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## **On the Elliptic Sturmian Theory for General Domains Compt 4 The Elliptic Sturmian The E. MÜLLER-PFEIFFER**

**-S** 

 

Per bekannte Vergleichssatz von Sturm und Picone fur gewohnliche, seibstadjungierte Differentialgleichungen zweiter Ordnung wird auf sclbstadjungierte elliptische Differentialglcichungen verailgemeinert. Dabei sind dàs Grundgebiet G und die Koeffizienten der Differential gleichung nicht notwendig-beschränkt, und es werden keine Regularitätsforderungen an den Rand  $\partial G$  gestellt.

*0:*

Известная теорема сравнения Штурма и Пиконе для обыкновенных самосопряженных дифференциальных уравнений второго порядка обобщается на самосопряженные эллиптические дифференциальные уравнения. При этом основная область  $G$  и коэффициенты дифференциального уравнения не обязательно ограничены, и условия регулярности для границы *дG* не требуются.

The well-known comparison theorem by Sturm and Picone for ordinary, self-adjoint, second The well-known comparison theorem by Sturm and Picone for ordinary, self-adjoint, second<br>order differential equations is extended to self-adjoint elliptic differential equations. The<br>basic domain G and the coefficients of basic domain  $G$  and the coefficients of the equation are not necessarily bounded, and no regularity hypotheses on the boundary  $\partial G$  are required.<br>Consider the differential equations regularity hypotheses on the boundary  $\partial G$  are required. *(and Picone split-adjoint equation are*  $\alpha$  *equined.<br>*  $(x \in [a, b]),$ The well-known comparison theorem by Sturm and Picone for ordina<br>
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regular

$$
\mathcal{A}_0 u \equiv -(P(x) u')' + Q(x) u = 0
$$
  

$$
\mathcal{A}_g u \equiv -(p(x) u')' + q(x) u = 0
$$
  $(x \in [a, b$ 

 $\mathcal{A}_q u \equiv -(p(x) u')' + q(x) u = 0$ <br>where  $P, p \in C^1[a, b]$  and  $Q, q \in C[a, b]$  are real-valued and  $P(x), p(x) > 0, x \in [a, b]$ . A wellknown version of the Sturm-Picone theorem is the following one (compare [8: Cor. 1], [13: *u* and the coefficients of the equation are not necessarily bounded, and no<br> *u* hypotheses on the boundary  $\partial G$  are required.<br> *u*  $\partial u = -(P(x) u')' + Q(x) u = 0$ <br>  $u \cdot \partial u = -(P(x) u')' + Q(x) u = 0$ <br>  $u \cdot \partial u = -(p(x) u')' + q(x) u = 0$ <br>  $p \in C^1[a, b]$ 

Theorem 1: *If there exists a real solution*  $u \neq 0$  of  $A_0u = 0$  such that

$$
u(a) = 0 = u(b)
$$
 and  $\int [p(u')^2 + qu^2] dx \leq 0$ ,

*then every real solution v of*  $A_qv = 0$  *is a constant multiple of u or has at least one zero in*  $(a, b)$ *.* 

In the following this theorem will be extended to self-adjoint, second order, elliptic differential equations. The present investigation complements the paper [11], where the extension of the following version of the Sturm-Picone theorem is handled. Theorem 1.5]).<br>
Theorem 1: If there exists a real solution  $u \neq 0$  of  $\mathcal{A}_{Q}u = 0$  such that<br>  $u(a) = 0 = u(b)$  and  $\int_a^b [p(u')^2 + qu^2] dx \leq 0$ ,<br>
then every real solution v of  $\mathcal{A}_{q}v = 0$  is a constant multiple of u or has Theorem 1: If there exists a real solution  $u \neq 0$  of<br>  $u(a) = 0 = u(b)$  and  $\int_a^b [p(u')^2 + qu^2] dx$ <br>
then every real solution v of  $\mathcal{A}_q v = 0$  is a constant mul<br>
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ential equati *a*<br> **hen every real solution v of**  $A_qv = 0$  **is a constant multiple of u or has a**<br>
In the following this theorem will be extended to self-adjoint, sec-<br>
of the following version of the Sturm-Picone theorem is handled.<br>
Th

Theorem 1': Suppose  $p(x) \leq P(x)$  and  $q(x) \leq Q(x)$ ,  $x \in [a, b]$ . If there exists a real solution  $u \neq 0$  *of*  $\mathcal{A}_Q u = 0$  *with*  $u(a) = 0 = u(b)$ , then every real solution *v of*  $\mathcal{A}_q v = 0$  has at least one <br>zero in (a, b) if<br>(I)  $q(x') < Q(x')$  for some  $x' \in [a, b]$  or

•

zero in  $(a, b)$  if<br>
(I)  $q(x') < Q(x')$  for some  $x' \in [a, b]$  or<br>
(II)  $p(x') < P(x')$  and  $Q(x') \neq 0$  for some  $x' \in (a, b)$ .<br>
Concerning the extensive investigations in the literature which deal with extending the Sturm-Picone theorem to the *n*-dimensional case we refer to the references in  $[3, 7, 13]$ .

Using the summation convention, let  $\mathcal{A}_{\bm{Q}},$   $\mathcal{\breve{A}}_{\bm{q}}$  denote the differential expressions

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\nUsing the summation convention, let 
$$
\mathcal{A}_0
$$
,  $\mathcal{A}_q$  denote the different  
\n
$$
\mathcal{A}_0 u \equiv -(P_{ij}u_{x_i})_{x_j} + Q(x) u \qquad (x = (x_1, ..., x_n) \in G \subseteq \mathbb{R}^n),
$$
\nwhere  
\n(i) G is a (possibly unbounded) domain in the Euclidean space **R**  
\nlarity hypotheses on the boundary  $\partial G$  are required;  
\n(ii) the coefficients  $P_{ij} = P_{ji}$ ,  $p_{ij} = p_{ji} \in C^1$   $(i = 1, ..., n)$  and *t*  
\nvalued and defined on *G*;  
\n(iii) the smallest eigenvalues  $E(x)$  and  $e(x)$  of the matrices  $(P_{ij})$ ;  
\nrespectively, are positive on *G*;  
\n(iv) there are positive constants  $c_1 < 1$  and  $c_2$  such that

where

(i)  $G$  is a (possibly unbounded) domain in the Euclidean space  $\mathbb{R}^n$  where no regularity hypotheses on the boundary  $\partial G$  are required;

(ii) the coefficients  $P_{ij} = P_{ji}$ ,  $p_{ij} = p_{ji} \in C^1$   $(i = 1, ..., n)$  and  $Q, q \in C$  are real-valued and defined on  $G$ ; 78 E. MÜLLER-PFEIFFER<br>
Using the summation convention, let  $\mathcal{A}_0$ ,  $\mathcal{A}_q$  denote the differential expressions<br>  $\mathcal{A}_0 u = -(P_{ij} u_{x_i})_{x_j} + Q(x) u$   $(x = (x_1, ..., x_n) \in G \subseteq \mathbb{R}^n)$ ,<br>
where<br>
(i) G is a (possibly unbounded) doma

(iii) the smallest eigenvalues  $E(x)$  and  $e(x)$  of the matrices  $(P_{ij})_{i,j=1}^n$  and respectively, are positive on  $G$ ;<br>(iv) there are positive constants  $c_1 < 1$  and  $c_2$  such that

$$
A_0 u = -(F_{ij} u_{x_i})_{x_i} + Q(x) u \qquad (x = (x_1, ..., x_n) \in G \subseteq \mathbb{R}^n),
$$
  
\n
$$
A_q u = -(p_{ij} u_{x_i})_{x_j} + q(x) u \qquad (x = (x_1, ..., x_n) \in G \subseteq \mathbb{R}^n),
$$
  
\nis a (possibly unbounded) domain in the Euclidean space  $\mathbb{R}^n$  where no regu-  
\napprotheses on the boundary  $\partial G$  are required;  
\nthe coefficients  $P_{ij} = P_{ji}$ ,  $p_{ij} = p_{ji} \in C^1$   $(i = 1, ..., n)$  and  $Q, q \in C$  are real-  
\nand defined on  $G$ ;  
\nthe smallest eigenvalues  $E(x)$  and  $e(x)$  of the matrices  $(P_{ij})_{i,j=1}^n$  and  $(p_{ij})_{i,j=1}^n$ ,  
\nively, are positive constants  $c_1 < 1$  and  $c_2$  such that  
\n
$$
|(Q^-\varphi, \varphi)| \le c_1 a_q + [\varphi, \varphi] + c_2 ||\varphi||^2 \qquad (1)
$$
\n
$$
(\varphi \in C_0^\infty(G)),
$$
\n
$$
|(q^-\varphi, \varphi)| \le c_1 a_q + [\varphi, \varphi] + c_2 ||\varphi||^2 \qquad (2)
$$
\n
$$
(\cdot, \cdot)
$$
 and  $||\cdot||$  denote the inner product and the norm of the Hilbert space  $L_2(G)$   
\n
$$
Q^-(x) = \min (Q(x), 0), \qquad Q^+(x) = \max (Q(x), 0),
$$
\n
$$
q^{-}(x) = \min (Q(x), 0) \qquad q^+(x) = \max (Q(x), 0).
$$

where  $\langle \cdot,\cdot\rangle$  and  $\| \cdot \|$  denote the inner product and the norm of the Hilbert space  $L_2(G)$ and

(IV) there are positive constants 
$$
c_1
$$
 < 1 and  $c_2$  such that  
\n
$$
|(Q^-\varphi, \varphi)| \leq c_1 a_0^+ [\varphi, \varphi] + c_2 ||\varphi||^2.
$$
\n( $\varphi \in C_0^\infty(G)$ ),  
\n
$$
|(q^-\varphi, \varphi)| \leq c_1 a_0^+ [\varphi, \varphi] + c_2 ||\varphi||^2
$$
\nwhere (·, ·) and  $||\cdot||$  denote the inner product and the norm of the Hilbert s  
\nand  
\n
$$
Q^-(x) = \min (Q(x), 0), \qquad Q^+(x) = \max (Q(x), 0),
$$
\n
$$
q^-(x) = \min (q(x), 0), \qquad q^+(x) = \max (q(x), 0),
$$
\n
$$
a_0^+ [\varphi, \varphi] = \int P_{ij} \varphi_{x_i} \overline{\varphi}_{x_j} dx + \int Q^+ \varphi \overline{\varphi} dx
$$
\n( $\varphi, \psi \in C_0^\infty(G)$ )  
\n
$$
a_0^+ [\varphi, \psi] = \int_{G} p_{ij} \varphi_{x_i} \overline{\varphi}_{x_j} dx + \int_{G} q^+ \varphi \overline{\varphi} dx.
$$
\n( $\varphi, \psi \in C_0^\infty(G)$ )  
\nIf  $\Omega$  is a subdomain of  $G$ , let the inner product and the norm of  $L_2(\Omega)$  b  
\nby (·, ·)  $\varphi$  and  $||\cdot||_{G}$ ; the index  $\Omega$  will be omitted when  $\Omega = G$ . It follows from  
\nthat the symmetric operators  $A_{Q,0}$  and  $A_{q,0}$ ,  
\n
$$
A_{Q,0}\varphi = \mathcal{A}_{Q}\varphi, \qquad A_{q,0}\varphi = \mathcal{A}_{q}\varphi \qquad (\varphi \in C_0^\infty(G))
$$
\nare bounded from below. Consequently, the sesquilinear forms  
\n
$$
a_0[\varphi, \psi] = (A_{Q,0}\varphi, \psi) = \int_{G} P_{ij} \varphi_{x_i} \overline{\varphi}_{x_j} dx + \int_{G} Q \varphi \overline{\psi} dx,
$$

If  $\Omega$  is a subdomain of G, let the inner product and the norm of  $L_2(\Omega)$  be denoted by  $(\cdot, \cdot)_{\Omega}$  and  $\|\cdot\|_{\Omega}$ ; the index  $\Omega$  will be omitted when  $\Omega = G$ . It follows from (1), (2) that the symmetric operators  $A_{q,0}$  and  $A_{q,0}$ ,  $\begin{array}{l} \displaystyle{ \begin{array}{l} \displaystyle{ \bar{\psi} \, dx,} \end{array} } \displaystyle{ \begin{array}{l} \displaystyle{ \begin{array}{l} \displaystyle{ \bar{\psi} \, dx,} \end{array} } \displaystyle{ \begin{array}{l} \displaystyle{ \begin{array}{l} \displaystyle{ \bar{\psi} \, d\bar{x},} \end{array} } \displaystyle{ \begin{array}{l} \displaystyle{ \begin{array}{l} \displaystyle{ \end{array} } \displaystyle{ \begin{array} \displaystyle{ \begin{array} \displaystyle{ \bar{\psi} \, d\bar{z}} \end{array} } } \displaystyle{ \begin{array} \displaystyle{ \end{array} } \display$ 

$$
A_{Q,0}\varphi = \mathcal{A}_Q\varphi, \qquad A_{q,0}\varphi = \mathcal{A}_q\varphi \qquad (\varphi \in C_0^{\infty}(G))
$$

$$
a_q + [\varphi, \psi] = \int_{G} p_{ij} \varphi_x \overline{\psi}_x dx + \int_{G} q^+ \varphi \overline{\psi} dx.
$$
\n
$$
[ \varphi, \psi \in C_0^{\infty} ]
$$
\nis a subdomain of  $G$ , let the inner product and the no,  $\cdot$ .)<sub>O</sub> and  $|| \cdot ||_{G}$ ; the index  $\Omega$  will be omitted when  $\Omega = G$ , the symmetric operators  $A_{Q,0}$  and  $A_{q,0}$ ,  
\n $A_{Q,0}\varphi = \mathcal{A}_Q\varphi$ ,  $A_{q,0}\varphi = \mathcal{A}_q\varphi$   $(\varphi \in C_0^{\infty}(G))$   
\nounded from below. Consequently, the sesquilinear form  
\n $a_Q[\varphi, \psi] = (A_{Q,0}\varphi, \psi) = \int_{G} P_{ij}\varphi_{z_i}\overline{\psi}_{z_j} dx + \int_{G} Q\varphi \overline{\psi} dx,$   
\n $a_q[\varphi, \psi] = (A_{q,0}\varphi, \psi) = \int_{G} p_{ij}\varphi_{z_i}\overline{\psi}_{z_j} dx + \int_{G} q\varphi \overline{\psi} dx,$   
\n $\in C_0^{\infty}(G)$  are closable [5: p. 318]. Let the corresponding  
\n $g[f, g]$  and  $\mathring{a}_q[f, g]$ , respectively. We shall always consid

 $(\varphi, \, \psi \in C_0^\infty(G))$  are closable [5: p. 318]. Let the corresponding closed forms be denoted by  $\mathring{a}_q[f,g]$  and  $\mathring{a}_q[f,g]$ , respectively. We shall always consider real-valued solutions of the equations  $\mathcal{A}_0u = 0$ - and  $\mathcal{A}_0u = 0$  which belong to  $C(G) \cap W^2_{2,\text{loc}}(G)$ , where  $W_{2,loc}^2(G)$  denotes the Sobolcv space of (complex-valued) functions the generalized. derivatives of which up to order two belong to  $L_2$  on compact subsets of G. Assuming that *u* is a non-trivial solution of one of these equations the set  $N_u = \{y \in G \mid u(y)\}$  $= 0$ . is said to be the *nodal contour* of *u.* By a theorem of MCNABB [10] in every derivatives of which up to order two belong to  $L_2$  on compact subsets of G. Assuming<br>that u is a non-trivial solution of one of these equations the set  $N_u = \{y \in G \mid u(y) = 0\}$  is said to be the *nodal contour* of u. By a derivatives of which up to order two belong to  $L_2$  on compact subsets of G. Assuming<br>that u is a non-trivial solution of one of these equations the set  $N_u = \{y \in G \mid u(y) = 0\}$  is said to be the *nodal contour* of u. By a and  $\partial G$  the domain  $G$  is divided into at least-two connected subdomains. Such a subdomain  $\Omega$  of  $\theta$  is said to be a *nodal domain* of  $u$ ; *u* has fixed sign in  $\Omega$  and  $u(x) = 0$ ,

 $x \in \partial\Omega \cap G$ . To formulate the following theorem we further require the set

Elliptic Sturmian Theory for General Domains  
\nG. To formulate the following theorem we further require the set  
\n
$$
D_q = \left\{ f \in W_{2,loc}^1(G) \mid \int_G p_{ij} f_x \bar{f}_{x_i} dx + \int_G q^+ |f|^2 dx < \infty \right\}.
$$
\n(3)  
\nIt a function  $f \in D_q$  is not necessarily contained in  $L_2(G)$ .

Note that a function  $f \in D_q$  is not necessarily contained in  $L_2(G)$ .

 

Theorem 2: *Let the hypotheses* (i)—(iv) from above be fulfilled. If there exists a *non-trivial solution*  $u \in D(\dot{a}_0)$  of  $\mathcal{A}_0u = 0$  such that  $u \in D(\dot{a}_0)$  and  $\dot{a}_0[u, u] \leq 0$ , then *every solution*  $v \in D_q$  *of*  $\mathcal{A}_q v = 0$  *is a constant multiple of u or changes sign in G.* 

Proof: Let  $\Omega$  be any nodal domain of  $u$ . (Possibly, G itself is a nodal domain of  $u$ .) Then the restriction  $u_{\rho}$  of *u* to  $\Omega$  belongs to  $D(\dot{a}_{q,\rho})$  as well as to  $D(\dot{a}_{q,\rho})$ , the domains Note that a function  $f \in D_q$  is not necessarily contained in  $L_2(G)$ .<br>
. Theorem 2: Let the hypotheses (i)-(iv) from above be fulfilled.<br>
non-trivial solution  $u \in D(a_0)$  of  $A_0u = 0$  such that  $u \in D(a_q)$  and *every solutio i*<sub>4</sub>*f*<sub>*x*</sub><sub>*dx*</sub> + *f*<sub>*q*</sub><sup>*t*</sup> |*f*|<sup>2</sup> *dx* < ∞ }.<br> *h* necessarily contained in *L*<sub>2</sub>(*G*).<br> *s* (i)-(iv) from above be fulfilled. If there ex<br> *d*<sub>Q</sub>u = 0 *such that*  $u \in D(\dot{a}_q)$  and  $\dot{a}_q[u, u] \leq 0$ <br> *s* a

*aQ•Q[92, ]* = *f - - '(,E C0 (Q)), -. - P] =1 -r- qqnp]dx . . -* respectively [11: Lemma]. Since *u<sup>Q</sup> E W <sup>1</sup> (Q).* and *CJIQUs? == 0, u. E D(A <sup>00</sup> ), -* 

where  $A_{Q,Q,0}\varphi = \mathcal{A}_Q\varphi$ ,  $\varphi \in C_0^{\infty}(\Omega)$ . Hence, it follows from  $u_Q \in D(d_{Q,Q})$  that  $u_q \in D(A_{Q,q})$ ,  $A_{Q,q}$  being the Friedrichs extension of  $A_{Q,Q,0}$ .  $u_q$  is an eigenfunction of  $A_{Q,Q}$  and  $\lambda = 0$  is the corresponding eigenvalue. Hence, we have  $a_{Q,Q}(u_Q, u_Q)$ <br>  $\lambda = (A_0u_Q, u_Q) = 0$ . Of course, we also have  $a_Q[u, u] = 0$ . By the help of (2), one<br>
can easily prove that  $a_q[u, u]$  is represented by<br>  $a_q$ respectively [11: Lenna]. Since  $u_q \in W_{2,10}^2$ <br>where  $A_{Q,\Omega,\Omega} \varphi = \mathcal{A}_Q \varphi$ ,  $\varphi \in C_0^{\infty}(\Omega)$ . Hence,<br> $u_q \in D(A_{Q,\Omega})$ ,  $A_{Q,\Omega}$  being the Friedrichs extens<br>of  $A_{Q,\Omega}$  and  $\lambda = 0$  is the corresponding eiger<br> $= (A_q u_q, u_q)$ From the of the space of the space of  $u_0 \in D$ <br>respectively.  $f = \int_{\mathcal{Q}} [P_{ij}\varphi_{x_i}\overline{\psi}]$ <br>  $f = \int_{\mathcal{Q}} [p_{ij}\varphi_{x_i}\overline{\psi}]$ <br>
Lemma]. S<br>
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S<br>
S<br>  $\varphi\varphi, \varphi \in C_0$ <br>
being the **F**<br>
is the corr<br>
of course,<br>  $\int_{\mathcal{Q}} (\varphi_{ij}u_{x_i}\overline{u}_{x_j})$ <br>  $\int_{\mathcal{Q}} (\varphi_{ij}u_{x$ 

$$
\dot{a}_q[u, u] = \int (p_{ij}u_{x_i}\dot{u}_{x_j} + qu^2) dx.
$$

Because  $d_q[u, u] \leq 0$  there exists at least one nodal domain  $\Omega$  of  $u$  such that

$$
d_q[u, u] = \int\limits_G (p_{ij}u_{x_i}\dot{u}_{x_j} + qu^2) dx.
$$
  
use  $d_q[u, u] \leq 0$  there exists at least one nodal domain  

$$
d_{q,0}[u_q, u_q] = \int\limits_G [p_{ij}(u_q)_{x_i}(u_q)_{x_j} + q(u_q)^2] dx \leq 0.
$$

At this point, without loss of generality, we can assume that  $u_0(x) > 0$ ,  $x \in \Omega$ . The following.two cases are possible: (a) this point, without loss of generality<br>
(i)  $a_{q,\Omega}[\varphi, \varphi] \ge 0$  for all  $\varphi \in C_0^{\infty}(\Omega)$ .<br>
(i)  $a_{q,\Omega}[\varphi, \varphi] \ge 0$  for all  $\varphi \in C_0^{\infty}(\Omega)$ .

(1) 
$$
a_{\alpha,\Omega}[\varphi,\varphi] \geq 0
$$
 for all  $\varphi \in C_0^{\infty}(\Omega)$ .

(II) 
$$
a_{a,\Omega}[\varphi_0, \varphi_0] < 0
$$
 for some  $\varphi_0 \in C_0^\infty(\Omega)$ .

(110*a<sub>0</sub>*, *a<sub>0</sub>*) = 0. Of course, we also have  $a_0(u, u) = 0$ . By the m<br>
n easily prove that  $a_q[u, u]$  is represented by<br>  $a_q[u, u] = \int_{0}^{1} (p_{ij}u_{x_i}\tilde{u}_{x_i} + qu^2) dx$ .<br>
ccause  $a_q[u, u] \le 0$  there exists at least one nodal doma *Case* I: In this case we have  $d_{q,\Omega}[u_q, u_q] = 0$ , and  $D(d_{q,\Omega}) = D(A_{q,\Omega}^{1/2})$  [5: p. 331],  $A_{q,0}$  being the Friedrichs extension of  $A_{q,0,0}$ ,  $A_{q,0,0}\varphi = \mathcal{A}_{q}\varphi$ ,  $\varphi \in C_0^{\infty}(\Omega)$ . It follows<br>from  $0 = a_{q,0}[u_0, u_0] = ||A_{q,0}^{1/2}u_0||^2$  that  $A_{q,0}u_0 = 0$ . Therefore  $u_0$  is also an eigen-4q,  $\mu$ a<sub>n</sub>,  $\mu$ <sub>a</sub><sub>n</sub>,  $\mu$ <sub>2</sub> *D*<sub>*i*</sub>,  $\mu$ <sub>2</sub> *A*<sub>*z*</sub> *p*<sub>1</sub>  $\mu$ <sub>2</sub> *A*<sub>*z*</sub> *p*<sub>1</sub>  $\mu$ <sub>2</sub>  $\mu$ <sub>2</sub>  $\geq$  0.<br>
4d this point, without loss of generality, we can assume that  $u_{\Omega}(x)$  > following two cases are possi following two cases are possible:<br>
(1)  $a_{q,\Omega}[\varphi, \varphi] \ge 0$  for all  $\varphi \in C_0^{\infty}(\Omega)$ .<br>
(II)  $a_{q,\Omega}[\varphi_0, \varphi_0] < 0$  for some  $\varphi_0 \in C_0^{\infty}(\Omega)$ .<br>
Case I: In this case we have  $a_{q,\Omega}[u_0, u_0]$ <br>  $A_{q,\Omega}$  being the Fried *W*: In this case we have  $\dot{a}_{q,0}[u_0, u_0] = 0$ , and  $D(\dot{a}_{q,0}) = D(A_q^{1/2})$ <br> *g* the Friedrichs extension of  $A_{q,0,0}$ ,  $A_{q,0,0}\varphi = A_q\varphi$ ,  $\varphi \in C_0^{\infty}(\Omega)$ <br> *i*  $\dot{a}_{q,0}[u_0, u_0] = ||A_q^{1/2}u_0||^2$  that  $A_{q,0}u_0 = 0$ .

function of  $A_{q,\Omega}$  and satisfies the equation  $A_{q}u_{\Omega} = 0$ .<br>If there exists a zero of *v* in G, by the theorem of McNABB the solution *v* changes sign [11]. Thus, we can assume in the following that  $v(x) > 0$ ,  $x \in G$ . Choose any

$$
w(x) = u\Omega(x*) v(x) - v(x*) u\Omega(x) \qquad (x \in \Omega).
$$

This is a solution of  $\mathcal{A}_q w = 0$  in  $\Omega$  with  $w(x^*) = 0$ . The identity  $w = 0$  implies  $v(x) = \text{const} \cdot u(x), x \in \Omega$ . If  $\Omega$  is a proper subdomain of G, then *v* vanishes on  $\partial\Omega \cap G$ , which implies that *v* changes sign in G. Because of the assumption  $v(x) > 0$  $(x \in G)$ , however, we have  $\Omega = G$  and  $v(x) = \text{const} \cdot u(x)$ ,  $x \in G$ . In the case where  $w \not\equiv 0$  this function must change sign in every neighbourhood of  $x^*$  in  $\Omega$  as repeatedly

remarked. Let  $\omega \subset \Omega$  be a subdomain such that  $w(x) < 0$  for  $x \in \omega$  and  $w(y) = 0$ for  $y \in \partial \omega \cap \Omega$ . We prove that  $z \in D(\dot{a}_{q,\Omega}),$ 

d. Let 
$$
\omega \subset \Omega
$$
 be a subdomain such that  $w(x)$   
\n $\omega \cap \Omega$ . We prove that  $z \in D(\dot{a}_{q,\Omega})$ ,  
\n
$$
z(x) = \begin{cases} w(x) & \text{for } x \in \omega, \\ 0 & \text{for } x \in \Omega \setminus \omega. \end{cases}
$$
  
\n
$$
\subset C_0^{\infty}(\Omega) \text{ be a real sequence with } \varphi_m \longrightarrow u_{\Omega},
$$

Let  $(\varphi_m) \subset C_0^{\infty}(\Omega)$  be a real sequence with  $\varphi_m \frac{1}{\varphi_0 \Omega}$   $\psi_0$ , that is [5: p. 313]

$$
z(x) = \begin{cases} \infty & \text{for } x \in \Omega \setminus \omega, \\ 0 & \text{for } x \in \Omega \setminus \omega. \end{cases}
$$
  
\n
$$
C_0^{\infty}(\Omega) \text{ be a real sequence with } \varphi_m \xrightarrow[\overline{a_0, \Omega]} u_{\Omega}, \text{ that is [5: p. 313]}
$$
  
\n
$$
a_{q,\Omega}[\varphi_m - \varphi_{m'}, \varphi_m - \varphi_{m'}] \xrightarrow[\overline{m}, \overline{m' \to \infty]} 0 \text{ and } ||\varphi_m - u_{\Omega}||_{\Omega} \xrightarrow[\overline{m \to \infty]} 0.
$$
  
\n
$$
= \tilde{u}_{\Omega}(x^*) v(x) - v(x^*) w(x) (x \in \Omega) \text{ and define}
$$

Let  $(\varphi_m) \subset C_0^{\infty}(\Omega)$  be a real sequence with  $\varphi_m \xrightarrow[\alpha_0, 0]{} u_\Omega$ , to  $\alpha_{q,\Omega}[\varphi_m - \varphi_m, \varphi_m - \varphi_m] \xrightarrow[m,m' \to \infty]{} 0$  and  $||\varphi_m$ <br>Set  $w_m(x) = u_\Omega(x^*) v(x) - v(x^*) \varphi_m(x)$  ( $x \in \Omega$ ) and define Set  $w_m(x) = u_{\Omega}(x^*) v(x) - v(x^*) \varphi_m(x)$  ( $x \in \Omega$ ) and define

$$
w_m^{-}(x) = \min (w_m(x), 0)
$$
 and  $\zeta_m(x) = \max (z(x), w_m^{-}(x)).$ 

Note that  $\zeta_m(x) = 0$  ( $x \in \Omega \setminus \omega$ ) and  $w, z, w_m, w_m$ ,  $\zeta_m \in W^1_{2,loc}(\Omega)$  [6: p. 50]. From  $w_m(x) > 0$  we have  $w_m(x) = 0$  and  $\zeta_m(x) = 0$ ,  $x \in \Omega \setminus \text{supp } \varphi_m$ . Therefore,  $\zeta_m \in D(\dot{a}_{q,\Omega})$ . We now prove that  $z \in L_2(\Omega)$  and  $\|\zeta_m - z\|_{\Omega} \to 0$ . It follows from Note that  $\zeta_m(x) = 0$   $(x \in \Omega \setminus \omega)$  and  $\zeta_m(x) = \max(z(x), w_m^{-}(x))$ .<br>
Note that  $\zeta_m(x) = 0$   $(x \in \Omega \setminus \omega)$  and  $w, z, w_m, w_m^{-1}, \zeta_m \in W^1_{2,loc}(\Omega)$  [6: p. 50].<br>  $w_m(x) > 0$  we have  $w_m^{-}(x) = 0$  and  $\zeta_m(x) = 0$ ,  $x \in \Omega \setminus \text{supp } \varphi_m$ . Then  $|v(x^*)|$ Let  $(\varphi_m) \subset C_0^{\infty}(\Omega)$  be a real sequence with  $\varphi_m \frac{1}{\varphi_n \Omega} \star u_{\Omega}$ , that is [5: p. 313]<br>  $\alpha_{q,\Omega}[\varphi_m - \varphi_{m}, \varphi_m - \varphi_{m}]\frac{1}{m,m' \to \infty} \star 0$  and  $||\varphi_m - u_{\Omega}||_{\Omega} \frac{1}{m \to \infty} \star 0$ .<br>
Set  $w_m(x) = \dot{u}_{\Omega}(x^*) v(x) - v(x^*) \varphi_m$  $\|w - w_m\| \le \|w - w_m\| = |v(x^*)| \|u_Q - \varphi_m\|$   $(x \in \omega)$  that  $\|w - w_m\| \le |v(x^*)|$ <br>  $\times \|u_Q - \varphi_m\|$  and, consequently,  $w - w_m^- \in L_2(\omega)$ . Thus, we have  $w \in L_2(\omega)$ <br>
because  $w_m^- \in L_2(\omega)$ . But this implies  $z \in L_2(\Omega)$ . To prove  $\|\zeta_m - z\|$ Lie (ym)  $\Box$  θ<sub>0</sub> (sz) so a central explicition with  $\theta_m \frac{1}{\alpha_{e,B}}$  v a<sub>2</sub>, such i<br>  $\alpha_{q,B}[\varphi_m - \varphi_{m'}, \varphi_m - \varphi_{m'}] \frac{1}{m,m' \to \infty}$  + 0 and  $||\varphi_m - u$ <br>
Set  $w_m(x) = u_0(x^*) v(x) - v(x^*) \varphi_m(x)$  (x ∈ Ω) and define<br>  $w_m^-(x) = \min (w_m(x),$ *d*  $w_m^-(x) = m$   $m, m \to \infty$ <br>  $w_m^-(x) = \min(w_m(x), 0)$  and  $\zeta_m(x) = \max(z(x), w_m^-(x))$ .<br>  $w_m^-(x) = \min(w_m(x), 0)$  and  $\zeta_m(x) = \max(z(x), w_m^-(x))$ .<br>  $d \zeta_m(x) = 0$  ( $x \in \Omega \setminus \omega$ ) and  $w, z, w_m, w_m^-, \zeta_m \in W_{2,loc}^1(\Omega)$  [6: p. 50]. From<br>  $0$  we have  $w_m^-(x) =$  $w - w_m^{-1} \le |w - w_m|$ <br>  $w - w_m^{-1} \le |w - w_m|$ <br>  $\times ||u_g - \varphi_m||_{\omega}$  and, conceause  $w_m^{-} \in L_2(\omega)$ . B<br>
stimate  $||\zeta_m - z||_2 = ||\zeta$ <br>
Now we prove that ther<br>  $a_{q,\Omega}[\zeta_m, \zeta_m] \le$ <br>
By using (3) with  $f$ <br>  $2_m = \sup p \varphi_m$  we obtai<br>  $a_{q,\Omega}[\zeta_m,$ estimate  $||\zeta_m - z||_0 = ||\zeta_m - z||_0 \le ||w_m - z||_0 = ||w_m - w_m||_0 \le |v(x^*)| ||\varphi_m - u_0||_0$ .<br>Now we prove that there exists a constant  $C > 0$  such that because  $w_m$ <sup>-</sup>  $\in$   $L_2(\omega)$ . But this implies  $z \in L_2(\Omega)$ . To prove  $\|\zeta_m - z\|_{\Omega} \to 0$  use the

$$
\dot{a}_{q,\Omega}[\zeta_m,\zeta_m] \leq C \qquad (m \in \mathbb{N}). \tag{4}
$$

By using (3) with  $f = v$  and the Schwarz inequality [5: p. 53] and setting.  $\Omega_m = \text{supp } \varphi_m$  we obtain

Now we prove that there exists a constant 
$$
C > 0
$$
 such that  $\ell = \lfloor \log_m - \log_m - \log_m \rfloor$  and  $\ell = \lfloor \ell (d-1) \rfloor |\psi_m - \omega_0| | \beta$ .  
\nNow we prove that there exists a constant  $C > 0$  such that  $\ell = \lfloor \log_m - \log_m \rfloor$  (4)  
\nBy using (3) with  $f = v$  and the Schwarz inequality [5: p. 53] and setting,  
\n $Q_m = \sup p \varphi_m$  we obtain  
\n $d_{q,0}[\zeta_m, \zeta_m] \leq \int p_{ij}(\zeta_m)_{z_i} (\zeta_m)_{z_i} dx + \int q^2 \zeta_m^2 dx$   
\n $\leq \int p_{ij} z_i z_{z_i} dx + \int p_{ij} (w_m^-)_{z_i} (w_m^-)_{z_i} dx + \int q^2 z^2 dx + \int q^4 (w_m^-)^2 dx$   
\n $\leq \int p_{ij} w_{z_i} w_{z_i} dx + \int p_{ij} (w_m)_{z_i} (w_m)_{z_i} dx + \int q^2 z^2 dx + \int q^4 w_m^2 dx$   
\n $\leq C_1 \left[ \int_{Q_m} p_{ij} (u_{\Omega})_{z_i} (u_{\Omega})_{z_j} dx + \int_{Q_m} p_{ij} v_{z_i} v_{z_j} dx + \int_{Q_m} p_{ij} (\varphi_m)_{z_i} (\varphi_m)_{z_j} dx \right]$   
\n $+ \int q^4 (u_{\Omega})^2 dx + \int q^4 v^2 dx + \int q^4 \varphi_m^2 dx$   
\n $\leq C_1 \left[ \int_{Q_m} p_{ij} (u_{\Omega})_{z_i} (u_{\Omega})_{z_j} + q^4 (u_{\Omega})^2 \right] dx$   
\n $+ \int [p_{ij}(\varphi_m)_{z_i} (\varphi_m)_{z_j} + q^4 \varphi_m^2 dx] + C_2.$   
\nIf follows from (2) that  
\n $a_{q,0}^* \left[ \varphi_m, \varphi_m \right] \leq (1 - c_1)^{-1} a_{q,0} \left[ \varphi_m, \varphi_m \right] + (1 - c_1)^{-1} c_2 ||\varphi_m||_2^2.$   
\nHence, in view of  
\n

It follows from (2) that

 $\frac{1}{2}$ 

•

 $a_{q,\Omega}[\varphi_m, \varphi_m] \to \dot{a}_{q,\Omega}[u_\Omega, u_\Omega] = 0$  and  $\|\varphi_m\|_{\Omega} \to \|u_\Omega\|_{\Omega}$ ,

 $C_3$   $(m \in N)$ . Finally, we have, for  $m \in N$ ,

$$
[u_q]_{\alpha} \circ p_m \circ p_m = \alpha_q \circ p[u_\alpha, u_\alpha] = 0 \quad \text{and} \quad ||\varphi_m||_{\alpha} \to ||u_\alpha||_{\alpha},
$$
  
ists a constant  $C_3$  such that  $a_q^+ \circ p[\varphi_m, \varphi_m] \leq C_3$   $(m \in N)$ . Final,  

$$
\int_{\Omega_m} [p_{ij}(u_\alpha)_{x_i}(u_\alpha)_{x_j} + q^+(u_\alpha)^\alpha] dx \leq \int_{\Omega} [p_{ij}u_{x_i}u_{x_j} + q^+u^\alpha] dx < \infty.
$$

Ξ

**S** 

By using these estimates in (5) we obtain (4). It now follows from  $\|\zeta_m - z\|_{\mathcal{Q}} \to$ and (4) that  $z \in D(\dot{a}_{q, \Omega})$  [5: Theorem 1.16/p. 315]. we obtain (4). It now follows from<br>
prem 1.16/p. 315].<br>
belongs to  $D(\dot{a}_{q,\omega})$ , the domain of the<br>  $dx + \int q\varphi \bar{\psi} dx$   $(\varphi, \psi \in C_0^{\infty}(\omega))$ 

The restriction  $z_{\omega}$  of z to  $\omega$  belongs to  $D(\dot{a}_{q,\omega})$ , the domain of the closure of the form

Elliptic Sturmian Theory for General  
\nng these estimates in (5) we obtain (4). It now follows for  
\nthat 
$$
z \in D(\dot{a}_{q,\Omega})
$$
 [5: Theorem 1.16/p. 315].  
\nrestriction  $z_{\omega}$  of z to  $\omega$  belongs to  $D(\dot{a}_{q,\omega})$ , the domain of t  
\n $a_{q,\omega}[\varphi, \psi] = \int p_{i,j}p_{z_{i}}\bar{\psi}_{z_{j}} dx + \int q\varphi\bar{\psi} dx \qquad (\varphi, \psi \in C_{0}^{\infty}(\omega))$   
\nmmal. Further we have  $z_{\omega} \in W_{2,\text{loc}}^{2}(\omega)$  and  $\mathcal{A}_{q}z_{\omega} = 0$ . Hence  
\nno of the Friedrichs extension  $A_{q,\omega}$  of  $A_{q,\omega,0}$ ,  $A_{q,\omega,0}\varphi = \mathcal{A}_{q}\varphi$ 

/

and (4) that  $z \in D(d_{q,\Omega})$  [5: Theorem 1.16/p. 315].<br>
The restriction  $z_{\omega}$  of z to  $\omega$  belongs to  $D(d_{q,\omega})$ , the domain of the closure of the<br>
form<br>  $a_{q,\omega}[\varphi, \psi] = \int p_{i} \varphi_{z_{i}} \overline{\psi}_{z_{i}} dx + \int q \varphi \overline{\psi} dx$   $(\varphi, \psi \in C_{0}$ = 0 and, consequently,  $d_{q,\rho}[z, z] = 0$ . This relation implies  $A_{q,\rho}^{1/2}z = 0$  which leads to  $A_{q,0}z = 0$ . Hence,  $z \in W^2_{2,loc}(\tilde{G})$  (see [2], for instance) and z is a solution of the equation  $A_qz = 0$ . By considering the property  $z(x) = 0$  ( $x \in \Omega \setminus \omega$ ) and using the unique continuation theorem for solutions of elliptic equations (see [4: p. 224 and Remarks 3/p. 203]) it follows that  $z = 0$  in  $\Omega$ . Finally,  $w(x) = 0$  for  $x \in \omega$  implies  $x(x) = 0$  for  $x \in \Omega$ . This proves that  $v(x) = \text{const} \cdot u(x)$ ,  $x \in \Omega$ . In the case where  $\Omega$ is a proper subdomain of G we have  $v(y) = 0$   $(y \in \partial \Omega \cap G)$  which implies that *v* iust change sign in *G.* But this case is impossible because of the assumption  $v(x) > 0$ ,  $x \in G$ . In the case  $\Omega = G$ , however, the solution v is a constant multiple  $a_{q,w}[\varphi, \psi] = \int p_{ij}p_{z_i}\overline{\psi}_{z_i} dx + \int q\varphi\overline{\psi} dx$  ( $\varphi$ <br>
[11: Lemma]. Further we have  $z_{\omega} \in W_{2,10c}^2(\omega)$  and  $\mathcal{A}_q$ <br>
function of the Friedrichs extension  $A_{q,w}$  of  $A_{q,w,0}$ ,  $A$ <br>  $\lambda = 0$  is the corresponding eig  $w(x) = 0$  for  $x \in \Omega$ . This proves that  $v(x) = \text{const} \cdot u(x)$ ,  $x \in \Omega$ . In the case where  $\Omega$ <br>is a proper subdomain of  $G$  we have  $v(y) = 0$  ( $y \in \partial \Omega \cap G$ ) which implies that  $v$ <br>must change sign in  $G$ . But this case is imposs

*Case* (II): We conclude as in [11]. There exists a function  $\varphi_0 \in C_0^{\infty}(\Omega)$  with  $a_{q,\Omega} [\varphi_0, \varphi_0] < 0$ . It follows from dist (supp  $\varphi_0, \partial \Omega$ ) > 0 that there exists a bounded domain  $\omega_0$  with supp  $\varphi_0 \subset \omega_0 \subset \overline{\omega}_0 \subset G$  belonging to the class  $C^{\infty}$  (definition in *Case* (II): We conclude as in [11]. There exists a function  $\varphi_0 \in C_0^{\infty}(\Omega)$  with  $a_{q,\Omega}[\varphi_0, \varphi_0] < 0$ . It follows from dist (supp  $\varphi_0, \partial\Omega$ ) > 0 that there exists a bounded domain  $\omega_0$  with  $\sup p \varphi_0 \subset \omega_0 \subset$  $|q(x)| \leq C_{\omega_0} < \infty$  ( $x \in \omega_0$ ). It follows from  $a_{q,\Omega}[\varphi_0, \varphi_0] < 0$  that the smallest eigenvalue  $\lambda_1$  of  $A_{q,\omega_0}$  is negative. The corresponding eigenfunction  $u_1$  belongs to the Sobolev space  $W_2^p(\omega_0)$  for every  $p < \infty$  [2: Theorem 24/p. 93]. Hence, by an embedding theorem [12],  $u_1 \in C^1(\overline{\omega}_0)$ . Further, we have  $u_1(x) = 0$ ,  $x \in \partial \omega_0$ . Without loss of generality, we can assume that  $u_1$  is real-valued and takes on positive values in a subdomain of  $\omega_0$ . (One can show that  $\lambda_1$  is a simple eigenvalue and that the (real) eigenfunction  $u_1$  has fixed sign in  $\omega_0$ .) Because  $v(x)$  is positive on  $\overline{\omega}_0$  there exists a uniquely determined  $\varepsilon > 0$  and a point  $x_{\varepsilon} \in \omega_0$  such that  $\varepsilon u_1(x) \leq v(x)$  $(x \in \omega_0)$  and  $\varepsilon u_1(x_{\varepsilon}) = v(x_{\varepsilon})$ . Thus, in a neighbourhood  $K_e(x_{\varepsilon}) = |x| |x - x_{\varepsilon}| < \varrho$  $\mathcal{L} \omega_0$  of  $x_e$  we have  $\mathcal{A}_0(\epsilon u_1) = \epsilon \lambda_1 u_1 \leq \mathcal{A}_0 v = 0$ . Hence, by the theorem of McNABB. [10],  $\epsilon u_1 < v$  or  $\epsilon u_1 = v$  on  $K_{\epsilon}(x_{\epsilon})$ . The case  $\epsilon u_1 < v$  is impossible because of  $\epsilon u_1(x_{\epsilon})$ <br>  $= v(x_{\epsilon})$ . From  $\mathcal{A}_q(\epsilon u_1) < \mathcal{A}_q v$ , on the other hand, it follows that the case  $\epsilon u_1 = v$ <br>
is also impossible.  $v(x_i)$ . From  $A_q(\varepsilon u_1) < A_q v$ , on the other hand, it follows that the case  $\varepsilon u_1 = v$  is also impossible. Hence, *v* must change sign in *G*  $\blacksquare$ in a subdomain of  $\omega_0$ . (One can show<br>
(real) eigenfunction  $u_1$  has fixed sign<br>
exists a uniquely determined  $\varepsilon > 0$ <br>  $(x \in \omega_0)$  and  $\varepsilon u_1(x_i) = v(x_i)$ . Thus, in<br>  $\subset \omega_0$  of  $x_\epsilon$  we have  $\mathcal{A}_q(\varepsilon u_1) = \varepsilon \lambda_1 u_1 <$ Set a uniquely determined  $\varepsilon > \varepsilon$   $\omega_0$ ) and  $\varepsilon u_1(x_t) = v(x_t)$ . Thus<br>  $\omega_0$  of  $x_t$  we have  $\mathcal{A}_q(\varepsilon u_1) = \varepsilon \lambda_1 u$ <br>  $\vert$ ,  $\varepsilon u_1 < v$  or  $\varepsilon u_1 = v$  on  $K_i(x_t)$ .<br>  $v(x_t)$ . From  $\mathcal{A}_q(\varepsilon u_1) < \mathcal{A}_q v$ , on  $v$ <br>  $v(x$ 

rem 2 was proved for uniformly elliptic (but not necessarily selfadjoint) equations by AHMAD *(vi)*  $\alpha$  *q*  $\alpha$  *x* as proved for uniformly ellaps are proved for uniformly ellaps are proved in the hypothese (v)  $p_{ij}(x) \xi_i \xi_j \leq P_{ij}(x) \xi_i \xi_j$ <br>
(v)  $q(x) \leq Q(x)$   $(x \in G)$ .<br> *there exists a nontrivial solut* **of 40**  $P$  is a constant multiple (but not necessarily selfadjo and Lazer [1].<br>
Corollary: Let the hypotheses (i) -(iv) be fulfilled and assum<br>
(v)  $p_{ij}(x) \xi_i \xi_j \leq P_{ij}(x) \xi_i \xi_j$  ( $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$ ;  $x \in G$ <br>
(vi) Under certain restrictions on the domain  $G$ , the coefficients and the set of s<br>
n 2 was proved for uniformly elliptic (but not necessarily selfadjoint) equat<br>
d LAZER [1].<br>
Corollary: Let the hypotheses (i)-(iv) be fulfi

Corollary: *Let the hypotheses (i) — (iv) be fulfilled and aseume* 

$$
(v) \quad p_{ij}(x) \xi_i \xi_j \leq P_{ij}(x) \xi_i \xi_j \qquad (\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n; \, x \in G),
$$

*If there exists a nontrivial solution*  $u \in D(\dot{a}_0)$  of  $\mathcal{A}_0u = 0$ , then every solution  $v \in D_q$  of  $\mathcal{A}_qv = 0$  is a constant multiple of u or changes sign in G. (vi)  $q(x) \leq Q(x)$   $(x \in G)$ .<br>  $\therefore$  there exists a nontrivial solution  $u \in D(\mathcal{A}_q v = 0$  is a constant multiple of u or ch<br>
Proof: Two cases are possible:<br>
(I)  $a_q[\varphi, \varphi] \geq 0$  for all  $\varphi \in C_0^{\infty}(\mathcal{G}),$ <br>
(II)  $a_q[\varphi, \varphi$ (v)  $p_{ij}(x) \xi_i \xi_j \leq P_{ij}(x) \xi_i \xi_j$   $(\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n; x \in \mathbb{R}^n$  *q*(x)  $q(x) \leq Q(x)$   $(x \in G)$ .<br> *If there exists a nontrivial solution*  $u \in D(d_0)$  *of*  $\mathcal{A}_qu = 0$ *, to*  $\mathcal{A}_qv = 0$  *is a constant multiple of u or change hen every sol*<br>,<br>,

(I) 
$$
a_o[\varphi, \varphi] \geq 0
$$
 for all  $\varphi \in C_0^{\infty}(\hat{G})$ 

*Case* (I): It follows from the hypotheses (v) and (vi) that  $0 \le a_g[\varphi, \varphi] \le a_g[\varphi, \varphi]$ **(***Case* (I): It follows from the hypotheses (v) and (vi) that  $0 \le a_q[\varphi, \varphi] \le a_q[\varphi, \varphi]$ <br>  $(\varphi \in C_0^{\infty}(G))$ . Hence, we have  $D(a_q) \subseteq D(a_q)$  and  $a_q[u, u] = 0$  implies  $a_q[u, u] = 0$ .<br>
Thus, Theorem 2 can be used. Thus, Theorem 2 can be used. *• Case (1):* It follows from the hypotheses (v) and (v) that  $0 \leq$ <br>  $(\varphi \in C_0^{\infty}(G))$ , Hence, we have  $D(d_Q) \subseteq D(d_q)$  and  $d_Q[u, u] = 0$  is Thus, Theorem 2 can be used.<br> *Case (II):* We can conclude as in the case (II) of the *a E.* MÜLLER-PFEIFFER<br> *a (G)*), Hence, we have  $D(\dot{a}_Q) \subseteq D(\dot{a}_q)$  and  $\dot{a}_Q[u, u] = 0$  is<br> *a corem* 2 can be used.<br> *I*): We can conclude as in the case (II) of the proof of<br> *cm* 3: *Let the hypotheses* (i) – (

*Case* (II): We can conclude as in the case (II) of the proof of Theorem 2

Theorem 3: Let the hypotheses (i)—(iii) and (1) be fulfilled and let  $u \in D(\dot{a}_0)$  be a

$$
a_q[\varphi,\psi] = \int\limits_G p_{ij} \varphi_{x_i} \overline{\psi}_{x_j} dx + \int\limits_G q \varphi \overline{\psi} dx \qquad (\varphi,\psi \in C_0^\infty(G))
$$

*is closable and that*  $u \in D(a_q)$ ,  $\dot{a}_q[f, g]$  being the closure of  $a_q[\varphi, \psi]$ . If  $\dot{a}_q[u, u] < 0$ , *then every solution v of*  $A_qv = 0$  *changes sign in G.* 

Proof: It follows from  $d_q[u, u] < 0$  that there exists a  $\varphi_0 \in C_0^{\infty}(G)$  with  $a_q[\varphi_0, \varphi_0]$  $< 0$ . Then we conclude as in the case (II) of the proof of Theorem 2

Theorem 4: Let hypotheses (i) and concerning the coefficients  $P_{ij}$   $(i, j = 1, ..., n)$ *and Q the hypothesis (ii) and (iii) be fulfilled. Assume that u and v are linearly in*<br> *dependent solutions of*  $\mathcal{A}_0 u = 0$ . Then, the nodal contour  $N_v$  of v intersects each bounded<br> *nodal domain*  $\Omega$  of u with  $\over$ *dependent solutions of*  $\mathcal{A}_0u = 0$ *. Then, the nodal contour*  $N_v$  *of v intersects each bounded nodal domain*  $\Omega$  *of u with*  $\overline{\Omega} \subset G$ . *Additionally, we have*  $\partial\Omega \cap N_v \neq \emptyset$ .<br>
Proof: Obviously, there are positive constants  $c_1 < 1$  and  $c_2$  such  $|(Q^-\varphi, \varphi)| \leq c_1 a_0^+ [\varphi, \varphi] + c_2 ||\varphi||^2 \qquad (\varphi \in C_0^\infty(\Omega)).$ <br>
We nd Q<br>epend<br>odal d<br>Pro<br>Ve ha *P* **P**  $\Omega$  *P Example 12 <i>P P* 

Proof: Obviously, there are positive constants  $c_1 < 1$  and  $c_2$  such that

$$
|(Q^- \varphi, \varphi)| \leq c_1 a_0^+ [\varphi, \varphi] + c_2 ||\varphi||^2 \qquad (\varphi \in C_0^\infty(\Omega)).
$$

We have  $u_q \in D(\dot{a}_{q,q})$  as remarked above. Further, concerning the solution *v* of  $A_0v = 0$  we have

$$
\int\limits_{\Omega} P_{ij}v_{x_i}v_{x_j}\,dx + \int\limits_{\Omega} Q^+v^2\,dx < \infty.
$$

Hence, Theorem 2 can be applied with  $\Omega$  in place of G. By assuming that  $v(x)$  $=$  const  $\cdot u(x)$  ( $x \in \Omega$ ) the unique continuation theorem for solutions of elliptic equations implies  $v(x) = \text{const} \cdot u(x)$ ,  $x \in G$ . Because u and v are linearly independent we have  $v \neq \text{const} \cdot u$  in  $\Omega$ . Hence, by Theorem 2, *v* changes sign in  $\Omega$ . To prove Notational the set of the set of the set of theorem 4 is an extension of the set of the set of the set of  $\mathcal{U}_q$ ,  $\varphi$  and  $\mathcal{U}_q(\varphi, \varphi)$  are positive constand  $|(\varphi - \varphi, \varphi)| \leq c_1 a_0^+ [\varphi, \varphi] + c_2 ||\varphi||^2$  (we have  $\partial\Omega \cap N_v + \emptyset$  assume  $\partial\Omega \cap N_v = \emptyset$ . Then there exists a nodal domain  $\Theta$  of  $v$  with  $\bar{\theta} \subset \Omega$ . This situation, however, is impossible because, conversely, the nodal con tour of *u* intersects  $\theta$  **I** 

Theorem 4 is an extension of Sturm's separation theorem to the *n*-dimensional case.

Example: Let

This student, however, is impossible because, conversely, the no  
intersects 
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$$
 is an extension of Sturm's separation theorem to the *n*-dimensional cap  
ple: Let  

$$
\mathcal{A}_Q u \equiv -(\sqrt{1-x^2} u') - \frac{u}{\sqrt{1-x^2}} = 0 \qquad (-1 < x < 1; P = p, Q = q).
$$

The hypotheses (i)  $-$  (iv) are fulfilled. Concerning the estimate (1) we refer to [9: Theorem 4]. A non-trivial solution *u* of  $\mathcal{A}_Qu = 0$  is  $u(x) = \sqrt{1 - x^2} (-1 < x < 1)$  which belongs to  $D(\dot{a}_Q)$ . Each linearly independent solution *v*,  $v(x) = C_1\sqrt{1-x^2} + C_2x$  (-1 < x < 1;  $\overline{C_2} = 0$ ) belongs to  $D_q$  and has a zero in  $(-1, 1)$ .  $\partial \Omega \cap N_y = \emptyset$  assume  $\partial \Omega \cap N_y = \emptyset$ . Then there exists a nodal domain  $\Theta$  of v with<br>  $\overline{\Theta} \subset \Omega$ . This situation, however, is impossible because, conversely, the nodal contour of u interests  $\Theta$  ■<br>
Theorem 4 is an e

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