Zeitschrift für Analysis
und ihre Anwendungen
Bd. 7 (1) 1988, S. 77–83

On the Elliptic Sturmian Theory for General Domains

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Der bekannte Vergleichssatz von Sturm und Picone für gewöhnliche, selbstadjungierte Differentialgleichungen zweiter Ordnung wird auf selbstadjungierte elliptische Differentialgleichungen verallgemeinert. Dabei sind das Grundgebiet G und die Koeffizienten der Differentialgleichung nicht notwendig beschränkt, und es werden keine Regularitätsforderungen an den Rand ∂G gestellt.

Известная теорема сравнения Штурма и Пиконе для обыкновенных самосопряженных дифференциальных уравнений второго порядка обобщается на самосопряженные эллиптические дифференциальные уравнения. При этом основная область G и коэффициенты дифференциального уравнения не обязательно ограничены, и условия регулярности для границы ∂G не требуются.

The well-known comparison theorem by Sturm and Picone for ordinary, self-adjoint, second order differential equations is extended to self-adjoint elliptic differential equations. The basic domain G and the coefficients of the equation are not necessarily bounded, and no regularity hypotheses on the boundary ∂G are required.

Consider the differential equations

$$\begin{aligned} \mathcal{A}_Q u &\equiv -(P(x)\,u')' + Q(x)\,u = 0 \\ \mathcal{A}_q u &\equiv -(p(x)\,u')' + q(x)\,u = 0 \end{aligned} \qquad (x \in [a, b]),$$

where $P, p \in C^1[a, b]$ and $Q, q \in C[a, b]$ are real-valued and $P(x), p(x) > 0$, $x \in [a, b]$. A well-known version of the Sturm-Picone theorem is the following one (compare [8: Cor. 1], [13: Theorem 1.5]).

Theorem 1: *If there exists a real solution $u \neq 0$ of $\mathcal{A}_Q u = 0$ such that*

$$u(a) = 0 = u(b) \quad \text{and} \quad \int_a^b [p(u')^2 + qu^2]\,dx \leqq 0,$$

then every real solution v of $\mathcal{A}_q v = 0$ is a constant multiple of u or has at least one zero in (a, b).

In the following this theorem will be extended to self-adjoint, second order, elliptic differential equations. The present investigation complements the paper [11], where the extension of the following version of the Sturm-Picone theorem is handled.

Theorem 1': *Suppose $p(x) \leqq P(x)$ and $q(x) \leqq Q(x)$, $x \in [a, b]$. If there exists a real solution $u \neq 0$ of $\mathcal{A}_Q u = 0$ with $u(a) = 0 = u(b)$, then every real solution v of $\mathcal{A}_q v = 0$ has at least one zero in (a, b) if*

(I) *$q(x') < Q(x')$ for some $x' \in [a, b]$ or*

(II) *$p(x') < P(x')$ and $Q(x') \neq 0$ for some $x' \in (a, b)$.*

Concerning the extensive investigations in the literature which deal with extending the Sturm-Picone theorem to the n-dimensional case we refer to the references in [3, 7, 13].

Using the summation convention, let $\mathcal{A}_Q$, $\mathcal{A}_q$ denote the differential expressions

$$\begin{aligned} \mathcal{A}_Q u &\equiv -(P_{ij}u_{x_i})_{x_j} + Q(x)\,u \\ \mathcal{A}_q u &\equiv -(p_{ij}u_{x_i})_{x_j} + q(x)\,u \end{aligned} \qquad \big(x = (x_1, \ldots, x_n) \in G \subseteqq \mathbf{R}^n\big),$$

where

(i) G is a (possibly unbounded) domain in the Euclidean space $\mathbf{R}^n$ where no regularity hypotheses on the boundary ∂G are required;

(ii) the coefficients $P_{ij} = P_{ji}$, $p_{ij} = p_{ji} \in C^1$ $(i = 1, \ldots, n)$ and $Q, q \in C$ are real-valued and defined on G;

(iii) the smallest eigenvalues $E(x)$ and $e(x)$ of the matrices $(P_{ij})_{i,j=1}^n$ and $(p_{ij})_{i,j=1}^n$, respectively, are positive on G;

(iv) there are positive constants $c_1 < 1$ and c_2 such that

$$|(Q^-\varphi, \varphi)| \leqq c_1 a_Q^+[\varphi, \varphi] + c_2 \|\varphi\|^2 \qquad (1)$$

$$\big(\varphi \in C_0^\infty(G)\big),$$

$$|(q^-\varphi, \varphi)| \leqq c_1 a_q^+[\varphi, \varphi] + c_2 \|\varphi\|^2 \qquad (2)$$

where $(\cdot, \cdot)$ and $\|\cdot\|$ denote the inner product and the norm of the Hilbert space $L_2(G)$ and

$$Q^-(x) = \min\big(Q(x), 0\big), \qquad Q^+(x) = \max\big(Q(x), 0\big),$$

$$q^-(x) = \min\big(q(x), 0\big), \qquad q^+(x) = \max\big(q(x), 0\big),$$

$$\begin{aligned} a_Q^+[\varphi, \psi] &= \int_G P_{ij}\varphi_{x_i}\overline{\psi}_{x_j}\,dx + \int_G Q^+\varphi\overline{\psi}\,dx \\ a_q^+[\varphi, \psi] &= \int_G p_{ij}\varphi_{x_i}\overline{\psi}_{x_j}\,dx + \int_G q^+\varphi\overline{\psi}\,dx. \end{aligned} \qquad \big(\varphi, \psi \in C_0^\infty(G)\big)$$

If Ω is a subdomain of G, let the inner product and the norm of $L_2(\Omega)$ be denoted by $(\cdot, \cdot)_\Omega$ and $\|\cdot\|_\Omega$; the index Ω will be omitted when $\Omega = G$. It follows from (1), (2) that the symmetric operators $A_{Q,0}$ and $A_{q,0}$,

$$A_{Q,0}\varphi = \mathcal{A}_Q\varphi, \qquad A_{q,0}\varphi = \mathcal{A}_q\varphi \qquad \big(\varphi \in C_0^\infty(G)\big)$$

are bounded from below. Consequently, the sesquilinear forms

$$a_Q[\varphi, \psi] = (A_{Q,0}\varphi, \psi) = \int_G P_{ij}\varphi_{x_i}\overline{\psi}_{x_j}\,dx + \int_G Q\varphi\overline{\psi}\,dx,$$

$$a_q[\varphi, \psi] = (A_{q,0}\varphi, \psi) = \int_G p_{ij}\varphi_{x_i}\overline{\psi}_{x_j}\,dx + \int_G q\varphi\overline{\psi}\,dx,$$

$\big(\varphi, \psi \in C_0^\infty(G)\big)$ are closable [5: p. 318]. Let the corresponding closed forms be denoted by $\mathring{a}_Q[f, g]$ and $\mathring{a}_q[f, g]$, respectively. We shall always consider real-valued solutions of the equations $\mathcal{A}_Q u = 0$ and $\mathcal{A}_q u = 0$ which belong to $C(G) \cap W_{2,\mathrm{loc}}^2(G)$, where $W_{2,\mathrm{loc}}^2(G)$ denotes the Sobolev space of (complex-valued) functions the generalized derivatives of which up to order two belong to L_2 on compact subsets of G. Assuming that u is a non-trivial solution of one of these equations the set $N_u = \{y \in G \mid u(y) = 0\}$ is said to be the *nodal contour* of u. By a theorem of McNabb [10] in every neighbourhood $K_\varrho(y) = \{x \mid |x - y| < \varrho\} \subseteqq G$ of a point $y \in N_u$ the non-trivial solution u changes sign [11]. If there exists a nodal contour $N_u \subset G$, by the sets N_u and ∂G the domain G is divided into at least two connected subdomains. Such a subdomain Ω of G is said to be a *nodal domain* of u; u has fixed sign in Ω and $u(x) = 0$,

$x \in \partial\Omega \cap G$. To formulate the following theorem we further require the set

$$D_q = \left\{f \in W^1_{2,\mathrm{loc}}(G) \;\middle|\; \int_G p_{ij} f_{x_i} \bar{f}_{x_j}\, dx + \int_G q^+ |f|^2\, dx < \infty\right\}. \tag{3}$$

Note that a function $f \in D_q$ is not necessarily contained in $L_2(G)$.

Theorem 2: *Let the hypotheses* (i)—(iv) *from above be fulfilled. If there exists a non-trivial solution* $u \in D(\dot{a}_Q)$ *of* $\mathcal{A}_Q u = 0$ *such that* $u \in D(\dot{a}_q)$ *and* $\dot{a}_q[u, u] \leqq 0$, *then every solution* $v \in D_q$ *of* $\mathcal{A}_q v = 0$ *is a constant multiple of* u *or changes sign in* G.

Proof: Let Ω be any nodal domain of u. (Possibly, G itself is a nodal domain of u.) Then the restriction u_Ω of u to Ω belongs to $D(\dot{a}_{Q,\Omega})$ as well as to $D(\dot{a}_{q,\Omega})$, the domains of the closures $\dot{a}_{Q,\Omega}[\cdot,\cdot]$ and $\dot{a}_{q,\Omega}[\cdot,\cdot]$ of the sesquilinear forms

$$\left.\begin{aligned} a_{Q,\Omega}[\varphi, \psi] &= \int_\Omega [P_{ij}\varphi_{x_i}\bar{\psi}_{x_j} + Q\varphi\bar{\psi}]\, dx \\ a_{q,\Omega}[\varphi, \psi] &= \int_\Omega [p_{ij}\varphi_{x_i}\bar{\psi}_{x_j} + q\varphi\bar{\psi}]\, dx \end{aligned}\right\} \quad (\varphi, \psi \in C_0^\infty(\Omega)),$$

respectively [11: Lemma]. Since $u_\Omega \in W^2_{2,\mathrm{loc}}(\Omega)$ and $\mathcal{A}_Q u_\Omega = 0$, $u_\Omega \in D(A^*_{Q,\Omega,0})$, where $A_{Q,\Omega,0}\varphi = \mathcal{A}_Q\varphi$, $\varphi \in C_0^\infty(\Omega)$. Hence, it follows from $u_\Omega \in D(\dot{a}_{Q,\Omega})$ that $u_\Omega \in D(A_{Q,\Omega})$, $A_{Q,\Omega}$ being the Friedrichs extension of $A_{Q,\Omega,0}$. u_Ω is an eigenfunction of $A_{Q,\Omega}$ and $\lambda = 0$ is the corresponding eigenvalue. Hence, we have $\dot{a}_{Q,\Omega}[u_\Omega, u_\Omega] = (A_Q u_\Omega, u_\Omega) = 0$. Of course, we also have $\dot{a}_Q[u, u] = 0$. By the help of (2), one can easily prove that $\dot{a}_q[u, u]$ is represented by

$$\dot{a}_q[u, u] = \int_G (p_{ij}u_{x_i}u_{x_j} + qu^2)\, dx.$$

Because $\dot{a}_q[u, u] \leqq 0$ there exists at least one nodal domain Ω of u such that

$$\dot{a}_{q,\Omega}[u_\Omega, u_\Omega] = \int_\Omega [p_{ij}(u_\Omega)_{x_i}(u_\Omega)_{x_j} + q(u_\Omega)^2]\, dx \leqq 0.$$

At this point, without loss of generality, we can assume that $u_\Omega(x) > 0$, $x \in \Omega$. The following two cases are possible:

(I) $a_{q,\Omega}[\varphi, \varphi] \geqq 0$ for all $\varphi \in C_0^\infty(\Omega)$.

(II) $a_{q,\Omega}[\varphi_0, \varphi_0] < 0$ for some $\varphi_0 \in C_0^\infty(\Omega)$.

Case I: In this case we have $\dot{a}_{q,\Omega}[u_\Omega, u_\Omega] = 0$, and $D(\dot{a}_{q,\Omega}) = D(A^{1/2}_{q,\Omega})$ [5: p. 331], $A_{q,\Omega}$ being the Friedrichs extension of $A_{q,\Omega,0}$, $A_{q,\Omega,0}\varphi = \mathcal{A}_q\varphi$, $\varphi \in C_0^\infty(\Omega)$. It follows from $0 = \dot{a}_{q,\Omega}[u_\Omega, u_\Omega] = \|A^{1/2}_{q,\Omega}u_\Omega\|^2$ that $A_{q,\Omega}u_\Omega = 0$. Therefore u_Ω is also an eigenfunction of $A_{q,\Omega}$ and satisfies the equation $\mathcal{A}_q u_\Omega = 0$.

If there exists a zero of v in G, by the theorem of McNabb the solution v changes sign [11]. Thus, we can assume in the following that $v(x) > 0$, $x \in G$. Choose any point $x^* \in \Omega$ and define w by

$$w(x) = u_\Omega(x^*)\, v(x) - v(x^*)\, u_\Omega(x) \qquad (x \in \Omega).$$

This is a solution of $\mathcal{A}_q w = 0$ in Ω with $w(x^*) = 0$. The identity $w \equiv 0$ implies $v(x) = \mathrm{const} \cdot u(x)$, $x \in \Omega$. If Ω is a proper subdomain of G, then v vanishes on $\partial\Omega \cap G$, which implies that v changes sign in G. Because of the assumption $v(x) > 0$ ($x \in G$), however, we have $\Omega = G$ and $v(x) = \mathrm{const} \cdot u(x)$, $x \in G$. In the case where $w \not\equiv 0$ this function must change sign in every neighbourhood of x^* in Ω as repeatedly remarked. Let $\omega \subset \Omega$ be a subdomain such that $w(x) < 0$ for $x \in \omega$ and $w(y) = 0$ for $y \in \partial\omega \cap \Omega$. We prove that $z \in D(\mathring{a}_{q,\Omega})$,

$$z(x) = \begin{cases} w(x) & \text{for} \quad x \in \omega, \\ 0 & \text{for} \quad x \in \Omega \setminus \omega. \end{cases}$$

Let $(\varphi_m) \subset C_0^\infty(\Omega)$ be a real sequence with $\varphi_m \xrightarrow[a_{q,\Omega}]{} u_\Omega$, that is [5: p. 313]

$$a_{q,\Omega}[\varphi_m - \varphi_{m'}, \varphi_m - \varphi_{m'}] \xrightarrow[m,m'\to\infty]{} 0 \quad \text{and} \quad \|\varphi_m - u_\Omega\|_\Omega \xrightarrow[m\to\infty]{} 0.$$

Set $w_m(x) = u_\Omega(x^*)\, v(x) - v(x^*)\, \varphi_m(x)$ $(x \in \Omega)$ and define

$$w_m^-(x) = \min\big(w_m(x), 0\big) \quad \text{and} \quad \zeta_m(x) = \max\big(z(x), w_m^-(x)\big).$$

Note that $\zeta_m(x) = 0$ $(x \in \Omega \setminus \omega)$ and $w, z, w_m, w_m^-, \zeta_m \in W^1_{2,\mathrm{loc}}(\Omega)$ [6: p. 50]. From $w_m(x) > 0$ we have $w_m^-(x) = 0$ and $\zeta_m(x) = 0$, $x \in \Omega \setminus \operatorname{supp} \varphi_m$. Therefore, $\zeta_m \in D(\mathring{a}_{q,\Omega})$. We now prove that $z \in L_2(\Omega)$ and $\|\zeta_m - z\|_\Omega \to 0$. It follows from $|w - w_m^-| \leqq |w - w_m| = |v(x^*)|\, |u_\Omega - \varphi_m|$ $(x \in \omega)$ that $\|w - w_m^-\|_\omega \leqq |v(x^*)| \times \|u_\Omega - \varphi_m\|_\omega$ and, consequently, $w - w_m^- \in L_2(\omega)$. Thus, we have $w \in L_2(\omega)$ because $w_m^- \in L_2(\omega)$. But this implies $z \in L_2(\Omega)$. To prove $\|\zeta_m - z\|_\Omega \to 0$ use the estimate $\|\zeta_m - z\|_\Omega = \|\zeta_m - z\|_\omega \leqq \|w_m^- - z\|_\omega = \|w_m^- - w_m\|_\omega \leqq |v(x^*)|\, \|\varphi_m - u_\Omega\|_\Omega$. Now we prove that there exists a constant $C > 0$ such that

$$\mathring{a}_{q,\Omega}[\zeta_m, \zeta_m] \leqq C \qquad (m \in \mathbb{N}). \tag{4}$$

By using (3) with $f = v$ and the Schwarz inequality [5: p. 53] and setting $\Omega_m = \operatorname{supp} \varphi_m$ we obtain

$$\begin{aligned}
\mathring{a}_{q,\Omega}[\zeta_m, \zeta_m] &\leqq \int_{\Omega_m} p_{ij}(\zeta_m)_{x_i} (\zeta_m)_{x_j}\, dx + \int_{\Omega_m} q^+\zeta_m^2\, dx \\
&\leqq \int_{\Omega_m} p_{ij} z_{x_i} z_{x_j}\, dx + \int_{\Omega_m} p_{ij}(w_m^-)_{x_i} (w_m^-)_{x_j}\, dx + \int_{\Omega_m} q^+ z^2\, dx + \int_{\Omega_m} q^+ (w_m^-)^2\, dx \\
&\leqq \int_{\Omega_m} p_{ij} w_{x_i} w_{x_j}\, dx + \int_{\Omega_m} p_{ij}(w_m)_{x_i} (w_m)_{x_j}\, dx + \int_{\Omega_m} q^+ w^2\, dx + \int_{\Omega_m} q^+ w_m^2\, dx \\
&\leqq C_1 \Big[\int_{\Omega_m} p_{ij}(u_\Omega)_{x_i} (u_\Omega)_{x_j}\, dx + \int_{\Omega_m} p_{ij} v_{x_i} v_{x_j}\, dx + \int_{\Omega_m} p_{ij}(\varphi_m)_{x_i} (\varphi_m)_{x_j}\, dx \\
&\qquad + \int_{\Omega_m} q^+ (u_\Omega)^2\, dx + \int_{\Omega_m} q^+ v^2\, dx + \int_{\Omega_m} q^+ \varphi_m^2\, dx \Big] \\
&\leqq C_1 \Big(\int_{\Omega_m} [p_{ij}(u_\Omega)_{x_i} (u_\Omega)_{x_j} + q^+ (u_\Omega)^2]\, dx \\
&\qquad + \int_{\Omega_m} [p_{ij}(\varphi_m)_{x_i} (\varphi_m)_{x_j} + q^+ \varphi_m^2]\, dx \Big) + C_2.
\end{aligned} \tag{5}$$

It follows from (2) that

$$a^+_{q,\Omega}[\varphi_m, \varphi_m] \leqq (1 - c_1)^{-1}\, a_{q,\Omega}[\varphi_m, \varphi_m] + (1 - c_1)^{-1}\, c_2\, \|\varphi_m\|_\Omega^2.$$

Hence, in view of

$$a_{q,\Omega}[\varphi_m, \varphi_m] \to \mathring{a}_{q,\Omega}[u_\Omega, u_\Omega] = 0 \quad \text{and} \quad \|\varphi_m\|_\Omega \to \|u_\Omega\|_\Omega,$$

there exists a constant C_3 such that $a^+_{q,\Omega}[\varphi_m, \varphi_m] \leqq C_3$ $(m \in N)$. Finally, we have, for $m \in N$,

$$\int_{\Omega_m} [p_{ij}(u_\Omega)_{x_i} (u_\Omega)_{x_j} + q^+ (u_\Omega)^2]\, dx \leqq \int_G [p_{ij} u_{x_i} u_{x_j} + q^+ u^2]\, dx < \infty.$$

By using these estimates in (5) we obtain (4). It now follows from $\|\zeta_m - z\|_\Omega \to 0$ and (4) that $z \in D(\dot{a}_{q,\Omega})$ [5: Theorem 1.16/p. 315].

The restriction z_ω of z to ω belongs to $D(\dot{a}_{q,\omega})$, the domain of the closure of the form

$$a_{q,\omega}[\varphi, \psi] = \int_\omega p_{ij}\varphi_{x_i}\overline{\psi}_{x_j}\, dx + \int_\omega q\varphi\overline{\psi}\, dx \qquad (\varphi, \psi \in C_0^\infty(\omega))$$

[11: Lemma]. Further we have $z_\omega \in W^2_{2;\mathrm{loc}}(\omega)$ and $\mathcal{A}_q z_\omega = 0$. Hence, z_ω is an eigenfunction of the Friedrichs extension $A_{q,\omega}$ of $A_{q,\omega,0}$, $A_{q,\omega,0}\varphi = \mathcal{A}_q\varphi$ $(\varphi \in C_0^\infty(\omega))$, and $\lambda = 0$ is the corresponding eigenvalue. Therefore, we have $\dot{a}_{q,\omega}[z_\omega, z_\omega] = (A_{q,\omega}z_\omega, z_\omega)_\omega = 0$ and, consequently, $\dot{a}_{q,\Omega}[z, z] = 0$. This relation implies $A^{1/2}_{q,\Omega}z = 0$ which leads to $A_{q,\Omega}z = 0$. Hence, $z \in W^2_{2,\mathrm{loc}}(G)$ (see [2], for instance) and z is a solution of the equation $\mathcal{A}_q z = 0$. By considering the property $z(x) = 0$ $(x \in \Omega \setminus \omega)$ and using the unique continuation theorem for solutions of elliptic equations (see [4: p. 224 and Remarks 3/p. 203]) it follows that $z \equiv 0$ in Ω. Finally, $w(x) = 0$ for $x \in \omega$ implies $w(x) = 0$ for $x \in \Omega$. This proves that $v(x) = \mathrm{const} \cdot u(x)$, $x \in \Omega$. In the case where Ω is a proper subdomain of G we have $v(y) = 0$ $(y \in \partial\Omega \cap G)$ which implies that v must change sign in G. But this case is impossible because of the assumption $v(x) > 0$, $x \in G$. In the case $\Omega = G$, however, the solution v is a constant multiple of u in G.

Case (II): We conclude as in [11]. There exists a function $\varphi_0 \in C_0^\infty(\Omega)$ with $a_{q,\Omega}[\varphi_0, \varphi_0] < 0$. It follows from dist (supp φ_0, $\partial\Omega) > 0$ that there exists a bounded domain ω_0 with supp $\varphi_0 \subset \omega_0 \subset \overline{\omega}_0 \subset G$ belonging to the class C^∞ (definition in [2: p. 28]). Let A_{q,ω_0} be the Friedrichs extension of the operator $A_{q,\omega_0,0}$, $A_{q,\omega_0,0}\varphi = \mathcal{A}_q\varphi$ $(\varphi \in C_0^\infty(\omega_0))$. The spectrum of A_{q,ω_0} is discrete because $0 < c_{\omega_0} \leqq e(x)$ and $|q(x)| \leqq C_{\omega_0} < \infty$ $(x \in \omega_0)$. It follows from $a_{q,\Omega}[\varphi_0, \varphi_0] < 0$ that the smallest eigenvalue λ_1 of A_{q,ω_0} is negative. The corresponding eigenfunction u_1 belongs to the Sobolev space $W_2{}^p(\omega_0)$ for every $p < \infty$ [2: Theorem 24/p. 93]. Hence, by an embedding theorem [12], $u_1 \in C^1(\overline{\omega}_0)$. Further, we have $u_1(x) = 0$, $x \in \partial\omega_0$. Without loss of generality, we can assume that u_1 is real-valued and takes on positive values in a subdomain of ω_0. (One can show that λ_1 is a simple eigenvalue and that the (real) eigenfunction u_1 has fixed sign in ω_0.) Because $v(x)$ is positive on $\overline{\omega}_0$ there exists a uniquely determined $\varepsilon > 0$ and a point $x_\varepsilon \in \omega_0$ such that $\varepsilon u_1(x) \leqq v(x)$ $(x \in \omega_0)$ and $\varepsilon u_1(x_\varepsilon) = v(x_\varepsilon)$. Thus, in a neighbourhood $K_\varrho(x_\varepsilon) = \{x \mid |x - x_\varepsilon| < \varrho\} \subset \omega_0$ of x_ε we have $\mathcal{A}_q(\varepsilon u_1) = \varepsilon\lambda_1 u_1 < \mathcal{A}_q v = 0$. Hence, by the theorem of McNabb [10], $\varepsilon u_1 < v$ or $\varepsilon u_1 = v$ on $K_\varepsilon(x_\varepsilon)$. The case $\varepsilon u_1 < v$ is impossible because of $\varepsilon u_1(x_\varepsilon) = v(x_\varepsilon)$. From $\mathcal{A}_q(\varepsilon u_1) < \mathcal{A}_q v$, on the other hand, it follows that the case $\varepsilon u_1 = v$ is also impossible. Hence, v must change sign in G ∎

Under certain restrictions on the domain G, the coefficients and the set of solutions, Theorem 2 was proved for uniformly elliptic (but not necessarily selfadjoint) equations by Ahmad and Lazer [1].

Corollary: *Let the hypotheses* (i)—(iv) *be fulfilled and assume*

(v) $p_{ij}(x)\,\xi_i\xi_j \leqq P_{ij}(x)\,\xi_i\xi_j$ $\quad (\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n;\ x \in G)$,

(vi) $q(x) \leqq Q(x)$ $\quad (x \in G)$.

If there exists a nontrivial solution $u \in D(\dot{a}_Q)$ *of* $\mathcal{A}_Q u = 0$, *then every solution* $v \in D_q$ *of* $\mathcal{A}_q v = 0$ *is a constant multiple of* u *or changes sign in* G.

Proof: Two cases are possible:

(I) $a_q[\varphi, \varphi] \geqq 0$ for all $\varphi \in C_0^\infty(G)$,

(II) $a_q[\varphi_0, \varphi_0] < 0$ for some $\varphi_0 \in C_0^\infty(G)$.

Case (I): It follows from the hypotheses (v) and (vi) that $0 \leqq a_q[\varphi, \varphi] \leqq a_Q[\varphi, \varphi]$ $(\varphi \in C_0^\infty(G))$. Hence, we have $D(\dot{a}_Q) \subseteqq D(\dot{a}_q)$ and $\dot{a}_Q[u, u] = 0$ implies $\dot{a}_q[u, u] = 0$. Thus, Theorem 2 can be used.

Case (II): We can conclude as in the case (II) of the proof of Theorem 2 ∎

Theorem 3: *Let the hypotheses* (i)—(iii) *and* (1) *be fulfilled and let* $u \in D(\dot{a}_Q)$ *be a non-trivial solution of* $\mathcal{A}_Q u = 0$. *Assume that the sesquilinear form*

$$a_q[\varphi, \psi] = \int_G p_{ij}\varphi_{x_i}\overline{\psi}_{x_j}\,dx + \int_G q\varphi\overline{\psi}\,dx \qquad (\varphi, \psi \in C_0^\infty(G))$$

is closable and that $u \in D(\dot{a}_q)$, $\dot{a}_q[f, g]$ *being the closure of* $a_q[\varphi, \psi]$. *If* $\dot{a}_q[u, u] < 0$, *then every solution* v *of* $\mathcal{A}_q v = 0$ *changes sign in* G.

Proof: It follows from $\dot{a}_q[u, u] < 0$ that there exists a $\varphi_0 \in C_0^\infty(G)$ with $a_q[\varphi_0, \varphi_0] < 0$. Then we conclude as in the case (II) of the proof of Theorem 2 ∎

Theorem 4: *Let hypotheses* (i) *and concerning the coefficients* P_{ij} $(i, j = 1, \ldots, n)$ *and* Q *the hypothesis* (ii) *and* (iii) *be fulfilled. Assume that* u *and* v *are linearly independent solutions of* $\mathcal{A}_Q u = 0$. *Then, the nodal contour* N_v *of* v *intersects each bounded nodal domain* Ω *of* u *with* $\overline{\Omega} \subset G$. *Additionally, we have* $\partial\Omega \cap N_v \neq \emptyset$.

Proof: Obviously, there are positive constants $c_1 < 1$ and c_2 such that

$$|(Q^-\varphi, \varphi)| \leqq c_1 a_Q^+[\varphi, \varphi] + c_2 \|\varphi\|^2 \qquad (\varphi \in C_0^\infty(\Omega)).$$

We have $u_\Omega \in D(\dot{a}_{Q,\Omega})$ as remarked above. Further, concerning the solution v of $\mathcal{A}_Q v = 0$ we have

$$\int_\Omega P_{ij} v_{x_i} v_{x_j}\,dx + \int_\Omega Q^+ v^2\,dx < \infty.$$

Hence, Theorem 2 can be applied with Ω in place of G. By assuming that $v(x) = \text{const} \cdot u(x)$ $(x \in \Omega)$ the unique continuation theorem for solutions of elliptic equations implies $v(x) = \text{const} \cdot u(x)$, $x \in G$. Because u and v are linearly independent we have $v \not\equiv \text{const} \cdot u$ in Ω. Hence, by Theorem 2, v changes sign in Ω. To prove $\partial\Omega \cap N_v \neq \emptyset$ assume $\partial\Omega \cap N_v = \emptyset$. Then there exists a nodal domain Θ of v with $\overline{\Theta} \subset \Omega$. This situation, however, is impossible because, conversely, the nodal contour of u intersects Θ ∎

Theorem 4 is an extension of Sturm's separation theorem to the n-dimensional case.

Example: Let

$$\mathcal{A}_Q u \equiv -\left(\sqrt{1 - x^2}\, u'\right)' - \frac{u}{\sqrt{1 - x^2}} = 0 \qquad (-1 < x < 1;\ P = p,\ Q = q).$$

The hypotheses (i)—(iv) are fulfilled. Concerning the estimate (1) we refer to [9: Theorem 4]. A non-trivial solution u of $\mathcal{A}_Q u = 0$ is $u(x) = \sqrt{1 - x^2}$ $(-1 < x < 1)$ which belongs to $D(\dot{a}_Q)$. Each linearly independent solution v, $v(x) = C_1\sqrt{1 - x^2} + C_2 x$ $(-1 < x < 1;\ C_2 \neq 0)$ belongs to D_q and has a zero in $(-1, 1)$.

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Manuskripteingang: 28. 04. 1986

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