**Generalized Solutions of the Cauchy Problem for a Ionlinear Functional Partial Differential Equation** 

J. TURO

Es vird em Satz über die Existenz und Eindeutigkeit.der verallgemeinerten Losung (in Sinne ,,fast überall") des Cauchy-Problems für nichtlineare Funktional-Differential-Gleichungen mit partiellen Ableitungen erster Qrdnung bewiesen.

Доказана теорема о существовании и единственности обобщенного решения (в смысле ,,почти всюду") задачи Коши для нелинейного дифференциально-функционального уравнения с частными производными первого порядка.

An existence and uniqueness theorem for the generalized solution (in the sense "almost everywhere") of the Cauchy problem for a nonlinear functional partial differential equation of first order is proved.

**1. Introduction.** Let us consider. the Cauchy problem

\n- Es wird ein Satz über die Existenz und Eindeutigkeit der verallgemeinerten Lösung (in Sinne, fast überall") des Cauchy-Problems für nichtlineare Funktional-Differential-Gleichungen mit partiellen Abletungen erster Ordnung bewiesen.
\n- Abasaana reopema o cyuecrosaanu v eguarreennocru oбоóquenuoro peuneu v (e cøbace, noogy") saqaau Kouu "длn nennnechroro gupepeenuañbho-dyukuuonañbioro ypabeneu u gaoaru w mopusu wepsoro noppawa.
\n- An existence and uniqueness theorem for the generalized solution (in the sense "almost everywhere") of the Cauchy problem for a nonlinear functional partial differential equation of first order is proved.
\n- Introduction. Let us consider the Cauchy problem
\n- $$
D_x u(x, y) = F(x, y, u(x, y), (Vu)(x, y), D_y u(x, y))
$$
\n
	\n- (a.e. in [0, a];  $y \in \mathbb{R}$ ),
	\n- $u(0, y) = \varphi(y)$   $(y \in \mathbb{R})$ ,
	\n- where  $D_x = \partial/\partial x$ ,  $D_y = \partial/\partial y$ , and V is an operator of Volterra type.
	\n- Equation (1) contains as particular cases  $((Vu)(x, y) = u(\alpha(x, y), \beta(x, y)))$  the differential equations with a retarded argument, the special cases of which arise in the theory of the distributions.
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\n

Equation (1) contains as particular cases  $((Vu)(x, y) = u(\alpha(x, y), \beta(x, y)))$  the differential equations with a retarded argument, the special cases of which arise in the theory of the distribution of wealth [5]. A few kinds of integral-differential equations can be obtained from (1) by specializing the operator *V.* For instance, problems arising from laser problems in Nonlinear

Optics are also<sup>t</sup> particular cases of the problem with  $(Vu)(x, y) = \int K(y - t) u(x, t) dt$  [1].

In recent papers P. BRANDI and R. CEPPITELLI [2], Z. KAMONT [6, *7],* and A. SAL-VADORI [8] have considered the existence and uniqueness of continuously differentiable solutions of problem  $(1)$ ,  $(2)$  under the assumption that  $F$  is differentiable. The aim of the present paper is to extend these results to a more general case where the given function *F* is not necessarily continuous and solutions of problem (1), (2) are *generalized* (or weak) in the sense "almost everywhere" (a.ê.). The method applied is of fixed point type. It is based on defining an operator, whose range consists of solutions of suitable equations without functional argument. By applying differential inequalities (see Lemma 1) it is proved that this operator is a contraction, and its fixed point is a solution of problem  $(1)$ ,  $(2)$ . The solution is local in x and global in y, and it is unique in a class of bounded functions, absolutely continuous in x and possessing Lipschitzian derivatives in y.<br>2. The auxiliary results. In the sequal we will use the existence theorem for the non linear pa Optics are also particular cases of the problem with  $(Vu)$  (2)<br>
In recent papers P. BRANDI and R. CEPPITELLI [2]<br>
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tiable solutions of problem (1), (2) under the assume<br> *(a.e. in [0, a]*;  $y \in \mathbb{R}$ ). The method is an operator, whose range consists al argument. By applying differenties operator is a contraction, and its solution is local in x and global in s, absolutely continuous in x

2. The auxiliary results. In the sequal we will use the existence theorem for the nonlinear partial differential equation

$$
D_x u(x, y) = f(x, y, u(x, y), D_y u(x, y)) \qquad (a.e. in [0, a]; y \in \mathbf{R}).
$$
 (3)

 

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Assumption  $H_1$ : Suppose that

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Assumption  $H_1$ : Suppose that<br>  $\mathbf{1}^\circ f(x, \cdot) \colon \mathbb{R}^3 \to \mathbb{R}$  is continuous, and derivativ<br>  $\cdot \to \mathbb{R}$  exist and are continuous for every  $x \in [0, a_1, a_2, a_3, b_1, b_2, c_3, d_1, d_2, d_3, d_3, d_4, d_5, d_6, d_7, d_$ 1<sup>o</sup>  $f(x, \cdot): \mathbb{R}^3 \to \mathbb{R}$  is continuous, and derivatives  $D_yf(x, \cdot), D_zf(x, \cdot), D_yf(x, \cdot)$ :  $\mathbf{R}^3 \to \mathbf{R}$  exist and are continuous for every  $x \in [0, a_0]$ ,  $a_0 > 0$ ;<br>  $2^\circ$   $f(\cdot, y, z, q)$ ,  $D_yf(\cdot, y, z, q)$ ,  $D_zf(\cdot, y, z, q)$ ,  $D_qf(\cdot, y, z, q)$ ;  $[0, a_0] \to \mathbf{R}$  are measur-Assumption  $\mathbf{H}_1$ : Suppose that <br>  $1^\circ$   $f(x, \cdot)$ :  $\mathbf{R}^3 \to \mathbf{R}$  is continuous, and derivatives  $D_yf(x, \cdot)$ ,  $D_zf(x, \cdot)$ ,  $D_qf(x, \cdot)$ :<br>  $\cdot^3 \to \mathbf{R}$  exist and are continuous for every  $x \in [0, a_0]$ ,  $a_0 > 0$ ;<br>  $2^\circ f(\$ 128 J. Tuno<br>
Assumption  $\mathbf{H}_1$ : Suppose that<br>  $\mathbf{1}^\circ f(x, \cdot) : \mathbf{R}^3 \to \mathbf{R}$  is continuous, and derivati<br>  $\mathbf{R}^3 \to \mathbf{R}$  exist and are continuous for every  $x \in [0, 2^\circ f(\cdot, y, z, q), D_yf(\cdot, y, z, q), D_zf(\cdot, y, z, q), D_yf(\cdot))$ <br>
a ppose that<br>
continuous, and derivatives  $D_y f(x, \cdot), D_z f(x, \cdot), D_q f(x, \cdot)$ :<br>
ontinuous for every  $x \in [0, a_0], a_0 > 0$ ;<br>  $y, z, q), D_z f(\cdot, y, z, q), D_q f(\cdot, y, z, q)$ :  $[0, a_0] \rightarrow \mathbb{R}$  are measur-<br>  $\mathbb{R}^3$ ;<br>
ble and integrable functions  $M_i,$ 

able for every  $(y, z, q) \in \mathbb{R}^3$ ;

3° there are measurable and integrable functions  $M_i$ ,  $L_i$ :  $[0, a_0] \rightarrow R_+ = [0, +\infty)$ <br>
3° there are measurable and integrable functions  $M_i$ ,  $L_i$ :  $[0, a_0] \rightarrow R_+ = [0, +\infty)$ <br>  $|f(x, y, z, q)| \leq M_0(x)$ ,  $|D_yf(x, y, z, q)| \leq M_1(x)$ ,<br>  $|$ 

$$
|f(x, y, z, q)| \leq M_0(x), \qquad |D_yf(x, y, z, q)| \leq M_1(x),
$$

$$
|D_1f(x, y, z, q)| \leq M_2(x), \qquad |D_qf(x, y, z, q)| \leq M_3(x)
$$

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Assumption  $H_1$ : Suppose that<br>  $1^{\circ} f(x, \cdot)$ :  $R^3 \rightarrow R$  is continuous, a<br>  $R^3 \rightarrow R$  exist and are continuous for e<br>  $2^{\circ} f(\cdot, y, z, q), D_y f(\cdot, y, z, q), D_z f(\cdot, y)$ <br>
able for every  $(y, z, q) \in R^3$ ;<br>  $3^{\circ}$  there are measu  $|f(x, y, z, q) - f(x, \overline{y}, \overline{z}, \overline{q})| \leq L_0(x) (|y - \overline{y}| + |z - \overline{z}| + |q - \overline{q}|).$  $|D_yf(x, y, z, q) - D_yf(x, \overline{y}, \overline{z}, \overline{q})| \leq L_1(x) (|y - \overline{y}| + |z - \overline{z}| + |q - \overline{q}|),$  $|D_x f(x, y, z, q) - D_x f(x, \bar{y}, \bar{z}, \bar{q})| \leq L_2(x) (|y - \bar{y}| + |z - \bar{z}| + |q - \bar{q}|),$  $|D_qf(x, y, z, q) - D_qf(x, \bar{y}, \bar{z}, \bar{q})| \leq L_3(x) (|y - \bar{y}| + |z - \bar{z}| + |q - \bar{q}|)$  $(i = 0, 1, 2, 3)$  such that, a.e. in  $[0, a_0]$ ,<br>  $|f(x, y, z, q)| \leq M_0(x),$   $|D_yf(x, y, z, z, |D_zf(x, y, z, q)|) \leq M_2(x),$   $|D_qf(x, y, z, q)| \leq M_2(x),$ <br>
and<br>  $|f(x, y, z, q) - f(x, \bar{y}, \bar{z}, \bar{q})| \leq L_0(x)$  ( $|y|$ <br>  $|D_yf(x, y, z, q) - D_yf(x, \bar{y}, \bar{z}, \bar{q})| \leq$  $|D_i f(x, y, z, q)| \le M_2(x),$   $|D_q f(x, y, z, q)| \le M_3(x)$ <br>
and<br>  $|f(x, y, z, q) - f(x, \bar{y}, \bar{z}, \bar{q})| \le L_0(x) (|y - \bar{y}| + |z - \bar{z}| + |D_y f(x, y, z, q) - D_y f(x, \bar{y}, \bar{z}, \bar{q})| \le L_1(x) (|y - \bar{y}| + |z - |D_z f(x, y, z, q) - D_z f(x, \bar{y}, \bar{z}, \bar{q})| \le L_2(x) (|y - \bar{y}| + |z - |D_q f(x,$  $|D_i f(x, y, z, q)| \le M_2(x),$ <br>
and<br>  $|f(x, y, z, q) - f(x, \overline{y}, \overline{z}, \overline{q})| \le$ <br>  $|D_y f(x, y, z, q) - D_y f(x, \overline{y}, \overline{z}, \overline{q})|$ <br>  $|D_i f(x, y, z, q) - D_i f(x, \overline{y}, \overline{z}, \overline{q})|$ <br>  $|D_q f(x, y, z, q) - D_q f(x, \overline{y}, \overline{z}, \overline{q})|$ <br>
for all  $(y, z, q), (\overline{y}, \overline{z}, \overline{$ 

 $4^\circ$  the initial function  $\varphi$  in (2) belongs to  $C^1(\mathbf{R}, \mathbf{R})$  ( $C^1(\mathbf{R}, \mathbf{R})$  denotes the set of all continuously differentiable functions on **R** into **R**), and there exist constants  $k_1, k_2 > 0$ 

for all 
$$
(y, z, q)
$$
,  $(\overline{y}, \overline{z}, \overline{q}) \in \mathbb{R}^3$ ;  
\n4° the initial function  $\varphi$  in (2) belongs to  $C^1(\mathbb{R}, \mathbb{R})$  ( $C^1(\mathbb{R}, \mathbb{R})$  denotes the set of all  
\ncontinuously differentiable functions on **R** into **R**), and there exist constants  $k_1, k_2 > 0$   
\nsuch that  $|\varphi'(y)| \leq k_1$  and  $|\varphi'(y) - \varphi'(\overline{y})| \leq k_2 |y - \overline{y}|$   $(y, \overline{y} \in \mathbb{R})$ .  
\nLet us define the constants  
\n
$$
K_1 = k_1 + (1 + k_1 + k_2) \int_0^a \left( (L_0 + \Omega(a_0) L_3 + M_3) \exp \int_0^x G dt \right) dx,
$$
\n
$$
K_2 = k_2 + (1 + k_1 + k_2) \int_0^a \left( (L_1 + \Omega(a_0) L_2 + M_2) \exp \int_0^x G dt \right) dx,
$$
\n
$$
g_0 = 1 - (1 + k_1 + k_2) \int_0^a \left( L_3 \exp \int_0^x G dt \right) dx,
$$
\nwhere  
\n
$$
G = L_0 + L_1 + L_3 + \Omega(a_0) (L_2 + L_3) + M_2 + M_3,
$$
\n
$$
\Omega(a_0) = \exp \int_0^a M_2 dt \left\{ k_1 + \int_0^a \left( M_1 \exp \left( -\int_0^t M_2 ds \right) \right) dt \right\}.
$$

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$$
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\n
$$
\Omega(a_0) = \exp \int_0^{a_0} M_2 \, dt \left\{ k_1 + \int_0^{a_0} \left( M_1 \exp \left( - \int_0^t M_2 \, ds \right) \right) dt \right\}.
$$
  
\nrem 1. [3, 4]. If Assumption  $H_1$  is satisfied, then there are a  
\nunction  $u: E_a = [0; a] \times \mathbb{R} \to \mathbb{R}$  satisfying equation (3)  
\nution is unique in that class of functions  $u: E_a \to \mathbb{R}$  for which  
\n**R** are absolutely continuous for every  $y \in \mathbb{R}$ , and  
\n
$$
|D_y u(x, y)| \leq \Omega(a_0),
$$
  
\n
$$
|u(x, y) - u(x, \bar{y})| \leq \frac{K_1}{g_0} |y - \bar{y}|, \qquad |D_y u(x, y) - D_y u(x, y)|
$$

**Theorem 1.**[3, 4]: If Assumption  $H_1$  is satisfied, then there are a constant  $a \in (0, a_0]$ . **Theorem 1.[3, 4]:** *If Assumption H<sub>1</sub>* is satisfied, then there are a constant  $a \in (0, a_0]$ <br>and a function  $u: E_a = [0, a] \times \mathbb{R} \to \mathbb{R}$  satisfying equation (3) and condition (2).<br>This solution is surized in that descrip *This solution is unique in that class of functions*  $u: E_a \to \mathbf{R}$  *for which*  $u(\cdot, y)$ *,*  $D_y u(\cdot, y)$ *:*  $\cdots$  *.<br>*  $\vdots$  $Q(a_0) = \exp \int_0^{a_0} M_2 dt \left\{ k_1 + \int_0^{a_0} \left( M_1 \exp \left( -\int_0^t M_2 ds \right) \right) dt \right\}.$ <br>
Theorem 1.[3, 4]: If Assumption  $H_1$  is satisfied, then there are a constant  $a \in (0, a_0)$ <br>
and a function  $u: E_a = [0, a] \times \mathbf{R} \to \mathbf{R}$  satisfying Theorem 1.[3, 4]: If Assumption  $H_1$  is satisfied, then there are a constant  $a \in (0, a_0]$ <br>and a function  $u : E_a = [0, a] \times \mathbb{R} \to \mathbb{R}$  satisfying equation (3) and condition (2).<br>This solution is unique in that class of fu  $\begin{aligned} \mathcal{L}(a_0) &= \exp\int_{0}^{1} M_2 \, dt \, \left\{ k_1 + k_2 \right\} \ \mathcal{L}(a_0) &= \exp\int_{0}^{1} M_2 \, dt \, \left\{ k_2 + k_3 \right\} \ \mathcal{L}(a_0) &= \exp\left\{ k_1 + k_2 \right\} \ \mathcal{L}(a_0) &= \exp\left\{ k_2 + k_3 \right\} \ \mathcal{L}(a_0) &= \exp\left\{ k_1 + k_2 \right\} \end{aligned}$  $G = L_0 + L_1 + L_3 + \Omega(a_0) \cdot (L_2 + L_3) + M_2 + M_3,$ <br>  $\Omega(a_0) = \exp \int_{1}^{a_0} M_2 dt \left\{ k_1 + \int_{0}^{a_0} \left( M_1 \exp \left( -\int_{0}^{t} M_2 ds \right) \right) dt \right\}$ <br>  $\text{reorem 1.}[3, 4]$ : *If Assumption H<sub>1</sub>* is satisfied, then there are a constant  $a \in (0, a)$  and *un* 

$$
|D_y u(x, y)| \leq \Omega(a_0),
$$
  
\n
$$
|u(x, y) - u(x, \overline{y})| \leq \frac{K_1}{g_0} |y - \overline{y}|,
$$
  
\n
$$
|D_y u(x, y) - D_y u(x, \overline{y})| \leq \frac{K_2}{g_0} |y - \overline{y}|
$$
  
\n
$$
(x, y), (x, \overline{y}) \in E_a.
$$

*for every*  $(x, y)$ ,  $(x, \bar{y}) \in E_a$ .

**Remark:** In the above theorem a is chosen sufficiently small such that  $g_0 > 0$ .

Set, for  $a, b > 0$ ,  $M \ge 0$ , and  $Ma \le b$ ,

$$
\begin{aligned}\n\text{Generalized Solu} \\
\text{for } a, b > 0, M \geq 0, \text{ and } Ma \leq b, \\
E_{ab} &= \{(x, y): 0 \leq x \leq a, |y| \leq b - Mx\}, \\
E_{ab} &= \{(x, y): 0 \leq x \leq a, |y| \leq b - Mx\},\n\end{aligned}
$$

for 
$$
a, b > 0
$$
,  $M \ge 0$ , and  $Ma \le b$ ,  
\n
$$
E_{ab} = \{(x, y): 0 \le x \le a, |y| \le b - Mx\},
$$
\n
$$
E_{xb} = \{(s, y) \in E_{ab}: s \le x\}, \qquad S_x = \{y: (x, y) \in E_{ab}\}.
$$
\nneed the following

We shall need the following

Set, for  $a, b > 0, M \ge 0$ , and  $Ma \le b$ ,<br>  $E_{ab} = \{(x, y): 0 \le x \le a, |y| \le b - Mx\}$ ,<br>  $E_{xb} = \{(s, y) \in E_{ab}: s \le x\}$ ,  $S_x = \{y: (x, y) \in E_{ab}\}$ .<br>
c shall need the following<br>
Lemma 1: *Suppose that*<br>  $1^{\circ}u: E_{ab} \rightarrow R$  is continuous, and  $D_xu$  e Lemma 1: Suppose that<br>  $1^\circ u: E_{ab} \to \mathbb{R}$  is continuous, and  $D_x u$  exists for a.e.  $x \in [0, a]$  and every  $y \in [-b$ <br>  $Mx, b - Mx]$ ;<br>  $2^\circ$  for every  $x \in [0, a]$ ,  $u(x, \cdot)$  fulfils a Lipschitz condition;<br>  $3^\circ$  there are constants  $+ Mx, b - Mx$ ; We shall<br>Lemm<br> $\begin{array}{l} 1^{\circ} u : E \\ + Mx, b \\ 2^{\circ} _{\circ} for e \\ 3^{\circ} there \\ \end{array}$ <br>*rhen the' d* 

 $2^{\circ}$  for every  $x \in [0, a]$ ,  $u(x, \cdot)$  fulfils a Lipschitz condition;

Set, for 
$$
a, b > 0
$$
,  $M \ge 0$ , and  $Ma \le b$ ,  
\n $E_{ab} = \{(x, y) : 0 \le x \le a, |y| \le b - Mx\},$   
\n $E_{xb} = \{(s, y) \in E_{ab} : s \le x\},$   $S_x = \{y : (x, y) \in E_{ab}\}.$   
\ne shall need the following  
\nLemma 1: Suppose that  
\n $1^{\circ} u : E_{ab} \rightarrow \mathbb{R}$  is continuous, and  $D_x u$  exists for a.e.  $x \in [0, a]$  and ever  
\n $Mx, b - Mx\};$   
\n $2^{\circ}$  for every  $x \in [0, a]$ ,  $u(x, \cdot)$  fulfils a Lipschitz condition;  
\n $3^{\circ}$  there are constants  $c_1, c_2 \ge 0$  such that  
\n $|D_x u(x, y)| \le c_1 |u(x, y)| + c_2 + M |D_y u(x, y)|$  (a.e. in [0, a];  
\n $|y| \le b - Mx).$   
\nthen the derivative  $\gamma'$  of the function  $\gamma$ ,  
\n $\gamma(x) = \max \{|u(s, t)| : (s, t) \in E_{xb}\}$   $(x \in [0, a]);$   
\n $3^{\circ} u = \sum_{i=1}^{m} |u(x, y)| + |v(x, y)|$   
\n $u(x, y) = \sum_{i=1}^{m} |u(x, y)| + |v(x, y)| + |v(x, y)|$   
\n $u(x, y) = \sum_{i=1}^{m} |u(x, y)| + |v(x, y)| + |v(x, y)|$   
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\n $u(x, y) = \sum_{i=1}^{m} |u(x, y)| + |v(x, y)| + |v(x, y)| + |v(x, y)|$   
\n<

Then the derivative  $\gamma'$  of the function  $\gamma$ ,

$$
\gamma(x) = \max \{|u(s, t)| : (s, t) \in E_{xb}\} \qquad (x \in [0, a]),
$$

*exists a.e. and*  $\gamma'(x) \leq c_1 \gamma(x) + c_2$  *a.e. in* [0, *a*].

**Proof:** Since  $\delta(x) = \max \{|u(x, y)| : y \in S_x\}$  is continuous on [0, *a*] (see [9]) and  $\gamma(x) = \max \{\delta(s) : s \in [0, x]\}, \gamma$  is continuous on  $[0, a]$  and  $\gamma'$  exists a.e. in  $[0, a]$ . Suppose that  $(\bar{x}, \bar{y}) \in E_{xb}$  is such that *Axial*  $x \in [0, a]$ *,*  $u(x, \cdot)$  *fulfi*<br> *Axial*  $x_1, c_2 \ge 0$  *sui*<br>  $|D_x u(x, y)| \le c_1 |u(x, y)| +$ <br>  $|y| \le b - Mx$ .<br> *Acrivative*  $\gamma'$  *of the function*<br>  $\gamma(x) = \max \{|u(s, t)| : (s, t) \in$ <br> *And*  $\gamma'(x) \le c_1 \gamma(x) + c_2$  *a.e.*<br>  $\therefore$  Since  $\delta(x) = \max \$  $\gamma(x) = \text{ma}$ <br>exists a.e. and  $\gamma'(x)$ <br>Proof: Since  $\delta(x)$ <br> $\gamma(x) = \text{max } \{\delta(s) : s$ <br>Suppose that  $(\bar{x}, \bar{y})$ <br> $\gamma(x) = |u|$ <br>and  $u_x(\bar{x}, \bar{y}), u_y(\bar{x}, \bar{y})$ <br>definitions of  $\gamma$  and *Mx*).<br>  $\gamma'$  of the function  $\gamma$ ,<br>  $x \{|u(s, t)| : (s, t) \in E_{xb}\}$   $(x \in [0, a])$ ;<br>  $\leq c_1 \gamma(x) + c_2 a.e. in [0, a]$ .<br>  $\gamma$  = max  $\{|u(x, y)| : y \in S_x\}$  is continuous on  $[0, a]$  (see [9]) and<br>  $\in [0, x]$ ,  $\gamma$  is continuous on  $[0, a]$  and  $\gamma'$ 

$$
\gamma(x) = |u(\overline{x}, \overline{y})|,
$$

definitions of  $\gamma$  and  $\delta$  it follows that  $\gamma(x) = \delta(\bar{x})$ . For  $h < 0$ , we get and  $u_x(\bar{x}, \bar{y}), u_y(\bar{x}, \bar{y}), \gamma'(x)$  exist. Thus we have  $0 \le \bar{x} \le \bar{x}$ . First, let  $\bar{x} \ne 0$ . From the

\n- \n f: Since 
$$
\delta(x) = \max \{|u(x, y)| : y \in S_x\}
$$
 is continuous on  $[0, a]$  (see [9]) and  $\max \{\delta(s) : s \in [0, x] \}$ ,  $\gamma$  is continuous on  $[0, a]$  and  $\gamma'$  exists a.e. in  $[0, a]$ .\n
\n- \n that  $(\bar{x}, \bar{y}) \in E_{xb}$  is such that\n
\n- \n $\gamma(x) = |u(\bar{x}, \bar{y})|$ ,\n  $\gamma'(x) = |u(\bar{x}, \bar{y})|$ ,\n  $\gamma'(x) = \delta(\bar{x})$ . For  $h < 0$ , we get\n
\n- \n $\gamma(x + h) \geq \delta(x + h)$  and\n
\n- \n $\gamma(x + h) = \gamma(x)$  and  $\frac{\gamma(x + h) - \gamma(x)}{h} \leq \frac{\delta(\bar{x} + h) - \delta(\bar{x})}{h}$ .\n
\n- \n by  $h \to 0^-$ , we obtain\n
\n- \n $\gamma'(x) \leq D_{\mathcal{O}}(x)$ \n
\n- \n he left-hand (lower Dini derivative). For  $\bar{x} = 0$  inequality (5) is certainly, since  $\gamma$  is constant in  $[0, x]$ . If  $(\bar{x}, \bar{y})$  is an interior point of  $E_{ab}$ , then we have\n
\n

Hence, by  $h$ 

$$
\nu'(x) \leq D_\nu(\delta x)
$$

*(D\_* is the left-hand<sup>'</sup>lower Dini derivative). For  $\bar{x} = 0$  inequality (5) is certainly satisfied, since  $\gamma$  is constant in [0, *x*]. If  $(\bar{x}, \bar{y})$  is an interior point of  $E_{ab}$ , then we have  $[9]$   $D_\nu(\bar{x}) \leq |D_x u(\bar{x}, \bar{y})|$ , and also  $|D_y u(\bar{x}, \bar{y})| = 0$ . Hence, by (5) and assumption 3<sup>o</sup> it follows that **•** *• • • • • • <i>• • • <i>• <i>• • • • <i>f <i><i>• <i>f <i>f*

$$
\gamma'(x) \leq D_{\tilde{z}}(\bar{x}) \leq |D_x u(x, y)| \leq c_1 |u(\bar{x}, \bar{y})| + c_2 = c_1 \gamma(x) + c_2.
$$

Now, suppose that  $(\bar{x}, \bar{y})$  is not an interior point of  $E_{ab}$ . Then  $\bar{y} = b - M\bar{x}$  or  $= -b + M\bar{x}$ . We consider only the first case. Assume also, that in (4) we have  $\gamma(x) = u(\bar{x}, \bar{y})$  (for  $\gamma(x) = -u(\bar{x}, \bar{y})$  the proof is quite similar). Let us consider the function  $\tilde{m}$  defined by  $\tilde{m}(x) = u(x, b - Mx)$ . Since  $\tilde{m}(x) \leq \delta(x)$ ,  $x \in [0, \bar{x}]$ , and  $\gamma'(x) \leq D_-\delta(\bar{x}) \leq |D_x u(x, y)| \leq c_1 |u(\bar{x}, \bar{y})| + c_2 = c_1 \gamma(x) + c_2.$ <br>Now, suppose that  $(\bar{x}, \bar{y})$  is not an interior point of  $E_{ab}$ . Then  $\bar{y} = b - M\bar{x}$  or  $\bar{y} = -b + M\bar{x}$ . We consider only the first case. Assume also, th [9]  $D_-\delta(\bar{x}) \le |D_x u(\bar{x}, \bar{y})|$ , and also  $|D_y u(\bar{x}, \bar{y})| = 0$ . Hence, by (5) and<br>it follows that<br> $\gamma'(x) \le D_-\delta(\bar{x}) \le |D_x u(x, y)| \le c_1 |u(\bar{x}, \bar{y})| + c_2 = c_1 \gamma(x) + c$ <br>Now, suppose that  $(\bar{x}, \bar{y})$  is not an interior point of  $E_{ab}$ . pose that  $(\bar{x}, \bar{y})$ <br> *- M* $\bar{x}$ . We consi<br> *T*,  $\bar{y}$ ) (for  $\gamma(x) = \tilde{n}$ <br>  $\tilde{n}$  defined by  $\tilde{n}$ <br>  $\tilde{v}$ ),<br>  $\tilde{n}(\bar{x} + h) - \tilde{m}(\bar{x})$ also |<br> *|D<sub>z</sub>u*(*x*,<br>
is not a<br>
der only<br>  $-u(\bar{x}, \bar{y})$ <br>  $\bar{n}(x) = u$ <br>  $\frac{\delta(\bar{x})}{\delta(x)}$  $\tilde{m}(\bar{x}) = \delta(\bar{x}),$ ows that<br>  $\gamma'(x) \leq D_-\delta(\bar{x}) \leq |D_xu(x, y)| \leq c_1 |u(\bar{x}, \bar{y})| + c_2 = c_1\gamma(x) + c_2.$ <br>
suppose that  $(\bar{x}, \bar{y})$  is not an interior point of  $E_{ab}$ . Then  $\bar{y} = b - M\bar{x}$  or<br>  $-b + M\bar{x}$ . We consider only the first case. Assume also, th

function *m* defined by 
$$
m(x) = u(x, b - mx)
$$
. Since  $m(x) \ge \tilde{m}(\bar{x}) = \delta(\bar{x})$ ,  
\n
$$
\frac{\tilde{m}(\bar{x} + h) - \tilde{m}(\bar{x})}{h} \ge \frac{\delta(\bar{x} + h) - \delta(\bar{x})}{h}
$$
 for  $h < 0$ .  
\nHence, we get  
\n
$$
\tilde{m}'(x) \ge D_0(\bar{x})
$$
.  
\n9 Analysis Bd. 7, Hett 2 (1988)

Hence, we get

$$
\tilde{m}'(x) \ge D_-\delta(\bar{x}).
$$

(4)

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From assumption 2° it follows that  $\tilde{m}'$  exists a.e. in [0, a] and  $\tilde{m}'(x) = D_x u(x, b - Mx)$ *— MD<sub>u</sub>* $u(x, b - Mx)$ . In particular, we have  $\tilde{m}'(\bar{x}) = D_x u(\bar{x}, \bar{y}) - M D_y u(\bar{x}, \bar{y})$ . Hence, and by (6), we obtain  $D_-\delta(\bar{x}) \leq D_xu(\bar{x}, \bar{y}) - MD_uu(\bar{x}, \bar{y})$ . The inequality together with assumption 3° yields  $-MD_yu(x, b - Mx)$ . In particular, we have  $m'$ <br>
Hence, and by (6), we obtain  $D_0(\bar{x}) \le D_xu(\bar{x}, \bar{y})$  -<br>
ogether with assumption 3° yields<br>  $D_0(\bar{x}) \le D_xu(\bar{x}, \bar{y}) - MD_yu(\bar{x}, \bar{y}) \le c_1 |u(\bar{x}, \bar{y})$ <br>
Hence and by (5) we get the asse

$$
D_-\delta(\bar{x}) \leq D_x u(\bar{x}, \bar{y}) - M D_y u(\bar{x}, \bar{y}) \leq c_1 |u(\bar{x}, \bar{y})| + c_2 = c_1 \gamma(x) + c_2.
$$

Hence and by (5) we get the assertion of the lemma I

3. The existence theorem. We denote by  $K(a, P, Q)$  the class of all continuous and bounded functions  $u: E_a \to \mathbf{R}$  satisfying the following conditions:

(i)  $u(\cdot, y), D_y u(\cdot, y) : [0, a] \to \mathbf{R}$  are absolutely continuous for every  $y \in \mathbf{R}$ ;

(ii) there are constants  $P, Q \geq 0$  such that

$$
|u(x, y) - u(x, \overline{y})| \le P |y - \overline{y}|, \qquad |D_y u(x, y) - D_y u(x, \overline{y})| \le Q |y - \overline{y}|,
$$

Assumption **<sup>112</sup> :** Suppose that

 $\mathbf{P} \in F(x, \cdot): \mathbb{R}^4 \to \mathbb{R}$  is continuous, and the derivatives  $D_y F(x, \cdot), D_z F(x, \cdot)$ ,  $D_pF(x, \cdot), D_qF(x, \cdot): \mathbf{R}^4 \to \mathbf{R}$  exist and are continuous for every  $x \in [0, a_0]$ ; for all  $(x, y)$ ,  $(x$ <br>  $1^{\circ}$   $F(x, \cdot)$ ;  $\mathbf{R}$ <br>  $D_p F(x, \cdot)$ ,  $D_q F(\cdot)$ <br>  $2^{\circ} F(\cdot, y, z, p)$ <br>  $2^{\circ} F(\cdot, y, z, p)$ <br>  $2^{\circ}$   $F(\cdot, y, z, p)$ <br>  $3^{\circ}$  there are<br>  $[0, a_0] \rightarrow \mathbf{R}_+$  (i<br>  $|F(x, y)|$ <br>  $|D_p F(z)|$ <br>  $|D_p F(z)|$ <br>
and<br>  $|F(x, y)| \leq l_0$ 

 $2^{\circ} F(\cdot, y, z, p, q), D_y F(\cdot, y, z, p, q), D_z F(\cdot, y, z, p, q), D_p F(\cdot, y, z, p, q), D_q F(\cdot, y, z, p, q)$  $\mathbf{1}^{\circ}$   $F(x, \cdot): \mathbf{K}^{\bullet} \to \mathbf{K}$  is continuous, and the derivatives  $D_y F(x, \cdot), D_z F(x, \cdot),$ <br>  $F(x, \cdot), D_q F(x, \cdot): \mathbf{R}^4 \to \mathbf{R}$  exist and are continuous for every  $x \in [0, a_0]$ ;<br>  $2^{\circ} F(\cdot, y, z, p, q), D_y F(\cdot, y, z, p, q), D_z F(\cdot, y, z, p$  $[2^{\circ}F(\cdot, y, z, p, q), D_yF(\cdot, y, z, p, q), D_zF(\cdot, y, z, p, q), D_pF(\cdot, y, z, p, q),$ <br>  $[p, q): [0, a_0] \rightarrow \mathbf{R}$  are measurable for every  $(y, z, p, q) \in \mathbf{R}^4$ ;<br>  $3^{\circ}$  there are a constant  $l_0 \ge 0$ , and measurable and integrable<br>  $[0, a_0] \rightarrow \mathbf{R}_$ 

for all 
$$
(x, y)
$$
,  $(x, \bar{y}) \in E_a$ .  
\nAssumption  $H_2$ : Suppose that  
\n $1^{\circ} F(x, \cdot)$ :  $\mathbb{R}^4 \to \mathbb{R}$  is continuous, and the derivatives  $D_y F(x, \cdot)$ ,  $D_z F(x, \cdot)$ ,  $D_x F(x, \cdot)$ ,  $D_y F(x, \cdot)$ :  $\mathbb{R}^4 \to \mathbb{R}$  exist and are continuous for every  $x \in [0, a_0]$ ;  
\n $2^{\circ} F(\cdot, y, z, p, q)$ ,  $D_y F(\cdot, y, z, p, q)$ ,  $D_x F(\cdot, y, z, p, q)$ ,  $D_p F(\cdot, y, z, p, q)$ ,  $D_q F(\cdot, p, q)$ :  $[0, a_0] \to \mathbb{R}$  are measurable for every  $(y, z, p, q) \in \mathbb{R}^4$ ;  
\n $3^{\circ}$  there are a constant  $l_0 \ge 0$ , and measurable and integrable functions  $m$   
\n $[0, a_0] \to \mathbb{R}_+$   $(i = 0, 1, 2, 3, 4; j = 1, 2, 3, 4)$  such that a.e. in  $[0, a_0]$   
\n $|F(x, y, z, p, q)| \le m_0(x)$ ,  
\n $|D_y F(x, y, z, p, q)| \le m_1(x)$ ,  $|D_x F(x, y, z, p, q)| \le m_2(x)$ ,  
\n $|D_p F(x, y, z, p, q)| \le m_3(x)$ ,  $|D_q F(x, y, z, p, q)| \le m_4(x)$ ,  
\nand  
\n $|F(x, y, z, p, q) - F(x, \overline{y}, \overline{z}, \overline{p}, \overline{q})|$   
\n $\leq l_0(|y - \overline{y}| + |z - \overline{z}| + |p - \overline{p}| + |q - \overline{q}|)$ ;  
\n $|D_y F(x, y, z, p, q) - D_y F(x, \overline{y}, \overline{z}, \overline{p}, \overline{q})|$ 

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Assumption H<sub>2</sub>: Suppose that  
\n
$$
P^c F(x, \cdot); P_x + \cdot \mathbf{B}
$$
 is continuous, and the derivatives  $D_y F(x, \cdot), D_z F(x, \cdot)$   
\n $D_y F(x, \cdot); P_y + \cdot \mathbf{B}$  is continuous for every  $x \in [0, a_0]$ ;  
\n $2^c F(\cdot, y, z, p, q), D_y F(\cdot, y, z, p, q), D_z F(\cdot, y, z, p, q), D_y F(\cdot, y, z, p, q), D_y F(\cdot, y, z, p, q)$   
\n $p, q): [0, a_0] \rightarrow \mathbf{R}$  are measurable for every  $(y, z, p, q) \in \mathbf{R}^4$ ;  
\n $3^c$  there are a constant  $l_0 \ge 0$ , and measurable and integrable functions  $m_i, l_j$   
\n $[0, a_0] \rightarrow \mathbf{R}_+ (i = 0, 1, 2, 3, 4; j = 1, 2, 3, 4)$  such that a.c. in  $[0, a_0]$   
\n $|F(x, y, z, p, q)| \le m_0(x)$ ,  
\n $|D_y F(x, y, z, p, q)| \le m_1(x)$ ,  $|D_x F(x, y, z, p, q)| \le m_2(x)$ ,  
\n $|D_y F(x, y, z, p, q)| \le m_3(x)$ ,  $|D_q F(x, y, z, p, q)| \le m_4(x)$ ,  
\nand  
\n $|F(x, y, z, p, q) - F(x, \overline{y}, \overline{z}, \overline{p}, \overline{q})|$   
\n $\le l_0(|y - \overline{y}| + |z - \overline{z}| + |p - \overline{p}| + |q - \overline{q}|)$ ,  
\n $|D_x F(x, y, z, p, q) - D_y F(x, \overline{y}, \overline{z}, \overline{p}, \overline{q})|$   
\n $\le l_4(x) (|y - \overline{y}| + |z - \overline{z}| + |p - \overline{p}| + |q - \overline{q}|)$ ,  
\n $|D_y F(x, y, z, p, q) - D_y F(x, \overline{y}, \overline{z}, \overline{p}, \overline{q})|$ 

for all  $(y, z, p, q)$ ,  $(\bar{y}, \bar{z}, \bar{p}, \bar{q}) \in \mathbb{R}^4$ ;<br> $4^{\circ}$   $(Vu) \cdot (y)$ :  $[0, a_0] \rightarrow \mathbb{R}$  is measurable for  $y \in \mathbb{R}$ ,  $u \in K(a_0, P, Q)$ , there exists  $D_{\nu}(Vu) \in C(E_{a_{\nu}}, \mathbb{R})$  for each  $u \in K(a_{0}, P, Q)$ , and there are measurable and integrable

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functions  $p, r_i : [0, a_0] \to \mathbb{R}^+$  (i = 0, 1) such that a.e. in [0, a<sub>0</sub>]

 $|(Vu)(x,y)-(Vu)(x,\bar{y})|\leq r_0(x)|y-\bar{y}|,$ 

*D*<br> *D*(*Vu) (x, y) - P<sub>(<i>x*)</sub> *D*<sub>*y*</sub>(*x*) *D*<sub>*y*</sub>(*Yu*) (*x, y*)  $\leq$  *P*<sub>(</sub>*x*),  $|D_y(Vu)(x, y) - (Vu)(x, \bar{y})| \leq r_0(x) |y - \bar{y}|$ ,<br> *D*<sub>*y*</sub>(*Yu*) (*x, y*)]  $\leq$  *p*(*x*),  $|D_y(Vu)(x, y) - D_y(Vu)(x, \bar{y})| \leq r_1(x) |y - \bar{y}|$ <br>
for

functions  $p, r_i : [0, a_0] \to \mathbb{R}_+$   $(i = 0, 1)$  such  $|(Vu)(x, y) - (Vu)(x, \bar{y})| \le r_0(x)$ <br> $|D_y(Vu)(x, y)| \le p(x), \qquad |D_y(Vu)|$ <br>for each  $u \in K(a_0, P, Q)$ ;<br> $\bar{u} \in K(a_0, P, Q)$ , where  $||u||_z = \sup \{|u(s, y)|\}$ <br> $6^{\circ}$  the initial function  $\varphi$  in (2) be  $5^\circ$  there is a constant  $s \geq 0$  such that  $||Vu - V\overline{u}||_x \leq s ||u - \overline{u}||_x$  for any *u*,  $\overline{u} \in K(a_0; P, Q)$ , where  $||u||_x = \sup \{|u(s, y)| : (s, y) \in E_x\}, E_x = [0, x] \times \overline{R};$ for each  $u \in K(a_0, P, Q)$ ;<br>  $\overline{u} \in K(a_0, P, Q)$ ,<br>  $\overline{u} \in K(a_0, P, Q)$ , where  $||u||_x = \sup \{|u(s, y)| : (s, y) \in E_x\}$ ,  $E_x = [0, x] \times \mathbb{R}$ ;<br>  $\overline{u} \in K(a_0, P, Q)$ , where  $||u||_x = \sup \{|u(s, y)| : (s, y) \in E_x\}$ ,  $E_x = [0, x] \times \mathbb{R}$ ;<br>  $\overline{u} \in K(a_0, E$ 

 $\overline{u} \in K(a_0, P, Q)$ , where  $||u||_z = \sup \{ |u(s, y)| : (s, y) \in E_z \}$ ,  $E_z =$ <br> $\overline{u} \in K(a_0, P, Q)$ , where  $||u||_z = \sup \{ |u(s, y)| : (s, y) \in E_z \}$ ,  $E_z =$ <br> $\overline{b}^{\circ}$  the initial function  $\varphi$  in (2) belongs to  $C^1(\mathbf{R}, \mathbf{R})$ , and<br> $k_0, k_1 >$  $k_0, k_1 > 0$  and  $k_2 \ge 0$  such that  $|\varphi(y)| \le k_0$ ,  $|\varphi'(y)| \le k_1$  and  $|\varphi'(y) - \varphi'(y)|$  $\leq k_2 |y - \bar{y}| (y, \bar{y} \in \mathbf{R}).$  $|Vu|(x, y) - (y)$ <br>  $|D_y(Vu)(x, y)|$ <br>
for each  $u \in K(a_0, P, Q)$ <br>
for each  $u \in K(a_0, P, Q)$ <br>  $\overline{u} \in K(a_0, P, Q)$ , where  $||\overline{u} \in K(a_0, P, Q)$ , where  $||\overline{u} \in K(a_0, P, Q)$ , where  $||\overline{u} \in K(a_0, P, Q)$ <br>  $\leq k_2 |y - \overline{y}| (y, \overline{y} \in \mathbb{R})$ .<br> *satisfied*  $\{P(u, y) | (x, y) | \leq p(x), |D_y(Vu)(x, y) - D_y(Vu)(x, y) \}$ <br>  $\{D_y(Vu)(x, y) | \leq p(x), |D_y(Vu)(x, y) - D_y(Vu)(x, y) \}$ <br> *i*  $\alpha \in K(a_0, P, Q)$ ;<br> *b* there is a constant  $s \geq 0$  such that  $||Vu - V\overline{u}||_z \leq s ||$ <br>  $\overline{u} \in K(a_0, P, Q)$ , where  $||u||_$ 

Leinma 2: *If Assumption*  $H_2$  is satisfied, then there exist constants  $a \in (0, a_0]$  and  $P, Q \geq 0$ , *such that for every*  $w \in K(a, P, Q)$  *there is a unique solution u*[w]  $\in K(a, P, Q)$  *of the equation* 

$$
D_x \tilde{u}(x, y) = F(x, y, u(x, y); (Vw) (x, y), D_y u(x, y)) \quad (a.e. in [0, a]; y \in \mathbf{R}) \tag{7}
$$

Proof: In order to prove this lemma, we show that all assumptions of Theorem 1 *are satisfied with*  $f(x, y, z, q) = F(x, y, z, (Vw) (x, y), q)$ *, where*  $w \in K(a, P, Q)$ *. From*  $H_2/1^\circ$ , 2° and 4° it follows that  $H_1/1^\circ$ , 2° are satisfied,  $H_1/3^\circ$  is satisfied with dition (2).<br>
order to prove this lemma, we show that all assumptions<br>
with  $f(x, y, z, q) = F(x, y, z, (Vw) (x, y), q)$ , where  $w \in K((4^{\circ} \text{ it follows that } H_1/1^{\circ}, 2^{\circ} \text{ are satisfied}, H_1/3^{\circ} \text{ is satisfied})$ <br>  $= m_0, \qquad M_1 = m_1 + m_3 p, \qquad M_2 = m_2, \qquad M_3 = m_4,$ <br>  $= l_$ 

\n- 1 a 2: If Assumption 
$$
H_2
$$
 is satisfied, then there exist constants  $a$ , such that for every  $w \in K(a, P, Q)$  there is a unique solution  $u(w)$   $D_x\tilde{u}(x, y) = F(x, y, u(x, y), (Vw) (x, y), D_yu(x, y))$  (a.e. in [0,  $I$  condition (2).
\n- 1. In order to prove this lemma, we show that all assumptions tied with  $f(x, y, z, q) = F(x, y, z, (Vw) (x, y), q)$ , where  $w \in K(a$  and  $4^{\circ}$  it follows that  $H_1/1^{\circ}$ ,  $2^{\circ}$  are satisfied,  $H_1/3^{\circ}$  is satisfied  $M_0 = m_0$ ,  $M_1 = m_1 + m_3p$ ,  $M_2 = m_2$ ,  $M_3 = m_4$ ,  $L_0 = l_0(1 + r_0)$ ,  $L_1 = (1 + r_0)(l_1 + pl_3) + m_3r_1$ ,  $L_2 = l_2(1 + r_0)$ ,  $L_3 = l_4(1 + r_0)$ .  $1$
\n- $H_1/4^{\circ}$  is covered by  $H_2/6^{\circ}$ . By Theorem 1 it follows that there  $u(w)$  satisfying equation (7) and condition (2). Moreover, this conditions
\n

At last,  $H_1/4^{\circ}$  is covered by  $H_2/6^{\circ}$ . By Theorem 1 it follows that there is a unique function  $u[w]$  satisfying equation (7) and condition (2). Moreover, this soluton satis-<br>fies the conditions  $H_2/1^{\circ}$ , 2° and 4° it follow<br>  $M_0 = m_0$ ,  $M$ <br>  $L_0 = l_0(1 + r_0)$ ,<br>  $L_2 = l_2(1 + r_0)$ ,<br>
At last,  $H_1/4^{\circ}$  is covered<br>
function  $u[w]$  satisfying c<br>
fies the conditions<br>  $|u[w](x, y) - u|$ 

*itifying condition* (2).  
\n*Proof*: In order to prove this lemma, we show that all assumptions of Theorem 1  
\n*of* in order to prove this lemma, we show that all assumptions of Theorem 1  
\n*of* 
$$
[a_1^T, g_2^T, g_3^T, g_4^T, g_5^T, g_6^T, g_7^T, g_7^T, g_8^T, g_9^T, g_9^T
$$

Let *P* and *Q* in the definition of  $K(a, P, Q)$  be constants satisfying the inequalities  $\tilde{K}_1 \leq P\tilde{g}_0$ ,  $\tilde{K}_2 \leq Q\tilde{g}_0$  which are certainly satisfied for a sufficiently small (for  $a \to 0$ 

where  $\,$ 

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132 J. Turo<br>these inequalities reduce to  $k_1 \le P$ ,  $k_2 \le Q$ ). Thus we have

$$
|u[w](x, y) - u[w](x, \bar{y})| \le P |y - \bar{y}|,
$$

*jD0u[w](x,y) -- Du[w] (x,* QIy -. Since  $u[w]$  is generated by characteristics [3], then writing the characteristic system for (7) it is easy to show that  $u[w]$  is bounded. Hence, we have  $u[w] \in K(a, P, Q)$  for

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these inequalities reduce to  $k_1 \le$ <br>  $|u[w](x, y) - u[w](x, \bar{y})$ <br>  $|D_yu[w](x, y) - D_yu[w]$ <br>
Since  $u[w]$  is generated by charac<br>
for (7) it is easy to show that  $u[w$ <br>
every  $w \in K(a, P, Q)$ <br>
Theorem 2: If Assumption  $H_2$ <br> Theorem 2: *If Assumption*  $H_2$  *is satisfied, then there are constants a, P, Q, O < a*  $\leq a_0$ , and a function  $u: E_a \to \mathbf{R}$ , satisfying equation (1) and condition (2). Fur*thermore, u is unique in the class*  $K(a, P, Q)$ *. y* to show that  $u[w]$  is bounded. He  $P, Q$   $\blacksquare$ <br>
: If Assumption H<sub>2</sub> is satisfied, then<br>
unction  $u: E_a \to \mathbf{R}$ , satisfying equ<br>
unique in the class  $K(a, P, Q)$ .<br>
us define the operator  $T$  on  $K(a,$ <br>
are given in Lemma

Proof: Let us define the operator *T* on  $K(a, P, Q)$  by  $(Tw)$   $(x, y) = u[w](x, y)$ , where  $a, P, Q$  are given in Lemma 2. It follows by Lemma 2 that  $T$  maps  $K(a, P, Q)$ into itself. Let us introduce the norm  $E_a \rightarrow \mathbf{R}$ , satisfying equation (1) and<br> *z* class  $K(a, P, Q)$ .<br>
he operator  $T$  on  $K(a, P, Q)$  by  $(Tw)$ <br>
Lemma 2. It follows by Lemma 2 tha<br>
e the norm<br>  $|\exp(-\lambda x),$  where  $\lambda > l_0(1 + s)$ .

 $||z|| = \sup_{(x,y)\in E_a} |z(x, y)| \exp(-\lambda x),$  where  $\lambda > l_0(1+s)$ .

By  $H_2 / 5^\circ$  we have

 $|(Vu)(x, y) - (V\overline{u})(x, y)| \exp(-\lambda x) \leq s \sup |u(s, y) - \overline{u}(s, y)| \exp(-\lambda x)$  $\lambda > l_0(1 +$ <br>
sup  $|u(s, y)|$ <br>
s  $||u - \overline{u}||$ *s* have<br> *u*)  $(x, y) - (V\bar{u}) (x, y) | \exp(-\lambda x) \leq s \sup_{(s, y) \in E_z} |u(s, y)|$ <br>  $s \sup_{(s, y) \in E_z} |u(s, y) - \bar{u}(s, y)| |\exp(-\lambda s)| \leq s ||u - \bar{u}||.$ *(3.y)* are given in Lemma 2.<br> *(3.y)* are given in Lemma 2.<br>  $\lim_{(x,y)\in E_a} |z(x, y)| \exp(-\lambda)$ <br>
have<br> *(3.y)*  $\infty$  *(Vu) (x, y)*  $\infty$ <br> *(3.y)*  $\infty$ <br>  $\lim_{(s,y)\in E_x} |u(s, y) - \overline{u}(s, y)|$ <br>  $\infty$ <br>  $\infty$  *Vu*||  $\leq s$  || $u - \overline{u}$ ||.

Hence  $||Vu - V\overline{u}|| \leq s ||u - \overline{u}||$ . Now, we prove that *T* is a contraction. Indeed, for any  $w, \overline{w} \in K(a, P, Q)$ , we have

$$
|D_x[(Tw) (x, y) - (T\overline{w}) (x, y)]| = |D_x[u[w] (x, y) - u[\overline{w}] (x, y)]|
$$
  
\n
$$
\leq l_0 |u[w] (x, y) - u[\overline{w}] (x, y)| + l_0 |(Vw) (x, y) - (V\overline{w}) (x, y)|
$$
  
\n
$$
+ l_0 |D_y[u[w] (x, y) - u[\overline{w}] (x, y)]|
$$
  
\n
$$
\leq l_0 |u[w] (x, y) - u[\overline{w}] (x, y)| + l_0 s ||w - \overline{w}|| \exp (\lambda x)
$$
  
\n
$$
+ l_0 |D_y[u[w] (x, y) - u[\overline{w}] (x, y)]|,
$$

and  $u[w](0, y) - u[\overline{w}](0, y) = 0$ . Hence, and by Lemma 1, we obtain  $\gamma'(x) \le l_0\gamma(x) + l_0 s ||w - \overline{w}|| \exp(\lambda x)$ , where  $\gamma(x) = \sup \{|u[w](s, t) - u[\overline{w}](s, t)| : (s, t)$  $\leq l_0 |u[w](x, y) - u[\overline{w}](x, y)| + l_0 s ||w - \overline{w}|| \exp(\lambda x)$ <br>+  $l_0 |D_y[u[w](x, y) - u[\overline{w}](x, y)]|$ ,<br>and  $u[w](0, y) - u[\overline{w}](0, y) = 0$ . Hence, and by Lemma 1, we obtain  $\gamma'(x)$ <br> $\leq l_0 \gamma(x) + l_0 s ||w - \overline{w}|| \exp(\lambda x)$ , where  $\gamma(x) = \sup \{|u[w](s, t) - u[\overline{w$  $u[w](0, y) - u[\overline{w}] (0, y) = 0$ . Hence, and by Lemma 1,  $v(x) + l_0 s ||w - \overline{w}|| \exp(\lambda x)$ , where  $v(x) = \sup \{|u[w](s, t) - t_0\}$  and consequently, by the extended Peano inequality [ $|v| - \overline{w}|| \exp(\lambda x)$ . Since this inequality is satisfied for e

$$
||Tw - T\overline{w}|| \leq l_0 s(\lambda - l_0)^{-1} ||w - \overline{w}||, \qquad l_0 s(\lambda - l_0)^{-1} < 1.
$$

Thus,  $T$  is a contraction. It is easily seen that the fixed point of the operator  $T$  satisfies condition (2) and equation (1) **<sup>I</sup>**

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