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On a Singular Part of an Unbounded Operator

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Es wird die Singularität eines Operators im Hilbertraum untersucht.

Исследуется сингулярность оператора в Гильбертовом пространстве.

The singularity of an operator in a Hilbert space is studied.

The concept of singularity of an operator is originally derived from the measure theory. We showed in [3] that a special operator, i.e., an unbounded derivation in operator algebras is always decomposed into the sum of a singular part and a normal part, and at the time JØRGENSEN [1] also proved that every unbounded operator in a Hilbert space has such a decomposition. Our method in [3] to obtain the decomposition theorem is different from Jørgensen's one. In this note we will make the concept of singularity of an unbounded operator in a Hilbert space more clear by using the idea of [3] and show some relations between singularity of an operator and its characteristic projection.

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A densely defined operator A in a Hilbert space $\mathcal{H}$ is said to be *singular* if, for each $x \in \mathcal{R}(A)$, there is a sequence $\{\xi_n\} \subset \mathcal{D}(A)$ such that $\xi_n \to 0$ and $A\xi_n \to x$ as $n \to \infty$. We denote the graph of A in the direct sum $\mathcal{H} \oplus \mathcal{H}$ by $G(A)$. Clearly, a non-trivial singular operator is not closable.

Examples: 1. *A singular operator* [4: p. 312]: Let $\{x_n\} \subset \mathcal{H}$ be an orthonormal basis and let $e \in \mathcal{H}$ be a vector which is not a finite linear combination of the x_n. Let $\mathcal{D}$ be the set of finite linear combinations of $\{x_n\}$ and e, and on $\mathcal{D}$ define an operator T by $T(\alpha e + \sum c_i x_i) = \alpha e$. Then T is singular with $\overline{G(T)} = \mathcal{H} \oplus \{e\}$; i.e., $\mathcal{D}(T^*) = \{e\}^\perp$.

2. *A non-closable but not singular operator*: Let T be the operator cited above in a Hilbert space $\mathcal{H}$, and let L be a bounded operator in $\mathcal{H}$ such that the restriction on $\mathcal{D}(T^*)$ does not vanish. Then the operator $T + L^*$ is non-closable but not singular.

Define

$$S_A = \{\xi \in \mathcal{H}: 0 \oplus \xi \in \overline{G(A)}\}.$$

It is then clear that the operator A is singular if and only if $S_A \supset \overline{\mathcal{R}(A)}$. If this is the case, S_A is closed and is just equal to $\overline{\mathcal{R}(A)}$. We define the *flip operator* on $\mathcal{H} \oplus \mathcal{H}$ by $V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Lemma 1: *For a densely defined operator A in $\mathscr{H}$, one has* $S_A = \mathscr{D}(A^*)^\perp$.

Proof: For $\xi \in \mathscr{D}(A^*)$, one has $-A^*\xi \oplus \xi = V(\xi \oplus A^*\xi)$. Since $VG(A^*) = G(A)^\perp$, it follows that $(-A^*\xi \oplus \xi, 0 \oplus \eta) = 0$ for all $\eta \in S_A$, so that $\mathscr{D}(A^*) \subset S_A{}^\perp$. On the other hand, take $\eta \in \mathscr{D}(A^*)^\perp$. It follows that $(-A^*\xi \oplus \xi, 0 \oplus \eta) = 0$ for all $\xi \in \mathscr{D}(A^*)$. Hence we have $0 \oplus \eta \in \big(VG(A^*)\big)^\perp = \big(G(A)^\perp\big)^\perp = \overline{G(A)}$. This means that $\eta \in S_A$ ∎

Theorem 2: *Let A be a densely defined linear operator in a Hilbert space $\mathscr{H}$. Then the following statements are equivalent:*

(i) *A is singular.*

(ii) $\mathscr{H} \oplus \mathscr{D}(A^*)^\perp = \overline{G(A)}$.

(iii) $A^* = 0|_{\mathscr{D}(A^*)}$, *that is*, Ker $(A^*) = \mathscr{D}(A^*)$.

Proof: Suppose A is singular and take $\eta \in \mathscr{D}(A^*)$. For each $\xi \in \mathscr{D}(A)$, there is a sequence $\{\xi_n\} \subset \mathscr{D}(A)$ such that $\xi_n \to 0$ and $A\xi_n \to A\xi$. Hence we have $(\xi, A^*\eta) = (A\xi, \eta) = \lim (A\xi_n, \eta) = \lim (\xi_n, A^*\eta) = 0$, the implication (i) $\Rightarrow$ (iii) follows. Suppose the statement (iii) holds. Since we have $\overline{G(A)} = VG(A^*)^\perp = \big(\{0\} \oplus \mathscr{D}(A^*)\big)^\perp = \mathscr{H} \oplus \mathscr{D}(A^*)^\perp$, the statement (ii) holds. The implication (ii) $\to$ (i) follows from the above Lemma ∎

A densely defined operator A in $\mathscr{H}$ is said to be *strict singular* if $\mathscr{D}(A^*) = \{0\}$. A strict singular operator is singular by Lemma 1 and it range is dense in $\mathscr{H}$. Conversely, it is easily seen that a singular operator with dense range is strict singular.

Example: Let T be a non-zero singular operator in $\mathscr{H}$ and let K be an isometry of $\mathscr{H}$ into $\mathscr{D}(T^*)^\perp$. Put $A = K^*T$. Then A is strict singular.

We next consider singularity of an operator in connection with quasi-affinity of unbounded operators.

Proposition 3: *Suppose there is a bounded linear operator X with dense range such that $XA \subset BX$. If A is singular (resp. strict singular), then B is singular (resp. strict singular).*

Proof: For any $\xi \in \mathscr{D}(B^*)$, one has $X^*B^*\xi = A^*X^*\xi = 0$, since $A^* = 0|_{\mathscr{D}(A^*)}$. It follows from the injectivity of X^* that $B^*\xi = 0$. Hence B is singular. Suppose now A is strict singular. Since $\mathscr{R}(A)$ is dense in $\mathscr{H}$, one has $\mathscr{R}(B) \supset B\big(X\mathscr{D}(A)\big) = XA\mathscr{D}(A)$, so that $\overline{\mathscr{R}(B)} \supset X\overline{\mathscr{R}(A)} = \mathscr{R}(X)$. Since $\mathscr{R}(X)$ is dense, it follows that B is singular with dense range. This means that B is strict singular ∎

Proposition 4 [1]: *Let A be a densely defined linear operator in a Hilbert space $\mathscr{H}$. Let p_A be the projection of $\mathscr{H}$ onto S_A. It follows that p_AA is singular with the adjoint domain equal to $\mathscr{D}(A^*)$ and $(I - p_A)A$ is closable.*

Proof: Since $(p_AA)^* = A^*p_A$, it follows from Lemma 1 that

$$\mathscr{D}\big((p_AA)^*\big) = \{x : p_Ax \in \mathscr{D}(A^*)\} = \{x : p_Ax = 0\} = S_A{}^\perp = \overline{\mathscr{D}(A^*)}.$$

This implies that p_AA is singular. Suppose $x_n \to 0$ and $(I - p_A)Ax_n \to x$. We have

$$(x, \xi) = \lim \big((I - p_A)Ax_n, \xi\big) = \lim (x_n, A^*\xi) = 0$$

for all $\xi \in \overline{\mathscr{D}(A^*)}$. Since $x \in \overline{\mathscr{D}(A^*)}$, it follows that $x = 0$ ∎

The above proposition shows that every densely defined operator A is decomposed into the sum of a singular operator and a closable operator by the projection p_A which is called *canonical projection*, and such a decomposition is called *canonical decompo-*

sition. In what follows, we denote the singular part (resp., the closable part) of A by A_s (resp. A_c). Of course, if A is closable, then $A = A_c$ and, moreover, if A is singular, then $A = A_s$.

Remarks: 1. The canonical projection p_A is the minimum among the projections q such that $(I - q)\,A$ is closable. In fact, for each $\xi \in \mathcal{H}$, there is a sequence $\{x_n\} \subset \mathcal{D}(A)$ such that $Ax_n \to p_A\xi$ with $x_n \to 0$ as $n \to \infty$. Since $(I - q)\,A$ is closable and $(I - q)\,Ax_n \to (I - q)\,p_A\xi$, it follows that $(I - q)\,p_A\xi = 0$, and so $p_A \leqq q$.

2. In general, it is known by the general theory of operators that $A^* \supset (A_c)^* + (A_s)^*$. In the canonical decomposition, however, it is easily checked that the equality in the above relation holds.

Corollary 1: *Let T be a densely defined operator in a Hilbert space $\mathcal{H}$. Then $\mathcal{D}(T^*)$ is closed if $(T_c)^*$ is bounded (with $\mathcal{D}\big((T_c)^*\big) = \mathcal{H}$).*

Proof: This follows from $\mathcal{D}(T^*) = \mathcal{D}\big((T_c)^*\big) \cap \mathcal{D}\big((T_s)^*\big)$, and $\overline{\mathcal{D}(T^*)} = \mathcal{D}\big((T_s)^*\big)$ ∎

Corollary 2: *Let T be an everywhere defined unbounded operator on a Hilbert space. Then $\mathcal{D}(T^*)$ is closed and T^* is continuous on $\mathcal{D}(T^*)$.*

Proof: This follows from Remark 2 and Corollary 1 ∎

Let A and B be operators in a Hilbert space $\mathcal{H}$. If there is a unitary operator U on $\mathcal{H}$ such that $UA = BU$ (that is, $U\mathcal{D}(A) = \mathcal{D}(B)$ and $UA\xi = BU\xi$ for all $\xi \in \mathcal{D}(A)$), then we say that A is *unitarily equivalent* to B with intertwining operator U.

Theorem 5: *Let A and B be densely defined linear operators in a Hilbert space $\mathcal{H}$. Suppose A is unitarily equivalent to B. Then the canonical projections p_A and p_B are unitarily equivalent.*

Proof: Let U be an inertwining operator to realize the unitarily equivalence between A and B. It follows that

$$B = U\big((I - p_A)\,A + p_AA\big)\,U^{-1} = U(I - p_A)\,U^{-1}B + Up_AU^{-1}B.$$

Since $(I - p_A)\,A$ is closable, it follows that $U(I - p_A)\,AU^{-1} = U(I - p_A)\,U^{-1}B$ is closable. Hence, by Remark 1, we have $Up_AU^{-1} \geqq p_B$. By applying the same argument for U^{-1}, we have $U^{-1}p_BU \geqq p_A$. This implies the theorem ∎

Let T be a densely defined linear operator in a Hilbert space $\mathcal{H}$. We write Q_T for the projection of $\mathcal{H} \oplus \mathcal{H}$ onto $\overline{G(T)}$, which is called the *characteristic projection* of T. Following [5] (also see [2]), Q_T is expressed in terms of a 2×2-matrix $Q_T = (q_{ij})$ of bounded linear operators on $\mathcal{H}$ as

$$Q_T = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \quad \text{with} \quad (q_{ij})^* = q_{ji} \quad \text{and} \quad \sum q_{ik}q_{kj} = q_{ij}.$$

In particular, $q_{12} = T^*(I - q_{22})$ and $I - q_{11} = T^*q_{21}$.

Theorem 6: *Let T be a densely defined linear operator in a Hilbert space $\mathcal{H}$ with characteristic projection $Q_T = (q_{ij})$. Then T is singular if and only if $q_{22}T = T$. If this is the case, $q_{22} = p_T$ (the canonical projection) and $Q_T = \begin{pmatrix} I & 0 \\ 0 & q_{22} \end{pmatrix}$.*

Proof: Suppose T is singular. Since $S_T \supset \mathcal{R}(T)$, it follows that $Q_T(0 \oplus T\xi) = 0 \oplus T\xi$ for all $\xi \in \mathcal{D}(T)$. This means that $q_{12}T\xi = 0$ and $q_{22}T\xi = T\xi$ for all $\xi \in \mathcal{D}(T)$. Hence $q_{22}T = T$. The converse is almost clear by the relation Ker $(I - q_{22})$

$\subset \operatorname{Ker}(q_{12})$. Moreover, it follows that $q_{12} = (T - q_{22}T)^* = 0$ and $q_{11} = I - T^*(q_{12})^* = I - (q_{12}T)^* = I$. Furthermore $q_{21} = (q_{12})^* = 0$. It follows from Theorem 4 that $q_{22} = p_T$ ∎

Corollary: *Keeping the same notation as above, T is strict singular if and only if the characteristic projection Q_T of T is the identity operator; that is, $q_{22} = I$.*

Proof: This follows from Theorems 2 and 6 ∎

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