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Nodal Domains for One- and Two-Dimensional Elliptic Differential Equations

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Für die Schrödinger-Gleichung $-\Delta u + q(x) u = 0$, $x = (x_1, x_2) \in \mathbf{R}^2$, wird für q eine Bedingung angegeben, so daß ein kreisförmiges Knotengebiet existiert. Entsprechend werden für die Sturm-Liouvillesche Gleichung Bedingungen formuliert, so daß ein Paar konjugierter Punkte existiert.

Для уравнения Шредингера $-\Delta u + q(x) u = 0$, $x = (x_1, x_2) \in \mathbf{R}^2$, дается условие на q чтобы существовала кругообразная область узлов. Соответственно формулируется условие для уравнения Штурма-Лиувилля чтобы существовала пара сопряжённых точек.

Concerning the Schrödinger equation $-\Delta u + q(x) u = 0$, $x = (x_1, x_2) \in \mathbf{R}^2$, a condition for q is formulated to ensure that there exists a nodal domain being a circle. Similarly, conditions for the Sturm-Liouville equation are obtained for the existence of a pair of conjugate points.

First consider the Schrödinger equation

$$-\Delta u + q(x) u = 0, \quad x = (x_1, x_2) \in \mathbf{R}^2, \ q \in C(\mathbf{R}^2). \tag{1}$$

A bounded domain $G \subset \mathbf{R}^2$ is said to be a *nodal domain* of (1) if there exists a nontrivial solution $u \in \mathring{W}_2^1(G) \cap W_2^2(G)$ of (1) ($\mathring{W}_2^1(G)$ and $W_2^2(G)$ are Sobolev spaces; $W_2^2(G)$ is the set of (complex-valued) functions the generalized derivatives of which up to order two belong to $L_2(G)$, $\mathring{W}_2^1(G)$ is the closure of $C_0^\infty(G)$ in the norm $\|\cdot\|_{1,G}$, $\|u\|_{1,G}^2 = \int_G (|u|^2 + |u_{x_1}|^2 + |u_{x_2}|^2)\,dx$). We prove the following

Theorem 1: *Let*

$$\limsup_{r\to\infty} \int_{|x|\le r} q(x)\,dx \leqq 0, \qquad 0 \neq q \in C(\mathbf{R}^2). \tag{2}$$

Then there exists a circle $K_R = \{x \in \mathbf{R}^2 \mid |x| < R\}$ being a nodal domain of (1).

Proof: The equation (1) possesses a nodal domain if there exists a function $\varphi \in C_0^\infty(\mathbf{R}^2)$ with [5]

$$\int_{\mathbf{R}^2} (|\nabla\varphi|^2 + q(x)\,|\varphi|^2)\,dx < 0, \qquad \nabla\varphi = (\varphi_{x_1}, \varphi_{x_2}). \tag{3}$$

In the following we construct a function φ with this property. By (2) there exists a point $y \in \mathbf{R}^2$ with $q(y) < 0$. Since q is continuous, there exist numbers $d > 0$ and $\varrho > 0$ such that $q(x) \leqq -d$, $|x - y| \leqq \varrho$. Define the functions v_0, δ by

$$v_0(x) = \begin{cases} 1, & |x| \leqq r_1, \\ 1 - \varepsilon \ln \dfrac{|x|}{r_1}, & r_1 \leqq |x| \leqq r_2, \\ 0, & |x| \geqq r_2, \end{cases}$$

$$\delta(x) = \begin{cases} h\left(1 - \frac{|x-y|}{\varrho}\right), & |x-y| \leqq \varrho, \\ 0, & |x-y| > \varrho, \end{cases}$$

where $\varepsilon > 0$, $r_2 = e^{1/\varepsilon} r_1$, $h > 0$. Let $r_1 > |y| + \varrho$. The function $v = v_0 + \delta$ belongs to $\mathring{W}_2^1(K_{r_2})$ and will be used as a test function for the quadratic form of (1). We have

$$\begin{aligned} \int\limits_{\mathbf{R}^2} (\nabla v)^2 \, dx &= \int\limits_{\mathbf{R}^2} (\nabla v_0)^2 \, dx + \int\limits_{\mathbf{R}^2} (\nabla \delta)^2 \, dx \\ &= \int\limits_0^{2\pi} \int\limits_{r_1}^{r_2} (\nabla v_0)^2 \, r \, dr \, d\varphi + h^2\pi = 2\pi\varepsilon^2 \ln \frac{r_2}{r_1} + h^2\pi \\ &= \pi(2\varepsilon + h^2) \end{aligned} \tag{4}$$

and

$$\int\limits_{\mathbf{R}^2} q(x) v^2(x) dx = \int\limits_{\mathbf{R}^2} q(x) v_0^2(x) dx + 2 \int\limits_{|x-y|\leqq\varrho} q(x)\delta(x) dx + \int\limits_{|x-y|\leqq\varrho} q(x)\delta^2(x) dx. \tag{5}$$

The items on the right-hand side of the last equality will be handled as follows. For the first item we have

$$\begin{aligned} \int\limits_{\mathbf{R}^2} q(x)\, v_0^2(x)\, dx &= \int\limits_{|x|\leqq r_1} q(x)\, dx + \int\limits_{r_1 \leqq |x| \leqq r_2} q(x)\, v_0^2(x)\, dx \\ &= \int\limits_{|x|\leqq r_1} q(x)\, dx + \int\limits_{r_1}^{r_2} \left[\int\limits_{\substack{0 \\ (|x|=r)}}^{2\pi} q(x)\, d\varphi \right] v_0^2(r)\, r\, dr. \end{aligned}$$

By setting $\int\limits_{(|x|=r)}^{2\pi} q(x)\, d\varphi = \bar{q}(r)$ it follows that

$$\int\limits_{\mathbf{R}^2} q(x)\, v_0^2(x)\, dx = \int\limits_{|x|\leqq r_1} q(x)\, dx + \int\limits_{r_1}^{r_2} \bar{q}(r)\, v_0^2(r)\, r\, dr. \tag{6}$$

The factor $v_0^2(r)$ is monotone decreasing on $[r_1, r_2]$ from $v_0^2(r_1) = 1$ to $v_0^2(r_2) = 0$. By the second mean value theorem of integral calculus there exists a point $s \in [r_1, r_2]$ such that

$$\int\limits_{r_1}^{r_2} \bar{q}(r)\, v_0^2(r)\, r\, dr = \int\limits_{r_1}^{s} \bar{q}(r)\, r\, dr = \int\limits_{r_1}^{s} \int\limits_{\substack{0 \\ (|x|=r)}}^{2\pi} q(x)\, r\, d\varphi\, dr = \int\limits_{r_1 \leqq |x| \leqq s} q(x)\, dx.$$

By using this we obtain

$$\int\limits_{\mathbf{R}^2} q(x)\, v_0^2(x)\, dx = \int\limits_{|x|\leqq s} q(x)\, dx. \tag{7}$$

The second item of (5) will be estimated as follows:

$$2 \int\limits_{|x-y|\leqq\varrho} q(x)\, \delta(x)\, dx \leqq -2d \int\limits_{|x-y|\leqq\varrho} \delta(x)\, dx = -\frac{2\pi}{3}\, d\varrho^2 h. \tag{8}$$

Because the third item of (5) is negative, by (7) and (8) we obtain

$$\int\limits_{\mathbf{R}^2} q(x)\, v^2(x)\, dx < \int\limits_{|x|\leqq s} q(x)\, dx - \frac{2\pi}{3}\, d\varrho^2 h. \tag{9}$$

By setting $h = \sqrt{\varepsilon}$ it follows from (4) and (9) that

$$\int_{\mathbf{R}^2} [(\nabla \tilde{v})^2 + q(x)\, v^2]\, dx < \int_{|x| \leq s} q(x)\, dx + 3\pi\varepsilon - \frac{2\pi}{3}\, d\varrho^2 \sqrt{\varepsilon}. \tag{10}$$

By (2) the parameter r_1 can be chosen so large that $\int_{|x| \leq s} q(x)\, dx \leq \pi\varepsilon$, $s \geq r_1(\varepsilon)$. Hence, it follows that

$$\int_{\mathbf{R}^2} [(\nabla v)^2 + q(x) v^2]\, dx < \sqrt{\varepsilon} \left(4\pi \sqrt{\varepsilon} - \frac{2\pi}{3}\, d\varrho^2\right). \tag{11}$$

By choosing ε sufficiently small the right-hand side takes on negative values. Hence, there exists a finite function $v \in \mathring{W}_2^1(\mathbf{R}^2)$ such that $\int_{\mathbf{R}^2} [(\nabla v)^2 + q(x)\, v^2]\, dx < 0$. The support of v is contained in a circle $K_{R'}$, $0 < R' < \infty$. Obviously, there exists a (real-valued) function $\varphi \in C_0^\infty(K_{R'})$ such that

$$\int_{\mathbf{R}^2} [(\nabla \varphi)^2 + q(x)\, \varphi^2]\, dx < 0 \tag{12}$$

also holds. Let $A_{R'}$ be the Friedrichs extension of the operator $A_{R',0}$,

$$A_{R',0}\psi = -\Delta\psi + q(x)\, \psi, \qquad \psi \in C_0^\infty(K_{R'}).$$

The spectrum of $A_{R'}$ is discrete. By (12) the lowest eigenvalue $\lambda_{R'}$ of $A_{R'}$ is negative. Let $u_{R'}$ be the corresponding eigenfunction. Because $u_{R'}$ belongs to the domain $D(A^*_{R',0})$ of the adjoint operator $A^*_{R',0}$ of $A_{R',0}$, we have $u_{R'} \in W_2^2(K_{R'})$ [2]. On the other hand, $u_{R'}$ belongs to the energy space of $A_{R',0}$. Because the energy norm

$$\left(\int_{K_{R'}} \left[|\nabla\psi|^2 + \left(q(x) - C_{R'}\right) |\psi|^2\right] dx\right)^{1/2}$$

$$C_{R'} = \min_{|x| \leq R'} q(x) - 1, \; \psi \in C_0^\infty(K_{R'}),$$

of $A_{R',0}$ is equivalent to the norm of $\mathring{W}_2^1(K_{R'})$, the function $u_{R'}$ also belongs to $\mathring{W}_2^1(K_{R'})$. Hence, we have $u_{R'} \in W_2^2(K_{R'}) \cap \mathring{W}_2^1(K_{R'})$. By Courant's variation principle $\lambda_{R'}$ is a monotone decreasing function of R', which tends to infinity when R' tends to zero [5]. Because $\lambda_{R'}$ is continuous (cf. [5]), there exists $R > 0$ with $\lambda_R = 0$. The eigenfunction u_R determines the circle K_R being a nodal domain of (1) ∎

Note that in the case $q = 0$ there does not exist a nodal domain of (1).

Corollary 1: *There exists a circle $K_R = \{x \mid |x| < R\}$ being a nodal domain of* (1) *if $q(-x) = -q(x)$, $x \in \mathbf{R}^2$, $q \neq 0$, holds.*

Proof: For every $r > 0$ we have $\int_{|x| \leq r} q(x)\, dx = 0$ and, consequently, assumption (2) is fulfilled ∎

Let the operator

$$A_0 = -\Delta + q(x), \qquad D(A_0) = C_0^\infty(\mathbf{R}^2),$$

be bounded from below and denote the Friedrichs extension of A_0 by A. Let $\sigma(A)$ be the spectrum of A.

Corollary 2: *If* (2) *holds, then $\sigma(A) \cap (-\infty, 0) \neq \emptyset$.*

The corollary immediately follows from (12). Suppose in the following that q is bounded and define the cube

$$Q_\varkappa(y) = \left\{x \mid |x_j - y_j| \leqq \frac{\varkappa}{2},\ j = 1, 2\right\} \qquad y = (y_1, y_2),\quad \varkappa > 0.$$

The essential spectrum $\sigma_e(A)$ of A is equal to $[0, \infty)$ if [3].

$$\lim_{|y|\to\infty} \int_{Q_\varkappa(y)} q(x)\,dx = 0 \qquad \text{for each (fixed) } \varkappa > 0. \tag{13}$$

Corollary 2 now leads to the following result.

Corollary 3: *If q is bounded and the hypotheses* (2) *and* (13) *are satisfied, then there exists at least one negative eigenvalue of the operator A.*

Corollary 3 is essentially due to SCHMINCKE [7: Th. 5].

Consider in the following the Sturm-Liouville equation

$$\left(-p(x)\,u'\right)' + q(x)\,u = 0, \qquad -\infty \leqq a < x < b \leqq \infty. \tag{14}$$

The points x_1 and x_2, $a < x_1 < x_2 < b$, are said to be *conjugate* with respect to (14) if there exists a nontrivial solution u of (14) with $u(x_1) = 0 = u(x_2)$.

Theorem 2: *Assume*

$$\int_a^c p^{-1}(x)\,dx = \infty = \int_c^b p^{-1}(x)\,dx, \qquad a < c < b,$$

and

$$\limsup_{\alpha\downarrow a\ \beta\uparrow b} \int_\alpha^\beta q(x)\,dx \leqq 0, \qquad q \neq 0. \tag{15}$$

Then there exists a pair of conjugate points $x_1 = a + \xi$, $x_2 = b - \xi$, $0 < \xi < \frac{b-a}{2}$, with respect to (14).

Proof: Let y be a point with $q(y) < 0$. There exist numbers $d > 0$ and $\varrho > 0$ such that $q(x) \leqq -d$, $|x - y| \leqq \varrho$. Choose points x_2 and x_3 with $a < x_2 < y - \varrho$ and $y + \varrho < x_3 < b$ and a $\varepsilon > 0$ arbitrarily. Then the points x_1 and x_4 are uniquely determined by

$$\varepsilon \int_{x_1}^{x_2} p^{-1}\,dx = 1, \quad a < x_1 < x_2, \qquad \text{and} \qquad \varepsilon \int_{x_3}^{x_4} p^{-1}\,dx = 1, \quad x_3 < x_4 < b.$$

Choose a number $h > 0$ and define a function v by

$$v(x) = \begin{cases} 0, & x \in (a, x_1] \cup (x_4, b), \\ \varepsilon \int_{x_1}^{x} p^{-1}\,dt, & x \in (x_1, x_2], \\ 1, & x \in (x_2, y - \varrho] \cup (y + \varrho, x_3], \\ h[1 - \varrho^{-1}|x - y|] + 1, & x \in (y - \varrho, y + \varrho], \\ 1 - \varepsilon \int_{x_3}^{x} p^{-1}\,dt, & x \in (x_3, x_4]. \end{cases}$$

It will be used for estimating the quadratic form $\int_a^b [p(v')^2 + qv^2]\,dx$: by choosing x_2, x_3, ε, and h in an analogous way as in the proof of Theorem 1 one can obtain the estimate $\int_a^b [p(v')^2 + qv^2]\,dx \leq 0$. Then there exist a function $\varphi \in C_0^\infty(a, b)$ and a number $\eta > 0$ with supp $\varphi \subseteqq [a + \eta, b - \eta]$ such that

$$\int_a^b [p(\varphi')^2 + q\varphi^2]\,dx < 0. \tag{16}$$

The smallest eigenvalue λ_η of the Friedrichs extension A_η of the operator $A_{\eta,0}$,

$$A_{\eta,0}\psi = -\big(p(x)\,\psi'\big)' + q(x)\,\psi, \qquad \psi \in C_0^\infty(a + \eta, b - \eta),$$

is negative. There exists a $\xi > \eta$ with $\lambda_\xi = 0$ (compare the proof of Theorem 1). Let u_ξ be the corresponding eigenfunction. It has the property $u_\xi(a + \xi) = 0 = u_\xi(b - \xi)$. Let $z = z(x)$ be the solution of (14) being identical with $u_\xi(x)$ on $[a + \xi, b - \xi]$. The points $a + \xi$ and $b - \xi$ are conjugate. This proves Theorem 2 ∎

There does not exist a pair of conjugate points with respect to (14) when $q = 0$. Otherwise, the solution $u = 1$ would vanish somewhere on (a, b) by Sturm's comparison theorem.

Theorem 2 improves results of Tipler [8: Th. 2] (the case $p = 1$, $a = -\infty$, $b = \infty$), Ahlbrandt, Hinton and Lewis [1: Th. 3.3] (the case $q \leqq 0$), and the author [4: Th. 1].

If $a = -\infty$, $b = \infty$, $p = 1$, $q(x) \to 0$, $|x| \to \infty$, the essential spectrum of the Friedrichs extension A of the operator A_0, $A_0\psi = -\psi'' + q(x)\,\psi$, $\psi \in C_0^\infty(-\infty, \infty)$, coincides with $[0, \infty)$. It follows from (15) that there exists a function $\varphi \in C_0^\infty(-\infty, \infty)$ satisfying (16). Hence, there exists at least one negative eigenvalue of A. This result is due to Schmincke [6: Lemma 8].

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