$.

Assumption (5.1) follows in the same way by (7.3) and Lemma 5. At last, one has to evaluate the expression $M(n; \mathfrak{G}_k)$ in the case of Markovian processes. In fact, $M(n; \mathfrak{G}_k) = \overline{M}(n; X_k)$ ∎

Corollary: *Assume additionally to the hypotheses of Theorem 8 that*

$$\mathrm{E}[|X_k|^r \mid X_{k-1}] \leqq M_r, \quad \mathrm{E}[|Z_k|^r] \leqq M_r^* \quad (r \in \mathbf{N} \setminus \{1\}; k \in \mathbf{N}). \tag{7.4}$$

Then one has for $f \in \mathrm{Lip}\,(\alpha; r; C)$, $\alpha \in (0, r]$, $\|V_{\varphi(n)S_n}f - V_Z f\| = \mathcal{O}(\varphi(n)^\alpha n^{\alpha/r})$.

Let us now apply Theorem 4a to a central limit theorem for Markovian processes.

Theorem 10: *Let* $(X_k, \mathfrak{G}_k)$ *be given as in Theorem 9. Let further* $(a_k) \subset \mathbf{R}$, *and* X^* *a standard normally distributed random variable. If condition* (7.2) *holds and* (7.3) *with* $P_{Z_k} = P_{a_k X^*}$, *then for* $f \in C$

$$\|V_{\varphi(n)S_n}f - V_{X^*}f\| \leqq 2c_{2,r}N_1\omega_r\left(\left[\frac{\varphi(n)^r}{(r-1)!}\overline{M}(n; X_k)\right]^{1/r}; f; C\right).$$

This theorems follows immediately by Theorems 4a) and 9 ∎

Corollary: *In the classical case, where* $\mathrm{Var}\, X_k = a_k^2$, *and* $\varphi(n) = A_n^{-1}$ *with* $A_n = (a_1^2 + \cdots + a_n^2)^{1/2}$, *one has for* $f \in \mathrm{Lip}\,(\alpha; r; C)$, $\alpha \in (0, r]$,

$$\|V_{A_n^{-1}S_n}f - V_{X^*}f\| = \mathcal{O}(A_n^{-\alpha}\overline{M}(n; X_k)^{\alpha/r}).$$

If in particular (7.4) *holds, and* $a_i = a_j = 1$, $i \neq j$, *then* $\|V_{n^{-1/2}S_n}f - V_{X^*}f\| = \mathcal{O}(n^{\alpha(2-r)/2r})$.

Let us also formulate a strong version of the central limit theorems for Markovian processes by using the results of Section 6.

Theorem 11: *Let* $(X_k, \mathfrak{G}_k)$ *be given as in Theorem 9, and* (a_k), A_n *and* X^* *as in Theorem* 10. *If conditions* (7.2) *and* (7.3) *hold, then*

$$\|F_{A_n^{-1}S_n} - F_{X^*}\| = \mathcal{O}(A_n^{-r/(r+1)}\overline{M}(n; X_k)^{1/(r+1)}).$$

The proof follows directly by Theorems 7 and 10 ∎

Corollary: *If additionally to the hypotheses of Theorem* 11 *condition* (7.4) *holds, and* $a_i = a_j = 1$, $i \neq j$, *then* $\|F_{n^{-1/2}S_n} - F_{X^*}\| = \mathcal{O}(n^{(2-r)/(2r+2)})$.

Taking instead of X^* the limiting random variable $Z = X_0$ with $P(X_0 = 0) = 1$, one can formulate a weak law of large numbers for Markovian processes as an application of Theorem 5.

Theorem 12: *Let* $(X_k, \mathfrak{G}_k)$ *be given as in Theorem 9 together with condition* (7.2). *If, instead of* (7.3),

$$\sum_{k=1}^{n} \mathrm{E}[X_k^j \mid X_{k-1}] = \mathcal{O}\left(\frac{\varphi(n)^r}{(r-1)!}\sum_{k=1}^{n}\mathrm{E}[|X_k|^r \mid X_{k-1}]\right),$$

then one has for $f \in C$

$$\|V_{\varphi(n)S_n}f - f(0)\| \leqq 2c_{2,r}N_1\omega_r\left(\left[\frac{\varphi(n)^r}{(r-1)!}\sum_{k=1}^{n}\mathrm{E}[|X_k|^r \mid X_{k-1}]\right]^{1/r}; f; C\right).$$

Remark: The counterparts of the theorems of this subsection that are equipped with o-rates, may be found in [8] or deduced from Theorem 1. Recall also the references to other authors in the introduction. It should further be mentioned that one could transform all theorems and results of this subsection for which the limiting random variable Z satisfies (6.1) into theorems dealing with strong convergence for the distribution functions, as carried out in Section 6. The weak law of large numbers is an exception since $Z = X_0$ does not fulfil (6.1).

C. Processes with dependent increments

This subsection is devoted to the behaviour of the process $\varphi(n)\, X_n = \varphi(n)\, Y_1 + \cdots + \varphi(n)\, Y_n$, described in Definition 4.

Theorem 13: *Let* (X_i) *be a Markovian process with dependent increments* (Y_i), $(Y_k, \mathfrak{G}_k)$ *being a sequence of couples with* $\mathfrak{G}_k := \mathfrak{A}(Y_1, \ldots, Y_{k-1})$. *Let Z be φ-decomposable with* $\mathrm{E}[Z] = 0$. *If furthermore* $\mathrm{E}[|Y_k|^r \mid \mathfrak{G}_k] < \infty$ *a.s. as well as* $\mathrm{E}[|Z_k|^r] < \infty$ *for* $k \in \mathrm{N}$ *and some* $r \in \mathrm{N} \setminus \{1\}$, *and*

$$\sum_{k=1}^{n} \{\mathrm{E}[Y_k{}^j \mid \mathfrak{G}_k] - \mathrm{E}[Z_k{}^j]\} = \mathcal{O}\left(\frac{\varphi(n)^r}{(r-1)!}\, M(n; \mathfrak{G}_k)\right)$$

$$(1 \leqq j \leqq r-1;\ n \to \infty),$$

then for each $f \in C$,

$$\|V_{\varphi(n)X_n} f - V_Z f\| \leqq 2c_{2,r} N_1 \omega_r \left(\left[\frac{\varphi(n)^r}{(r-1)!}\, M(n; \mathfrak{G}_k)\right]^{1/r}; f; C\right).$$

The proof follows by Lemma 8 and Theorem 2, as did Theorem 9 ∎

Remark: Concludingly it should be mentioned that it is also possible to formulate Theorem 13 particularly in the instance of independent increments. In this case all questions concerning dependence properties are superflous, and the $\mathfrak{G}_k$ may be chosen to be $\mathfrak{G}_k = \{\Omega, \varnothing\}$, all $k \in \mathrm{N}$. Preciser explanations can be found in [8].

REFERENCES

[1] Bauer, H.: Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie, 2. Aufl. Berlin: De Gruyter 1974.

[2] Bolthausen, E.: On the rates of convergence in a random central limit theorem and in the central limit theorem for Markoff chains. Z. Wahrsch. Verw. Gebiete **38** (1977), 279–286.

[3] Bradley, R. C.: Multilinear forms and measures of dependence between random variables. J. Multiv. Anal. **16** (1985), 335–367.

[4] Bradley, R. C.: On the central limit question under absolute regularity. Ann. Prob. **13** (1985), 1314–1325.

[5] Breimann, L.: Probability. Reading: Addison-Wesley 1968.

[6] Butzer, P. L., and H. Berens: Semi-groups of Operators and Approximation. Berlin: Springer-Verlag 1967.

[7] Butzer, P. L., and L. Hahn: General theorems on rates of convergence in distribution of random variables. I: General limit theorems. J. Multiv. Anal. 8 (1978), 181–201.

[8] Butzer, P. L., and H. Kirschfink: General limit theorems with o-rates and Markov processes under pseudo-moment conditions. Z. Anal. Anw. (in print).

[9] Butzer, P. L., Kirschfink, H., and D. Schulz: An extension of the Lindeberg-Trotter operator-theoretic approach to limit theorems for dependent random variables. II: Approximation theorems with $\mathcal{O}$-rates, applications to martingale difference arrays. Acta Sci. Math. (Szeged) **51** (1988) (in print).

[10] Butzer, P. L., Hahn, L., and M. T. Roeckerath: Central limit theorem and weak law of large numbers with rates for martingales in Banach spaces. J. Multiv. Anal. **13** (1983), 287–301.

[11] BUTZER, P. L., and D. SCHULZ: General Random Sum Limit Theorems for Martingales with Large $\mathcal{O}$-Rates. Z. Anal. Anw. 2 (1983), 97—109.
[12] BUTZER, P. L., and D. SCHULZ: The random martingale central limit theorem and weak law of large numbers with o-rates. Acta Sci. Math. (Szeged) 45 (1983), 81—94.
[13] CHUNG, K. L.: Lectures from Markov Processes to Brownian Motion. New York: Springer-Verlag 1982.
[14] DOOB, J. L.: Stochastic Processes, 3d ed. New York: John Wiley & Sons 1960.
[15] GAENSSLER, P., and W. STUTE: Wahrscheinlichkeitstheorie. Berlin: Springer-Verlag 1977.
[16] GORDIN, M. I., and B. A. LIFSHITZ: The central limit theorem for stationary Markov processes. Dokl. Akad. Nauk SSSR 4 (1978), 392—394.
[17] GUDYNAS, P.: Refinements of the central limit theorem for a homogeneous Markov chain. Lith. Math. J. 22 (1982), 36—45.
[18] GUDYNAS, P.: Approximation of distributions of sums of dependent random variables with values in Banach space. Lit. Mat. Sbornik 23 (1983) 3, 3—21.
[19] HEINRICH, L.: Über Wahrscheinlichkeiten großer Abweichungen von Summen Markovsch verbundener Zufallsvektoren bei Nichterfüllung der Cramérschen Bedingung. Math. Nachr. 108 (1982), 179—196.
[20] KARLIN, S., and H. M. TAYLOR: A First Course in Stochastic Processes, 2d ed. New York: Academic Press 1975.
[21] LAHA, R. G., and V. K. ROHATGI: Probability Theory. New York: John Wiley & Sons 1979.
[22] LANDERS, D., and L. ROGGE: On the rate of convergence in the central limit theorem for Markov chains. Z. Wahrsch. Verw. Gebiete 35 (1976), 57—63.
[23] LIFSHITS, B. A.: On the central limit theorem for Markov chains. Theor. Prob. Appl. 23 (1978), 279—296.
[24] LIFSHITS, B. A.: Invariance principle for weakly dependent variables. Theor. Prob. Appl. 24 (1985), 33—40.
[25] NAGAEV, S. V.: More exact statements of limit theorems for homogeneous Markov chains. Theor. Prob. Appl. 6 (1961), 62—81.
[26] O'BRIEN, G. L.: Limit theorems for sums of chain-dependent processes. J. Appl. Prob. 11 (1974), 582—587.
[27] PADITZ, L.: Über die Annäherung von Summenverteilungsfunktionen gegen unbegrenzt teilbare Verteilungsfunktionen in der Terminologie der Pseudomente. Wiss. Z. Techn. Univ. Dresden 27 (1978), 1129—1133.
[28] SAZONOV, V. V., and V. V. ULYANOV: On the accuracy of normal approximation. J. Multiv. Anal. 12 (1982), 371—384.
[29] SIRAZHDINOV, S. KH., and SH. K. FORMANOV: On estimates of the rate of convergence in the central limit theorem for homogeneous Markov chains. Theor. Prob. Appl. 28 (1985), 229—239.
[30] ZOLOTAREV, A. M.: Approximation of the distributions of sums of independent random variables with values in infinite dimensional spaces. Theor. Prob. Appl. 21 (1976), 721 to 737.
[31] ZOLOTAREV, V. M.: Probability metrics. Theor. Prob. Appl. 28 (1985), 278—302.
[32] ZOLOTAREV, V. M.: On pseudomoments. Theor. Prob. Appl. 23 (1978), 269—278.

Manuskripteingang: 20. 03. 1987

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