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On $\mathscr{B}_{p,k}$-Boundedness and Compactness of Linear Pseudo-Differential Operators

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Es werden Kriterien der Beschränktheit und Kompaktheit linearer Pseudodifferentialoperatoren $L(X, D)$ in den Hörmander-Räumen $\mathscr{B}_{p,k}$ erörtert. Gewisse Bedingungen an das Symbol $L(x, \xi)$ von $L(X, D)$ sollen dabei garantieren, daß $L(X, D)$ den Schwartz-Raum $\mathscr{S}$ in sich abbildet und die formal Transponierte $L'(X, D)$: $\mathscr{S} \to \mathscr{S}$ existiert. Eine Charakterisierung der Beschränktheit des Operators $L'(X, D)$: $\mathscr{B}_{1,kk^\sim} \to \mathscr{B}_{1,k}$ wird hergeleitet, desgleichen eine hinreichende Bedingung für die Beschränktheit des Operators $L'(X, D)$: $\mathscr{B}_{p,kk^\sim} \to \mathscr{B}_{p,k}$ mit $p \in [1, \infty)$. Schließlich wird die Kompaktheit der stetigen Erweiterung $L'(X, D)$: $\mathscr{B}_{p,kk^\sim}(G) \to \mathscr{B}_{p,k}$ erörtert, wobei G eine offene beschränkte Menge des $\mathbf{R}^n$ und $\mathscr{B}_{p,kk^\sim}(G)$ die Vervollständigung von $C_0^\infty(G)$ bezüglich der $\mathscr{B}_{p,kk^\sim}$-Norm ist.

Обсуждаются критерии ограниченности и компактности линейных псевдодифференциальных операторов $L(X, D)$ в пространствах Хёрмандера $\mathscr{B}_{p,k}$. При этом некоторые условия на символ $L(x, \xi)$ от $L(X, D)$ должны обеспечивать чтобы $L(X, D)$ отображал пространство Шварца $\mathscr{S}$ в себя и чтобы существовал формально сопряженное $L'(X, D)$: $\mathscr{S} \to \mathscr{S}$. Выводятся характеризация ограниченности оператора $L'(X, D)$: $\mathscr{B}_{1,kk^\sim} \to \mathscr{B}_{1,k}$ и достаточное условие ограниченности оператора $L'(X, D)$: $\mathscr{B}_{p,kk^\sim} \to \mathscr{B}_{p,k}$ с $p \in [1, \infty)$. Наконец, обсуждается компактность непрерывного расширения $L'(X, D)$: $\mathscr{B}_{p,kk^\sim}(G) \to \mathscr{B}_{p,k}$, где G открытое ограниченное множество в $\mathbf{R}^n$ и $\mathscr{B}_{p,kk^\sim}(G)$ дополнение $C_0^\infty(G)$ относительно $\mathscr{B}_{p,kk^\sim}$-нормы.

Boundedness and compactness arguments in the Hörmander spaces $\mathscr{B}_{p,k}$ for linear pseudo-differential operators $L(X, D)$ are considered. The symbol $L(x, \xi)$ of $L(X, D)$ is assumed to obey appropriate temperate criteria, which guarantee that $L(X, D)$ maps the Schwartz class $\mathscr{S}$ into itself and that the formal transpose $L'(X, D)$: $\mathscr{S} \to \mathscr{S}$ exists. A characterization for the boundedness of the operator $L'(X, D)$: $\mathscr{B}_{1,kk^\sim} \to \mathscr{B}_{1,k}$ is obtained. A sufficient condition for the boundedness of the operator $L'(X, D)$: $\mathscr{B}_{p,kk^\sim} \to \mathscr{B}_{p,k}$ with $p \in [1, \infty)$ is established as well. Finally, the compactness of the continuous extension of $L'(X, D)$: $\mathscr{B}_{p,kk^\sim}(G) \to \mathscr{B}_{p,k}$ is studied, where G is an open bounded set in $\mathbf{R}^n$ and where $\mathscr{B}_{p,kk^\sim}(G)$ is (essentially) the completion of $C_0^\infty(G)$ with respect to the $\mathscr{B}_{p,kk^\sim}$-norm.

1. Introduction

Let $m, \varrho, \delta \in \mathbf{R}$ be such that $0 < \varrho \leq 1$ and $0 \leq \delta < 1$. Define the class $S^m_{\varrho,\delta}$ of $C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$-mappings $L(\cdot, \cdot)$ via the requirement: $L(\cdot, \cdot)$ lies in $S^m_{\varrho,\delta}$ if and only if the estimate

$$|D_x^\alpha D_\xi^\beta L(x, \xi)| \leqq C_{\alpha,\beta}(1 + |\xi|)^{m+\delta|\alpha|-\varrho|\beta|} \quad (x, \xi \in \mathbf{R}^n; \alpha, \beta \in \mathbf{N}_0^n)$$

is valid. It holds a very extensive theory concerning the L^p-boundedness of respective pseudo-differential operators $L(x, D)$, that is, linear operators defined by

$$\big(L(X, D) \varphi\big)(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} L(x, \xi) (\mathscr{F}\varphi)(\xi)\, e^{i\langle \xi, x\rangle}\, d\xi, \tag{1.1}$$

where $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is the Fourier transform in the Schwartz space $\mathcal{S}$ and where $L(\cdot, \cdot) \in S^m_{\varrho,\delta}$. For the results and for their generalizations we refer to [1, 3–6].

Let $\mathcal{K}$ be the class of weight functions given in [2: 34] and let $p \in [1, \infty)$. Define a norm $\|\cdot\|_{p,k}$ by

$$\|\Phi\|_{p,k} \doteq \left(\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} |(\mathcal{F}\Phi)(\xi)\, k(\xi)|^p \, d\xi\right)^{1/p} \quad (\Phi \in \mathcal{S}).$$

Choose weight functions $k, k^{\sim} \in \mathcal{K}$. In this contribution we consider the validity of the inequality

$$\|L'(X, D)\, \varphi\|_{p,k} \leqq C\, \|\varphi\|_{p,kk^{\sim}} \qquad (\varphi \in \mathcal{S}), \tag{1.2}$$

where $L'(X, D): \mathcal{S} \to \mathcal{S}$ is the formally transpose operator of a linear pseudo-differential operator $L(X, D): \mathcal{S} \to \mathcal{S}$. The validity of (1.2) means that $L'(X, D)$ has a bounded extension from $\mathcal{B}_{p,kk^{\sim}}$ into $\mathcal{B}_{p,k}$, where these Banach spaces are defined as in [2: 36]. We consider also some arguments for the compactness of the appropriate continuous extensions of $L'(X, D)$.

For the first instance we establish an (algebraic) criterion, which guarantees the fact that $L(X, D)$ maps $\mathcal{S}$ into itself. In addition, a condition for the existence of the formal transpose $L'(X, D): \mathcal{S} \to \mathcal{S}$ of $L(X, D)$ is given (cf. Theorems 2.1 and 2.2). After that a characterization for the validity of (1.2) with $p = 1$ is shown for a certain class of operators (cf. Theorem 3.3). The validity of (1.2) with general $p \in [1, \infty)$ is considered as well. Furthermore, the compactness of the continuous extension $\bar{L}_p': \mathcal{B}_{p,kk^{\sim}}(G) \to \mathcal{B}_{p,k}$ of $L'(X, D)$ is investigated, when G is an open bounded set in $\mathbf{R}^n$. Here $\mathcal{B}_{p,kk^{\sim}}(G)$ is (essentially) the completion of $C_0^\infty(G)$ with respect to the $\|\cdot\|_{p,kk^{\sim}}$-norm.

2. Preliminaries

2.1 For the unexplaned notions about the distribution theory we refer to the monograph [2: 1–25]. Let G be an open set in $\mathbf{R}^n$. The class $\mathcal{K}$ of weight functions k, the Banach spaces $\mathcal{B}_{p,k}$ and the Frechet spaces $\mathcal{B}^{\text{loc}}_{p,k}(G)$ with $p \in [1, \infty)$ are defined as in [2: 34–45]. The norm in $\mathcal{B}_{p,k}$ is then given by

$$\|u\|_{p,k} = \left(\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} |\mathcal{F}(u)(\xi)\, k(\xi)|^p \, d\xi\right)^{1/p}.$$

The topology in $\mathcal{B}^{\text{loc}}_{p,k}(G)$ is defined by the semi-norms $u \to \|\psi u\|_{p,k}$, $\psi \in C_0^\infty(G)$. Let $\mathcal{B}^{\wedge}_{p,k}(G)$ be the completion of $C_0^\infty(G)$ with respect to the $\|\cdot\|_{p,k}$-norm. Then $\mathcal{B}^{\wedge}_{p,k}(G)$ can be imbedded into $\mathcal{B}_{p,k}$ via the injection $\mathcal{J}: \mathcal{B}^{\wedge}_{p,k}(G) \to \mathcal{B}_{p,k}$ given by $\mathcal{J}(T)(\varphi) \doteq \lim \varphi_n(\varphi)$ for $\varphi \in C_0^\infty$, where $\{\varphi_n\}$ is a representative of T. Denote $\mathcal{B}_{p,k}(G) = \mathcal{J}(\mathcal{B}^{\wedge}_{p,k}(G))$. The norm in $\mathcal{B}_{p,k}(G)$ is defined by $\|\mathcal{J}(T)\|_{p,k} = \|T\|_{p,k}$.

Let $L(X, D)$ be a linear pseudo-differential operator on the Schwartz class $\mathcal{S}$ defined by the relation ($x \in G$)

$$\big(L(X, D)\, \varphi\big)(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} L(x, \xi)\, (\mathcal{F}\varphi)(\xi)\, e^{i\langle \xi, x\rangle} d\xi. \tag{2.1}$$

Here $L(\cdot, \cdot) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$. In the sequel we give a condition for $L(\cdot, \cdot)$, under which $L(X, D)$ maps $\mathcal{S}$ into itself.

Theorem 2.1: *Suppose that there exists a number $\varrho \in \mathbf{R}$ with $0 \leqq \varrho < 1$ such that for each pair $(\alpha, \beta) \in \mathbf{N}_0^n \times \mathbf{N}_0^n$ one can find constants $C_{\alpha,\beta} > 0$, $N_\alpha \in \mathbf{R}$ and $N_{\alpha,\beta} \in \mathbf{R}$ with which*

$$|D_x^\alpha D_\xi^\beta L(x, \xi)| \leqq C_{\alpha,\beta} k_{N_\alpha + \varrho|\beta|}(x)\, k_{N_{\alpha,\beta}}(\xi) \quad \big((x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n\big),$$

where k_s $(s \in \mathbf{R})$ is defined by $k_s(\xi) = (1 + |\xi|^2)^{s/2}$. Then $L(X, D)$ given by (2.1) is an operator from $\mathcal{S}$ into $\mathcal{S}$.

Proof: Let $\varphi \in \mathcal{S}$. By applying the Lebesgue Dominated Convergence Theorem one sees easily that $L(X, D)\,\varphi \in C^\infty(\mathbf{R}^n)$ and that

$$D_x^\alpha\big(L(X, D)\,\varphi\big)(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} D_x^\alpha\big(L(x, \xi)\, e^{i(\xi,x)}\big)\, (\mathcal{F}\varphi)(\xi)\, d\xi.$$

Furthermore, for every $x \in \mathbf{R}^n$ and $\gamma, \tau \in \mathbf{N}_0^n$ one has

$$\begin{aligned}
&(2\pi)^n \left|x^\gamma D_x^\tau\big(L(X, D)\,\varphi\big)(x)\right| \\
&= \left|\int_{\mathbf{R}^n} x^\gamma D_x^\tau\big(L(x, \xi)\, e^{i(\xi,x)}\big)\, (\mathcal{F}\varphi)(\xi)\, d\xi\right| \\
&\leqq \sum_{\beta \leqq \tau} \binom{\tau}{\beta} \left|\int_{\mathbf{R}^n} x^\gamma \big(D_x^\beta L(x, \xi)\big)\, \xi^{\tau-\beta} (\mathcal{F}\varphi)(\xi)\, e^{i(\xi,x)}\, d\xi\right| \\
&\leqq \sum_{\beta \leqq \tau} \binom{\tau}{\beta} \sum_{\alpha \leqq \gamma} \binom{\gamma}{\alpha} \left|\int_{\mathbf{R}^n} \big((-D_\xi)^\alpha D_x^\beta L(x, \xi)\big)\, D_\xi^{\gamma-\alpha} \mathcal{F}(D^{\tau-\beta}\varphi)(\xi)\, e^{i(\xi,x)}\, d\xi\right| \\
&\leqq \sum_{\beta \leqq \tau} \sum_{\alpha \leqq \gamma} \binom{\tau}{\beta} \binom{\gamma}{\alpha} \int_{\mathbf{R}^n} |D_\xi^\alpha D_x^\beta L(x, \xi)|\, |\mathcal{F}(x^{\gamma-\alpha} D^{\tau-\beta}\varphi)(\xi)|\, d\xi \\
&\leqq \sum_{\beta \leqq \tau} \sum_{\alpha \leqq \gamma} \binom{\tau}{\beta} \binom{\gamma}{\alpha} C_{\beta,\alpha} k_{N_\beta + \varrho|\alpha|}(x) \int_{\mathbf{R}^n} |\mathcal{F}(x^{\gamma-\alpha} D^{\tau-\beta}\varphi)(\xi)\, k_{N_{\beta,\alpha}}(\xi)|\, d\xi.
\end{aligned}$$

Let now γ and τ be fixed. Then we obtain

$$\left|x^\alpha x^\gamma D_x^\tau\big(L(X, D)\,\varphi\big)(x)\right| \leqq C_{\alpha,\gamma,\tau,\varphi} k_{N(\tau) + \varrho(|\alpha| + |\gamma|)}(x),$$

where $N(\tau) = \max\{N_\beta : \beta \leqq \tau\}$. Hence for every $N \in \mathbf{N}$ one can find a constant $C_N > 0$ such that

$$\left|x^\gamma D_x^\tau\big(L(X, D)\,\varphi\big)(x)\right| \leqq C_N k_{N(\tau) + \varrho|\gamma| + (\varrho - 1)N}(x).$$

By choosing N large enough we see that $\sup\left\{\left|x^\gamma D_x^\tau\big(L(X, D)\,\varphi\big)(x)\right| : x \in \mathbf{R}^n\right\} < \infty$, which completes the proof ∎

Suppose that $L(\cdot) \in C^\infty(\mathbf{R}^n)$ and that $|D_\xi^\beta L(\xi)| \leqq C_\beta k_{N_\beta}(\xi)$ $(\xi \in \mathbf{R}^n)$. Then by Theorem 2.1 the operator $L(D)$ defined by

$$\big(L(D)\,\varphi\big)(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} L(\xi)\, (\mathcal{F}\varphi)(\xi)\, e^{i(\xi,x)}\, d\xi$$

maps $\mathcal{S}$ into itself.

2.2 We say that $L'(X, D): \mathcal{S} \to \mathcal{S}$ is a *formally transpose operator* (or a *formal transpose*) of the operator $L(X, D): \mathcal{S} \to \mathcal{S}$ when

$$\big(L(X, D)\varphi\big)(\psi) := \int_{\mathbf{R}^n} \big(L(X, D)\varphi\big)(x)\,\psi(x)\,dx = \varphi\big(L'(X, D)\psi\big)$$

holds for all $\varphi, \psi \in \mathcal{S}$. A sufficient criterion for the existence of $L'(X, D)$ is given in the following

Theorem 2.2: *Suppose that the operator $L(X, D)$ defined by (2.1) maps $\mathcal{S}$ into itself and that one can find a number $\delta \in \mathbf{R}$ with $0 \leqq \delta < 1$ such that for each pair $(\alpha, \beta) \in \mathbf{N}_0^n \times \mathbf{N}_0^n$ there exist constants $N_\beta \in \mathbf{R}$ and $N_{\alpha,\beta} \in \mathbf{R}$ with which for all $x, \xi \in \mathbf{R}^n$*

$$|D_x^\alpha D_\xi^\beta L(x, \xi)| \leqq C_{\alpha,\beta} k_{N_\beta + \delta|\alpha|}(\xi)\, k_{N_{\alpha,\beta}}(x). \tag{2.2}$$

Then there exists the formal transpose $L'(X, D)$ of $L(X, D)$.

Proof: Let $\psi \in \mathcal{S}$. By changing the roles of x and ξ one sees in virtue of (2.2) and due to the proof of Theorem 2.1 that $L(D, \zeta)\psi \in \mathcal{S}$, where

$$\big(L(D, \zeta)\psi\big)(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} L(x, \xi)\,(\mathcal{F}\psi)(x)\,e^{i(x,\xi)}\,dx.$$

For all $\varphi, \psi \in \mathcal{S}$ we obtain by the Fubini's Theorem and by the Parseval's identity (here we denote $\check{\psi}(x) = \psi(-x)$)

$$\begin{aligned}
\big(L(X, D)\varphi\big)(\psi) &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} L(x, \xi)\,(\mathcal{F}\varphi)(\xi)\,e^{i(\xi,x)}\,d\xi \right) \psi(x)\,dx \\
&= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} (\mathcal{F}\varphi)(\xi) \left(\int_{\mathbf{R}^n} L(x, \xi)\,\psi(x)\,e^{i(\xi,x)}\,dx \right) d\xi \\
&= \int_{\mathbf{R}^n} (\mathcal{F}\varphi)(\xi)\,\big(L(D, \zeta)(\mathcal{F}\check{\psi})\big)(\xi)\,d\xi \\
&= \int_{\mathbf{R}^n} \varphi(y)\,\mathcal{F}\big(L(D, \zeta)(\mathcal{F}\check{\psi})\big)(y)\,dy = \varphi\big(\mathcal{F}\big(L(D, \zeta)(\mathcal{F}\check{\psi})\big)\big).
\end{aligned}$$

The operations are legitimate since the function

$$(x, \xi) \to |L(x, \xi)\,(\mathcal{F}\varphi)(\xi)\,\psi(x)\,e^{i(\xi,x)}|$$

is by (2.2) integrable in $\mathbf{R}^n \times \mathbf{R}^n$. Hence there exists $L'(X, D): \mathcal{S} \to \mathcal{S}$ and $L'(X, D)\psi = \mathcal{F}\big(L(D, \zeta)(\mathcal{F}\check{\psi})\big)$ ∎

The existence of $L'(X, D)$ implies that $L(X, D)$ and $L'(X, D)$ have continuous extensions from the dual space $\mathcal{S}'$ of $\mathcal{S}$ into $\mathcal{S}'$ (here $\mathcal{S}'$ is equipped with the weak dual topology). Our aim in the next Chapters 3—4 is to seek criterions under which $L'(X, D)$ (and $L(X, D)$) has a continuous extension from $\mathcal{B}_{p,k\tilde{k}}$ into $\mathcal{B}_{p,k}$ and from $\mathcal{B}_{p,k\tilde{k}}$ into $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)$.

3. Characterization of boundedness in spaces $\mathcal{B}_{1,k}$

3.1 In the sequel we consider the validity of the following inequality: There exists a constant $C > 0$ such that

$$\|L'(X, D)\varphi\|_{1,k} \leqq C\,\|\varphi\|_{1,k\tilde{k}} \qquad (\varphi \in \mathcal{S}). \tag{3.1}$$

Here k and $k^{\sim}$ are weight functions belonging to $\mathscr{K}$.

Lemma 3.1: *Suppose that one can find numbers $\delta, \varrho \in \mathbf{R}$ with $0 \leqq \delta, \varrho < 1$ such that for each pair $(\alpha, \beta) \in \mathbf{N}_0^n \times \mathbf{N}_0^n$ there exist constants $C_{\alpha,\beta} > 0$, $N_\alpha \in \mathbf{R}$ and $N_\beta \in \mathbf{R}$ with which*

$$|D_x^\alpha D_\xi^\beta L(x, \xi)| \leqq C_{\alpha,\beta} k_{N_\alpha + \varrho|\beta|}(x)\, k_{N_\beta + \delta|\alpha|}(\xi) \qquad (x, \xi \in \mathbf{R}^n). \tag{3.2}$$

Then for each $\varphi \in \mathscr{S}$

$$\mathscr{F}\big(L'(X, D)\, \varphi\big)\,(\eta) = \mathscr{F}\big(\varphi L(x, -\eta)\big)\,(\eta) \qquad (\eta \in \mathbf{R}^n). \tag{3.3}$$

Proof: In virtue of Theorems 2.1 and 2.2, $L(X, D)$ maps $\mathscr{S}$ into itself and there exists the formal transpose $L'(X, D)\colon \mathscr{S} \to \mathscr{S}$ of $L(X, D)$. Let $0 \leqq \Theta \in C_0^\infty$, $(\mathscr{F}\Theta)\,(0) = 1$. Define functions Θ_j' through the relation $\Theta_j'(x) = j^n \Theta(jx)$. Choose $\Theta_j \in \mathscr{S}$, $\mathscr{F}\Theta_j = \Theta_j'$. Then for each $x, \eta \in \mathbf{R}^n$

$$\begin{aligned}(2\pi)^n \big(L(X, D)\,(\Theta_j\, \mathrm{e}^{-\mathrm{i}(\eta,x)})\big)\,(x) &= \int_{\mathbf{R}^n} L(x, \xi)\, \Theta_j'(\xi + \eta)\, \mathrm{e}^{\mathrm{i}(\xi,x)}\, d\xi \\ &= \int_{\mathbf{R}^n} L(x, -\eta + \gamma)\, \Theta_j'(\gamma)\, \mathrm{e}^{\mathrm{i}(-\eta+\gamma,x)}\, d\gamma \\ &= \int_{\mathbf{R}^n} L\left(x, -\eta + \frac{\tau}{j}\right) \Theta(\tau)\, \mathrm{e}^{\mathrm{i}\left(-\eta+\frac{\tau}{j},x\right)}\, d\tau.\end{aligned}$$

In virtue of (3.2) for all $j \in \mathbf{N}$ one has (with some $N > 0$)

$$\begin{aligned}&\left| L\left(x, -\eta + \frac{\tau}{j}\right) \Theta(\tau)\, \mathrm{e}^{\mathrm{i}\left(-\eta+\frac{\tau}{j},x\right)} \right| \\ &\leqq C_{0,0} k_N(x)\, k_N\left(-\eta + \frac{\tau}{j}\right) |\Theta(\tau)| \leqq C' k_N(x)\, k_N(-\eta)\, k_N(\tau)\, |\Theta(\tau)|,\end{aligned} \tag{3.4}$$

where the right-hand side is integrable in $\mathbf{R}^n$. Hence due to the Lebesgue Dominated Convergence Theorem we see that for all $x \in \mathbf{R}^n$

$$\begin{aligned}&\big(L(X, D)\,(\Theta_j\, \mathrm{e}^{-\mathrm{i}(\eta,x)})\big)\,(x) \\ &\to L(x, -\eta)\, \mathrm{e}^{-\mathrm{i}(\eta,x)} \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \Theta(\tau)\, d\tau = \frac{1}{(2\pi)^n} L(x, -\eta)\, \mathrm{e}^{-\mathrm{i}(\eta,x)}.\end{aligned}$$

Furthermore, in view of (3.4) one has

$$\begin{aligned}&\left|\big(L(X, D)\,(\Theta_j\, \mathrm{e}^{-\mathrm{i}(\eta,x)})\big)\,(x)\right| \\ &\leqq C' k_N(-\eta)\, k_N(x) \int_{\mathbf{R}^n} k_{N_0}(\tau)\, |\Theta(\tau)|\, d\tau = C'' k_N(-\eta)\, k_N(x).\end{aligned}$$

Hence with each $\eta \in \mathbf{R}^n$ we obtain

$$\varphi\big(L(X, D)\,(\Theta_j\, \mathrm{e}^{-\mathrm{i}(\eta,x)})\big) \to \frac{1}{(2\pi)^n}\, \varphi\big(L(\cdot, -\eta)\, \mathrm{e}^{-\mathrm{i}(\eta,\cdot)}\big). \tag{3.5}$$

In according to the definition of Θ_j one sees

$$(2\pi)^n\, \Theta_j(x) = \int_{\mathbf{R}^n} j^n \Theta(jy)\, \mathrm{e}^{\mathrm{i}(x,y)}\, dy = \int_{\mathbf{R}^n} \Theta(z)\, \mathrm{e}^{\mathrm{i}(x,z/j)}\, dz \to (\mathscr{F}\Theta)\,(0) = 1$$

and

$$(2\pi)^n |\Theta_j(x)| \leqq \int_{\mathbf{R}^n} |\Theta(z)| \, dz = 1$$

so that we get the convergence

$$\big(L'(X, D)\varphi\big)(\Theta_j e^{-i(\eta,\cdot)}) \to \frac{1}{(2\pi)^n} \big(L'(X, D)\varphi\big)(e^{-i(\eta,\cdot)})$$

$$= \frac{1}{(2\pi)^n} \mathcal{F}\big(L'(X, D)\varphi\big)(\eta). \tag{3.6}$$

Since for all $j \in \mathbf{N}$ and $\eta \in \mathbf{R}^n$ the equality

$$\big(L'(X, D)\varphi\big)(\Theta_j e^{-i(\eta,\cdot)}) = \varphi\big(L(X, D)(\Theta_j e^{-i(\eta,\cdot)})\big)$$

holds, we get by (3.5) and by (3.6) that

$$\mathcal{F}\big(L'(X, D)\varphi\big)(\eta) = \varphi\big(L(\cdot, -\eta) e^{-i(\eta,\cdot)}\big) = \mathcal{F}\big(\varphi L(\cdot, -\eta)\big)(\eta) \quad \blacksquare$$

Let $\Phi \in C_0^\infty$, $0 \leqq \Phi \leqq 1$ and $\Phi(x) = 1$ for $x \in \overline{B(0, 1)}$. Define functions $\Phi_l \in C_0^\infty$ by $\Phi_l(x) = \Phi(x/l)$. Then we obtain

Theorem 3.2: *Suppose that one can find numbers $\delta, \varrho \in \mathbf{R}$ with $0 \leqq \delta, \varrho < 1$ such that for each pair $(\alpha, \beta) \in \mathbf{N}_0^n \times \mathbf{N}_0^n$ there exist constants $C_{\alpha,\beta} > 0$, $N_\alpha \in \mathbf{R}$ and $N_\beta \in \mathbf{R}$ with which the inequality (3.2) holds. Furthermore, we suppose that there exists a constant $M > 0$ such that for all $l \in \mathbf{N}$ and $\xi \in \mathbf{R}^n$*

$$\int_{\mathbf{R}^n} \big|\mathcal{F}\big(\Phi_l L(\cdot, -\eta)\big)(\eta - \xi) k(\eta)\big| \, d\eta \leqq M(kk^\sim)(\xi). \tag{3.7}$$

Then one can find a constant $C > 0$ such that

$$\|L'(X, D)\varphi\|_{1,k} \leqq C \|\varphi\|_{1,kk^\sim} \qquad (\varphi \in \mathcal{S}).$$

Proof: In virtue of Lemma 3.1 we get for all $\varphi \in \mathcal{S}$, $l \in \mathbf{N}$ and $\eta \in \mathbf{R}^n$

$$\big|\mathcal{F}\big(L'(X, D)(\Phi_l\varphi)\big)(\eta)\big| = \Big|\int_{\mathbf{R}^n} \varphi(x) \Phi_l(x) L(x, -\eta) e^{-i(\eta,x)} \, dx\Big|$$

$$= \int_{\mathbf{R}^n} (\mathcal{F}\varphi)(\xi) \mathcal{F}\big(\overline{\Phi_l L(\cdot, -\eta)} e^{i(\eta,x)}\big)(\xi) \, d\xi$$

$$= \int_{\mathbf{R}^n} |(\mathcal{F}\varphi)(\xi) \mathcal{F}\big(\Phi_l L(\cdot, -\eta)\big)(\eta - \xi)| \, d\xi.$$

Define functions $f_{l,j}\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by

$$f_{l,j}(\xi, \eta) = |\mathcal{F}\big(\Phi_l L(\cdot, -\eta)\big)(\eta - \xi)(\mathcal{F}\Theta_j')(\eta)| \, k(\eta).$$

Then $f_{l,j}$ is by (3.2) continuous. In addition, due to (3.7) and the inequality

$$|(\mathcal{F}\Theta_j')(\eta)| = \Big|j^n \int_{\mathbf{R}^n} \Theta(jx) e^{-i(x,\eta)} \, dx\Big|$$

$$= \Big|\int_{\mathbf{R}^n} \Theta(z) e^{-i(z/j,\eta)} \, dz\Big| \leqq \int_{\mathbf{R}^n} |\Theta(z)| \, dz = 1,$$

we obtain that

$$\int_{\mathbf{R}^n} f_{l,j}(\xi, \eta) \, d\eta \leqq \Big(\int_{\mathbf{R}^n} |\Theta(z)| \, dz\Big) M(kk^\sim)(\xi) = M(kk^\sim)(\xi)$$

for all $\xi \in \mathbf{R}^n$. Furthermore, the function

$$g_{l,j}: \mathbf{R}^n \to \mathbf{R}, \qquad g_{l,j}(\xi) = \int_{\mathbf{R}^n} f_{l,j}(\xi, \eta)\, d\eta$$

is continuous and

$$\int_{\mathbf{R}^n} g_{l,j}(\xi)\, |(\mathcal{F}\varphi)(\xi)|\, d\xi \leqq (2\pi)^n M \|\varphi\|_{1,kk^\sim}.$$

Hence due to the Fubini's Theorem

$$\begin{aligned}
&(2\pi)^n \left\|\big(L'(X, D)(\Phi_l\varphi)\big) * \Theta_j'\right\|_{1,k} \\
&= \int_{\mathbf{R}^n} \left|\mathcal{F}\big(L'(X, D)(\Phi_l\varphi)\big)(\eta)\,(\mathcal{F}\Theta_j')(\eta)\right| k(\eta)\, d\eta \\
&= \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} \left|(\mathcal{F}\varphi)(\xi)\, \mathcal{F}\big(\Phi_l L(\cdot, -\eta)\big)(\eta - \xi)\right| d\xi\right) |\mathcal{F}(\Theta_j')(\eta)|\, k(\eta)\, d\eta \\
&= \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} \left|\mathcal{F}\big(\Phi_l L(\cdot, -\eta)\big)(\eta - \xi)\,(\mathcal{F}\Theta_j')(\eta)\, k(\eta)\right| d\eta\right) |(\mathcal{F}\varphi)(\xi)|\, d\xi \\
&\leqq M \|\varphi\|_{1,kk^\sim}. \qquad (3.8)
\end{aligned}$$

Since $\big(L'(X, D)(\Phi_l\varphi)\big) * \Theta_j' \to L'(X, D)(\Phi_l\varphi)$ in $\mathcal{B}_{1,k}$ as $j \to \infty$ (cf. [2: 42]) we obtain the assertion ∎

3.2 In this subsection we characterize the validity of the inequality (3.7) with the validity of the inequality (3.1).

Theorem 3.3: *Suppose that one can find numbers $\delta, \varrho \in \mathbf{R}$ with $0 \leqq \delta, \varrho < 1$ such that for each pair $(\alpha, \beta) \in \mathbf{N}_0^n \times \mathbf{N}_0^n$ there exist constants $C_{\alpha,\beta} > 0$, $N_\alpha \in \mathbf{R}$ and $N_\beta \in \mathbf{R}$ with which the inequality (3.2) holds. Then one can find a constant $C > 0$ such that*

(i) $\qquad \|L'(X, D)\varphi\|_{1,k} \leqq C \|\varphi\|_{1,kk^\sim} \qquad (\varphi \in \mathcal{S})$

if and only if one can find a constant $M > 0$ such that

(ii) $\qquad \int_{\mathbf{R}^n} \left|\mathcal{F}\big(\Phi_l L(\cdot, -\eta)\big)(\eta - \xi)\, k(\eta)\right| d\eta \leqq M(kk^\sim)(\xi) \qquad (l \in \mathbf{N}, \xi \in \mathbf{R}^n).$

Proof: Theorem 3.2 implies that (i) follows from (ii). On the other hand, suppose that (i) is valid. In virtue of Lemma 3.1 we obtain for every $\varphi \in \mathcal{S}$, $\xi \in \mathbf{R}^n$ and $l \in \mathbf{N}$

$$\begin{aligned}
&\int_{\mathbf{R}^n} \left|\mathcal{F}\big(L(\cdot, -\eta)\,\Phi_l\big)(\eta - \xi)\right| k(\eta)\, d\eta = \int_{\mathbf{R}^n} \left|\mathcal{F}\big(L(\cdot, -\eta)\,\Phi_l\, e^{i(\xi,\cdot)}\big)(\eta)\right| k(\eta)\, d\eta \\
&= \int_{\mathbf{R}^n} \left|\mathcal{F}\big(L'(X, D)(\Phi_l\, e^{i(\xi,\cdot)})\big)(\eta)\, k(\eta)\right| d\eta = (2\pi)^n \|L'(X, D)(\Phi_l\, e^{i(\xi,\cdot)})\|_{1,k} \\
&\leqq (2\pi)^n C \|\Phi_l\, e^{i(\xi,\cdot)}\|_{1,kk^\sim} = C \int_{\mathbf{R}^n} |(\mathcal{F}\Phi_l)(\eta - \xi)\,(kk^\sim)(\eta)|\, d\eta \\
&= l^n C \int_{\mathbf{R}^n} \left|(\mathcal{F}\Phi)\big(l(\eta - \xi)\big)\,(kk^\sim)(\eta)\right| d\eta \\
&= C \int_{\mathbf{R}^n} \left|(\mathcal{F}\Phi)(\tau)\,(kk^\sim)\left(\xi + \frac{\tau}{l}\right)\right| d\tau \\
&\leqq C \int_{\mathbf{R}^n} \left|(\mathcal{F}\Phi)(\tau)\, M_{kk^\sim}\left(\frac{\tau}{l}\right)\right| d\tau\, (kk^\sim)(\xi) \leqq (2\pi)^n C C_1 \|\Phi\|_{1,k_{N_1}}\, (kk^\sim)(\xi)
\end{aligned}$$

where $C_1 > 0$ and $N_1 > 0$ are chosen so that

$$M_{kk^\sim}(\tau) := \sup_{\eta \in \mathbf{R}^n} \{(kk^\sim)(\tau + \eta)/(kk^\sim)(\eta)\} \leqq C_1 k_{N_1}(\tau).$$

This completes the proof ∎

4. On boundedness in spaces $\mathcal{B}_{p,k}$

4.1 Let $p \in (1, \infty)$ and $k \in \mathcal{K}$ be given. Define $p' \in (1, \infty)$ and $k^\vee \in \mathcal{K}$ by $1/p + 1/p' = 1$ and by $k^\vee(\xi) = k(-\xi)$. Denote the dual space of $\mathcal{B}_{p,k}$ by $\mathcal{B}_{p,k}^*$. Then for each $L \in \mathcal{B}_{p,k}^*$ one can find an element $l \in \mathcal{B}_{p',1/k^\vee}$ such that $L\varphi = l(\varphi)$ $(\varphi \in \mathcal{S})$ and $\|L\| = \|l\|_{p',1/k^\vee}$. On the other hand with each $l \in \mathcal{B}_{p',1/k^\vee}$ the linear form $L: \mathcal{S} \to \mathbf{C}$ defined by $L\varphi = l(\varphi)$ has a continuous extension $\mathcal{B}_{p,k} \to \mathbf{C}$ (cf. [2: 42]). Hence one sees easily the following

Lemma 4.1: *Suppose that the operator $L(X, D)$ defined by (2.1) maps $\mathcal{S}$ into itself and that the formal transpose $L'(X, D): \mathcal{S} \to \mathcal{S}$ exists. Then, for $p > 1$,*

$$\|L'(X, D)\varphi\|_{p,k} \leqq C \|\varphi\|_{p,kk^\sim} \qquad (\varphi \in \mathcal{S})$$

if and only if

$$\|L(X, D)\varphi\|_{p',1/(kk^\sim)^\vee} \leqq C \|\varphi\|_{p',1/k^\vee} \qquad (\varphi \in \mathcal{S}).$$

Furthermore, we have

Theorem 4.2: *Suppose that one can find numbers $\delta, \varrho \in \mathbf{R}$ with $0 \leqq \delta, \varrho < 1$ such that for each pair $(\alpha, \beta) \in \mathbf{N}_0^n \times \mathbf{N}_0^n$ there exist constants $C_{\alpha,\beta} > 0$, $N_\alpha \in \mathbf{R}$ and $N_\beta \in \mathbf{R}$ with which the inequality (3.2) holds. Furthermore, suppose that there exist constants $M > 0$ and $K > 0$ such that for all $l \in \mathbf{N}$*

$$\int_{\mathbf{R}^n} \mathcal{F}\left|\left(\Phi_l L(\cdot, -\eta)\right)(\eta - \xi)\, k(\eta)\right| d\eta \leqq M(kk^\sim)(\xi) \qquad (\xi \in \mathbf{R}^n)$$

and

$$\int_{\mathbf{R}^n} \left|\mathcal{F}\left(\Phi_l L(\cdot, -\eta)\right)(\eta - \xi)\right| \frac{d\xi}{|(kk^\sim)(\xi)|} \leqq K \frac{1}{k(\eta)} \qquad (\eta \in \mathbf{R}^n). \tag{4.1}$$

Then one can find a constant $C > 0$ (which is independent of $p \in [1, \infty)$) such that

$$\|L'(X, D)\varphi\|_{p,k} \leqq C \|\varphi\|_{p,kk^\sim} \qquad (\varphi \in \mathcal{S}).$$

Proof: Let the functions Θ_j' be as in the proof of Lemma 3.1. Define functions $h_{l,j}: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by

$$h_{l,j}(\xi, \eta) = \left|\mathcal{F}\left(\Phi_l L(\cdot, -\eta)\right)(\eta - \xi)\,(\mathcal{F}\Theta_j')(\eta)\,\frac{k(\eta)}{(kk^\sim)(\xi)}\right|.$$

Then we have (cf. the proof of Theorem 3.2)

$$(2\pi)^n \left|\mathcal{F}\left(L'(X, D)(\Phi_l\varphi)\right)(\eta)\,(\mathcal{F}\Theta_j')(\eta)\right| k(\eta)$$

$$\leqq \int_{\mathbf{R}^n} |(\mathcal{F}\varphi)(\xi)\,(kk^\sim)(\xi)|\,\left|\mathcal{F}\left(\Phi_l L(\cdot, -\eta)\right)(\eta - \xi)\,(\mathcal{F}\Theta_j')(\eta)\right| \frac{k(\eta)}{(kk^\sim)(\xi)}\, d\xi$$

$$= \int_{\mathbf{R}^n} h_{l,j}(\xi, \eta)^{1/p'}\, h_{l,j}(\xi, \eta)^{1/p}\, |(\mathcal{F}\varphi)(\xi)\,(kk^\sim)(\xi)|\, d\xi$$

$$\leqq \|h_{l,j}(\cdot, \eta)^{1/p'}\|_{p'} \left(\int_{\mathbf{R}^n} h_{l,j}(\xi, \eta)\, |(\mathcal{F}\varphi)(\xi)\,(kk^\sim)(\xi)|^p\, d\xi\right)^{1/p}.$$

Hence we see that

$$\|L'(X, D)(\Phi_l\varphi) * \Theta_j'\|_{p,k}^p$$
$$\leqq \frac{K^{p/p'}}{(2\pi)^{np+n}} \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} h_{l,j}(\xi, \eta) \, |(\mathscr{F}\varphi)(\xi)(kk^{\sim})(\xi)|^p \, d\xi \right) d\eta$$
$$= \frac{K^{p/p'}}{(2\pi)^{np+n}} \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} h_{l,j}(\xi, \eta) \, d\eta \right) |(\mathscr{F}\varphi)(\xi)(kk^{\sim})(\xi)|^p \, d\xi$$
$$\leqq \frac{K^{p/p'}}{(2\pi)^{np}} M \|\varphi\|_{p,kk^{\sim}}^p \leqq \frac{(K+M)^{p/p'+1}}{(2\pi)^{np}} \|\varphi\|_{p,kk^{\sim}}^p.$$

This implies our assertion ∎

4.2 Let G be an open set in $\mathbf{R}^n$. We apply Theorem 4.2 to obtain a criterion for the existence of a continuous extension of $L(X, D)$ from $\mathscr{B}_{p,1/k^{\vee}}$ into $\mathscr{B}^{\mathrm{loc}}_{p,1/(kk^{\sim})^{\vee}}(G)$. Let $p \in [1, \infty)$.

Corollary 4.3: *Suppose that one can find a number $\delta \in \mathbf{R}$ with $0 \leqq \delta < 1$ such that for each pair $(\alpha, \beta) \in \mathbf{N}_0^n \times \mathbf{N}_0^n$ and for each $\psi \in C_0^\infty(G)$ there exist $C_{\alpha,\beta,\psi} > 0$ and $N_{\beta,\psi} \in \mathbf{R}$ with which*

$$\sup_{x \in \operatorname{supp}\psi} |D_x^\alpha D_\xi^\beta L(x, \xi)| \leqq C_{\alpha,\beta,\psi} k_{N_{\beta,\psi}+\delta|\alpha|}(\xi) \qquad (\xi \in \mathbf{R}^n).$$

Furthermore, suppose that there exists $C_\psi > 0$ such that for all $\xi, \eta \in \mathbf{R}^n$

$$\left.\begin{aligned} &\int_{\mathbf{R}^n} \left|\mathscr{F}\big(\psi L(\cdot, -\tau)\big)(\tau - \xi)\right| k(\tau) \, d\tau \leqq C_\psi (kk^{\sim})(\xi) \\ &\int_{\mathbf{R}^n} \left|\mathscr{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \tau)\right| \frac{d\tau}{(kk^{\sim})(\tau)} \leqq C_\psi \frac{1}{k(\eta)} \end{aligned}\right\} \tag{4.2}$$

Then $L(X, D)$ defined by (2.1) is an operator from $\mathscr{S}$ into $C^\infty(G)$ and it has a continuous extension from $\mathscr{B}_{p,1/k^{\vee}}$ into $\mathscr{B}^{\mathrm{loc}}_{p,1/(kk^{\sim})^{\vee}}(G)$.

Proof: Let $Q_\psi(X, D)$ be the operator (2.1) with the symbol $\psi(x) L(x, \xi)$, where $\psi \in C_0^\infty(G)$. Then $Q_\psi(x, \xi)$ satisfies (3.2) so that the first assertion is obvious. Since $\Phi_l\psi = \psi$ for $l \in \mathbf{N}$ large enough, the conditions (3.7) and (4.1) hold for $Q_\psi(X, D)$. Theorem 4.2 implies the existence of a constant $C_\psi' > 0$ (which is independent of $p \in [1, \infty)$) such that

$$\|Q_\psi'(X, D)\varphi\|_{p,k} \leqq C_\psi' \|\varphi\|_{p,kk^{\sim}} \qquad (\varphi \in \mathscr{S}).$$

In view of Theorem 4.1 one has for $p' \in (1, \infty)$

$$\|Q_\psi(X, D)\varphi\|_{p',1/(kk^{\sim})^{\vee}} \leqq C_\psi' \|\varphi\|_{p',1/k^{\vee}} \qquad (\varphi \in \mathscr{S}).$$

Since C_ψ' does not depend on $p' \in (1, \infty)$, this inequality implies

$$\|\psi L(X, D)\varphi\|_{p,1/(kk^{\sim})^{\vee}} \leqq C_\psi' \|\varphi\|_{p,1/k^{\vee}} \qquad (\varphi \in \mathscr{S})$$

and then $L(X, D)$ has the stated extension ∎

A sufficient condition for the validity of (4.2) is the following one: for each $\psi \in C_0^\infty(G)$ and $|\alpha| \leqq [N + L + n + \varepsilon]$ (with $\varepsilon > 0$)

$$\sup_{x \in \operatorname{supp}\psi} |D_x^\alpha L(x, -\xi)| \leqq C_{\alpha,\psi} k^{\sim}(\xi) \qquad (\xi \in \mathbf{R}^n), \tag{4.3}$$

where N and $L \in \mathbf{R}$ are chosen such that for all $\xi, \eta \in \mathbf{R}^n$

$$k(\xi + \eta) \leqq Ck_N(\xi)\, k(\eta) \quad \text{and} \quad k^{\sim}(\xi + \eta) \leqq Ck_L(\xi)\, k^{\sim}(\eta)$$

(cf. the proof of Theorem 2.1). Furthermore, one sees for $p \in [1, \infty)$

Corollary 4.4: *Suppose that one can find a number $\delta \in \mathbf{R}$ with $0 \leqq \delta < 1$ such that for each pair $(\alpha, \beta) \in \mathbf{N}_0^n \times \mathbf{N}_0^n$ there exist constants $C_{\alpha,\beta} > 0$ and $N_\beta \in \mathbf{R}$ with which*

$$\sup_{x \in \operatorname{supp}\psi} |D_x^\alpha D_\xi^\beta L(x, \xi)| \leqq C_{\alpha,\beta} k_{N_\beta + \delta|\alpha|}(\xi) \qquad (\xi \in \mathbf{R}^n),$$

where $\psi \in C_0^\infty$. Furthermore, suppose that for $|\alpha| \leqq [N + L + n + 1]$

$$\sup_{x \in \operatorname{supp}\psi} |D_x^\alpha L(x, -\xi)| \leqq C_\alpha k^{\sim}(\xi) \qquad (\xi \in \mathbf{R}^n).$$

Then the operator $\big(\psi L(x, D)\big)'$ has a continuous extension from $\mathscr{B}_{p,kk^{\sim}}$ into $\mathscr{B}_{p,k}$.

For example the operators defined by (1.1), where $L(x, \xi) \in S^m_{0,0}$ *satisfy the assumptions* (3.2) *and* (4.3) *with* $k^{\sim} = k_m$.

5. Criteria for compactness

5.1 Let G be an open bounded set in $\mathbf{R}^n$. We are going to investigate the compactness of the continuous extension $\bar{L}'\colon \mathscr{B}_{1,kk^{\sim}}(G) \to \mathscr{B}_{1,k}$ of the formal transpose $L'\colon C_0^\infty(G) \to \mathscr{S}$ of an operator $L\colon \mathscr{S} \to C^\infty(G)$ defined by (2.1), that is, L' satisfies the relation $(L'\varphi)(\psi) = \varphi(L\psi)$ for all $\varphi \in C_0^\infty(G)$ and $\psi \in \mathscr{S}$. We give also a sufficient condition for the compactness of the continuous extension $\bar{L}_p'\colon \mathscr{B}_{p,kk^{\sim}}(G) \to \mathscr{B}_{p,k}$ of L'. In the following we keep the open set $G \subset \mathbf{R}^n$ fixed. Let $\psi \in C_0^\infty$, $\psi(x) \doteq 1$ in the closure $\bar{G}$ of G.

Lemma 5.1: *Suppose that one can find a number $\delta \in \mathbf{R}$ with $0 \leqq \delta < 1$ such that for each pair $(\alpha, \beta) \in \mathbf{N}_0^n \times \mathbf{N}_0^n$ there exist constants $C_{\alpha,\beta} > 0$ and $N_\beta \in \mathbf{R}$ with which*

$$\sup_{x \in \operatorname{supp}\psi} |D_x^\alpha D_\xi^\beta L(x, \xi)| \leqq C_{\alpha,\beta} k_{N_\beta + \delta|\alpha|}(\xi) \qquad (\xi \in \mathbf{R}^n). \tag{5.1}$$

In addition, we assume that there exists a constant $M > 0$ such that

$$\int_{\mathbf{R}^n} \left|\mathscr{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi)\, k(\eta)\right| d\eta \leqq M(kk^{\sim})(\xi) \qquad (\xi \in \mathbf{R}^n). \tag{5.2}$$

Then L maps $\mathscr{S}$ into $C^\infty(G)$, the formal transpose $L'\colon C_0^\infty(G) \to \mathscr{S}$ exists, L' has a continuous extension $\bar{L}'\colon \mathscr{B}_{1,kk^{\sim}}(G) \to \mathscr{B}_{1,k}$ and

$$\|\bar{L}'\| \leqq \frac{1}{(2\pi)^n} \sup_{\xi \in \mathbf{R}^n} \frac{1}{(kk^{\sim})(\xi)} \int_{\mathbf{R}^n} \left|\mathscr{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi)\, k(\eta)\right| d\eta.$$

Proof: Let $Q(x, \xi) = \psi(x)\, L(x, \xi)$. Then due to (5.1) the condition (3.2) is valid for $Q(x, \xi)$ (with $N_\alpha = 0$ and $\varrho = 0$). Replacing in the proof of Theorem 3.2 Φ_l with ψ one sees that (cf. (3.8))

$$\|Q'(X, D)\, \varphi\|_{1,k} \leqq \frac{M}{(2\pi)^n} \|\varphi\|_{1,kk^{\sim}}.$$

Since for all $\varphi \in C_0^\infty(G)$ and $\Phi \in \mathcal{S}$ one has $(Q'\varphi)(\Phi) = \varphi(Q\Phi) = \varphi(L\Phi)$ (where we denoted $Q = Q(X, D)$ and $Q' = Q'(X, D)$) and since M can be chosen to be

$$\sup_{\xi \in \mathbf{R}^n} \frac{1}{(kk^\sim)(\xi)} \int_{\mathbf{R}^n} \left|\mathcal{F}\big(\psi(L(\cdot, -\eta))(\eta - \xi)\big) k(\eta)\right| d\eta,$$

the proof is ready ∎

Remark: The validity of (5.1) is sufficient to imply that L maps $\mathcal{S}$ into $C^\infty(G)$ and that $L' : C_0^\infty(G) \to \mathcal{S}$ exists.

Lemma 5.2: *Suppose that* (5.1) *is valid for* $L(x, \xi)$. *Then for each fixed* $R > 0$ *the quantity*

$$P(R, N) = \sup_{|\xi| \leq R} \frac{1}{(kk^\sim)(\xi)} \int_{|\eta| > N} \left|\mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi) k(\eta)\right| d\eta$$

is tending to zero with $N \to \infty$.

Proof: Since $1/(kk^\sim)$ is continuous, it suffices to show that

$$\sup_{|\xi| \leq R} \int_{|\eta| > N} \left|\mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi) k(\eta)\right| d\eta \xrightarrow[N \to \infty]{} 0.$$

For each $\alpha \in \mathbf{N}_0^n$ one gets

$$\left|(\eta - \xi)^\alpha \mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi)\right| = \left|\mathcal{F}\big(D_x^\alpha(\psi L(\cdot, -\eta))\big)(\eta - \xi)\right|$$

$$\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \left|\mathcal{F}\big(D^{\alpha-\gamma}\psi D_x^\gamma L(x, -\eta)\big)(\eta - \xi)\right|$$

$$\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \sup_{x \in \operatorname{supp}\psi} |D_x^\gamma L(x, -\eta)| \, \|D^{\alpha-\gamma}\psi\|_{L^1}$$

$$\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} C_{\gamma,0} k_{N_0 + \delta|\gamma|}(\eta) \, \|D^{\alpha-\gamma}\psi\|_{L^1} \leq C_\alpha k_{N_0 + \delta|\alpha|}(\eta)$$

with some $C_\alpha > 0$. Hence for each $m \in \mathbf{N}$ there exists $C_m > 0$ such that

$$\left|\mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi)\right| \leq C_m k_{N_0 + (\delta - 1)m}(\eta) \, k_m(\xi).$$

Choose m so large that $N_0 + N_1 + (\delta - 1)m \leq -(n + 1)$, where $N_1 \in \mathbf{R}$ such that $k \leq C k_{N_1}$. Then we obtain

$$\sup_{|\xi| \leq R} \int_{|\eta| > N} \left|\mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi) k(\eta)\right| d\eta$$

$$\leq C_m C \sup_{|\xi| \leq R} k_m(\xi) \int_{|\eta| > N} k_{-(n+1)}(\eta) d\eta \xrightarrow[N \to \infty]{} 0.$$

This proves the assertion ∎

Let $\Phi \in C_0^\infty\big(B(0, 2)\big)$, $0 \leq \Phi \leq 1$ and $\Phi(x) = 1$ for all $x \in \overline{B(0, 1)}$. Define $\Phi_j \in C_0^\infty$ by $\Phi_j(x) = \Phi(x/j)$. Furthermore, let $L_j(x, \xi) = L(x, \xi) \Phi_j(\xi)$ and $Q_j(x, \xi) = \psi(x) \times L(x, \xi) \Phi_j(\xi)$. Supposing that $L(x, \xi)$ satisfies (5.1)—(5.2), one sees that $L_j(x, \xi)$ satisfies (5.1)—(5.2) as well. Hence due to Lemma 5.1 the continuous extension $\bar{L}_j' : \mathcal{B}_{1,kk^\sim}(G) \to \mathcal{B}_{1,k}$ exists. We have

Lemma 5.3: *Suppose that* (5.1) *is valid for* $L(x, \xi)$ *and that*

$$\frac{1}{(kk^{\sim})(\xi)} \int_{\mathbf{R}^n} \left|\mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi)\, k(\eta)\right| d\eta \xrightarrow[|\xi|\to\infty]{} 0. \tag{5.3}$$

Then

$$\|\bar{L}_j' - \bar{L}'\| \xrightarrow[j\to\infty]{} 0. \tag{5.4}$$

Proof: The symbol $L_j(x, \xi) - L(x, \xi)$ satisfies (5.1)—(5.2). Hence due to Lemma 5.1

$$(2\pi)^n \|\bar{L}_j' - \bar{L}'\| = (2\pi)^n \|\overline{(L_j - L)'}\|$$

$$\leq \sup_{\xi\in\mathbf{R}^n} \frac{1}{(kk^{\sim})(\xi)} \int_{\mathbf{R}^n} \left|\mathcal{F}\big(\psi(L_j(\cdot, -\eta) - L(\cdot, -\eta))\big)(\eta - \xi)\right| k(\eta)\, d\eta$$

$$= \sup_{\xi\in\mathbf{R}^n} \frac{1}{(kk^{\sim})(\xi)} \int_{\mathbf{R}^n} \left|\big(1 - \Phi_j(-\eta)\big)\, \mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi)\, k(\eta)\right| d\eta. \tag{5.5}$$

Let $\varepsilon > 0$. Choose $R > 0$ so large that

$$\frac{1}{(kk^{\sim})(\xi)} \int_{\mathbf{R}^n} \left|\mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi)\, k(\eta)\right| d\eta < \varepsilon \qquad (|\xi| > R).$$

Furthermore, choose $N \in \mathbf{N}$ such that $P(R, j) < \varepsilon$ $(j \geq N)$, where $P(R, j)$ is the quantity defined in Lemma 5.2. Then one gets by (5.5)

$$(2\pi)^n \|\bar{L}_j' - \bar{L}'\|$$

$$\leq \sup_{|\xi|>R} \frac{1}{(kk^{\sim})(\xi)} \int_{\mathbf{R}^n} \left|\big(1 - \Phi_j(-\eta)\big)\, \mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi)\, k(\eta)\right| d\eta$$

$$+ \sup_{|\xi|\leq R} \frac{1}{(kk^{\sim})(\xi)} \int_{\mathbf{R}^n} \left|\big(1 - \Phi_j(-\eta)\big)\, \mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi)\, k(\eta)\right| d\eta$$

$$\leq \varepsilon + \sup_{|\xi|\leq R} \frac{1}{(kk^{\sim})(\xi)} \int_{|\eta|>j} \left|\mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi)\, k(\eta)\right| < 2\varepsilon(2\pi)^n$$

for all $j \geq N$. Thus $\bar{L}_j'$ is converging to $\bar{L}'$ ∎

5.2 In this subsection we show that $\bar{L}_j'$ is compact for each $j \in \mathbf{N}$. Hence the compactness of $\bar{L}'$ follows from (5.4).

Lemma 5.4: *Suppose that* (5.1) *is valid for* $L(x, \xi)$. *Let* $Q_j(x, \xi)$ *be defined as in the Subsection* 5.1. *Then the formal transpose* Q_j' *of the operator* (2.1) *corresponding the symbol* $Q_j(x, \xi)$ *has a continuous extension* $\bar{Q}_j' : \mathcal{B}_{1,kk^{\sim}} \to \mathcal{B}_{1,k}$ *and* $\bar{Q}_j'$ *is compact (with each* $k, k^{\sim} \in \mathcal{K}$*).*

Proof: a) For each $(\alpha, \beta) \in \mathbf{N}_0^n \times \mathbf{N}_0^n$ the symbol $Q_j(x, \xi)$ satisfies

$$|D_x^\alpha D_\xi^\beta Q_j(x, \xi)|$$

$$\leqq \sum_{\gamma \leqq \alpha} \sum_{\tau \leqq \beta} \binom{\alpha}{\gamma} \binom{\beta}{\tau} |(D_x^\gamma \psi)(x)| \, |(D_\xi^\tau \Phi_j)(\xi)| \, |D_x^{\alpha-\gamma} D_\xi^{\beta-\tau} L(x, \xi)|$$

$$= \sum_{\gamma \leqq \alpha} \sum_{\tau \leqq \beta} \binom{\alpha}{\gamma} \binom{\beta}{\tau} |(D_x^\alpha \psi)(x)\, (D_\xi^\tau \Phi_j)(\xi)| \, C_{\alpha-\gamma, \beta-\tau} k_{N_{\beta-\tau} + \delta|\alpha-\gamma|}(\xi)$$

and then $Q_j(x, \xi)$ obeys the assumptions of Corollary 4.4 (with respect to each k, $k^{\sim} \in \mathcal{K}$). Hence by Corollary 4.4 the continuous extension $\bar{Q}_j' : \mathcal{B}_{1,kk^{\sim}} \to \mathcal{B}_{1,k}$ exists.

b) Let $\{u_m\} \subset \mathcal{B}_{1,kk^{\sim}}$ be a sequence such that $\|u_m\|_{1,kk^{\sim}} \leqq C$ $(m \in \mathbf{N})$. Then there exists a $\varphi_m \in \mathcal{S}$ such that

$$\|u_m - \varphi_m\|_{1,kk^{\sim}} \leqq 1/m. \tag{5.6}$$

We shall show that $\{\varphi_m\}$ possesses a subsequence $\{\varphi_m'\}$ so that $\|Q_j'\varphi_m - f\|_{1,k} \to 0$ with $m \to \infty$ (with some $f \in \mathcal{B}_{1,k}$). This implies

$$\|\bar{Q}_j' u_m' - f\|_{1,k} \leqq \|\bar{Q}_j' u_m' - Q_j'\varphi_m'\|_{1,k} + \|Q_j'\varphi_m' - f\|_{1,k}$$

$$\leqq \|\bar{Q}_j'\| \, \|u_m' - \varphi_m'\|_{1,kk^{\sim}} + \|Q_j'\varphi_m' - f\|_{1,k} \xrightarrow[m\to\infty]{} 0,$$

and then $\bar{Q}_j'$ is compact.

c) Define $f_m : \mathbf{R}^n \to \mathbf{C}$ with $f_m(\xi) = \mathcal{F}(Q_j'\varphi_m)(\xi)$. Then $f_m \in \mathcal{S} \subset C^\infty(\mathbf{R}^n)$. Furthermore, $\{f_m\}$ is uniformly bounded: in virtue of (3.3) one has

$$|\mathcal{F}(Q_j'\varphi_m)(\xi)|$$

$$= \left|\mathcal{F}\big(\varphi_m Q_j(x, -\xi)\big)(\xi)\right| = \frac{1}{(2\pi)^n} \left| \int_{\mathbf{R}^n} (\mathcal{F}\varphi_m)(\eta)\, \mathcal{F}\big(Q_j(x, -\xi)\big)(\xi - \eta)\, d\eta \right|$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} |\mathcal{F}(\varphi_m)(\eta)| \, (kk^{\sim})(\eta) \int_{\mathbf{R}^n} \left| \frac{\mathcal{F}\big(\psi L(x, -\xi)\, \Phi_j(-\xi)\big)(\xi - \eta)}{(kk^{\sim})(\eta)} \right| d\eta$$

$$\leqq \|\varphi_m\|_{1,kk^{\sim}} \, \Phi_j(-\xi) \int_{\mathbf{R}^n} \left| \frac{\mathcal{F}\big(\psi L(x, -\xi)\big)(\xi - \eta)}{(kk^{\sim})(\eta)} \right| d\eta. \tag{5.7}$$

As in the proof of Lemma 4.2 one sees that

$$E := \sup_{\xi \in \mathbf{R}^n} \Phi_j(-\xi) \int_{\mathbf{R}^n} \left| \frac{\mathcal{F}\big(\psi L(x, -\xi)\big)(\xi - \eta)}{(kk^{\sim})(\eta)} \right| d\eta < \infty.$$

Hence by (5.6) and by (5.7) $|f_m(\xi)| \leqq (C + 1)\, E$ $(m \in \mathbf{N}, \xi \in \mathbf{R}^n)$.

d) We show that $\{f_m\}$ is equicontinuous. Let $\xi, \xi_0 \in \mathbf{R}^n$. Then

$$|f_m(\xi) - f_m(\xi_0)| \leqq \sup_{\tau \in \mathbf{R}^n} \sum_{i=1}^n |(D_i f_m)(\tau)| \, |\xi - \xi_0|.$$

Hence it is sufficient to verify that

$$\sup_{\tau \in \mathbf{R}^n} |(D_i f_m)(\tau)| \leqq C' \quad (m \in \mathbf{N}). \tag{5.8}$$

In fact we obtain

$$|D_i f_m(\xi)| = |D_i \mathcal{F}(Q_j'\varphi_m)(\xi)| = \left|D_i \mathcal{F}\big(\varphi_m Q_j(x, -\xi)\big)(\xi)\right|$$
$$= \left|D_i \int_{\mathbf{R}^n} \varphi_m(x)\, \psi(x)\, L(x, -\xi)\, \Phi_j(-\xi)\, e^{-i(\xi,x)}\, dx\right|$$
$$\leq \left|\int_{\mathbf{R}^n} \varphi_m \psi D_i\big(L(x, -\xi)\, \Phi_j(-\xi)\big)\, e^{-i(\xi,x)}\, dx\right|$$
$$+ \left|\int_{\mathbf{R}^n} \varphi_m \psi L(x, -\xi)\, \Phi_j(-\xi)\, \xi_i\, e^{-i(\xi,x)}\, dx\right|$$
$$= \left|\mathcal{F}\big(\varphi_m \psi D_i(L(x, -\xi)\, \Phi_j(-\xi))\big)(\xi)\right| + \left|\mathcal{F}\big(\varphi_m \psi L(x, -\xi)\, \Phi_j(-\xi)\, \xi_i\big)(\xi)\right|,$$

and then (5.8) follows with the same conclusions as done in the part c.

e) Since $\{f_m\}$ is uniformly bounded and $\{f_m\}$ is a equicontinuous set, the Ascoli-Arzela Theorem implies that one can find a subsequence $\{f_m'\}$ so that $\{f_m'\}$ is uniformly convergent on every compact subset of $\mathbf{R}^n$. Hence we obtain

$$\int_{\mathbf{R}^n} |\mathcal{F}(Q_j'\varphi_m' - Q_j'\varphi_l')(\xi)\, k(\xi)|\, d\xi$$
$$= \int_{B(0,2j)} |f_m'(\xi) - f_l'(\xi)|\, k(\xi)\, d\xi \xrightarrow[m,l\to\infty]{} 0$$

and then $\|Q_j'\varphi_m' - f\|_{1,k} \to 0$ with some $f \in \mathcal{B}_{1,k}$ ∎

Let G and ψ be as above. Then we obtain

Theorem 5.5: *Suppose that* (5.1) *is valid for* $L(x, \xi)$ *and let* $L_j(x, \xi) = L(x, \xi)\, \Phi_j(\xi)$. *Then the continuous extension* $\bar{L}_j' : \mathcal{B}_{1,kk^\sim}(G) \to \mathcal{B}_{1,k}$ *exists and* $\bar{L}_j'$ *is compact.*

Proof: Due to Lemma 5.4 the continuous extension $\bar{Q}_j' : \mathcal{B}_{1,kk^\sim} \to \mathcal{B}_{1,k}$ exists and $\bar{Q}_j'$ is compact. Hence it is easy to see that the continuous extension $\bar{L}_j'$ exists and that $\bar{L}_j' u = \bar{Q}_j' u$ for each $u \in \mathcal{B}_{1,kk^\sim}(G)$. This completes the proof ∎

Corollary 5.6: *Suppose that* (5.1) *and* (5.3) *are valid for* $L(x, \xi)$. *Then the continuous extension* $\bar{L}' : \mathcal{B}_{1,kk^\sim}(G) \to \mathcal{B}_{1,k}$ *exists and* $\bar{L}'$ *is compact.*

Proof: $\bar{L}_j' : \mathcal{B}_{1,kk^\sim}(G) \to \mathcal{B}_{1,k}$ is compact for each $j \in \mathbf{N}$. Since the space of all compact operators $K : \mathcal{B}_{1,kk^\sim}(G) \to \mathcal{B}_{1,k}$ is closed in the space of all bounded operators $T : \mathcal{B}_{1,kk^\sim}(G) \to \mathcal{B}_{1,k}$, the Lemma 5.3 proves the assertion ∎

5.3 Let G be an open bounded subset in $\mathbf{R}^n$ and let $\psi \in C_0^\infty$, $\psi(x) = 1$ for $x \in \bar{G}$. Choose an open bounded set G' such that supp $\psi \subset G'$ and let $\psi' \in C_0^\infty$, $\psi'(x) = 1$ for $x \in \bar{G}'$. Assume that

$$\sup_{x \in \operatorname{supp}\psi'} |D_x^\alpha D_\xi^\beta L(x, \xi)| \leq C_{\alpha,\beta} k_{N_\beta + \delta|\alpha|}(\xi) \qquad (\xi \in \mathbf{R}^n). \tag{5.9}$$

Our goal is to show that the compactness of $\bar{L}' : \mathcal{B}_{1,kk^\sim}(G') \to \mathcal{B}_{1,k}$ implies the condition (5.3). Let $\{\xi_j\} \subset \mathbf{R}^n$, $|\xi_j| \to \infty$, and

$$u_j \in C_0^\infty(G'), \qquad u_j(x) = \psi(x)\, e^{i(\xi_j,x)}/(kk^\sim)(\xi_j). \tag{5.10}$$

Lemma 5.7: *Suppose that* (5.1) *is valid for* $L(x, \xi)$. *Then*

$$\left|\mathcal{F}\big(\psi L(x, -\xi)\big)(\xi - \xi_j)\right|/(kk^\sim)(\xi_j) \xrightarrow[j\to\infty]{} 0 \qquad (\xi \in \mathbf{R}^n).$$

Proof: In virtue of (5.1) we obtain for each $\alpha \in \mathbf{N}_0^n$

$$\left|(\xi - \xi_j)^\alpha \mathcal{F}\big(\psi L(x, -\xi)\big) (\xi - \xi_j)\right| = \left|\mathcal{F}\big(D_x^\alpha(\psi L(x, -\xi))\big) (\xi - \xi_j)\right|$$
$$\leqq \sum_{\gamma \leqq \alpha} \binom{\alpha}{\gamma} \sup_{x \in \operatorname{supp} \psi} |D_x^\gamma L(x, -\xi)| \, \|D_x^{\alpha-\gamma}\psi\|_{L^1} \leqq C_{\psi,\alpha} k_{N_0+\delta|\alpha|}(\xi)$$

with some constant $C_{\psi,\alpha} > 0$. Hence one sees that for each $N \in \mathbf{N}$ there exists a $C_N > 0$ such that

$$\left|\mathcal{F}\big(\psi L(x, -\xi)\big) (\xi - \xi_j)\right| \leqq C_N k_{N_0+(\delta+1)N}(\xi)\, k_{-N}(\xi_j).$$

This proves the Lemma ∎

We are now ready to show

Theorem 5.8: *Suppose that* (5.9) *is valid for* $L(x, \xi)$. *Furthermore, suppose that the continuous extension* $\bar{L}' : \mathcal{B}_{1,kk^\sim}(G) \to \mathcal{B}_{1,k}$ *exists and that* $\bar{L}'$ *is compact. Then the convergence* (5.3) *is valid.*

Proof: a) Let $u_j \in C_0^\infty(G')$ be defined by (5.10). We must show that there exists a subsequence $\{\xi_j'\}$ of $\{\xi_j\}$ so that

$$\frac{1}{(kk^\sim)(\xi_j')} \int_{\mathbf{R}^n} \left|\mathcal{F}\big(\psi L(\cdot, -\eta)\big) (\eta - \xi_j')\, k(\eta)\right| d\eta \xrightarrow[j\to\infty]{} 0.$$

For all $j \in \mathbf{N}$ one has

$$\|u_j\|_{1,kk^\sim} = \frac{\dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbf{R}^n} |(\mathcal{F}\psi)(\eta - \xi_j)\,(kk^\sim)(\eta)|\, d\eta}{(kk^\sim)(\xi_j)} \leqq \|\psi\|_{1,M_{kk^\sim}}.$$

Since $\bar{L}'$ is compact, there exists a subsequence $\{u_j'\}$ of $\{u_j\}$ such that

$$\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \left|\big(\mathcal{F}(L'u_j')(\xi) - (\mathcal{F}f)(\xi)\big)\, k(\xi)\right| d\xi$$
$$= \|L'u_j' - f\|_{1,k} \xrightarrow[j\to\infty]{} 0. \tag{5.11}$$

b) We show that $f = 0$. In virtue of (5.11) there is a subsequence $\{u_j''\}$ of $\{u_j'\}$ such that $(L'u_j'')(\xi) \to (\mathcal{F}f)(\xi)$ a.e. in $\mathbf{R}^n$. On the other hand, by Lemma 5.7 and by (3.3) for each $\xi \in \mathbf{R}^n$

$$\mathcal{F}(L'u_j'')(\xi) = \mathcal{F}\big(L'(X, D)(\psi\, e^{i(\xi_j'', x)})\big)(\xi)/(kk^\sim)(\xi_j)$$
$$= \mathcal{F}\big(\psi L(x, -\xi)\big)(\xi - \xi_j)/(kk^\sim)(\xi_j) \xrightarrow[j\to\infty]{} 0.$$

Hence $(\mathcal{F}f)(\xi) = 0$ a.e. in $\mathbf{R}^n$ and then $f = 0$.

c) In view of (5.11) and (3.3) one obtains

$$\frac{1}{(2\pi)^n} \frac{1}{(kk^\sim)(\xi_j)} \int_{\mathbf{R}^n} \left|\mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi_j')\, k(\eta)\right| d\eta$$
$$= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \left| \mathcal{F}\left(L'(X, D) \frac{\psi\, e^{i(\xi_j', x)}}{(kk^\sim)(\xi_j')}\right)(\eta)\, k(\eta) \right| d\eta = \|L'u_j'\|_{1,k} \xrightarrow[j\to\infty]{} 0$$

and then the proof is ready ∎

Remark: With the same kind of conclusions one sees the following result: *Suppose that* (5.1) *is valid for* $L(x, \xi)$ *and that*

$$\int_{\mathbf{R}^n} \left|\mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi)\, k(\eta)\right| d\eta \leqq M(kk^{\sim})(\xi) \qquad (\xi \in \mathbf{R}^n)$$

and

$$\int_{\mathbf{R}^n} \left|\frac{\mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \tau)}{(kk^{\sim})(\tau)}\right| d\tau \leqq K\, \frac{1}{k(\eta)} \qquad (\eta \in \mathbf{R}^n).$$

Furthermore, assume that either

$$\frac{1}{(kk^{\sim})(\xi)} \int_{\mathbf{R}^n} \left|\mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \xi)\, k(\eta)\right| d\eta \xrightarrow[|\xi|\to\infty]{} 0$$

or

$$k(\eta) \int_{\mathbf{R}^n} \left|\frac{\mathcal{F}\big(\psi L(\cdot, -\eta)\big)(\eta - \tau)}{(kk^{\sim})(\tau)}\right| d\tau \xrightarrow[|\eta|\to\infty]{} 0.$$

Then the continuous extension $\bar{L}_p' : \mathcal{B}_{p,kk^{\sim}}'(G) \to \mathcal{B}_{p,k}$ *exists and* $\bar{L}_p'$ *is compact.*

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