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 (4)

Nodal Domains for One- and Two-Dimensional Elliptic Differential Equations

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Für die Schrödinger-Gleichung $-\Delta u + q(x) u = 0$, $x = (x_1, x_2) \in \mathbb{R}^2$, wird für q eine Bedingung angegeben, so daß ein kreisförmiges Knotengebiet existiert. Entsprechend werden für die Sturm-Liouvillesche Gleichung Bedingungen formuliert, so daß ein Paar konjugierter Punkte existiert.

Для уравнения Шредингера $-2u + q(x) u = 0$, $x = (x_1, x_2) \in \mathbb{R}^2$, дается условие на q чтобы существовала кругообразная область узлов. Соответственно формулируется условие для уравнения Штурма-Лиувилля чтобы существовала пара сопряжённых точек.

Concerning the Schrödinger equation $-4u + q(x)u = 0$, $x = (x_1, x_2) \in \mathbb{R}^2$, a condition for q is formulated to ensure that there exists a nodal domain being a circle. Similarly, conditions for the Sturm-Liouville equation are obtained for the existence of a pair of conjugate points;

First consider the Schrödinger equation

$$
- \Delta u + q(x) u = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2, q \in C(\mathbb{R}^2).
$$

A bounded domain $G \subset \mathbb{R}^2$ is said to be a *nodal domain* of (1) if there exists a nontrivial solution $u \in \mathring{W}_2^1(G) \cap W_2^2(G)$ of (1) $(\mathring{W}_2^1(G)$ and $W_2^2(G)$ are Sobolev spaces; $W_2^2(G)$ is the set of (complex-valued) functions the generalized derivatives of which up to order two belong to $L_2(G)$, $\hat{W}_2^{-1}(G)$ is the closure of $C_0^{\infty}(G)$ in the norm $\|\cdot\|_{1,G}$, $\|u\|_{1,G}^2 = \int (|u|^2 + |u_{x_1}|^2 + |u_{x_1}|^2) dx$. We prove the following

Theorem 1: Let

$$
\limsup_{r \to \infty} \int_{|x| \le r} q(x) dx \le 0, \qquad 0 \ne q \in C(\mathbb{R}^2).
$$
 (2)

Then there exists a circle $K_R = \{x \in \mathbb{R}^2 \mid |x| < R\}$ being a nodal domain of (1).

Proof; The equation (1) possesses a nodal domain if there exists a function $\varphi \in C_0^{\infty}(\mathbf{R}^2)$ with [5]

$$
\int\limits_{\mathbf{R}^2} \left(|\nabla \varphi|^2 + q(x) |\varphi|^2 \right) dx \leq 0, \qquad \nabla \varphi = (\varphi_{x_1}, \varphi_{x_2}). \tag{3}
$$

In the following we construct a function φ with this property. By (2) there exists a point $y \in \mathbb{R}^2$ with $q(y) < 0$. Since q is continuous, there exist numbers $d > 0$ and $q > 0$ such that $q(x) \leq -d$, $|x - y| \leq q$. Define the functions v_0 , δ by

$$
v_0(x) = \begin{cases} 1, & |x| \leq r_1, \\ 1 - \varepsilon \ln \frac{|x|}{r_1}, & r_1 \leq |x| \leq r_2, \\ 0, & |x| \geq r_2, \end{cases}
$$

$$
\delta(x) = \begin{cases} h\left(1 - \frac{|x - y|}{\varrho}\right), \ |x - y| \leq \varrho, \\ 0, \qquad |x - y| > \varrho, \end{cases}
$$

where $\varepsilon > 0$, $r_2 = e^{1/\varepsilon}r_1$, $h > 0$. Let₁ $r_1 > |y| + \varrho$. The function $v = v_0 + \delta$ belongs to \hat{W}_2 ¹(K_r) and will be used as a test function for the quadratic form of (1). We have

$$
\delta(x) = \begin{cases}\nh\left(1 - \frac{|x - y|}{\rho}\right), \ |x - y| \leq \varrho, \\
0, & |x - y| > \varrho,\n\end{cases}
$$
\nwhere $\varepsilon > 0$, $r_2 = e^{1/r_1}$, $h > 0$. Let $r_1 > |y| + \varrho$. The function $v = v_0 + \delta$ belongs
\nto $\mathcal{W}_2^{-1}(K_{r_1})$ and will be used as a test function for the quadratic form of (1). We have
\n
$$
\int_{\mathbb{R}^2} (\nabla v)^2 dx = \int_{\mathbb{R}^2} (\nabla v_0)^2 dx + \int_{\mathbb{R}^2} (\nabla \delta)^2 dx
$$
\n
$$
= \int_{0}^{2\pi} \int_{r_1}^{r_1} (\nabla v_0)^2 r dr d\varphi + h^2 \pi = 2\pi \varepsilon^2 \ln \frac{r_2}{r_1} + h^2 \pi
$$
\n
$$
= \pi (2\varepsilon + h^2)
$$
\nand
\n
$$
\int_{\mathbb{R}^2} q(x) v^2(x) dx = \int_{\mathbb{R}^2} q(x) v_0^2(x) dx + 2 \int_{|x - y| \leq \varrho} q(x) \delta(x) dx + \int_{|x - y| \leq \varrho} q(x) \delta^2(x) dx.
$$
\n(f) The items on the right-hand side of the last equality will be handled as follows. For
\nthe first item we have
\n
$$
\int_{\mathbb{R}^2} q(x) v_0^2(x) dx = \int_{|x| \leq r_1} q(x) dx + \int_{r_1 \leq |x| \leq r_1} q(x) v_0^2(x) dx
$$
\n
$$
= \int_{|x| \leq r_1} q(x) dx + \int_{r_1}^{r_1} \int_{0}^{2\pi} q(x) d\varphi \Big| v_0^2(r) r dr.
$$

$$
\int\limits_{\mathbf{R}^3} q(x)v^2(x)dx = \int\limits_{\mathbf{R}^3} q(x)v_0^2(x)dx + 2\int\limits_{|x-y|\leq \varrho} q(x)\delta(x)dx + \int\limits_{|x-y|\leq \varrho} q(x)\delta^2(x)dx.
$$
 (5)

The items on the right-hand side of the last equality will be handled as follows. For' the first item we have

$$
\int_{\mathbf{R}^2} q(x) v_0^2(x) dx = \int_{|x| \le r_1} q(x) dx + \int_{r_1 \le |x| \le r_1} q(x) v_0^2(x) dx
$$
\n
$$
= \int_{|x| \le r_1} q(x) dx + \int_{r_1}^{r_1} \left[\int_{0}^{2\pi} q(x) d\varphi \right] v_0^2(r) r dr.
$$
\nBy setting $\int_{|x|=r_1}^{2\pi} q(x) d\varphi = \overline{q}(r)$ it follows that

\n
$$
\lim_{|x|=r_1} \int_{|x| \le r_1} q(x) dx = \int_{r_1}^{r_1} q(x) u_0^2(r) r dr.
$$
\nThe factor $v_0^2(r)$ is monotone decreasing on $[r_1, r_2]$ from $v_0^2(r_1) = 1$ to $v_0^2(r_2) = 0$. By the second mean value theorem of integral calculus there exists a point $s \in [r_1, r_2]$.

 $(|x|=r)$

$$
\int_{t} q(x) v_0^{2}(x) dx = \int_{|x| \leq r_1} q(x) dx + \int_{r_1}^{r_1} \overline{q}(r) v_0^{2}(r) r dr.
$$
 (6)

The factor $v_0^2(r)$ is monotone decreasing on $[r_1, r_2]$ from $v_0^2(r_1) = 1$ to $v_0^2(r_2) = 0$. By
the second mean value theorem of integral calculus there exists a point $s \in [r_1, r_2]$
such that
 $\int_{r_1}^{r_1} \overline{q}(r) v_0$ such that By setting $\int_{(|x|=r)}^{2\pi} q(x) \, d\varphi = \overline{q}(r)$ it follows $\int_{(|x|=r)}^{(|x|=r)} q(x) \, dx = \int_{|x| \le r_1} q(x)$

The factor $v_0^2(r)$ is monotone decrease the second mean value theorem of such that $\int_{r_1}^{r_2} \overline{q}(r) v_0^2(r) r \, dr = \int_{r$ $\begin{align*}\n\begin{array}{l}\n\text{(i)} = 0. \text{By} \\
\text{(ii)} = 0. \text{By} \\
\text{(iii)} = 0. \text{By} \\
\text{(iv)} = 0. \end{array}\n\end{align*}$

By setting
$$
\int_{(1x)=r}^{2\pi} q(x) d\varphi = \overline{q}(r)
$$
 it follows that
\n
$$
\int_{(1x)=r}^{r_1} q(x) v_0^2(x) dx = \int_{|x| \le r_1} q(x) dx + \int_{r_1}^{r_1} \overline{q}(r) v_0^2(r) r dr.
$$
\nThe factor $v_0^2(r)$ is monotone decreasing on $[r_1, r_2]$ from $v_0^2(r_1) = 1$ to $v_0^2(r_2) = 0$. By the second mean value theorem of integral calculus there exists a point $s \in [r_1, r_2]$ such that
\n
$$
\int_{r_1}^{r_1} \overline{q}(r) v_0^2(r) r dr = \int_{r_1}^{s} \overline{q}(r) r dr = \int_{r_1}^{s} \int_{(1x)=r_1}^{s} q(x) r d\varphi dr = \int_{r_1 \le |x| \le s} q(x) dx.
$$
\nBy using this we obtain
\n
$$
\int_{\mathbb{R}^1} q(x) v_0^2(x) dx = \int_{|x| \le s} q(x) dx.
$$
\n
$$
\int_{|x-y| \le e} q(x) \delta(x) dx \le -2d \int_{|x-y| \le e} \delta(x) dx = -\frac{2\pi}{3} d e^2 h.
$$
\n
$$
\int_{|x-y| \le e} q(x) v^2(x) dx < \int_{|x| \le s} q(x) dx - \frac{2\pi}{3} d e^2 h.
$$
\n(8) Because the third item of (5) is negative, by (7) and (8) we obtain
\n
$$
\int_{\mathbb{R}^1} q(x) v^2(x) dx < \int_{|x| \le s} q(x) dx - \frac{2\pi}{3} d e^2 h.
$$
\n(9)

r
y using
he secol

$$
\int_{\mathbf{R}^1} q(x) \, v_0^{\,2}(x) \, dx = \int_{|x| \le s} q(x) \, dx. \tag{7}
$$

The second item of (5) will be estimated as follows:

$$
\int_{r_1}^{r_1} \overline{q}(r) v_0^2(r) r dr = \int_{r_1}^{r_1} \overline{q}(r) r dr = \int_{r_1}^{s} \int_{(1x)-r_1}^{x} q(x) r d\varphi dr = \int_{r_1 \le |x| \le s} q(x) dx.
$$
\nthus we obtain

\n
$$
\int_{\mathbb{R}^1} q(x) v_0^2(x) dx = \int_{|x| \le s} q(x) dx.
$$
\nand item of (5) will be estimated as follows:

\n
$$
2 \int_{|x-y| \le e} q(x) \delta(x) dx \le -2d \int_{|x-y| \le e} \delta(x) dx = -\frac{2\pi}{3} d e^2 h.
$$
\n(8)

\n
$$
|x-y| \le e
$$
\nthe third item of (5) is negative, by (7) and (8) we obtain

\n
$$
\int_{\mathbb{R}^1} q(x) v^2(x) dx < \int_{|x| \le s} q(x) dx - \frac{2\pi}{3} d e^2 h.
$$
\n(9)

$$
\int_{\Omega} q(x) \, v^2(x) \, dx < \int_{|x| \leq s} q(x) \, dx - \frac{2\pi}{3} \, d\varrho^2 h. \tag{9}
$$

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\nBy setting
$$
h = \sqrt{\varepsilon}
$$
 it follows from (4) and (9) that

\n\n
$$
\int_{\mathbb{R}^4} [(\nabla \tilde{v})^2 + q(x) v^2] \, dx < \int_{|x| \le s} q(x) \, dx + 3\pi \varepsilon - \frac{2\pi}{3} \, d\varrho^2 \sqrt{\varepsilon}.
$$
\n

\n  By (2) the parameter r_1 can be chosen so large that

\n\n
$$
\int_{|x| \le s} [(\nabla v)^2 + q(x) v^2] \, dx < \sqrt{\varepsilon} \left(4\pi \sqrt{\varepsilon} - \frac{2\pi}{3} \, d\varrho^2 \right).
$$
\n

\n  By choosing ε sufficiently small the right-hand side takes on negative values. Hence, there exists a finite function $v \in W_2^1(\mathbb{R}^2)$ such that

\n\n
$$
\int_{\mathbb{R}^4} [(\nabla v)^2 + q(x) v^2] \, dx < 0.
$$
\n

$$
\int_{\mathbb{R}^3} [(\nabla \tilde{v})^2 + q(x) v^2] dx < \int_{|x| \le s} q(x) dx + 3\pi \varepsilon - \frac{2\pi}{3} d\varrho^2 \sqrt{\varepsilon}.
$$
\n(10)
\ne parameter r_1 can be chosen so large that $\int_{|x| \le s} q(x) dx \le \pi \varepsilon$, $s \ge r_1(\varepsilon)$. Hence,
\nthat
\n
$$
\int_{\mathbb{R}^3} [(\nabla v)^2 + q(x) v^2] dx < \sqrt{\varepsilon} \left(4\pi \sqrt{\varepsilon} - \frac{2\pi}{3} d\varrho^2 \right).
$$
\n(11)
\n
$$
\int_{\mathbb{R}^3} [(\nabla v)^2 + q(x) v^2] dx < \sqrt{\varepsilon} \left(4\pi \sqrt{\varepsilon} - \frac{2\pi}{3} d\varrho^2 \right).
$$
\n(12)
\n
$$
\int_{\mathbb{R}^3} [(\nabla v)^2 + q(x) v^2] dx < 0.
$$
\nThe
\n f is contained in a circle $K_{R'}, 0 < R' < \infty$. Obviously, there exists a (real-
\nfunction $\varphi \in C_0^\infty(K_R)$ such that
\n
$$
\int_{\mathbb{R}^3} [(\nabla \varphi)^2 + q(x) \varphi^2] dx < 0
$$
\n(12)
\n
$$
\int_{\mathbb{R}^3} [(\nabla \varphi)^2 + q(x) \varphi^2] dx < 0
$$
\n(12)

• By choosing *E* sufficiently small the right-hand side takes on negative values. Hence, there exists a finite function $v \in \mathring{W}_2^1(\mathbb{R}^2)$ such that $\int [(\nabla v)^2 + q(x) v^2] dx < 0$. The

 \sup support of *v* is contained in a circle $K_{R'}, 0 < R' < \infty$. Obviously, there exists a (realvalued) function $\varphi \in C_0^\infty(K_{R'})$ such that

$$
\int [(\nabla \varphi)^2 + q(x) \varphi^2] dx < 0
$$
\nR.

\ns. Let $A_{R'}$ be the Friedrich

\n
$$
A_{R'0}\psi = -\Delta \psi + q(x) \psi,
$$
\ntrum of $A_{R'}$ is discrete. By (1)

valued) function $\varphi \in C_0$ ² (R ^{*R*}) such that
 $\iint_R [(\nabla \varphi)^2 + q(x) \varphi^2] dx < 0$

also holds. Let $A_{R'}$ be the Friedrichs extension of the operator $A_{R',0}$,

function
$$
\varphi \in C_0^{\infty}(K_R)
$$
 such that
\n
$$
\int [(\nabla \varphi)^2 + q(x) \varphi^2] dx < 0
$$
\n
\nR
\nS. Let $A_{R'}$ be the Friedrichs extension of the
\n $A_{R',0}\varphi = -d\psi + q(x) \psi$, $\psi \in C_0^{\infty}(K_{R'})$.
\n
\n $\lim_{R \to \infty} \text{ of } A_{R'}$ is discrete. By (12) the lowest eig

The spectrum of A_R ^{*i*} is discrete. By (12) the lowest eigenvalue λ_R of A_R ^{*i*} is negative. Let $u_{R'}$ be the corresponding eigenfunction. Because $u_{R'}$ belongs to the domain $D(A_{R',0}^*)$ of the adjoint operator $\overline{A}_{R',0}^*$ of $A_{R',0}$, we have $u_{R'} \in W_2^2(K_{R'})$ [2]. On the other hand, $u_{R'}$ belongs to the energy space of $A_{R',0}$. Because the energy norm. the corresponding to the edge of the edge of $\int_{K_{R'}}^{\infty}$ **also holds.** Let $A_{R'}$ be the Friedrichs extension of the operator $A_{R',0}$,
 $A_{R',0}\psi = -d\psi + q(x) \psi$, $\psi \in C_0^{\infty}(K_R)$.

The spectrum of A_R is discrete. By (12) the lowest eigenvalue λ_R of A_R is negative. Let u_R

 $+$ $(q(x) - C_{R'}) |\psi|^2 dx^{1/2}$ $\int_{\mathbb{R}^3} [(\nabla \varphi)^2 + q(\nabla \varphi)^2] d\mathbf{R}$

S. Let A_R be
 $A_{R',0}\psi = -A_V$

rum of A_R is decorresponding

igoint operator

gs to the ener
 $\int_{K_R} [|\nabla \psi|^2 + (\nabla_R \psi)^2] d\mathbf{R}$
 $C_{R'} = \min_{|x| \le R'} q(x)$ */* $C_{R'} = \min_{|x| \le R'} q(x) - 1, \, \psi \in C_0^{\infty}(K_{R'})$,

of $A_{R'0}$ is equivalent to the norm of $\mathring{W}_2^{-1}(K_{R'})$, the function $u_{R'}$ also belongs to $\mathring{W}_2^{-1}(K_{R'})$. a monotone decreasing function of R' , which tends to infinity when R' tends to zero [5]. Because λ_R is continuous (cf. [5]), there exists $R > 0$ with $\lambda_R = 0$. The eigenfunction u_R determines the circle K_R being a nodal domain of (1) $C_R = \min_{|x| \leq R'} q(x) - 1$, $\psi \in C_0^{\infty}(K_R)$,

of $A_{R'0}$ is equivalent to the norm of $W_2^{-1}(K_{R'})$, the function u_R also belongs to $W_2^{-1}(K_R)$.

Hence, we have $u_R \in W_2^2(K_R) \cap W_2^{-1}(K_R)$. By Courant's variation principle Nodal Dominia for Elliptic

Ey setting $h = \sqrt{\epsilon}$ it follows from (4) and (9) that
 $\int_{\mathbb{R}^2} [(\nabla \hat{v})^2 + q(x) v^2] dx \leq \int_{\mathbb{R}^2} q(x) dx + 3x e$
 $\int_{\mathbb{R}^2} [(\nabla \hat{v})^2 + q(x) v^2] dx \leq \int_{\mathbb{R}^2} [q(x) dx + 3x e -$
 $\int_{\mathbb{R}^2} [$

Note that in the case $q = 0$ there does not exist a nodal domain of (1).

Corollary 1: There exists a circle $K_R = \{x \mid |x| < R\}$ being a nodal domain of (1) $if \, q(-x) = -q(x), \, x \in \mathbb{R}^2, \, q \neq 0, \, holds.$ Note that in the case $q = 0$ there does not

Corollary 1: There exists a circle K_R

if $q(-x) = -q(x), x \in \mathbb{R}^2, q \neq 0$, holds.

Proof: For every $r > 0$ we have \int

(2) is fulfilled **i**

Let the operator
 $A_0 = -A + q(x), \qquad D$ function u_R determines the circle I

Note that in the case $q = 0$ there do

Corollary 1: There exists a circle i
 i $q(-x) = -q(x), x \in \mathbb{R}^2, q \neq 0, I$

Proof: For every $r > 0$ we ha

(2) is fulfilled \blacksquare

Let the

Proof: For every $r>0$ we have $\int q(x) dx = 0$ and, consequently, assumption

Let the operator

$$
A_0 = -A + q(x), \qquad D(A_0) = C_0^{\infty}(\mathbf{R}^2),
$$

be bounded from below and denote the Friedrichs extension of A_0 by A . Let $\sigma(A)$ be the spectrum of A .

(12)

 $\label{eq:2} \frac{1}{\sqrt{2}}\sum_{i=1}^n\frac{1}{\sqrt{2}}\int_{\mathbb{R}^n} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\int_{\mathbb{R}^n} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\int_{\mathbb{R}^n} \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{\$

The corollary immediately follows froni (12). Suppose in the following that *q is* bounded and define the cube URIT IS also the following that q is

ollary immediately follows from (12). Suppose in the following that q is
 $f(x,y) = \left\{ x \mid |x_j - y_j| \leq \frac{x}{2}, j = 1, 2 \right\}$ $y = (y_1, y_2), x > 0.$

tial spectrum $\sigma_e(A)$ of A is equal to $[$ **138** E. MÜLLER-PFEIFFER
 Fig. concluded and define the cube
 $Q_x(y) = \left\{ x \mid |x_j - y_j| \leq \frac{x}{2}, j = 1, 2 \right\}$ $y = (y_1, y_2), x$

The essential spectrum $\sigma_e(A)$ of A is equal to $[0, \infty)$ if $[3]$.
 $\lim_{|y| \to \infty} \int_{a_x(y)} g(x) dx = 0$ f

$$
Q_x(y) = \left\{ x \mid |x_j - y_j| \le \frac{x}{2}, \ j = 1, 2 \right\} \quad y = (y_1, y_2), \ x > 0.
$$

The essential spectrum $\sigma_e(A)$ of A is equal to $[0, \infty)$ if $[3]$. •

$$
\lim_{|y| \to \infty} \int_{Q_x(y)} q(x) dx = 0 \quad \text{for each (fixed)} x > 0.
$$

Corollary 2 now leads to the following result'.

Corollary 3: Ifq is bounded and the hypotheses (2) *and* (13) *are èatis/ied, then there* $\lim_{|y|\to\infty} \int_{Q_x(y)} q(x) dx = 0$ for each (fixed) $x > 0$.
Corollary 2 now leads to the following result.
Corollary 3: If q is bounded and the hypotheses (2) and
exists at least one negative eigenvalue of the operator A.
Coroll

Consider in the following the Sturm-Liouville equation

$$
(-p(x) u')' + q(x) u = 0, \qquad -\infty \leq a < x < b \leq \infty. \tag{14}
$$

Q_x(y) = $\left\{x \mid |x_j - y_j| \leq \frac{x}{2}, j = 1, 2\right\}$ *y* = $\left\{y_1, y_2\right\}, x > 0$.

thial spectrum $\sigma_e(A)$ of *A* is equal to $[0, \infty)$ if [3]
 $\lim_{|y| \to \infty} \int_{\alpha(x)} g(x) dx = 0$ for each (fixed) $x > 0$. (13)

2 now leads to the fol The points x_1 and x_2 , $a < x_1 < x_2 < b$, are said to be *conjugate* with respect to (14) if there exists a nontrivial solution *u* of (14) with $u(x_1) = 0 = u(x_2)$.

Theorem 2: *Assume*

$$
-p(x) u' + q(x) u = 0, \quad -\infty \leq a < x < b
$$

\n*x*₁ and *x*₂, *a* < *x*₁ < *x*₂ < *b*, are said to be *conju*
\nlists a nontrivial solution *u* of (14) with *u*(*x*₁) = 0
\n*m* 2: Assume
\n
$$
p^{-1}(x) dx = \infty = \int_{c}^{b} p^{-1}(x) dx, \quad a < c < b,
$$

•
•
•

Corollary 2 now leads to the following result.
\nCorollary 3: If q is bounded and the hypotheses (2) and (13) are satisfied, then there exists at least one negative eigenvalue of the operator A.
\nCorollary 3 is essentially due to SCHMINCKE [7: Th. 5].
\nConsider in the following the Sturm-Liouville equation
\n
$$
(-p(x) u')' + q(x) u = 0, \quad -\infty \le a < x < b \le \infty.
$$
\n(14)
\nThe points x_1 and x_2 , $a < x_1 < x_2 < b$, are said to be conjugate with respect to (14)
\nif there exists a nontrivial solution u of (14) with $u(x_1) = 0 = u(x_2)$.
\nTheorem 2: Assume
\n
$$
\int_a^b p^{-1}(x) dx = \infty = \int_a^b p^{-1}(x) dx, \quad a < c < b,
$$
\nand
\n
$$
\int_a^b p^{-1}(x) dx = \int_a^b p^{-1}(x) dx, \quad a < c < b,
$$
\nand
\n
$$
\int_a^b p^{-1}(x) dx = \int_a^b p^{-1}(x) dx, \quad a < c < b,
$$
\n(15)
\n
$$
\int_a^b p^{-1}(x) dx = \int_a^b p^{-1}(x) dx, \quad a < c < b,
$$
\n(16)
\n
$$
\int_a^b p^{-1}(x) dx = 0, \quad a \ne 0.
$$
\nThen there exists a pair of conjugate points $x_1 = a + \xi, x_2 = b - \xi, 0 < \xi < \frac{b - a}{2}$,
\nwith respect to (14).

Proof: Let *y* be a point with $q(y) < 0$. There exist numbers $d > 0$ and $\varrho > 0$
such that $q(x) \le -d$, $|x - y| \le \varrho$. Choose points x_2 and x_3 with $a < x_2 < y - \varrho$
and $y + \varrho < x_3 < b$ and $a \varepsilon > 0$ arbitrarily. Then the po such that $q(x) \leq -d$, $|x - y| \leq \varrho$. Choose points x_2 and x_3 with $a < x_2 < y - \varrho$ $y + e < x_3 < b$ and a $\varepsilon > 0$ arbitrarily. Then the points x_1 and x_4 are uniquely 'determined by Proof: Let y be a point with $q(y) < 0$. There exist num
such that $q(x) \le -d$, $|x - y| \le 0$. Choose points x_2 and x_1
and $y + \varrho < x_3 < b$ and a $\varepsilon > 0$ arbitrarily. Then the points
determined by
 $\varepsilon \int_{x_1}^{x_1} p^{-1} dx = 1$,

$$
\varepsilon \int_{x_1}^{x_1} p^{-1} dx = 1, \quad a < x_1 < x_2, \quad \text{and} \quad \varepsilon \int_{x_1}^{x_1} p^{-1} dx = 1, \quad x_3 < x_4 < b.
$$

$$
\limsup_{a \to a} \int_{f b} q(x) dx \le 0, \qquad q \ne 0.
$$
\nThen there exists a pair of conjugate points $x_1 = a + \xi$, $x_2 = b - \xi$, $0 < \xi < \frac{b - \xi}{2}$ with respect to (14).
\nProof: Let y be a point with $q(y) < 0$. There exist numbers $d > 0$ and $q \ge 0$ such that $q(x) \le -d$, $|x - y| \le q$. Choose points x_2 and x_3 with $a < x_2 < y$ and $y + \varrho < x_3 < b$ and a $\varepsilon > 0$ arbitrarily. Then the points x_1 and x_4 are unique determined by
\n
$$
\varepsilon \int_{x_1}^{x_1} p^{-1} dx = 1, \qquad a < x_1 < x_2, \qquad \text{and} \qquad \varepsilon \int_{x_2}^{x_1} p^{-1} dx = 1, \qquad x_3 < x_4 < \frac{1}{x_1}
$$
\nChoose a number $h > 0$ and define a function v by
\n
$$
\varepsilon \int_{x_1}^{0} p^{-1} dx
$$
, $x \in (a, x_1] \cup (x_4, b),$ \n
$$
v(x) = \begin{cases} 0, & x \in (a, x_1] \cup (x_4, b), \\ \varepsilon \int_{x_1}^{x_1} p^{-1} dt, & x \in (x_1, x_2], \\ 1, & x \in (x_2, y - \varrho] \cup (y + \varrho, x_3], \\ 1, & x \in (x_3, x_4]. \end{cases}
$$
\nIt will be used for estimating the quadratic form $\int_{a}^{b} [p(v')^2 + qv^2] dx$: by choose

It will be used for estimating the quadratic form $\int [p(v')^2 + qv^2] dx$: by choosing x_2, x_3, ε and *h* in an analogous way as in the proof of Theorem 1 one can obtain the estimate $\int_{a}^{b} [p(v')^{2} + qv^{2}] dx < 0$. Then there exist a function $\varphi \in C_{0}^{\infty}(a, b)$ and a number $\eta > 0$ with supp $\varphi \leq [a + \eta, b - \eta]$ such that. • Nodal Domains for Elliptic Differential Equations 139

imate $\int_a^b [p(v')^2 + qv^2] dx < 0$. Then there exist a function $\varphi \in C_0^{\infty}(a, b)$ and a

mber $\eta > 0$ with supp $\varphi \leq [a + \eta, b - \eta]$ such that
 $\int_a^b [p(\varphi')^2 + q\varphi^2] dx <$ *f* $[p(v')^2 + qv^2] dx < 0$. Then there exist a function $\varphi \in C_0^{\infty}$
 $\int_a^b [p(v')^2 + qv^2] dx < 0$. Then there exist a function $\varphi \in C_0^{\infty}$
 > 0 with supp $\varphi \leq [a + \eta, b - \eta]$ such that
 $\int_a^b [p(\varphi')^2 + q\varphi^2] dx < 0$.

Lest e \int_{a}^{a} > 0 with supp $\varphi \leq [a + \eta, b - \eta]$ such that
 $\int_{a}^{b} [p(\varphi')^2 + q\varphi^2] dx < 0$.

lest eigenvalue λ_{η} of the Friedrichs extension A_{η} of the o
 $A_{\eta,0}\psi = -\left(p(x) \psi'\right)' + q(x) \psi$, $\psi \in C_0^{\infty}(a + \eta, b - \eta)$,
 ψe

$$
\int [p(\varphi')^2 + q\varphi^2] dx < 0.
$$

The smallest eigenvalue λ_n of the Friedrichs extension A_n of the operator $A_{n,0}$,

$$
A_{n,0}\psi = -(p(x)\psi')' + q(x)\psi, \qquad \psi \in C_0^{\infty}(a+\eta, b-\eta),
$$

is negative. There exists a $\xi > \eta$ with $\lambda_{\xi} = 0$ (compare the proof of Theorem 1). Let u_i be the corresponding eigenfunction. It has the property $u_i(a + \xi) = 0 = u_i(b - \xi)$. Let $z = z(x)$ be the solution of (14) being identical with $u_i(x)$ on $[a + \xi, b - \xi]$. The points $a + \xi$ and $b - \xi$ are conjugate. This proves Theorem 2 **points a** $\int_a^b [p(\varphi')^2 + q\varphi^2] dx < 0.$ (16)

The smallest eigenvalue λ_{η} of the Friedrichs extension A_{η} of the operator $A_{\eta,0}$,
 $A_{\eta,0}\psi = -(p(x) \psi')' + q(x) \psi$, $\psi \in C_0^{\infty}(a + \eta, b - \eta)$,

is negative. There exist Le smallest eigenvalue λ_{η} of the Friedrichs extension A_{η} of the operator $A_{\eta,0}$,
 $A_{\eta,0}\psi = -\left(p(x)\psi'\right) + q(x)\psi$, $\psi \in C_0^{\infty}(a + \eta, b - \eta)$,

negative. There exists a $\xi > \eta$ with $\lambda_{\xi} = 0$ (compare the proof o

There does not exist a pair of conjugate points with respect to (14) when $q = 0$. Otherwise, the solution $u = 1$ would vanish somewhere on (a, b) by Sturm's comparison theorem.
Theorem 2 improves results of TIPLER [8: Th. points $a + \xi$ and $b - \xi$ are conjugate. This prove.

There does not exist a pair of conjugate points with the solution $u = 1$ would vanish somewhere on (a, b) by

Theorem 2 improves results of TIPLER [8: Th. 2] (the

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0), and the author [4: Th..1].

There does not exist a pair
 e solution $u = 1$ would vanis

Theorem 2 improves results

ANDT, HINTON and LEWIS [

If $a = -\infty$, $b = \infty$, $p =$

tension A of the operator A₍ BRANDT, HINTON and Lewis [1: Th. 3.3] (the case $q \le 0$), and the author [4: Th. 1].

If $a = -\infty$, $b = \infty$, $p = 1$, $q(x) \to 0$, $|x| \to \infty$, the essential spectrum of the Friedrichs

extension A of the operator A_0 , $A_0\psi$ extension *A* of the operator A_0 , $A_0\psi = -\psi'' + q(x)\psi$, $\psi \in C_0^{\infty}(-\infty, \infty)$, coincides with $[0, \infty)$.
It follows from (15) that there exists a function $\varphi \in C_0^{\infty}(-\infty, \infty)$ satisfying (16). Hence, there exists at least one negative eigenvalue of A. This result is due to SCHMINCKE [6: Lemma 8]. •

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