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On Solvability of Linear Partial Differential Equations in Local Spaces $\mathcal{B}^{\mathrm{loc}}_{p,k}(G)$

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Es sei $L(x, D) = \sum_{|\sigma| \leq r} a_\sigma(x) D^\sigma$ ein linearer partieller Differentialoperator mit $C^\infty(G)$-Koeffizienten a_σ, wobei G ein offenes Teilgebiet in $\mathbf{R}^n$ ist. Ferner bezeichne $\Lambda^{\sim}_{p,k}(G)$ die minimale abgeschlossene Realisierung von $L(x, D)$ in dem lokalen Hörmander-Raum $\mathcal{B}^{\mathrm{loc}}_{p,k}(G)$. Es wird die Abgeschlossenheit der Wertebereiche $R(\Lambda^{\sim}_{p,k}(G))$ des Operators $\Lambda^{\sim}_{p,k}(G)$ und $R(\Lambda^{\sim +}_{p,k}(G))$ des dualen Operators $\Lambda^{\sim +}_{p,k}(G)$ untersucht. Unter anderem werden notwendige und hinreichende Bedingungen für die Abgeschlossenheit von $R(\Lambda^{\sim}_{p,k}(G))$ angegeben. Die Surjektivität des Operators $\Lambda^{\sim}_{p,k}(G)$ wird ebenfalls charakterisiert. Als Anwendung wird eine hinreichende Bedingung für die Abgeschlossenheit von $R(\Lambda^{\sim}_{2,1}(B(0, R)))$, wo $B(0, R)$ die offene Kugel in $\mathbf{R}^n$ ist, unter gewissen Apriori-Abschätzungen für den formal transponierten Operator $L'(x, D)$ von $L(x, D)$ hergeleitet.

Пусть $L(x, D) = \sum_{|\sigma| \leq r} a_\sigma(x) D^\sigma$ линейный дифференциальный оператор с частными производными, где a_σ — $C^\infty(G)$-коэффициенты и G — открытая область в $\mathbf{R}^n$. Обозначим $\Lambda^{\sim}_{p,k}(G)$ минимальную замкнутую реализацию оператора $L(x, D)$ в локальном пространстве Хёрмандера $\mathcal{B}^{\mathrm{loc}}_{p,k}(G)$. Исследуется замкнутость областей значений $R(\Lambda^{\sim}_{p,k}(G))$ оператора $\Lambda^{\sim}_{p,k}(G)$ и $R(\Lambda^{\sim +}_{p,k}(G))$ дуального оператора $\Lambda^{\sim +}_{p,k}(G)$. Даются между прочим необходимые и достаточные условия замкнутости $R(\Lambda^{\sim}_{p,k}(G))$. Характеризуется также суръективность оператора $\Lambda^{\sim}_{p,k}(G)$. Как применение дается достаточное условие замкнутости $R(\Lambda^{\sim}_{2,1}(B(0, R)))$, где $B(0, R)$ открытый шар в $\mathbf{R}^n$, при некоторых априорных оценках для формально транспонированного оператора $L'(x, D)$ к $L(x, D)$.

Let $L(x, D) = \sum_{|\sigma| \leq r} a_\sigma(x) D^\sigma$ be a linear partial differential operator with $C^\infty(G)$-coefficients a_σ, where G is an open subset in $\mathbf{R}^n$. Denote by $\Lambda^{\sim}_{p,k}(G)$ the minimal closed realization of $L(x, D)$ in the local Hörmander space $\mathcal{B}^{\mathrm{loc}}_{p,k}(G)$. The closedness of the range $R(\Lambda^{\sim}_{p,k}(G))$ and of the range $R(\Lambda^{\sim +}_{p,k}(G))$ of the dual operator $\Lambda^{\sim +}_{p,k}(G)$ is considered. Among other things, one shows necessary and sufficient conditions for the closedness of $R(\Lambda^{\sim}_{p,k}(G))$. The surjectivity of $\Lambda^{\sim}_{p,k}(G)$ is also characterized. As an application a sufficient condition for the closedness of $R(\Lambda^{\sim}_{2,1}(B(0, R)))$, where $B(0, R)$ is the open ball in $\mathbf{R}^n$, is established, when certain a priori estimates for the formal transpose $L'(x, D)$ of $L(x, D)$ hold.

1. Introduction

Let G be an open subset in $\mathbf{R}^n$ and let $L(x, D)$ be a linear partial differential operator with $C^\infty(G)$-coefficients. Furthermore, denote by $\mathcal{B}^{\mathrm{loc}}_{p,k}(G)$ the local subspace of the distribution space $\mathcal{D}'(G)$ for whose elements T it holds

$$\int_{\mathbf{R}^n} |F(\psi T)(\xi)\, k(\xi)|^p \, d\xi < \infty \qquad \text{for all } \psi \in C_0^\infty(G),$$

where $F: \mathcal{S}' \to \mathcal{S}'$ is the Fourier transform and where k is chosen from a class $\mathcal{K}$ of weight functions. Then one is able to construct the *minimal closed realization* $\Lambda^{\sim}_{p,k}(G)$:

$\mathcal{B}^{\mathrm{loc}}_{p,k}(G) \to \mathcal{B}^{\mathrm{loc}}_{p,k}(G)$ and the *maximal* (*closed*) *realization* $\Lambda'^{\#}_{p,k}(G)\colon \mathcal{B}^{\mathrm{loc}}_{p,k}(G) \to \mathcal{B}^{\mathrm{loc}}_{p,k}(G)$ of $L(x, D)$. The operators $\Lambda^{\sim}_{p,k}(G)$ and $\Lambda'^{\#}_{p,k}(G)$ are the same when $L(D) = L(x, D)$ has constant coefficients (cf. Theorem 2.1). The identity of the operators $\Lambda^{\sim}_{p,k}(G)$ and $\Lambda'^{\#}_{p,k}(G)$ is an analogical notion with the essential maximality of $L(x, D)$ considered in the global case (cf. [3, 4]).

There are several kind of (algebraic) criteria under which for each $f \in C_0^\infty(V_x)$ the distributional equation

$$\Lambda'^{\#}_{p,k}(V_x)\, u = f, \qquad u \in \mathcal{B}^{\mathrm{loc}}_{p,k}(V_x), \tag{1.1}$$

is solvable in some neighbourhood V_x of $x \in G$ (cf. [13], for example). In the case when the equality $\Lambda^{\sim}_{p,k}(V_x) = \Lambda'^{\#}_{p,k}(V_x)$ holds and when the range $R\big(\Lambda^{\sim}_{p,k}(V_x)\big)$ is closed in $\mathcal{B}^{\mathrm{loc}}_{p,k}(V_x)$ one gets from the validity of (1.1) that

$$R\big(\Lambda^{\sim}_{p,k}(V_x)\big) = R\big(\Lambda'^{\#}_{p,k}(V_x)\big) = \mathcal{B}^{\mathrm{loc}}_{p,k}(V_x). \tag{1.2}$$

One knows that for the operators $L(D)$ with constant coefficients the validity of (1.2) implies a connection (a so-called $L(D)$-*convexivity*) between the operator $L(D)$ and the open set $V_x \subset \mathbf{R}^n$ (cf. [5: pp. 41—59], [10: pp. 57—91] and [7]).

In this contribution one deals with the closedness of $R\big(\Lambda^{\sim}_{p,k}(G)\big)$ (for $p \in (1, \infty)$). Applying the theory of linear densily defined closed operators (in the Frechet spaces $\mathcal{B}^{\mathrm{loc}}_{p,k}(G)$) we show a characterization of the closedness of $R\big(\Lambda^{\sim}_{p,k}(G)\big)$ (cf. Section 4.3). Also a characterization for the surjectivity of $\Lambda^{\sim}_{p,k}(G)$ is given (cf. Section 4.4). Theorem 4.11 shows that the $L(x, D)$-convexivity (cf. [11: p. 391]) together with the validity of the inequation (with $\gamma > 0$)

$$\left\|\Lambda^{\#}_{p',1/k^{\vee}}(G)\, u\right\|_{p',1/k^{\vee}} \geqq \gamma\, \|u\|_{p',1/k^{\vee}} := \left(\int_{\mathbf{R}^n} |(Fu)\,(\xi)\,(1/k^{\vee}(\xi))|^{p'}\, d\xi\right)^{1/p'}$$

for all $u \in D\big(\Lambda^{\#}_{p',1/k^{\vee}}(G)\big) \cap \varepsilon'(G)$ is a sufficient criterion to imply the surjectivity of $\Lambda^{\sim}_{p,k}(G)$. Here $p' \in (1, \infty)$ and $k^{\vee} \in \mathcal{K}$ are defined by $1/p + 1/p' = 1$ and $k^{\vee}(\xi) = k(-\xi)$. The operator $\Lambda^{\#}_{p',1/k^{\vee}}(G)\colon \mathcal{B}^{\mathrm{loc}}_{p',1/k^{\vee}}(G) \to \mathcal{B}^{\mathrm{loc}}_{p',1/k^{\vee}}(G)$ is the maximal realization of the formal transpose $L'(x, D)$ of $L(x, D)$. In Section 4.5 we give some applications.

2. Preliminaries

2.1. For the standard notions about the distribution theory we refer to [5: pp. 1—33]. Let $\mathcal{K}$ be the totality of weight functions as in [5: p. 34]. Suppose that $p \in [1, \infty)$ and $k \in \mathcal{K}$. Furthermore, let G be an open subset in $\mathbf{R}^n$. Then the linear space $\mathcal{B}^{\mathrm{loc}}_{p,k}(G)$ is defined as a subspace of $\mathcal{D}'(G)$ such that the distribution T belongs to $\mathcal{B}^{\mathrm{loc}}_{p,k}(G)$ if and only if the quantity

$$\|\psi T\|_{p,k} := \left(\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} |F(\psi T)\,(\xi)\, k(\xi)|^p\, d\xi\right)^{1/p}$$

is finite for each $\psi \in C_0^\infty(G)$ (cf. [5: pp. 42—45]). Here F is the Fourier transform from the space $\mathcal{S}'$ of all tempered distributions into itself.

Let $\{K_j\}$ be some sequence of compact subsets of G satisfying $K_j \subset \operatorname{int} K_{j+1}$ and $\bigcup_j K_j = G$. Choose functions $\psi_j \in C_0^\infty(G)$ such that $\psi_j(x) = 1$, $x \in K_j$. Then the space $\mathcal{B}^{\mathrm{loc}}_{p,k}(G)$ equipped with the topology defined by the denumerable number of semi-norms q_j such that $q_j(u) = \|\psi_j u\|_{p,k}$ is a Frechet space. Define new semi-norms

$p_j\colon \mathcal{B}_{p,k}^{\mathrm{loc}}(G) \to \mathbf{R}$ by the relation

$$p_j(u) = \sum_{l=1}^{j} \|\psi_l u\|_{p,k}. \tag{2.1}$$

Then one has $p_j \leqq p_{j+1}$ and the topology τ' defined by the semi-norms q_j is equivalent to the topology τ defined by the semi-norms p_j. The metric, which defines the sopology of $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)$ can be chosen to be a mapping $d(\cdot,\cdot)\colon \mathcal{B}_{p,k}^{\mathrm{loc}}(G) \times \mathcal{B}_{p,k}^{\mathrm{loc}}(G) \to \mathbf{R}$ tuch that

$$d(u, v) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{p_j(u-v)}{1 + p_j(u-v)}. \tag{2.2}$$

Furthermore, the space $C_0^\infty(G)$ is dense in the space $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)$.

2.2. Let $L(x, D)$ be a linear partial differential operator with $C^\infty(G)$-coefficients, that is,

$$L(x, D) = \sum_{|\sigma| \leqq r} a_\sigma(x) D^\sigma, \qquad a_\sigma \in C^\infty(G).$$

Define a linear (dense) operator $\Lambda_{p,k}(G)\colon \mathcal{B}_{p,k}^{\mathrm{loc}}(G) \to \mathcal{B}_{p,k}^{\mathrm{loc}}(G)$ with the requirement

$$D(\Lambda_{p,k}(G)) = C_0^\infty(G), \qquad \Lambda_{p,k}(G)\,\varphi = L(x, D)\,\varphi \qquad \text{for } \varphi \in C_0^\infty(G).$$

Denote the formal transpose $\sum_{|\sigma| \leqq r} (-D)^\sigma \big(a_\sigma(x)\,(\cdot)\big)$ of $L(x, D)$ by $L'(x, D)$. Since for all $\varphi, \psi \in C_0^\infty(G)$ and for each $u \in \mathcal{B}_{p,k}^{\mathrm{loc}}(G)$ one has

$$\varphi(L\psi) := \int_G \varphi(x)\,\big(L(x, D)\,\psi\big)(x)\,dx = (L'\varphi)(\psi)$$

and

$$|u(\varphi)| \leqq \|\varphi\|_{p',1/k^\vee}\,\|\psi_l u\|_{p,k}, \tag{2.3}$$

where $l \in \mathbf{N}$ such that $\operatorname{supp}\varphi \subset K_{l-1}$, one sees that $\Lambda_{p,k}(G)$ is closable in $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)$. Here $p' \in (1, \infty]$ and $k^\vee \in \mathcal{K}$ are chosen so that $1/p + 1/p' = 1$ and $k^\vee(\xi) = k(-\xi)$. The norm $\|\varphi\|_{p,k}$ is defined by

$$\|\varphi\|_{p,k} = \begin{cases} \left(\dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbf{R}^n} |(F\varphi)(\xi)\,k(\xi)|^p\,d\xi\right)^{1/p} & \text{for } p < \infty, \\ \sup\limits_{\xi \in \mathbf{R}^n} |(F\varphi)(\xi)\,k(\xi)| & \text{for } p = \infty. \end{cases}$$

Let $\Lambda_{p,k}^{\sim}(G)\colon \mathcal{B}_{p,k}^{\mathrm{loc}}(G) \to \mathcal{B}_{p,k}^{\mathrm{loc}}(G)$ be the smallest closed extension of $\Lambda_{p,k}(G)$.

Furthermore we define a linear operator $\Lambda_{p,k}'^{\#}(G)\colon \mathcal{B}_{p,k}^{\mathrm{loc}}(G) \to \mathcal{B}_{p,k}^{\mathrm{loc}}(G)$ by

$$\left.\begin{aligned} D\big(\Lambda_{p,k}'^{\#}(G)\big) = \{u \in \mathcal{B}_{p,k}^{\mathrm{loc}}(G) \mid &\text{ there exists an element } f \in \mathcal{B}_{p,k}^{\mathrm{loc}}(G) \\ &\text{ such that } u\big(L'(x, D)\,\varphi\big) = f(\varphi) \text{ for all } \varphi \in C_0^\infty(G)\}, \\ \Lambda_{p,k}'^{\#}(G)\,u = f \qquad & \end{aligned}\right\}.$$

Then in virtue of (2.3) $\Lambda_{p,k}'^{\#}(G)$ is a closed operator and $\Lambda_{p,k}^{\sim}(G) \subset \Lambda_{p,k}'^{\#}(G)$ (in other words, $\Lambda_{p,k}'^{\#}(G)$ is an extension of $\Lambda_{p,k}^{\sim}(G)$). For the operator $L(D) = \sum_{|\sigma| \leqq r} a_\sigma D^\sigma$ with constant coefficients $a_\sigma \in \mathbf{C}$ we obtain

Theorem 2.1: *Let $L(D)$ be the linear partial differential operator with constant coefficients and let G be an open set in $\mathbf{R}^n$. Then one has for $p \in [1, \infty)$ and $k \in \mathcal{K}$ the* *equality*

$$\Lambda_{p,k}^{\sim}(G) = \Lambda_{p,k}'^{\#}(G). \tag{2.4}$$

Proof: It is sufficient to show that $\Lambda_{p,k}'^{\#}(G) \subset \Lambda_{p,k}^{\sim}(G)$. We suppose that u lies in $D(\Lambda_{p,k}'^{\#}(G))$ and that $\Lambda_{p,k}'^{\#}(G)\,u = f$. Let $\theta \in C_0^\infty$ such that $(F\theta)\,(0) = 1$. Define $\theta_j \in C_0^\infty$ through the relation $\theta_j(x) = j^n\theta(jx)$. Furthermore, let the functions $\psi_j \in C_0^\infty(G)$ be defined as in the Section 2.1. Then one sees that for each $\psi \in C_0^\infty(G)$ the convergence

$$\|\psi((\psi_j u) * \theta_j - u)\|_{p,k} + \|\psi(L(D)\,((\psi_j u) * \theta_j) - f)\|_{p,k} \to 0$$

with $j \to \infty$ holds. Hence u lies in $D(\Lambda_{p,k}^{\sim}(G))$ and $\Lambda_{p,k}^{\sim}(G)\,u = f$, which proves the theorem ∎

The equality (2.4) holds also when $L(x, D)$ is a formally hypoelliptic operator with $C^\infty(G)$-coefficients (cf. the regularity result of [5: p. 176]).

3. The dual operator of $\Lambda_{p,k}^{\sim}(G)$

3.1. Let $\mathscr{B}_{p,k}$ denote the linear subspace of $\mathscr{S}'$ for whose elements u the Fourier transform Fu lies in $L_{\text{loc}}^1(\mathbf{R}^n)$ and the quantity

$$\|u\|_{p,k} = \left(\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} |(Fu)\,(\xi)\,k(\xi)|^p\,d\xi\right)^{1/p}$$

is finite. Then the mapping $u \to \|u\|_{p,k}$ is a norm in $\mathscr{B}_{p,k}$. Equipped with this norm $\mathscr{B}_{p,k}$ becomes a Banach space. Using the reflexivity of $L^p(\mathbf{R}^n)$ one sees that $\mathscr{B}_{p,k}$ is a reflexive space for $p > 1$. Furthermore the space C_0^∞ is dense in $\mathscr{B}_{p,k}$. The dual space $\mathscr{B}_{p,k}^+$ of $\mathscr{B}_{p,k}$ can be characterized in the following way (in the sequel we assume that $p \in (1, \infty)$).

Lemma 3.1: *Suppose that V lies in $\mathscr{B}_{p,k}^+$. Then there exists a unique $v \in \mathscr{B}_{p',1/k^\vee}$ such that $V\varphi = v(\varphi)$ for all $\varphi \in \mathscr{S}$.*

On the other hand, assume that v is in $\mathscr{B}_{p',1/k^\vee}$. Then the linear form $V: \mathscr{S} \to \mathbf{C}$ defined by $V\varphi = v(\varphi)$ can be uniquely continuously be extended onto the whole space $\mathscr{B}_{p,k}$. In addition one has $\|V\| = \|v\|_{p',1/k^\vee}$.

For further properties of spaces $\mathscr{B}_{p,k}$ we refer to [5: pp. 36—42].

Let $\{K_j\}$ be a sequence of compact subsets as in the Section 2.1. Denote by $\mathscr{B}_{p,k}^c(K_j)$ the subspace of $\mathscr{B}_{p,k}$ defined by

$$\mathscr{B}_{p,k}^c(K_j) = \mathscr{B}_{p,k} \cap \varepsilon'(K_j).$$

Here $\varepsilon'(K_j)$ is the space of all distributions $u \in \mathscr{D}'(\mathbf{R}^n)$ with support (which is denoted by supp u) contained in K_j. Since K_j is closed in $\mathbf{R}^n$ one sees that $\mathscr{B}_{p,k}^c(K_j)$ is a closed subspace of $\mathscr{B}_{p,k}$. Hence $\mathscr{B}_{p,k}^c(K_j)$ is a Banach space and $\mathscr{B}_{p,k}^c(K_j) \subset \mathscr{B}_{p,k}^c(K_{j+1})$. Furthermore one has

$$\mathscr{B}_{p,k}^c(G) := \mathscr{B}_{p,k} \cap \varepsilon'(G) = \bigcup_{j=1}^{\infty} \mathscr{B}_{p,k}^c(K_j).$$

Equipp the space $\mathscr{B}_{p,k}^c(G)$ with the inductive limit topology of Banach spaces $\mathscr{B}_{p,k}^c(K_j)$ (cf. [11: pp. 126—149]). We need the following

Lemma 3.2: *Suppose that U lies in the dual space $\mathscr{B}_{p,k}^{\text{loc}}(G)^+$ of the space $\mathscr{B}_{p,k}^{\text{loc}}(G)$, where $p \in (1, \infty)$ and $k \in \mathscr{K}$. Then there exists a unique $u \in \mathscr{B}_{p',1/k^\vee}^c(G)$ such that $U\varphi = u(\varphi)$ for all $\varphi \in C_0^\infty(G)$.*

On the other hand, suppose that u is in $\mathscr{B}^c_{p',1/k^\vee}(G)$. Then the linear form $U: C_0^\infty(G) \to \mathbf{C}$ defined by $U\varphi = u(\varphi)$ can be uniquely continuously be extended onto the whole space $\mathscr{B}^{loc}_{p,k}(G)$.

Proof: Suppose that U lies in $\mathscr{B}^{loc}_{p,k}(G)^+$. Then there exist $C > 0$ and $j \in \mathbf{N}$ such that

$$|Uw| \leqq C \, \|\psi_j w\|_{p,k} \qquad \text{for all } w \in \mathscr{B}^{loc}_{p,k}(G) \tag{3.1}$$

(cf. [11: p. 64]). Define a linear form $U': \mathscr{B}_{p,k} \to \mathbf{C}$ by $U'v = U(v|_G)$, where $v|_G$ denotes the restriction of v on G. Then in virtue of (3.1) one has

$$|U'v| \leqq C \, \|\psi_j v\|_{p,k} \leqq C \, \|\psi_j\|_{1,M_k} \|v\|_{p,k} \qquad \text{for all } v \in \mathscr{B}_{p,k}, \tag{3.2}$$

where $M_k \in \mathscr{K}$ is defined as in [5: p. 34]. Thus U' lies in $\mathscr{B}^+_{p,k}$. In view of Lemma 3.1 there exists an $u \in \mathscr{B}_{p',1/k^\vee}$ such that $U'\varphi = u(\varphi)$ for all $\varphi \in C_0^\infty$. Let $\eta_j \in C_0^\infty(G)$ be such that $\eta_j(x) = 1$ for $x \in \operatorname{supp} \psi_j$. Then by (3.2) $U'\varphi = U'(\eta_j\varphi)$ for all $\varphi \in C_0^\infty$ and then $u = \eta_j u$ lies in $\mathscr{B}^c_{p',1/k^\vee}(G)$.

On the other hand, assume that u lies in $\mathscr{B}^c_{p',1/k^\vee}(G)$. Let $j \in \mathbf{N}$ be such that $\psi_j u = u$. In virtue of (2.3) we obtain

$$|U\varphi| = |(\psi_j u)(\varphi)| = |u(\psi_j\varphi)| \leqq \|u\|_{p',1/k^\vee} \, \|\psi_j\varphi\|_{p,k}$$

for all $\varphi \in C_0^\infty(G)$ and then the proof is complete ∎

Using general properties of LF-spaces (cf. [11: pp. 126–149]) one sees also

Lemma 3.3: *Suppose that V is in the dual space $\mathscr{B}^c_{p',1/k^\vee}(G)^+$ of $\mathscr{B}^c_{p',1/k^\vee}(G)$. Then there exists a unique element $v \in \mathscr{B}^{loc}_{p,k}(G)$ such that $V\varphi = v(\varphi)$ for all $\varphi \in C_0^\infty(G)$.*

Conversely, the linear form $V: C_0^\infty(G) \to \mathbf{C}$ such that $V\varphi = w(\varphi)$ where $w \in \mathscr{B}^{loc}_{p,k}(G)$ has a unique continuous extension on $\mathscr{B}^c_{p',1/k^\vee}(G)$.

In virtue of Lemma 3.2 there exists a linear bijection $\lambda: \mathscr{B}^c_{p',1/k^\vee}(G) \to \mathscr{B}^{loc}_{p,k}(G)^+$ such that

$$(\lambda u)(\varphi) = u(\varphi) \qquad \text{for all } \varphi \in C_0^\infty(G).$$

Similarly by Lemma 3.3 one sees that there exists a linear bijection $\varkappa: \mathscr{B}^{loc}_{p,k}(G) \to \mathscr{B}^c_{p',1/k^\vee}(G)^+$ such that

$$(\varkappa v)(\varphi) = v(\varphi) \qquad \text{for all } \varphi \in C_0^\infty(G).$$

3.2. Define a linear operator $\Gamma^\#_{p,k}(G): \mathscr{B}^c_{p,k}(G) \to \mathscr{B}^c_{p,k}(G)$ through the requirement

$$\left.\begin{aligned} D(\Gamma^\#_{p,k}(G)) = \{v \in \mathscr{B}^c_{p,k}(G) \mid & \text{ there exists an element } h \in \mathscr{B}^c_{p,k}(G) \\ & \text{such that } v(L(x, D)\varphi) = h(\varphi) \text{ for all } \varphi \in C_0^\infty\}, \\ \Gamma^\#_{p,k}(G)\, v = h. & \end{aligned}\right\}$$

The connection between the dual operator $\Lambda^\sim_{p,k}(G)^+: \mathscr{B}^{loc}_{p,k}(G)^+ \to \mathscr{B}^{loc}_{p,k}(G)^+$ and the operator $\Gamma^\#_{p',1/k^\vee}(G)$ is given (for $p > 1$) in the following

Theorem 3.4: *The operators $\Lambda^\sim_{p,k}(G)^+$ and $\Gamma^\#_{p',1/k^\vee}(G)$ satisfy the relation $\Gamma^\#_{p',1/k^\vee}(G) = \lambda^{-1} \circ \Lambda^\sim_{p,k}(G)^+ \circ \lambda$.*

Proof: Suppose that v lies in $D(\Gamma^\#_{p',1/k^\vee}(G))$ and that $\Gamma^\#_{p',1/k^\vee}(G)\, v = h$. Let V and H be the images $V = \lambda v$ and $H = \lambda h$. Then one has $V\varphi = v(\varphi)$ and $H\varphi = h(\varphi)$ for all $\varphi \in C_0^\infty(G)$. Hence we obtain $V(\Lambda_{p,k}(G)\varphi) = v(L(x, D)\varphi) = h(\varphi) = H\varphi$ for all $\varphi \in C_0^\infty(G)$ and then by the definition of the smallest closed extension $\Lambda^\sim_{p,k}(G)$ of

$\Lambda_{p,k}^{\sim}(G)$ one sees that $V(\Lambda_{p,k}^{\sim}(G)\,u) = Hu$ for all $u \in D(\Lambda_{p,k}^{\sim}(G))$. Thus V lies in $D(\Lambda_{p,k}^{\sim}(G)^+)$ and $\Lambda_{p,k}^{\sim}(G)^+ V = H$. This proves that $\Gamma^{\#}_{p',1/k^{\vee}}(G) \subset \lambda^{-1} \circ \Lambda_{p,k}^{\sim}(G)^+ \circ \lambda$. The converse can be seen with the similar conclusions ∎

4. Closedness of ranges $R(\Lambda_{p,k}^{\sim}(G))$ and $R(\Gamma^{\#}_{p',1/k^{\vee}}(G))$

4.1. We assume further that p lies in the interval $(1, \infty)$ and that k is in $\mathcal{K}$. The *weak⁺-topology* in the dual space $\mathcal{B}^{\mathrm{loc}}_{p,k}(G)^+$ is a locally convex (Hausdorff) topology defined by the semi-norms

$$p_u(U) = |U(u)|, \qquad u \in \mathcal{B}^{\mathrm{loc}}_{p,k}(G).$$

The *strong topology* in $\mathcal{B}^{\mathrm{loc}}_{p,k}(G)^+$ is a locally convex (Hausdorff) topology defined by the semi-norms

$$p_B(U) = \sup_{u \in B} \{|U(u)|\}, \qquad B \subset \mathcal{B}^{\mathrm{loc}}_{p,k}(G) \text{ bounded}.$$

The following lemma is an immediate consequence of the theory of LF-spaces (cf. [2: pp. 37–53]).

Lemma 4.1: *The range $R(\Gamma^{\#}_{p',1/k^{\vee}}(G))$ is closed in $\mathcal{B}^{c}_{p',1/k^{\vee}}(G)$ if and only if for each $j \in \mathbb{N}$ the linear subspace*

$$R(\Gamma^{\#}_{p',1/k^{\vee}}(G)) \cap \mathcal{B}^{c}_{p',1/k^{\vee}}(K_j)$$

is closed in $\mathcal{B}_{p',1/k^{\vee}}$.

Let p_j be the semi-norms defined by (2.1) and let $d(\cdot,\cdot)$ be the metric (2.2). Then one has

Lemma 4.2: *For each $\varrho > 0$ there exists a number $j(\varrho) \in \mathbb{N}$ such that*

$$\{u \in \mathcal{B}^{\mathrm{loc}}_{p,k}(G) \mid p_{j(\varrho)}(u) < \varrho/2\}$$
$$\subset B_d(0, \varrho) := \{u \in \mathcal{B}^{\mathrm{loc}}_{p,k}(G) \mid d(u, 0) < \varrho\}.$$

Proof: The proof follows by a simple calculation by taking into account that $p_j \leqq p_{j+1}$ and that $\sum_{j=1}^{\infty} 1/2^j = 1$ ∎

Furthermore we need the next

Lemma 4.3: *Let M be a subset of $\mathcal{B}^{\mathrm{loc}}_{p,k}(G)^+$ such that for each $u \in \mathcal{B}^{\mathrm{loc}}_{p,k}(G)$ there exists a constant $C_u > 0$ with which*

$$|U(u)| \leqq C_u \quad \text{for all } U \in M. \tag{4.1}$$

Then one can find a constant $C > 0$ and a number $j \in \mathbb{N}$ such that

$$|U(u)| \leqq C\, \|\psi_j u\|_{p,k} \qquad \text{for all } U \in M \text{ and } u \in \mathcal{B}^{\mathrm{loc}}_{p,k}(G).$$

Proof: In virtue of the Uniform Boundedness Theorem and by (4.1) there exists a number $\varrho > 0$ such that

$$\sup_{U \in M} |U(u)| \leqq 1 \qquad \text{for all } u \in B_d(0, \varrho).$$

Let $j(\varrho)$ be as in Lemma 4.2. Furthermore, let u be in $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)$. If $p_{j(\varrho)}(u) \neq 0$, one sees that $(\varrho/4)\, u/p_{j(\varrho)}(u)$ lies in $B_d(0, \varrho)$ and then $|U(u)| \leqq (4/\varrho)\, p_{j(\varrho)}(u)$. If $p_{j(\varrho)}(u) = 0$, one sees that qu belongs to $B_d(0, \varrho)$ with each $q > 0$ (since $p_{j(\varrho)}(qu) = 0$). Hence $|U(u)| \leqq 1/q$ for all $q > 0$, which implies that $|U(u)| = 0$. Thus one has

$$|U(u)| \leqq \frac{4}{\varrho}\, p_{j(\varrho)}(u) \leqq \frac{4}{\varrho} \sum_{j=1}^{j(\varrho)} \|\psi_{j(\varrho)}\|_{1,M_k}\, \|\psi_l u\|_{p,k}$$

where $l \in \mathbf{N}$ is so large that $\operatorname{supp} \psi_j \subset \operatorname{int} K_l$ for all $j \in \{1, \ldots, j(\varrho)\}$. This completes the proof ∎

4.2. A subset M of $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)^+$ is said to be *sequentially weak$^+$-closed* if it satisfies the following condition: Let $\{U_n\}$ be a sequence on M such that $U_n \to U$ with some $U \in \mathcal{B}_{p,k}^{\mathrm{loc}}(G)^+$ with respect to the weak$^+$-topology. Then U lies in M. We now show

Theorem 4.4: *The range $R\big(\Lambda_{p,k}^{\sim}(G)^+\big)$ is sequentially weak$^+$-closed if and only if the range $R\big(\Gamma_{p',1/k^\vee}^{\#}(G)\big)$ is closed.*

Proof: A. Suppose that $R\big(\Gamma_{p',1/k^\vee}^{\#}(G)\big)$ is closed. Let $\{U_n\} = \{\Lambda_{p,k}^{\sim}(G)^+ W_n\}$ be a sequence such that with some $U \in \mathcal{B}_{p,k}^{\mathrm{loc}}(G)^+$ one has

$$U_n(v) \to U(v) \qquad \text{for each } v \in \mathcal{B}_{p,k}^{\mathrm{loc}}(G). \tag{4.2}$$

In virtue of Lemma 4.3 there exist $C > 0$ and $j \in \mathbf{N}$ with which

$$|U_n(v)| \leqq C\, \|\psi_j v\|_{p,k} \qquad \text{for all } n \in \mathbf{N} \text{ and } v \in \mathcal{B}_{p,k}^{\mathrm{loc}}(G). \tag{4.3}$$

Let u_n be a distribution in $\mathcal{B}_{p',1/k^\vee}^{c}(G)$ such that $u_n = \lambda^{-1}(U_n)$ and let $u = \lambda^{-1}(U)$. Then by (4.3)

$$\operatorname{supp} u_n \subset \operatorname{supp} \psi_j \tag{4.4}$$

and

$$\|u_n\|_{p',1/k^\vee} \leqq C\, \|\psi_j\|_{1,M_k} \qquad \text{for all } n \in \mathbf{N}. \tag{4.5}$$

In according to (4.4) there exists $m \in \mathbf{N}$ so that $\{u_n \mid n \in \mathbf{N}\} \cup \{u\} \subset \mathcal{B}_{p',1/k^\vee}^{c}(K_m)$. Since $\mathcal{B}_{p',1/k^\vee}^{c}(K_m)$ as a closed subspace of a reflexive space $\mathcal{B}_{p',1/k^\vee}$ is reflexive one sees by (4.5) that there exists a subsequence $\{u_n'\}$ of $\{u_n\}$ and an element u' of $\mathcal{B}_{p',1/k^\vee}^{c}(K_m)$ such that

$$V(u_n') \to V(u') \qquad \text{for each } V \in \mathcal{B}_{p',1/k^\vee}^{c}(K_m)^+. \tag{4.6}$$

In view of Mazur's Lemma one can find elements u_n'' from $\{u_n'\}$ such that for each $j \in \mathbf{N}$ there exist $n_j \in \mathbf{N}$ and $\alpha_n^j \geqq 0$ with which

$$\left\| \sum_{n=1}^{n_j} \alpha_n^j u_n'' - u' \right\|_{p',1/k^\vee} \leqq 1/j \tag{4.7}$$

and

$$\sum_{n=1}^{n_j} \alpha_n^j = 1 \qquad \text{for each } j \in \mathbf{N}.$$

In virtue of Theorem 3.4 the elements u_n lie in $R\big(\Gamma_{p',1/k^\vee}^{\#}(G)\big)$ and then by the assumption and by (4.7) u' lies in $R\big(\Gamma_{p',1/k^\vee}^{\#}(G)\big)$. Furthermore, in view of (4.6) we have $u_n'(\varphi) \to u'(\varphi)$ for each $\varphi \in C_0^\infty(G)$. Hence by (4.2) $u = u'$. Let $w \in \mathcal{B}_{p',1/k^\vee}^{c}(G)$ be such that $\Gamma_{p',1/k^\vee}^{\#}(G)\, w = u$. Then one sees that $\lambda w \in D\big(\Lambda_{p,k}^{\sim}(G)^+\big)$ and $\Lambda_{p,k}^{\sim}(G)^+ (\lambda w) = U$.

B. On the other hand suppose that $R(\Lambda_{p,k}^{\sim}(G)^+)$ is sequentially weak$^+$-closed. Let $\{u_n\}$ be a sequence in $R(\Gamma_{p',1/k^\vee}^{\#}(G)) \cap \mathcal{B}_{p',1/k^\vee}^{c}(K_m)$ and u in $\mathcal{B}_{p',1/k^\vee}^{c}(G)$ such that $\|u_n - u\|_{p',1/k^\vee} \to 0$ with $n \to \infty$. Furthermore, let $U_n \in \mathcal{B}_{p,k}^{\mathrm{loc}}(G)^+$ and $U \in \mathcal{B}_{p,k}^{\mathrm{loc}}(G)^+$ be defined by $U_n = \lambda u_n$ and $U = \lambda u$. Then due to Lemma 3.4 the functional U_n lies in the range $R(\Lambda_{p,k}^{\sim}(G)^+)$. In addition one has

$$|U_n(v) - U(v)| \geqq \|u_n - u\|_{p',1/k^\vee} \, \|\psi_{m+1}u\|_{p,k} \qquad \text{for all } v \in \mathcal{B}_{p,k}^{\mathrm{loc}}(G).$$

This shows that $\{U_n\}$ is converging to U in the weak$^+$-topology and then $U = \Lambda_{p,k}^{\sim}(G)^+ W$ with some $W \in D(\Lambda_{p,k}^{\sim}(G)^+)$. Hence by Theorem 3.4 u lies in $R(\Gamma_{p',1/k^\vee}^{\#}(G))$. This completes the proof ∎

4.3. In virtue of the general theory of linear densely defined operators in Frechet spaces we get

Lemma 4.5: *The range $R(\Lambda_{p,k}^{\sim}(G))$ is closed in $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)$ if and only if the range $R(\Lambda_{p,k}^{\sim}(G)^+)$ is weak$^+$-closed in $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)^+$.*

For the proof cf. [1]. In addition we have

Lemma 4.6: *The range $R(\Lambda_{p,k}^{\sim}(G)^+)$ is weak$^+$-closed in $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)^+$ if and only if it is sequentially weak$^+$-closed.*

Proof: Since $C_0^\infty(G)$ is dense in $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)$ one sees that the spaces $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)$ are separable. Hence the convex set $R(\Lambda_{p,k}^{\sim}(G)^+)$ is weak$^+$-closed if and only if it is sequentially weak$^+$-closed (cf. [6: p. 273]) ∎

Combining Theorem 4.4 and Lemmas 4.5 and 4.6 we get

Theorem 4.7: *The range $R(\Lambda_{p,k}^{\sim}(G))$ is closed in $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)$ if and only if the range $R(\Gamma_{p',1/k^\vee}^{\#}(G))$ is closed in $\mathcal{B}_{p',1/k^\vee}^{c}(G)$.*

Remark: The assertion of Theorem 4.7 can be shown also by noting that $\lambda: \mathcal{B}_{p',1/k^\vee}^{c}(G) \to \mathcal{B}_{p,k}^{\mathrm{loc}}(G)^+$ is a homeomorphism when $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)^+$ is equipped with the strong dual topology. The continuity of λ is easily seen. For $p \in (1, \infty)$ the spaces $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)$ are reflexive and then $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)^+$ equipped with the strong dual topology is bornological (cf. [6: p. 400]). Hence also the continuity of λ^{-1} can be shown (cf. [12: p. 46]).

Since λ is a homeomorphism one sees that $R(\Lambda_{p,k}^{\sim}(G)^+)$ is strongly closed if and only if $R(\Gamma_{p',1/k^\vee}^{\#}(G))$ is closed. Furthermore, one knows that $R(\Lambda_{p,k}^{\sim}(G))$ is closed if and only if $R(\Lambda_{p,k}^{\sim}(G)^+)$ is strongly closed (since $p \in (1, \infty)$) (cf. [1: p. 84]).

We still give some characterizations of the closedness of $R(\Lambda_{p,k}^{\sim}(G))$.

Lemma 4.8: *For every $v \in \mathcal{B}_{p,k}^{\mathrm{loc}}(G)$ and $f \in \mathcal{B}_{p',1/k^\vee}^{c}(G)$ one has $(\varkappa v)(f) = (\lambda f)(v)$.*

Proof: Let v be in $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)$. Then there exists a sequence $\{\varphi_n\} \subset C_0^\infty(G)$ such that $\varphi_n \to v$ in $\mathcal{B}_{p,k}^{\mathrm{loc}}(G)$ and so we get

$$\lim_{n\to\infty} f(\varphi_n) = (\lambda f)(v) \qquad \text{for each } f \in \mathcal{B}_{p',1/k^\vee}^{c}(G). \tag{4.8}$$

Furthermore, let $\{\psi_m\} \subset C_0^\infty(\mathbf{R}^n)$ be a sequence such that $\|\psi_m - f\|_{p',1/k^\vee} \to 0$ with $m \to \infty$. Choose $\eta \in C_0^\infty(G)$ such that $\eta(x) \equiv 1$ in an open neighbourhood of supp f. Then we have $\|\eta\psi_m - f\|_{p',1/k^\vee} \to 0$ with $m \to \infty$. For every $n \in \mathbf{N}$ and $\varphi \in C_0^\infty(\mathbf{R}^n)$ we obtain

$$f(\varphi_n) = \lim_{m\to\infty} (\eta\psi_m)(\varphi_n) = \lim_{m\to\infty} \varphi_n(\eta\psi_m) = (\varkappa\varphi_n)(f) \tag{4.9}$$

and

$$|(\varkappa\varphi_n)(\eta\varphi) - (\varkappa v)(\eta\varphi)|$$
$$= |(\eta\varphi_n - \eta v)(\varphi)| \leqq \|\eta\varphi_n - \eta v\|_{p,k} \|\varphi\|_{p',1/k^\vee}$$

and then

$$|(\varkappa\varphi_n)(f) - (\varkappa v)(f)| \leqq \|\eta\varphi_n - \eta v\|_{p,k} \|f\|_{p',1/k^\vee}.$$

Thus by (4.9) and (4.8) we obtain the assertion ∎

Theorem 4.9: *The range $R(\Lambda_{p,k}^{\sim}(G))$ is closed in $\mathcal{B}_{p,k}^{\text{loc}}(G)$ if and only if*

$$R(\Gamma_{p',1/k^\vee}^{\#}(G)) = \{f \in \mathcal{B}_{p',1/k^\vee}^{c}(G) \mid V(f) = 0 \quad \textit{for all } V \in \varkappa(N(\Lambda_{p,k}^{\sim}(G)))\}. \tag{4.10}$$

Proof: The range $R(\Lambda_{p,k}^{\sim}(G))$ is closed if and only if

$$R(\Lambda_{p,k}^{\sim}(G)^+) = \{F \in \mathcal{B}_{p,k}^{\text{loc}}(G)^+ \mid Fv = 0 \quad \text{for all } v \in N(\Lambda_{p,k}^{\sim}(G))\} \tag{4.11}$$

(cf. [1: p. 57]). Suppose that $R(\Lambda_{p,k}^{\sim}(G))$ is closed. Let f be in $R(\Gamma_{p',1/k^\vee}^{\#}(G))$. Then by Theorem 3.4 one has $\lambda f \in R(\Lambda_{p,k}^{\sim}(G)^+)$ and so by (4.11) $(\lambda f)(v) = 0$ for all $v \in N(\Lambda_{p,k}^{\sim}(G))$. In virtue of Lemma 4.8 $(\varkappa v)(f) = (\lambda f)(v) = 0$ for all $v \in N(\Lambda_{p,k}^{\sim}(G))$. Similarly one sees that if $f \in \mathcal{B}_{p',1/k^\vee}^{c}(G)$ such that $(\varkappa v)(f) = 0$ for all $v \in N(\Lambda_{p,k}^{\sim}(G))$, then $\lambda f \in R(\Lambda_{p,k}^{\sim}(G)^+)$ and so $f \in R(\Gamma_{p',1/k^\vee}^{\#}(G))$. Hence (4.10) is valid.

On the other hand, in the same way one sees that (4.10) implies (4.11) and so the validity of (4.10) implies the closedness of $R(\Lambda_{p,k}^{\sim}(G))$. This finishes the proof ∎

Remark: Because $R(\Lambda_{p,k}^{\sim}(G))$ is closed if and only if

$$R(\Lambda_{p,k}^{\sim}(G)) = \{f \in \mathcal{B}_{p,k}^{\text{loc}}(G) \mid V(f) = 0 \quad \text{for all} \quad V \in N(\Lambda_{p,k}^{\sim}(G)^+)\}$$

(cf. [1]) and because by Theorem 3.4 one has

$$N(\Lambda_{p,k}^{\sim}(G)^+) = \lambda(N(\Gamma_{p',1/k^\vee}^{\#}(G)))$$

one also sees that $R(\Lambda_{p,k}^{\sim}(G))$ is closed if and only if the relation

$$R(\Lambda_{p,k}^{\sim}(G)) = \{f \in \mathcal{B}_{p,k}^{\text{loc}}(G) \mid V(f) = 0 \text{ for all } V \in \lambda(N(\Gamma_{p',1/k^\vee}^{\#}(G)))\}$$

holds.

4.4. This section considers the validity of the relation

$$R(\Lambda_{p,k}^{\sim}(G)) = \mathcal{B}_{p,k}^{\text{loc}}(G).$$

Let $\overline{R(\Lambda_{p,k}^{\sim}(G))}$ denote the closure of $R(\Lambda_{p,k}^{\sim}(G))$. We begin with

Lemma 4.10: *The relation*

$$\overline{R(\Lambda_{p,k}^{\sim}(G))} = \mathcal{B}_{p,k}^{\text{loc}}(G) \tag{4.12}$$

is valid if and only if the relation

$$N(\Gamma_{p',1/k^\vee}^{\#}(G)) = \{0\} \tag{4.13}$$

holds.

Proof: In virtue of Theorem 3.4 one sees that

$$N\left(\Lambda_{p,k}^{\sim}(G)^{+}\right) = \lambda\left(N\left(\Gamma_{p',1/k^{\vee}}^{\#}(G)\right)\right). \tag{4.14}$$

Suppose that (4.12) holds. Let u lie in $N\left(\Gamma_{p',1/k^{\vee}}^{\#}(G)\right)$. For every $g \in \mathcal{B}_{p,k}^{\text{loc}}(G)$ one can find a sequence $\{g_n\} \subset R\left(\Lambda_{p,k}^{\sim}(G)\right)$ such that $g_n \to g$ in $\mathcal{B}_{p,k}^{\text{loc}}(G)$. Hence we obtain by (4.14)

$$(\lambda u)(g) = \lim_{n\to\infty} (\lambda u)\, g_n = 0 \qquad \text{for all } g \in \mathcal{B}_{p,k}^{\text{loc}}(G)$$

and then $u = 0$. This shows the validity of (4.13).

Conversely, suppose that (4.13) is valid. Assume that (4.12) doesn't hold. Then one can find an element U from $\mathcal{B}_{p,k}^{\text{loc}}(G)^{+}$ such that $U \neq 0$ but $U(g) = 0$ for every $g \in \overline{R\left(\Lambda_{p,k}^{\sim}(G)\right)}$ (cf. [11: pp. 181–194]). Hence U lies in $N\left(\Lambda_{p,k}^{\sim}(G)^{+}\right) = \lambda\left(N\left(\Gamma_{p',1/k^{\vee}}^{\#}(G)\right)\right)$ and $U \neq 0$. This is a contradiction ∎

Combining Theorem 4.7 and Lemma 4.10 we obtain

Theorem 4.11: *The relation $R\left(\Lambda_{p,k}^{\sim}(G)\right) = \mathcal{B}_{p,k}^{\text{loc}}(G)$ is valid if and only if the following two conditions hold:*

$$\left.\begin{aligned} &R\left(\Gamma_{p',1/k^{\vee}}^{\#}(G)\right) \textit{ is closed in } \mathcal{B}_{p',1/k^{\vee}}^{c}(G) \\ &N\left(\Gamma_{p',1/k^{\vee}}^{\#}(G)\right) = 0. \end{aligned}\right\} \tag{4.15}$$

The following two criteria are sufficient to imply the validity of (4.15): For each $j \in \mathbf{N}$ there exists an $m \in \mathbf{N}$ such that the inclusion $\operatorname{supp} \Gamma_{p',1/k^{\vee}}^{\#}(G)\, u \subset K_j$ implies that $\operatorname{supp} u \subset K_m$ and, with some $\gamma > 0$,

$$\left\|\Gamma_{p',1/k^{\vee}}^{\#}(G)\, u\right\|_{p',1/k^{\vee}} \geqq \gamma\, \|u\|_{p',1/k^{\vee}} \qquad \text{for all } u \in D\left(\Gamma_{p',1/k^{\vee}}^{\#}(G)\right).$$

4.5. We shall establish some applications in the case when $p = 2$ and $k \equiv 1$. Denote the open ball $B(0, R)$, $R > 0$, by B. Let B_l (with $l \in \mathbf{N}$, $l > 1/R$) be a subset of B defined by $B_l = B\left(0, R - (1/l)\right)$. Furthermore, let $L_B'^{\sim}: L_2(B) \to L_2(B)$ be the minimal extension of $L'(x, D)$ in $L_2(B)$. We show

Theorem 4.12: *Suppose that the partial differential operator $L(x, D)$ with $C^\infty(B)$-coefficients obeys the following conditions:*

(i) *$\Gamma_{2,1}^{\#}(B)\, v = L_B'^{\sim} v$ for all $v \in D\left(\Gamma_{2,1}^{\#}(B)\right)$,*

(ii) *there exist constants $C_1 > 0$ and $C_2 \geqq 0$ such that*

$$\|L'(x, D)\, \varphi\|_0 := \|L'(x, D)\, \varphi\|_{2,1} \geqq C_1\, \|\varphi\|_{2,k^{\sim}} - C_2\, \|\varphi\|_0 \tag{4.16}$$

for all $\varphi \in C_0^\infty(B)$, where $k^{\sim} \in \mathcal{K}$ such that $k^{\sim}(\xi) \to \infty$ with $|\xi| \to \infty$,

(iii) *for each $\eta \in \mathbf{R}^n$ there exist constants $C > 0$ and $t_0 \geqq 0$ such that*

$$\|e^{t\langle\eta,\cdot\rangle}\varphi\|_0 \leqq C\, \|e^{t\langle\eta,\cdot\rangle} L'(x, D)\, \varphi\|_0 \tag{4.17}$$

for all $\varphi \in C_0^\infty(B)$ and $t \geqq t_0$.

Then the range $R\left(\Lambda_{2,1}^{\sim}(B)\right)$ is closed in $\mathcal{B}_{2,1}^{\text{loc}}(B) = L_2^{\text{loc}}(B)$.

Proof: **A.** Let v be in $D(L_B'^{\sim})$; then there exists a sequence $\{\varphi_n\} \subset C_0^\infty(B)$ such that $\|\varphi_n - v\|_0 + \|L'(x, D)\, \varphi_n - L_B'^{\sim} v\|_0 \to 0$ with $n \to \infty$. Thus in virtue of (4.17)

$$\|e^{t\langle\eta,\cdot\rangle} v\|_0 \leqq C\, \|e^{t\langle\eta,\cdot\rangle} L_B'^{\sim} v\|_0 \qquad \text{for all } v \in D(L_B'^{\sim}) \text{ and } t \geqq t_0. \tag{4.18}$$

Suppose that $L_B'^{\sim} v$ lies in $\mathscr{B}^c_{2,1}(\bar{B}_m)$. Let x_0 be in $B \setminus \bar{B}_{m+1}$ and let α be in the interval $(0, 1)$ such that $R - (1/m) - \alpha(R - (1/m + 1)) < 0$. Furthermore, choose $\varepsilon > 0$ such that $(x_0, x) \geqq \alpha |x_0|^2$ for all $x \in B(x_0, \varepsilon)$ and that $B(x_0, \varepsilon) \subset B$. Then we obtain by (4.18)

$$e^{\alpha t|x_0|^2} \Big(\int_{B(x_0,\varepsilon)} |v|^2 \Big)^{1/2} \leqq \|e^{t(x_0,\cdot)} v\|_0 \leqq C_{x_0} \|e^{t(x_0,\cdot)} L_B'^{\sim} v\|_0$$
$$\leqq C_{x_0} e^{t|x_0|(R-(1/m))} \|L_B'^{\sim} v\|_0$$

and so

$$\Big(\int_{B(x_0,\varepsilon)} |v|^2 \Big)^{1/2} \leqq C_{x_0} e^{t|x_0|(R-(1/m)-\alpha(R-(1/m+1)))} \|L_B'^{\sim} v\|_0.$$

Letting $t \to \infty$ we find that $v(x) = 0$ a.e. in $B(x_0, \varepsilon)$. Hence v lies in $\mathscr{B}^c_{2,1}(\bar{B}_{m+1})$.

B. In virtue of (4.16) the range $R(L_B'^{\sim})$ of $L_B'^{\sim}$ is closed in $L_2(B)$ (cf. [9]). Suppose that $\Gamma^{\#}_{2,1}(B) v_n \to f$ in $\mathscr{B}^c_{2,1}(\bar{B}_m)$. Then by (i) we have that $L_B'^{\sim} v_n \to f$ in $L_2(B)$ and so f lies in $R(L_B'^{\sim})$. Let v be the solution of $L_B'^{\sim} v = f$. Since f lies in $\mathscr{B}^c_{2,1}(\bar{B}_m)$, the Part A implies that v belongs to $\mathscr{B}^c_{2,1}(\bar{B}_{m+1})$. Hence f lies in $R(\Gamma^{\#}_{2,1}(B))$. The assertion follows from Theorem 4.7 ∎

Remark: Suppose that there exists a $k \in \mathscr{K}$ such that $\|L'(x, D) \varphi\|_0 \leqq C \|\varphi\|_{2,k}$ for all $\varphi \in C_0^\infty(B)$ and $D(\Gamma^{\#}_{2,1}(B)) \subset \mathscr{B}_{2,k}$. Then (i) holds. The operators with constant coefficients satisfy the condition (i) (this follows from Theorem 2.1, for example). The inequality (4.16) holds for the operators with constant coefficients if and only if $L^{\sim}(\xi) := \Big(\sum_{|\alpha| \leqq r} |L^{(\alpha)}(\xi)|^2 \Big)^{1/2} \geqq \gamma k^{\sim}(\xi)$ with $\gamma > 0$ (cf. [9]). The assumption (ii) can be replaced with $\|L'(x, D) \varphi\|_0 \geqq \gamma \|\varphi\|_0$ for all $\varphi \in C_0^\infty(B)$ (which holds for all operators with constant coefficients, since B is bounded).

Furthermore we have

Theorem 4.13: *Let G be a bounded open set in $\mathbf{R}^n$ and let $L(D)$ be a non-trivial partial differential operator with constant coefficients. Then we have*

$$\|e^{t(\eta,\cdot)} \varphi\|_0 \leqq C \|e^{t(\eta,\cdot)} L(D) \varphi\|_0$$

for all $\varphi \in C_0^\infty(G)$, $t \geqq 0$ and $\eta \in \mathbf{R}^n$.

Proof: Let $\eta \in \mathbf{R}^n$ and let $t \geqq 0$. Then we obtain for all $\varphi \in C_0^\infty(G)$ and $\xi \in \mathbf{R}^n$

$$|F(e^{t(\eta,\cdot)}\varphi)(\xi)| = \Big| \int_{\mathbf{R}^n} e^{t(\eta,y)} \varphi(y) e^{-i(\xi,y)} dy \Big|$$
$$= |(\mathscr{L}\varphi)(\xi + it\eta)|$$
$$\leqq C(L) \int_{|\sigma| \leqq 1} |(\mathscr{L}\varphi)(\xi + it\eta + \sigma) L(\xi + it\eta + \sigma)| d\sigma$$
$$= C(L) \int_{|\sigma| \leqq 1} |F(e^{i(\sigma,\cdot)+t(\eta,\cdot)} L(D) \varphi)(\xi)| d\sigma,$$

where $\mathscr{L}\varphi$ is the Fourier-Laplace transform of φ and where $d\sigma$ denotes the Lebesgue measure in $\mathbf{R}^{2n}$ (cf. [12: p. 186]). By $C(L)$ we denoted a constant which depends only on $L(D)$. Let $\phi \in C_0^\infty$ such that $\phi(x) = 1$ for all $x \in G$. Then one has by the Parseval Theorem

$$\|e^{t(\eta,\cdot)}\varphi\|_0 \leqq C(L) \Big(\int_{|\sigma| \leqq 1} 1 \Big)^{1/2} \|e^{i(\sigma,\cdot)}\phi\|_{1,1} \|e^{t(\eta,\cdot)} L(D) \varphi\|_0$$

(cf. [5: p. 39]). In virtue of the Paley-Wiener Theorem we find that $\|e^{i(\sigma,\cdot)}\phi\|_{1,1} \leqq C$ or all $|\sigma| \leqq 1$ (cf. [9]) and then the proof is ready ∎

Let G be an open set in $\mathbf{R}^n$ and let $L(x, D)$ be a partial differential operator with $C^\infty(G)$-coefficients. Furthermore, let I denote an open interval in $\mathbf{R}$. Using the idea of [8: pp. 358—362] we define an operator $\mathbf{L}_\eta$ in $C_0^\infty(I \times G)$ by

$$(\mathbf{L}_\eta u)(s, x) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbf{R}} \int_{\mathbf{R}^n} L(x, \xi + \mathrm{i}\, |\tau|\, \eta)\, (\bar{F}u)(\tau, \xi)\, \mathrm{e}^{\mathrm{i}(\langle x,\xi\rangle + \tau s)}\, d\tau\, d\xi,$$

where $\bar{F}$ is the Fourier transform in $\mathbf{R} \times \mathbf{R}^n$.

Theorem 4.14: *Suppose that for each $\eta \in \mathbf{R}^n$ there exists a constant $\gamma > 0$ such that*

$$\|\mathbf{L}_\eta u\|_0 := \|\mathbf{L}_\eta u\|_{L_2(I\times G)} \geqq \gamma \sum_{\alpha \neq 0} \|D_s^{|\alpha|} L^{(\alpha)}(x, D_x)\, u\|_0 \tag{4.19}$$

for all $u \in C_0^\infty(I \times G)$, where $L^{(\alpha)}(x, \xi) = (\partial^{|\alpha|}/\partial \xi^\alpha)\, L(x, \xi)$. Then for each $\eta \in \mathbf{R}^n$ there exist constants $C > 0$ and $t_0 \geqq 0$ such that

$$\sum_{\alpha \neq 0} t^{|\alpha|}\, \|\mathrm{e}^{t\langle \eta, \cdot\rangle} L^{(\alpha)}(x, D)\, \varphi\|_0 \leqq C\, \|\mathrm{e}^{t\langle \eta, \cdot\rangle} L(x, D)\, \varphi\|_0 \tag{4.20}$$

for all $\varphi \in C_0^\infty(G)$ and $t \geqq t_0$.

Proof: A. Choose a function $\phi \in C_0^\infty(I) \setminus \{0\}$ and define

$$u(s, x) = \phi(s)\, \varphi(x)\, \mathrm{e}^{\mathrm{i}ts}\, \mathrm{e}^{t\langle \eta, x\rangle},$$

where $\varphi \in C_0^\infty(G)$, $t \geqq 1$ and $\eta \in \mathbf{R}^n$. Then we have

$$\begin{aligned}(\bar{F}u)(\tau, \xi) &= \Big(\int_{\mathbf{R}} \phi(s)\, \mathrm{e}^{\mathrm{i}ts}\, \mathrm{e}^{-\mathrm{i}\tau s}\, ds \Big) \Big(\int_{\mathbf{R}^n} \varphi(x)\, \mathrm{e}^{t\langle \eta, x\rangle}\, \mathrm{e}^{-\mathrm{i}\langle \xi, x\rangle}\, dx \Big) \\ &= (F_1\phi)(\tau - t)\, (\mathcal{L}\varphi)(\xi + \mathrm{i}t\eta),\end{aligned}$$

where F_1 is the Fourier transform in $\mathbf{R}$. Hence we obtain by the Taylor's formula

$$\begin{aligned}&(\mathbf{L}_\eta u)(s, x) \\ &= \frac{1}{(2\pi)^{n+1}} \int_{\mathbf{R}} \int_{\mathbf{R}^n} L(x, \xi + \mathrm{i}\, |\tau|\, \eta)\, (\mathcal{L}\varphi)(\xi + \mathrm{i}t\eta)\, (F_1\phi)(\tau - t)\, \mathrm{e}^{\mathrm{i}(\langle x,\xi\rangle + \tau s)}\, d\tau\, d\xi \\ &= \Big(\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} L(x, \xi + \mathrm{i}t\eta)\, (\mathcal{L}\varphi)(\xi + \mathrm{i}t\eta)\, \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}\, d\xi \Big) \Big(\frac{1}{2\pi} \int_{\mathbf{R}} (F_1\phi)(\tau - t)\, \mathrm{e}^{\mathrm{i}\tau s}\, d\tau \Big) \\ &\quad + \frac{1}{(2\pi)^{n+1}} \sum_{\alpha \neq 0} \frac{1}{\alpha!} \Big(\int_{\mathbf{R}^n} L^{(\alpha)}(x, \xi + \mathrm{i}t\eta)\, (\mathcal{L}\varphi)(\xi + \mathrm{i}t\eta)\, \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}\, d\xi \Big) \\ &\quad \times \Big(\int_{\mathbf{R}} \mathrm{i}^{|\alpha|} \eta^\alpha (|\tau| - t)^{|\alpha|}\, (F_1\phi)(\tau - t)\, \mathrm{e}^{\mathrm{i}\tau s}\, d\tau \Big).\end{aligned}$$

We have

$$\begin{aligned}&\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} L^{(\alpha)}(x, \xi + \mathrm{i}t\eta)\, (\mathcal{L}\varphi)(\xi + \mathrm{i}t\eta)\, \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}\, d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} L^{(\alpha)}(x, \xi)\, (F\varphi)(\xi)\, \mathrm{e}^{\mathrm{i}\langle \xi - \mathrm{i}t\eta, x\rangle}\, d\xi \\ &= \big(L^{(\alpha)}(x, D)\, \varphi \big)(x)\, \mathrm{e}^{t\langle \eta, x\rangle}\end{aligned}$$

(cf. [11: pp. 309–310]) and

$$\left| \int_{\mathbf{R}} i^{|\alpha|} \eta^{\alpha} (|\tau| - t)^{|\alpha|} (F_1 \phi) (\tau - t) e^{i\tau s} d\tau \right|$$
$$\leqq |\eta^{\alpha}| \int_{\mathbf{R}} |(\tau - t)^{|\alpha|} (F_1 \phi) (\tau - t)| d\tau = |\eta^{\alpha}| \|D^{|\alpha|}\phi\|_{1,1}.$$

Thus we obtain

$$\|\mathbf{L}_{\eta} u\|_0 \leqq \|e^{t(\eta,\cdot)} L(x, D) \varphi\|_0 \|\phi\|_0$$
$$+ \frac{1}{2\pi} \sum_{\alpha \neq 0} \frac{1}{\alpha} |\eta^{\alpha}| m(I) \|D^{|\alpha|}\phi\|_{1,1} \|e^{t(\eta,\cdot)} L^{(\alpha)}(x, D) \varphi\|_0$$
$$\leqq \|e^{t(\eta,\cdot)} L(x, D) \varphi\|_0 \|\phi\|_0 \frac{C}{t} \sum_{\alpha \neq 0} t^{|\alpha|} \|e^{t(\eta,\cdot)} L^{(\alpha)}(x, D) \varphi\|_0, \tag{4.21}$$

where $m(I)$ is the measure of I and where C is large enough.

B. We now estimate the right-hand side of (4.19). For each $\alpha \neq 0$ we get

$$\|D_s^{|\alpha|}(\phi e^{its}) - t^{|\alpha|}\phi e^{its}\|_0 = \left\| \sum_{k<|\alpha|} \binom{|\alpha|}{k} (D_s^{|\alpha|-k}\phi) t^k e^{its} \right\|_0$$
$$\leqq \sum_{k<|\alpha|} \binom{|\alpha|}{k} \|D_s^{|\alpha|-k}\phi\|_0 t^k \leqq C_\alpha t^{|\alpha|} t^{-1}$$

and then

$$\sum_{\alpha \neq 0} \|D_s^{|\alpha|} L^{(\alpha)}(x, D_x) u\|_0$$
$$= \sum_{\alpha \neq 0} \|D_s^{|\alpha|}(\phi e^{its})\|_0 \|L^{(\alpha)}(x, D) (\varphi e^{t(\eta,\cdot)})\|_0$$
$$\geqq \sum_{\alpha \neq 0} t^{|\alpha|}(\|\phi\|_0 - C_\alpha t^{-1}) \|L^{(\alpha)}(x, D) (\varphi e^{t(\eta,\cdot)})\|_0$$
$$\geqq C \sum_{\alpha \neq 0} t^{|\alpha|} \|L^{(\alpha)}(x, D) (\varphi e^{t(\eta,\cdot)})\|_0 \tag{4.22}$$

(with $C > 0$) for all t large enough (say $t \geqq t_0'$). Let $g = \varphi e^{t(\eta,\cdot)}$. Then we find that

$$\sum_{\alpha \neq 0} t^{|\alpha|} \|e^{t(\eta,\cdot)} L^{(\alpha)}(x, D) (g e^{-t(\eta,\cdot)})\|_0$$
$$\leqq \sum_{\alpha \neq 0} \sum_{|\beta| \leqq r - |\alpha|} \frac{1}{\beta!} t^{|\alpha+\beta|} |\eta^{\beta}| \|L^{(\alpha+\beta)}(x, D) g\|_0$$
$$\leqq C \sum_{\alpha \neq 0} t^{|\alpha|} \|L^{(\alpha)}(x, D) g\|_0$$

and so by (4.22) (with a suitable $C > 0$)

$$\sum_{\alpha \neq 0} \|D_s^{|\alpha|} L^{(\alpha)}(x, D_x) u\|_0 \geqq C \sum_{\alpha \neq 0} t^{|\alpha|} \|e^{t(\eta,\cdot)} L^{(\alpha)}(x, D) \varphi\|_0 \tag{4.23}$$

for $t \geqq t_0'$. Hence one finds by (4.19), (4.21) and (4.23) that

$$\|\phi\|_0 \|e^{t(\eta,\cdot)} L(x, D) \varphi\|_0 \geqq C \sum_{\alpha \neq 0} t^{|\alpha|} \|e^{t(\eta,\cdot)} L^{(\alpha)}(x, D) \varphi\|_0$$

for t large enough (say $t \geqq t_0$). This completes the proof ∎

Remark: Suppose that there exists a $\alpha \in \mathbf{N}_0^n$ such that $L^{(\alpha)}(x, D) = q$, where $q \in C^\infty(B)$ so that $|q| \geqq \gamma > 0$. Then one gets (4.17) from (4.20).

From the theory presented in [8: pp. 358–370] one can show sufficient algebraic conditions for the validity of (4.19).

Suppose that $L(x, D)$ is an elliptic partial differential operator with real-analytic and $C^\infty(\overline{G})$-coefficients. Then using the Holmgren's uniqueness Theorem (cf. [8: p. 358]) and the method we applied in the proof of Theorem 4.12 one sees that $R(\Lambda_{2,1}^{\sim}(G))$ is closed when G is an open bounded set in $\mathbf{R}^n$. In the same way (by using [8: Cor. 2.9/p. 369) one sees that $R(\Lambda_{2,1}^{\sim}(\overline{G}))$ is closed when $L(x, D)$ is a second order elliptic partial differential operator with $C^\infty(\overline{G})$-coefficients such that a_σ is real-valued for $|\sigma| = r$ and when G is an open bounded set in $\mathbf{R}^n$.

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