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# On the Extension of $L^2$ Holomorphic Functions II

By

Takeo OHSAWA*

## Introduction

Let $(X, ds^2)$ be a complete Hermitian manifold of dimension $n$, and let $\varphi$ be a real-valued $C^\infty$ function on $X$. By the theory of L. Hörmander [H], a $\bar{\partial}$-closed form $u$ on $X$ is $\bar{\partial}$-exact if it satisfies the estimate

$$|(u, v)_\varphi| \leq C_u(\|\bar{\partial}v\|_\varphi + \|\bar{\partial}_\varphi^* v\|_\varphi)$$

for any compactly supported $C^\infty$ form $v$, where $C_u$ is a number independent of $v$, and in many cases the estimate is true for $C_u$=const. $\|u\|_\varphi$. In our previous work [O-T], we have established a new $L^2$-inequality involving the $\bar{\partial}$ operator, in which the estimation for $C_u$ is more elaborate. As a consequence, it enabled us to prove the following.

**Theorem.** *Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $H \subset \mathbb{C}^n$ be a complex hyperplane. Then, every $L^2$ holomorphic function on $D \cap H$ has an $L^2$ holomorphic extension to $D$.*

The purpose of the present paper is to formulate and prove a generalized $L^2$ extension theorem from higher codimensional submanifolds which includes our previous result as a special case, by using our new $L^2$ inequality.

Our main result is as follows.

**Theorem.** *Let $X$ be a Stein manifold of dimension $n$, $Y \subset X$ a closed complex submanifold of codimension $m$, and $(E, h)$ a Nakano-semipositive vector bundle over $X$. Let $\varphi$ be any plurisubharmonic function on $X$ and let $s_1, \cdots, s_m$ be holomorphic functions on $X$ vanishing on $Y$. Then, given a holomorphic $E$-valued $(n-m)$-form $g$ on $Y$ with*

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* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.

$$|\int_Y e^{-\varphi} h(g)\wedge\bar{g}|<\infty ,$$

*there exists for any* $\varepsilon>0$, *a holomorphic E-valued n-form* $G_\varepsilon$ *on* $X$ *which coincides with* $g\wedge ds_1\wedge\cdots\wedge ds_m$ *on* $Y$ *and satisfies*

$$(1)\qquad |\int_X e^{-\varphi}(1+|s|^2)^{-m-\varepsilon} h(G_\varepsilon)\wedge\bar{G}_\varepsilon|\leq\varepsilon^{-1} C_m|\int_Y e^{-\varphi} h(g)\wedge\bar{g}|,$$

*where* $|s|^2=\sum_{i=1}^{m}|s_i|^2$ *and* $C_m$ *is a positive number which depends only on m.*

To see that one cannot drop $\varepsilon$ in (1), it suffices to consider the case $X=\boldsymbol{C}$, $Y=\{0\}$, $\varphi=0$ and $g=1$.

Among a few direct consequences of Theorem, the following two observations might be of interest.

**Corollary 1.** *Let X be a weakly* 1*-complete manifold of dimension n which admits a positive line bundle, let s be a holomorphic function on X such that* $ds\neq 0$ *on* $Y:=s^{-1}(0)$, *and let* $(E, h)$ *be a Nakano semipositive vector bundle over X. Then the restriction map*

$$\Gamma(X, \mathcal{O}_X(K_X\otimes E))\rightarrow\Gamma(Y, \mathcal{O}_Y(K_X\otimes E))$$

*is surjective.*

**Corollary 2.** *Let Y be a pure dimensional closed complex submanifold of* $\boldsymbol{C}^N$, *let* $\Omega$ *be a bounded domain of holomorphy, and* $\varphi$ *a plurisubharmonic function on* $\Omega$. *Then, for any holomorphic function f on* $\Omega\cap Y$ *with*

$$\int_{\Omega\cap Y} e^{-\varphi}|f|^2 dV_Y<\infty$$

*there exists a holomorphic extension F to* $\Omega$ *such that*

$$\int_\Omega e^{-\varphi}|F|^2 dV\leq A\int_{\Omega\cap Y} e^{-\varphi}|f|^2 dV_Y .$$

*Here A depends on Y and* $\sup\{||z||;\ z\in\Omega\}$, but does not depend on $f$.

In §1 we improve the estimates shown in [O–T] and [O–2], so that one can dispense with auxiliary complete Kähler metrics which we needed before. We shall prove Theorem in §2 by solving $\bar{\partial}$-equations on a family of strongly pseudoconvex domains and taking a limit of solutions. In case $m=1$, it amounts to solve the equation

$$\bar{\partial}u=2\pi i g\wedge[Y]$$

with an appropriate $L^2$ estimate, where $[Y]$ denotes the (1, 1)-current associated to $Y$. For $m\geq 2$ the limit equation is hard to describe in the framework of distributions, and it might be interesting to know its legitimate description.

## §1. Notations and Preliminaries

Let $(X, ds^2)$ be a Kähler manifold of dimension $n$, $(E, h)$ a Hermitian vector bundle over $X$ and $\varphi$ a $C^\infty$ real valued function on $X$. We shall use the following notations.

$$C^{p,q}(X, E) = \{C^\infty\ E\text{-valued } (p, q)\text{-forms on } X\}$$
$$C_0^{p,q}(X, E) = \{f\in C^{p,q}(X, E);\ \operatorname{supp} f\Subset X\}$$
$$L_\varphi^{p,q}(X, E) = \{\text{measurable } E\text{-valued } (p, q)\text{-forms } f \text{ on } X \text{ satisfying } \int_X e^{-\varphi}|f|^2\, dV_X<\infty\},$$

where $|f|$ denotes the length of $f$ and $dV_X$ denotes the volume form.
We denote by $\bar\partial$: $L_\varphi^{p,q}(X, E)\to L_\varphi^{p,q+1}(X, E)$ the complex exterior derivative of type (0, 1) defined on

$$\operatorname{Dom}\bar\partial := \{f\in L_\varphi^{p,q}(X, E);\ \bar\partial f\in L_\varphi^{p,q+1}(X, E)\}.$$

The adjoint of $\bar\partial$ will be denoted by $\bar\partial_\varphi^*$.

Let $(f, g)_\varphi$ be the inner product of $f$, $g\in L_\varphi^{p,q}(X, E)$ associated to the norm

$$\|f\|_\varphi := \left(\int_X e^{-\varphi}|f|^2\, dV_X\right)^{1/2}.$$

For a $(p, q)$-form $f$ on $X$ we denote by $e(f)$ the left multiplication by $f$ in the exterior algebra of differential forms on $X$. The (pointwise) adjoint of $e(f)$ is denoted by $e(f)^*$. We shall denote by $\omega$ the fundamental form of $ds^2$, and put $\Lambda=e(\omega)^*$. The curvature form of $h$ will be denoted by $\Theta=\sum_{\alpha,\beta}\Theta_{\alpha\bar\beta\nu}^{\kappa}\, dz^\alpha\wedge d\bar z^\beta$. The left multiplication by $\Theta$ to $E$-valued forms is well-defined and denoted by $e(\Theta)$. $(E, h)$ is said to be Nakano semipositive if the Hermitian form $\sum_{\alpha,\beta,\nu,\mu}(\sum_\kappa \Theta_{\alpha\bar\beta\nu}^{\kappa} h_{\kappa\bar\mu})\, \xi^{\alpha\nu}\, \overline{\xi^{\beta\mu}}$ is semipositive. We note that $(f, g)_\varphi=i^n(-1)^{n(n-1)/2}\int_X e^{-\varphi}\, h(f)\wedge\bar g$ if $f$, $g\in L_\varphi^{n,0}(X, E)$. In particular $L_\varphi^{n,0}(X, E)$ does not depend on the choice of $ds^2$. We say $X$ is a weakly 1-complete manifold if there exists a $C^\infty$ plurisubharmonic function $\varphi$: $X\to\mathbf{R}$ such that $X_c:=\{x\in X;\ \varphi(x)<c\}$ is relatively compact for any $c\in\mathbf{R}$. For the basic materials on weakly 1-complete manifolds, see [O–1].

**Lemma 1.** *Let $D \Subset X$ be a strongly pseudoconvex domain with $C^\infty$ smooth boundary, let $\varphi_0$ be a $C^\infty$ defining function of $D$, and let $\psi$ be a nonnegative $C^\infty$ function defined on $\bar{D}$. Then, for any $\varepsilon>0$ and a compact subset $K\subset D$, there exists a $C^\infty$ function $\psi_K$ on $D$ satisfying the following properties.*

(i) $\psi \geqq \psi_K$ *and* $\psi-\psi_K=\varepsilon$ *on* $K$.

(ii) $\inf_D \psi_K=0$ *and* $D^c:=\{x\in D;\ \psi_K(x)\geqq c\}$ *is compact for all* $c\in(-\infty, 0)$.

(iii) $|d\psi_K \wedge d\varphi_0|\leqq C$, *where $C$ does not depend on the choice of $K$.*

(iv) $\psi-\psi_K$ *is plurisubharmonic.*

*Proof.* Since $D$ is strongly pseudoconvex, we may assume that $\varphi_0$ is strictly plurisubharmonic on a neighbourhood of $\partial D$. Let $\delta$ be any positive number satisfying $\delta<-\sup_K \varphi_0$ and let $\lambda_\delta$ be a $C^\infty$ function on $(-\infty, 0)$ such that $\lambda_\delta(t)$, $\lambda_\delta'(t)$, $\lambda_\delta''(t)\geqq 0$ for all $t$, $\lambda_\delta'(t)=-t^{-1}$ on $(-\delta/2, 0)$, and $\lambda_\delta(t)=0$ on $(-\infty, -\delta)$. Then, for any $\varepsilon>0$ and $\tau>0$ there exists a $\delta_0>0$ such that the function

$$\Phi^\tau_{\delta,\varepsilon}:=\tau\lambda_\delta(\varphi_0)-\psi-\varepsilon$$

satisfies

$$(*)\qquad \partial\bar{\partial}\Phi^\tau_{\delta,\varepsilon}\geqq\frac{-\tau}{2\varphi_0}\partial\bar{\partial}\varphi_0+\frac{1}{4\tau}\partial\Phi^\tau_{\delta,\varepsilon}\bar{\partial}\Phi^\tau_{\delta,\varepsilon}\quad\text{on}\quad D\backslash D_{-\delta/2}$$

if $0<\delta<\delta_0$.

Note that $\Phi^\tau_{\delta,\varepsilon}<-\varepsilon/2$ on $D_{-\delta/2}$ if $\tau<-\frac{\varepsilon}{2}(\log 2)^{-1}$. Let $\chi\colon \boldsymbol{R}\to\boldsymbol{R}$ be a $C^\infty$ increasing function such that $\chi(t)=t$ on $(-\infty, -\varepsilon/2)$ and $\chi(t)=-t^{-1}$ on $(2, \infty)$. If we put $\psi_K=-\chi(\Phi^\tau_{\delta,\varepsilon})$ for $\delta\ll\tau<-\frac{\varepsilon}{2}(\log 2)^{-1}$, then $\psi_K$ satisfies (i) through (iv). In fact, (i), (ii), (iii) are trivial and (iv) follows from $(*)$.

**Proposition 2.** *Let $D\Subset X$ be a strongly pseudoconvex domain with $C^\infty$ boundary, and let $\psi$ be a nonnegative $C^\infty$ function defined on $\bar{D}$. Then, for any $C^\infty$ function $\varphi$ on $\bar{D}$ and a $C^\infty$ $E$-valued $(n, q)$-form $u$ on $\bar{D}$ with $\overline{*u}|_{\partial D}=0$,*

$$(2)\qquad \|\sqrt{\psi}\,\bar{\partial}^*_\varphi u\|^2_{\varphi,D}+\|\sqrt{\psi}\,\bar{\partial}u\|^2_{\varphi,D} \geqq (ie(\psi(\partial\bar{\partial}\varphi+\Theta)-\partial\bar{\partial}\psi)\Lambda u, u)_{\varphi,D}+2\mathrm{Re}(e(\bar{\partial}\psi)\bar{\partial}^*_\varphi u, u)_{\varphi,D},$$

*where $*$ denotes the Hodge's star operator,*

$$\|\sqrt{\psi}\,\bar{\partial}^*_\varphi u\|^2_{\varphi,D}=\int_D \psi e^{-\varphi}|\bar{\partial}^*_\varphi u|^2 dV_X,\quad etc.$$

*Proof.* Let $K\subset D$ be any compact subset and let $\psi_K$ be chosen for $\psi$ as in Lemma 1. Then, for any $u$ as above,

$$(**)\qquad \|\sqrt{\psi_K}\,\bar\partial_\varphi^* u\|_{\varphi,D}^2+\|\sqrt{\psi_K}\,\bar\partial u\|_{\varphi,D}^2 \geqq (ie(\psi_K(\partial\bar\partial\varphi+\Theta)-\partial\bar\partial\psi_K)\,\Lambda u,\,u)_{\varphi,D} + (e(\bar\partial\psi_K)\,\bar\partial_\varphi^* u,\,u)_{\varphi,D}+(\bar\partial e(\bar\partial\psi_K)^* u,\,u)_{\varphi,D}$$

(cf. [O–2] §1, (6)).

Since $\overline{*u}\,|_{\partial D}=0$,

$$(\bar\partial e(\bar\partial\psi_K)^* u,\,u)_{\varphi,D}=(u,\,e(\bar\partial\psi_K)\,\bar\partial_\varphi^* u)_{\varphi,D}\,.$$

By (i) and (iii),

$$\|e(\bar\partial\psi_K)^* u-e(\bar\partial\psi)^* u\|_{\varphi,D}\leqq \text{const.}\ \|u\|_{\varphi,D\setminus K}\,.$$

By (iv),

$$(ie(\psi_K(\partial\bar\partial\varphi+\Theta)-\partial\bar\partial\psi_K)\,\Lambda u,\,u)_{\varphi,D} \geqq (ie(\psi\partial\bar\partial\varphi-\partial\bar\partial\psi)\,\Lambda u,\,u)_{\varphi,D}\,.$$

Thus, taking the limit of the inequality (**) we obtain the desired estimate.

In order to apply the estimate (2) effectively we have to digress a bit into linear algebra.

Let $V$ be a complex vector space of dimension $n$ and let $s_1$ be a Hermitian form on $V$. Let $V^*\otimes\boldsymbol{C}=V_+^*\oplus V_-^*$ be the decomposition into the $\pm\sqrt{-1}$-eigenspaces of the complex structure and let $V_{s_1}^{*,q}$ be the subspace of $(\overset{n}{\bigwedge}V_+^*)\otimes(\overset{q}{\bigwedge}V_-^*)\subset\overset{n+q}{\bigwedge}(V^*\otimes\boldsymbol{C})$ spanned by the vectors $u\wedge(\bar u_1\wedge\cdots\wedge\bar u_q)$, where $u\in\overset{n}{\bigwedge}V_+^*$ and $u_k(\xi)=0$ for $1\leqq k\leqq q$ on $\{\xi\in V\otimes\boldsymbol{C};\ s_1(\xi,\xi)=0\}$. Let $\{v_1,\cdots,v_n\}$ be a basis of $V_+^*$ such that $s_1=\sum_{\alpha=1}^{l}v_\alpha\otimes\bar v_\alpha-\sum_{\beta=l+1}^{m}v_\beta\otimes\bar v_\beta$. Then $V_{s_1}^{*,q}$ is spanned by $u\wedge(\overline{v_{i_1}}\wedge\cdots\wedge\overline{v_{i_q}})$, where $u\in\overset{n}{\bigwedge}V_+^*$ and $1\leq i_1<\cdots<i_q\leq m$. The star operator

$$*_{s_1}:\ V_{s_1}^{*,q}\to\overset{n-q}{\bigwedge}(V^*\otimes\boldsymbol{C})$$

is defined as a uniquely determined linear map which satisfy

$$*_{s_1}(v_1\wedge\cdots\wedge v_n\wedge\overline{v_{j_1}}\wedge\cdots\wedge\overline{v_{j_q}}) = v_{j_{q+1}}\wedge\cdots\wedge v_{j_n}\circ\operatorname{sgn}\begin{bmatrix}1\cdots n\\ j_1\cdots j_n\end{bmatrix}\left(\frac{-i}{2}\right)^n(-n)^{n(n-1)/2} \times\varepsilon_{j_1}\cdots\varepsilon_{j_q}\,.$$

Here $\varepsilon_j=1$ if $1\leq j\leq l$, $\varepsilon_j=-1$ if $l+1\leq j\leq m$ and $\varepsilon_j=0$ if $m<j\leq n$.

Then we have a nondegenerate pairing

$$\begin{array}{ccc} V^{*,q}_{s_1} \times V^{*,q}_{s_1} & \to & \bigwedge^{2n}(V^*\otimes \boldsymbol{C}) \\ \Downarrow & & \Downarrow \\ (u, v) & \mapsto & u\wedge\overline{*_{s_1} v} \end{array}$$

Let $\omega_{s_1}$ be the imaginary part of $s_1$ and denote by $e(\omega_{s_1})$ the multiplication by $\omega_{s_1}$. Let $s_2$ be any positive Hermitian form on $V$. We denote by $e(\omega_{s_1})^*$ the adjoint of $e(\omega_{s_1})$ with respect to $s_2$. We put

$$\langle u, v\rangle_{s_2}\frac{\omega_{s_2}^n}{n!} = u\wedge\overline{*_{s_2} v}$$

$$\langle u, v\rangle_{s_1}\frac{\omega_{s_2}^n}{n!} = u\wedge\overline{*_{s_1} v} \quad (v\in V^{*,q}_{s_1}).$$

Then we have

$$\langle v, v\rangle_{s_1} = \langle e(\omega_{s_2})\, e(\omega_{s_1})^{-1} v, v\rangle_{s_2} \quad \text{for} \quad v\in V^{*,1}_{s_1}.$$

Here $e(\omega_{s_1})^{-1}$ denotes the inverse map of $e(\omega_{s_1})\colon \bigwedge^{n-1} V^*_+ \to V^{*,1}_{s_1}$. If $s_1$ is semi-positive, then

$$|\langle u, v\rangle_{s_2}|^2 \leq \langle e(\omega_{s_1})\, e(\omega_{s_2})^* u, u\rangle_{s_2}\langle v, v\rangle_{s_1} \tag{3}$$

for any $u\in\bigwedge^{n+1}(V^*\otimes\boldsymbol{C})$ and $v\in V^{*,1}_{s_1}$.

Let $(W, h_1)$ be another Hermitian vector space. Then the inner product $\langle v, v\rangle_{s_1}$ is naturally extended to $W\otimes V^{*,1}_{s_1}$, which will be also denoted by $\langle v, v\rangle_{s_1}$. We have similar estimates as (3) for the elements of $W\otimes(\bigwedge^{n+1} V^*\otimes\boldsymbol{C})$ and $W\otimes V^{*,1}_{s_1}$.

Thus we have the following inequality for the bundle valued forms.

**Proposition 3.** *Let $\sigma$ be a semipositive* (1, 1)*-form on $X$ and let $u, v\in L^{n,1}_{\varphi}(X, E)$. If $v(x)\in E_x\otimes(T_{X,x})^{*,1}_{\sigma(x)}$ for any $x\in X$, then*

$$|(u, v)_{\varphi}|^2 \leq (e(\sigma)\Lambda u, u)_{\varphi}\int_X e^{-\varphi}\langle v, v\rangle_{\sigma}\ dV_X$$

*and*

$$\int_X e^{-\varphi}\langle v, v\rangle_{\sigma}\, dV_X = (e(\omega)\, e(\sigma)^{-1} v, v)_{\varphi}.$$

To simplify the notation we set

$$\begin{aligned} &L^{n,q}_{\varphi}(X, E)_{\sigma} \\ &\quad = \{v\in L^{n,q}_{\varphi}(X, E);\ v(x)\in(T_{X,x})^{*,q}_{\sigma} \text{ for any } x\in X\}. \end{aligned}$$

**Proposition 4.** *Let $D\Subset X$ be a strongly pseudoconvex domain with $C^\infty$-smooth* *boundary, $\psi$ a nonnegative $C^\infty$ function on $\bar{D}$, and $\varphi$ a $C^\infty$ function on $\bar{D}$. Suppose that $(E, h)$ is Nakano semipositive and there exists a positive locally bounded function $\eta$ on $D$ such that*

$$\sigma(\eta) := \psi\partial\bar{\partial}\varphi - \partial\bar{\partial}\psi - \eta^2 |\partial\psi|^{-2} \partial\psi\bar{\partial}\psi$$

*is semipositive. Then, for any $C^\infty$ $E$-valued $(n, 1)$-form $u$ on $\bar{D}$ with $\overline{*u}|_{\partial D}=0$ and $v \in L^{n,1}_\varphi(D, E)_{\sigma(\eta)}$,*

$$|(u, v)_{\varphi,D}|^2 \leq (e(\omega)\, e(\sigma(\eta))^{-1} v, v)_{\varphi,D}$$
$$\times (\|(\sqrt{\psi} + \eta|\partial\psi|)\, \bar{\partial}^*_\varphi u\|^2_{\varphi,D} + \|\sqrt{\psi}\, \bar{\partial} u\|^n_{\varphi,D}) .$$

## §2. Proof of Theorem

Let the notations be as in the introduction. Since $X$ is a Stein manifold one can find a decreasing sequence of $C^\infty$ plurisubharmonic functions $\{\varphi_\mu\}_{\mu=1}^\infty$ which converges to $\varphi$ almost everywhere. Hence it suffices to prove Theorem in case $\varphi$ is $C^\infty$. Moreover we may assume that $ds_1 \wedge \cdots \wedge ds_m \neq 0$ everywhere. In fact, take an analytic subset $Z \subset X$ of codimension one such that

$$Z \supset \{x;\, ds_1 \wedge \cdots \wedge ds_m|_x = 0\} .$$

Then it suffices to show the extendability of $g \wedge ds_1 \wedge \cdots \wedge ds_m$ to $X \backslash Z$, since the apparent singularity along $Z$ is improper in virtue of the $L^2$ condition.

As in [O–T] we fix an increasing family of strongly pseudoconvex domains $X_1 \Subset X_2 \Subset \cdots \Subset X_\mu \Subset \cdots$ with $C^\infty$ smooth boundaries such that

$$X = \bigcup_{\mu=1}^\infty X_\mu .$$

Then it suffices to find the extensions to $X_\mu$, since one obtains a desired extension as a weak limit of a subsequence of the extensions to $X_\mu$ ($\mu \to \infty$).

Let $G$ be an arbitrary holomorphic extension of $g \wedge ds_1 \wedge \cdots \wedge ds_m$ to $X$. It certainly exists since $X$ is a Stein manifold. Let $\chi: \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function satisfying $\chi(t)=1$ on $(-\infty, 1/2)$ and $\chi(t)=0$ on $(1, \infty)$. For any $\delta>0$ we put

$$G^{[\delta]} = \begin{cases} \chi(|s|^2/\delta^2)\, G \\ \quad \text{on} \quad \{x \in X;\ |s(x)| < \delta\} \\ 0 \quad \text{otherwise.} \end{cases}$$

Then $G^{[\delta]}$ is a $C^\infty$ extension of $g \wedge ds_1 \wedge \cdots \wedge ds_m$. We put

$$v^\delta = \bar{\partial} G^{[\delta]} .$$

Note that $v^\delta=0$ on a neighbourhood of $Y$. Taking an arbitrary Kähler metric $ds_0^2$ of $X$, we fix a metric $ds^2$ of $X$ by

$$ds^2 = ds_0^2+2\partial\bar\partial \log (1+|s|^2).$$

Then, for any $\mu$ one can find a sufficiently small $\delta_\mu$ such that

$$\int_{X_\mu} e^{-\varphi}|s|^{-2m}\delta^2|v^\delta|^2\,dV_X \le C_m\int_Y e^{-\varphi}|g|^2\,dV_Y \quad \text{if} \quad \delta<\delta_\mu. \tag{4}$$

Here $C_m$ depends only on $m$. Let $\lambda$: $\boldsymbol{R}\to\boldsymbol{R}$ be a $C^\infty$ function satisfying $\lambda'(t)\geq 0$, $\lambda''(t)\geq 0$, $\sup \lambda'(t)\leq 1$, $\sup \lambda''(t)<2/3$, and

$$\lambda(t) = \begin{cases} 0 & \text{on} \quad (-\infty, 0) \\ t-1 & \text{on} \quad (2, \infty). \end{cases}$$

We put $\psi_0=\lambda(-\log(|s|^2+\delta^2))$. Then $\partial\bar\partial\psi_0=-\partial\bar\partial\log(|s|^2+\delta^2)$ if $|s|^2+\delta^2<e^{-2}$, $\partial\bar\partial\psi_0=0$ if $|s|^2+\delta^2>1$, and

$$\partial\bar\partial\psi_0\leq\frac{8e^4}{3}\,\partial\bar\partial \log(|s|^2+1)$$

if $e^{-2}\leq|s|^2+\delta^2\leq 1$.

On the other hand

$$\partial\psi_0\,\bar\partial\psi_0\leq(|s|^2+1)^2(|s|^2+\delta^2)^{-1}\,\partial\bar\partial\log(|s|^2+1).$$

Thus $|\partial\psi_0|^2$ is estimated from above by $(|s|^2+1)^2(|s|^2+\delta^2)^{-1}$. For any $\varepsilon>0$ we put

$$\psi_\varepsilon = \psi_0+\varepsilon^{-1}\left(\frac{8e^4}{3}+4\right).$$

Then $|\partial\psi_\varepsilon|^2\leq(|s|^2+1)^2(|s|^2+\delta^2)^{-1}$ and

$$\psi_\varepsilon\leq\alpha_\varepsilon(|s|^2+\delta^2)^{-1},$$

where

$$\alpha_\varepsilon = \sup_{|s|<1}(|s|^2+\delta^2)\left(-\log(|s|^2+\delta^2)+\varepsilon^{-1}\left(\frac{8e^4}{3}+4\right)\right).$$

We put

$$\varphi_\varepsilon = \varphi+2m\log|s|+\varepsilon\log(|s|^2+1).$$

Then

$$\sigma_\varepsilon := \psi_\varepsilon \partial\bar{\partial}\varphi_\varepsilon - \partial\bar{\partial}\psi_\varepsilon - |\partial\psi_\varepsilon|^{-2}\partial\psi_\varepsilon \bar{\partial}\psi_\varepsilon$$

is semipositive and

$$\sigma_\varepsilon \geq \partial\bar{\partial} \log(|s|^2+\delta^2)$$

on $\{x; |s(x)|<\delta\}$.

Combining it with (4) we see that there exists a $\delta'_\mu>0$ such that

$$(e(\omega)\, e(\sigma_\varepsilon)^{-1} v^\delta, v^\delta)_{\varphi_\varepsilon, X_\mu} \leq 4C_m \int_Y e^{-\varphi_\varepsilon}|g|^2 dV_Y \quad \text{if} \quad \delta<\delta'_\mu.$$

Therefore by Proposition $\psi$, if $\delta<\delta'_\mu$

$$|(u, v^\delta)_{\varphi_\varepsilon, X_\mu}|^2 \leq 4C_m \int_Y e^{-\varphi_\varepsilon}|g|^2 dV_Y \times \{||(\sqrt{\psi_\varepsilon}+|\partial\psi_\varepsilon|)\,\bar{\partial}^*_{\varphi_\varepsilon} u||^2_{\varphi_\varepsilon, X_\mu} + ||\sqrt{\psi_\varepsilon}\,\bar{\partial}u||^2_{\varphi_\varepsilon, X_\mu}\}$$

for any $C^\infty$ $E$-valued $(n, 1)$-form $u$ on $\bar{X}_\mu$ with $\overline{*u}|_{\partial X_\mu}=0$. Since the same estimate also holds for $u\in \text{Dom}\,\bar{\partial}^*_{\varphi_\varepsilon} \cap \text{Dom}\,\bar{\partial} \cap L^{n,1}_{\varphi_\varepsilon}(X_\mu, E)$, there exists a solution $b^\delta_\varepsilon$ to the equation $\bar{\partial}((\sqrt{\psi_\varepsilon}+|\partial\psi_\varepsilon|)\, b^\delta_\varepsilon)=v^\delta$ with

$$||b^\delta_\varepsilon||^2_{\varphi_\varepsilon, X_\mu} \leq 4C_m \int_Y e^{-\varphi_\varepsilon}|g|^2 dV_Y$$

(cf. [H]).

We put

$$G_\varepsilon := G^{[\delta]} - (\sqrt{\psi_\varepsilon}+|\partial\psi_\varepsilon|)\, b^\delta_\varepsilon.$$

Then $G_\varepsilon$ is a holomorphic extension of $g\wedge ds_1 \wedge \cdots \wedge ds_m$ to $X_\mu$. The verification of the $L^2$ estimate is left to the reader.

*Proof of Corollary* 1. Let $\varphi: X\to \mathbf{R}$ be any $C^\infty$ plurisubharmonic exhaustion function and let $(B, a)$ be a positive line bundle over $X$. Then, for any $c\in\mathbf{R}$, $X_c := \{x; \varphi(x)<c\}$ is embeddable into a projective space by holomorphic sections of $B^m (m=m(c)\gg 0)$. In particular there exists a proper analytic subset $Z_c \subset X_c$ such that $Z_c \not\supset Y$ and $X_c\backslash Z_c$ is a Stein manifold. Let $g\wedge ds \in \Gamma(Y, \mathcal{O}_Y(K_X\otimes E))$ and choose a convex increasing $C^\infty$ function $\lambda$ such that

$$g\in L^{n-1,0}_{\lambda(\varphi)}(Y, E).$$

Applying Theorem to the manifolds $X_c\backslash Z_c \supset Y\cap X_c\backslash Z_c$ and $g|Y\cap X_c\backslash Z_c$, we have extensions of $g\wedge ds$ to $X_c\backslash Z_c$ whose norms in $L^{n,0}_{\lambda(\varphi)}(X_c\backslash Z_c, E)$ are dominated by const. $\|g\|_{\lambda(\varphi)}$. Since the singularities along $Z_c$ are improper, by taking a weak limit of these extended forms in $L^{n,0}_{\lambda(\varphi)}(X, E)$ we obtain a holomorphic extension of $g\wedge ds$ to $X$.

*Remark* 1. The assumption that $X$ admits a positive line bundle was only used to ensure the existence of the divisors $Z_c$. Hence one can replace the existence of a positive bundle in the hypothesis by the existence of a Zariski dense Stein open subset $\Omega\subset X$ such that $X\backslash\Omega$ does not contain any connected component of $Y$.

*Remark* 2. Corollary 1 may be regarded as an extension of Kazama-Nakano's vanishing theorem on weakly 1-complete manifolds (cf. [$O$–1]). In fact, if $E$ is Nakano-positive then $H^1(X, \mathcal{O}(K_X\otimes E))=0$ so that the surjectivity of the above map follows immediately. H. Skoda [$S$–2] has established a similar surjectivity theorem on weakly 1-complete manifolds as a generalization of his $L^2$ corona theorem on pseudoconvex domains in $\boldsymbol{C}^n$ (cf. [$S$–1]).

*Proof of Corollary* 2. Let $w_1, \cdots, w_k$ be holomorphic functions on $\boldsymbol{C}^N$ which generate the stalks of the ideal sheaf of $Y$ at each point of $\bar{\Omega}$. Let $m=\text{codim } Y$. Then for each $m$-tuple $(w_{i_1}, \cdots, w_{i_m})$ we apply Theorem as follows. Let $\sum_I\subset \boldsymbol{C}^N$ $(I=(i_1, \cdots, i_m))$ be an analytic subset of codimension one which contains the set $\{x\in Y;\, dw_{i_1}\wedge\cdots\wedge dw_{i_m}(x)=0\}$, and let $\sigma_I$ be a defining function of $\sum_I$. Then, by Rückert's theorem there exists a $p\in\boldsymbol{N}$ such that for all $I$

$$\sigma_I^p\, dz_1\wedge\cdots\wedge dz_N=g_I\wedge dw_{i_1}\wedge\cdots\wedge dw_{i_m} \quad \text{on} \quad Y$$

for some holomorphic $(N-m)$-form $g_I$ on $Y$. Here $(z_1, \cdots, z_N)$ denotes the coordinate of $\boldsymbol{C}^N$. Then $f\sigma_I^p\, dz_1\wedge\cdots\wedge dz_N$ has an extension $G_I$ to $\Omega$ with

$$\left|\int_\Omega e^{-\varphi}\, G_I\wedge\bar{G}_I\right|\le C_I\left|\int_{Y\cap\Omega} e^{-\varphi}|f|^2 g_I\wedge\bar{g}_I\right| ,$$

where $C_I$ does not depend on $f$. Let $\eta_I$ be holomorphic functions on $\boldsymbol{C}^N$ satisfying

$$\sum_{\substack{I\subset\{1,\cdots,k\}\\ \sharp I=m}} \sigma_I^p\,\eta_I = 1 \quad \text{on} \quad Y .$$

Then we define a function $F$ by

$$F\, dz_1\wedge\cdots\wedge dz_n = \sum_{\substack{I\subset\{1,\cdots,k\}\\ \sharp I=m}} \eta_I\, G_I .$$

Clearly $F$ is an extension of $f$ with desired properties.

*Remark.* Corollary 2 is easily generalized to relatively compact pseudoconvex domains of Stein manifolds. The detail is left to the reader.

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