Steady State Solutions of the Satsuma-Mimura Diffusion Equation

L. V. WOLFERSDORF

Die stationären Lösungen der Diffusionsgleichung von Satsuma und Mimura werden durch Zurückführung auf eine gewöhnliche komplexe Differentialgleichung erster Ordnung abgeleitet.

Установившиеся решения диффузионного уравнения Сатсума и Мимура выводятся с помощью редукции на обыкновенное комплексное дифференциальное уравнение первого порядка.

The steady state solutions of the diffusion equation of Satsuma and Mimura are derived by means of reducing to an ordinary complex differential equation of first order.

Introduction. In their papers $[3-5]$ J. Sarsuma and M. MIMURA investigated a class of nonlinear nonlocal diffusion equations involving singular integral terms. They developed an exact linearization method for these equations and derived some interesting particular solutions in explicit form by this method. In the present paper we rederive the steady state solutions of Satsuma and Mimura in systematic way by reducing the steady equations to complex differential equations of first order. We find that apart_i from an obvious invariant shifting transformation the steady state solutions found by Satsuma and Mimura are the only ones with a prescribed asymptotic behaviour at infinity. Besides we obtain new periodic steady state solutions for some corresponding equations with variable diffusion coefficient.

1. Coth-type kernel. At first we deal with the equation

$$
du_{xx} - (Tu \cdot u)_x = 0, \qquad d > 0,
$$
\n
$$
(1)
$$

where

 $(Tu)(x) = \frac{1}{2\delta} \int u(\xi) \coth \frac{\pi}{2\delta} (\xi - x) d\xi, \quad \delta > 0.$

We are looking for sufficiently smooth solutions $u = u(x)$ of (1) having vanishing limits $u(\pm \infty) = 0$ and $u_x(\pm \infty) = 0$ as $x \to \pm \infty$. Let $w(z) = u(x, y) + iv(x, y)$, $z = x + iy$, be a holomorphic function in the strip $\Pi: -\infty < x < \infty$, $0 < y < \delta$ with Hölder continuous boundary values on $y = 0$ and $y = \delta$, where $u(x, 0) = u(x)$ and

 $u(x, \delta) = 0, \quad -\infty < x < \infty,$ (2)

and there exist the uniform (with respect to $y \in [0, \delta]$) limits $v(\pm \infty)$ satisfying $v(+\infty) = -v(-\infty)$ and $v_x(\pm\infty)$ satisfying $v_x(+\infty) = v_x(-\infty)$ as $x \to \pm\infty$. Then (cf. $[6: § 3.16]$)

$$
w(z) = \frac{1}{2\delta i} \int\limits_{-\infty}^{\infty} u(\xi) \coth \frac{\pi}{2\delta} (\xi - z) d\xi \quad \text{in } \Pi
$$

 (3)

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with the boundary values $v(x, 0) = -(Tu)(x)$. Therefore integrating equation (1),
we obtain the boundary relation $du_x + vu = 0$ on $y = 0$ and due to (2) also on $y = \delta$, i.e. we have the boundary conditions Re $[dw' - (i/2) w^2] = 0$ on $y = 0$, $y = \delta$ for the function (3). Hence this function obeys the differential equation L. v. WOLFERSDORF

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ave the boundary conditions Re $[dw'-$

ion (3). Hence this function obeys the c
 $w'-\frac{1}{2d}w^2 = iK_0$ in Π

al constant $K_0(=v_x(\$

$$
w' - \frac{1}{2d} w^2 = iK_0 \qquad \text{in } \Pi \tag{4}
$$

with a real constant $K_0(= v_x(\pm \infty) + (1/2d) v^2(\pm \infty)$. We have to determine holomorphic solutions *w* of (4) in $\tilde{\Pi}$, (Hölder) continuous in $\tilde{\Pi}$ with the supposed behaviour at infinity and satisfying additionally the boundary condition (2).

As can be easily seen for $K_0 = 0$, no non-trivial solutions to (4) of such kind exist. Therefore putting $K_0 = -(1/2d) K_1^2$ with $K_1 \neq 0$ real or purely imaginary, we get the differential equation when a teat constant $\log_2(-\frac{x_2 + \cdots + x_n}{x})$ + ($x_1(x) = (x_1)(x_2 - \cdots + x_n)$) is the infinity and satisfying additionally the boundary conditions that infinity and satisfying additionally the boundary conditions.
Therefore put

$$
\frac{w'}{w^2-K_1{}^2}=\frac{1}{2d}
$$

with the general solution

e putting
$$
K_0 = -(1/2d) K_1^2
$$
 with $K_1 \neq 0$ real or p
ential equation

$$
\frac{w'}{w^2 - K_1^2} = \frac{1}{2d}
$$
general solution

$$
w(z) = K_1 \frac{1 + C e^{(K_1/d)t}}{1 - C e^{(K_1/d)t}}
$$
, C a complex number.

The supposed behaviour for *w* at infinity requires K_1 purely imaginary, i.e. $K_1 = \mathrm{i} dK$,

$$
w(z) = \mathrm{i} dK \frac{1 + \varrho \, \mathrm{e}^{\mathrm{i} z - Kz}}{1 - \varrho \, \mathrm{e}^{\mathrm{i} z - Kz}},
$$

where we put $C = \varrho e^{i\alpha}$ with $\varrho > 0$ and α arbitrary real. From (5) there follows

$$
w(z) = K_1 \frac{1 + C e^{(K_1/d)1z}}{1 - C e^{(K_1/d)1z}}, \qquad C \text{ a complex number}
$$

approsed behaviour for *w* at infinity requires K_1 purely
0, and

$$
w(z) = i dK \frac{1 + \rho e^{i\alpha - Kz}}{1 - \rho e^{i\alpha - Kz}},
$$

$$
w = \rho e^{i\alpha} \text{ with } \rho > 0 \text{ and } \alpha \text{ arbitrary real. For}
$$

$$
u(x, 0) = \frac{-2dK\rho \sin \alpha e^{-Kx}}{1 + \rho^2 e^{-2Kx} - 2\rho e^{-Kx} \cos \alpha}
$$

$$
u(x, \delta) = \frac{-2dK\rho \sin (\alpha - K\delta) e^{-Kx}}{1 + \rho^2 e^{-2Kx} - 2\rho e^{-Kx} \cos \alpha}.
$$
force (2) implies that $\alpha = K\delta$ or $\alpha = K\delta + \pi$. Further
theirity of *w* in *H* and its continuity in \overline{H} . This

and

$$
u(x,\delta) = \frac{-2dK\varrho\sin{(\alpha-K\delta)}}{1+\varrho^2e^{-2Kx}-2\varrho\,e^{-Kx}\cos{\alpha}}.
$$

Therefore (2) implies that $\alpha = K\delta$ or $\alpha = K\delta + \pi$. Further it remains to secure the analyticity of *w* in Π and its continuity in $\overline{\Pi}$. This demands that $1 - \varrho e^{i\alpha - Kz} \neq 0$ in $\overline{\Pi}$ which is fulfilled in case $\alpha = K\delta + \pi$ only if additionally $0 < K\delta < \pi$. That is, the only solutions of (1) with the supposed behaviour at infinity are put $C = \rho e^{4\alpha}$ with $\rho > 0$ and α arbitrary real. From (5) there follows
 $u(x, 0) = \frac{-2dK\rho \sin \alpha e^{-Kx}}{1 + \rho^2 e^{-2Kx} - 2\rho e^{-Kx} \cos \alpha}$
 $u(x, \delta) = \frac{-2dK\rho \sin (\alpha - K\delta) e^{-Kx}}{1 + \rho^2 e^{-2Kx} - 2\rho e^{-Kx} \cos \alpha}$.
 $v(2)$ implies that $\alpha =$ and
 $u(x, \delta) = \frac{-2dK\varrho \sin{(\alpha - K\delta)}}{1 + \varrho^2 e^{-2Kx} - 2\varrho e^{-Kx} \cos{\alpha}}.$

Therefore (2) implies that $\alpha = K\delta$ or $\alpha = K\delta + \pi$. Further it remains to

analyticity of w in \varPi and its continuity in \varPi . This demands that $1 - \varrho \$ *dishiffilled in case* $\alpha = K\delta + \pi$ only if additionally $0 < K\delta < \pi$. That $\alpha = \theta e^{K\delta} + \pi$ only if additionally $0 < K\delta < \pi$. That is fulfilled in case $\alpha = K\delta + \pi$ only if additionally $0 < K\delta < \pi$. That is fulfilled in case

$$
u(x) = \frac{2dK_{\varrho}\sin K\delta e^{-Kx}}{1+e^{2}e^{-2Kx}+2\varrho e^{-Kx}\cos K\delta}
$$

or putting $\rho = e^{K\beta}$, β arbitrary real,

which is fulfilled in case
$$
\alpha = K\delta + \pi
$$
 only if additionally $0 < K\delta < \pi$. That
only solutions of (1) with the supposed behaviour at infinity are

$$
u(x) = \frac{2dK\rho \sin K\delta e^{-Kx}}{1 + \rho^2 e^{-2Kx} + 2\rho e^{-Kx} \cos K\delta}
$$
ting $\rho = e^{K\beta}$, β arbitrary real.

$$
u(x) = dK \frac{\sin K\delta}{\cos K\delta + \cosh K(x - \beta)}, \quad 0 < K < \pi/\delta.
$$
 (6)

. **V •** These are (with $\beta = 0$) the solutions of J. SATSUMA and M. MIMURA [5] (see also [3, 4]).

2. Cauchy kernel. In the limiting case $\delta \rightarrow +\infty$ of (1) we have the equation

$$
u_{xx} - (Su \cdot u)_x = 0, \qquad d > 0
$$

(5)

where

(Su)
$$
(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\xi)}{\xi - x} d\xi
$$
.
the area looking for sufficient

Again ,we are looking for sufficiently smooth solutions *u* of (7) having vanishing limits $u(\pm \infty) = 0$ and $u_x(\pm \infty) = 0$ as $x \to \pm \infty$. The corresponding holomorphic function $w(z) = u + iv$ in the upper half-plane $\Pi: -\infty < x < \infty$, $y > 0$ with the boundary values $u(x, 0) = u(x)$ should be bounded at infinity suc function $w(z) = u + iv$ in the upper half-plane $\Pi: -\infty < x < \infty, y > 0$ with the boundary values $u(x, 0) = u(x)$ should be bounded at infinity, such that $v = 0$ at Steady State Solutions ...

where
 $(Su)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\xi)}{\xi - x} d\xi$.

Again we are looking for sufficiently smooth solutions u of (7) having vanis

limits $u(\pm \infty) = 0$ and $u_x(\pm \infty) = 0$ as $x \to \pm \infty$. The corre Steady State Solutions ... 167
 $(Su)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\xi)}{\xi - x} d\xi$

e are looking for sufficiently smooth solutions u of (7) having vanishing
 $\pm \infty$) = 0 and $u_x(\pm \infty) = 0$ as $x \to \pm \infty$. The corresponding holom Again we are looking for sufficiently smooth solutions u of (7) having vanishilm
limits $u(\pm \infty) = 0$ and $u_x(\pm \infty) = 0$ as $x \to \pm \infty$. The corresponding holomorp
function $w(z) = u + i v$ in the upper half-plane $\Pi: -\infty < x < \infty$

Then
\n
$$
w(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{u(\xi)}{\xi - z} d\xi
$$
\nin Π \nthe boundary values $v(x, 0) = -(\beta u)(x)$. Therefore integrating equation (7),
\nin the boundary condition Re [$dw' - (i/2) w^2$] = 0 on $y = 0$ for the function
\nace this function again obeys the differential equation (4).
\nIn be seen from above now for $K_0 \neq 0$ no solutions to (4) of the required form
\nherefore putting $K_0 = 0$ we get the differential equation
\n
$$
\frac{w'}{w^2} = \frac{i}{2d}
$$
\n
$$
w(z) = \frac{2di}{z - C}, \qquad C \text{ a complex number.}
$$
\n
$$
C = \alpha - i\beta \text{ with } \beta > 0 \text{ and } \alpha \text{ arbitrary real, the function (9) is holomorphic\nd (Hölder) continuous in \overline{H} with $w(\infty) = 0$. That is, we have the solutions
$$

with the boundary values $v(x, 0) = -(Su)(x)$. Therefore integrating equation (7), we obtain the boundary condition Re $\left[\frac{dw'}{(-1)^2}\right]$ w^2 = 0 on $y = 0$ for the function (8). Hence this function again obeys the differential equation (4). boundary values $u(x, 0) = u(x)$ should be bounded at infinity such that $v = 0$ a
infinity. Then
 $w(z) = \frac{1}{\pi i} \int_{\zeta}^{\infty} \frac{u(\xi)}{\xi - z} d\xi$ in Π
with the boundary values $v(x, 0) = -(\beta u)(x)$. Therefore integrating equation (7) **infinity.** Then
 $w(z) = \frac{1}{\pi i}$

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we obtain the boundary

(8). Hence this function

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exist. Therefore putt
 $\frac{w'}{w^2} = \frac{1}{2d}$

with the general solu
 $w(z) = \frac{2di}{z-1}$

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As can be seen from above now for $K_0 \neq 0$ no solutions to (4) of the required form exist. Therefore putting $K_0 = 0$ we get the differential equation ω). Hence this
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$$
\frac{w'}{w^2}=\frac{1}{2d}
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with the general solution

$$
\frac{w'}{w^2} = \frac{1}{2d}
$$

with the general solution

$$
w(z) = \frac{2di}{z - C}, \qquad C \text{ a complex number.}
$$

Putting $C = \alpha - i\beta$ with $\beta > 0$ and α arbitrary real, the function (9) is holomorphic

in *II* and (Hölder) continuous in \overline{I} with $w(\infty) = 0$. That is, we have the solutions $u(x) = \frac{2d\beta}{(x - \alpha)^2 + \beta^2}$, $\beta > 0$. (10)

These are (with $\alpha = 0$) again the solutions of J. Satsuma and M. MIMURA [5] (and [3,4]). $(x - \alpha)^2 + \beta^2$
 re (with $\alpha = 0$) again the solution
 rt kernel. In the case of periodic so
 $du_{ss} - (Hu \cdot u)_s = 0, \quad d > 0,$
 $u(s)$ occurs, where

3. Hilbert kernel. In the case of periodic Solutions of (1) (with period 2π) the equation

$$
du_{ss} - (Hu \cdot u)_s = 0, \qquad d > 0, \tag{11}
$$

for $u = u(s)$ occurs, where

re (with
$$
\alpha = 0
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) again the solutions of
rt **kernel.** In the case of periodic Soluti:
 $du_{ss} - (Hu \cdot u)_s = 0$, $d > 0$,
 $u(s)$ occurs, where
 (Hu) $(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\sigma) \cot \frac{\sigma - s}{2} d\sigma$.

For sufficiently smooth solutions u of (11) we introduce the Schwarz integral W of u (cf. [1: Chap. I, § 6]). $W(z) = U + iV$ is a holomorphic function in the unit disk $G: |z| < 1$ with the boundary values $U(e^{is}) = u(s)$ of U and $V(e^{is}) = -(Hu)(s)$ of for $u = u(s)$ occurs, where
 $(Hu)(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\sigma) \cot \frac{\sigma - s}{2} d\sigma$.

For sufficiently smooth solutions u of (11) we introduce the Schwarz integral $W \cdot 0$ *u* (ef. [1 : Chap. I, § 6]). $W(z) = U + iV$ is a holomorphic (*Hu*) (*s*) = $\frac{1}{2\pi} \int u(\sigma) \cot \frac{\sigma - s}{2} d\sigma$.

For sufficiently smooth solutions *u* of (11) we introduce the Schwarz integral *W* of *u* (cf. [1 : Chap. I, § 6]). *W*(*z*) = *U* + i*V* is a holomorphic function in the condition Re $\left[idzW' - (i/2) W^2 \right] = 0$ on *I* for *W*. Therefore *W* satisfies the differential equation
 $dzW' = \frac{1}{2} W^2 + K_0$ in *G* (12) **3. Hilbert kernel.** In the
 $du_{ss} - (Hu \cdot u)$

for $u = u(s)$ occurs, whe
 $(Hu)(s) = \frac{1}{2\pi}$

For sufficiently smooth

(cf. [1: Chap. I, § 6]). I
 $G: |z| < 1$ with the bou

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satisfies the condition V

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 $\frac{\sigma - s}{2} d\sigma$.
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integrating equa
 0 on Γ for \overline{W} .

in G
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For sufficiently smooth solutions *u* of (11) we introduce the Schwarz integral *W* of (cf. [1 : Chap. I, § 6]). *W*(*z*) = *U* + i*V* is a holomorphic function in

$$
dzW'=\frac{1}{2}W^2+K_0 \qquad \text{in } G
$$

As can be easily seen for $K_0 = 0$ no non-trivial regular solutions to (12) in G exist. Therefore putting $K_0 = -(1/2) K^2$, $K > 0$, we get the differential equation s can be easily s

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be easily scen for $K_0 = 0$
putting $K_0 = - (1/2) K$
 $\frac{W'}{W^2 - K^2} = \frac{1}{2dz}$ 168 L. v. WOLFERSDORF

As can be easily seen for $K_0 = 0$ no non-t

Therefore putting $K_0 = -(1/2) K^2$, $K > 0$,
 $\frac{W'}{W^2 - K^2} = \frac{1}{2dz}$

with the general solution
 $W(z) = K \frac{1 + Cz^{K/d}}{1 - Cz^{K/d}}$, C a con

$$
\frac{W'}{W^2 - K^2} = \frac{1}{2dz}
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L. v. WOLFERSDORF
\nbe easily seen for
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K_0 = 0
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 no non-trivial regular sol-
\ne putting $K_0 = -(1/2) K^2$, $K > 0$, we get the differ
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\frac{W'}{W^2 - K^2} = \frac{1}{2dz}
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\ngeneral solution
\n
$$
W(z) = K \frac{1 + Cz^{K/d}}{1 - Cz^{K/d}}, \qquad C \text{ a complex number.}
$$
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$$
\text{ation represents a holomorphic function in } G \text{ with (H-\nres on } \Gamma \text{ if } K = nd, n \in \mathbb{N} \text{ i.e.}
$$

This function represents a holomorphic function in G with (Hölder) continuous boundary values on Γ if $K = nd, n \in \mathbb{N}$, i.e.

As can be easily seen for
$$
K_0 = 0
$$
 no non-trivial regular solutions to (12) in G exist.
\nTherefore putting $K_0 = -(1/2) K^2$, $K > 0$, we get the differential equation
\n
$$
\frac{W'}{W^2 - K^2} = \frac{1}{2dz}
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h \text{ the general solution}
$$
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$$
W(z) = K \frac{1 + Cz^{K/d}}{1 - Cz^{K/d}}, \qquad G \text{ a complex number.}
$$
\nis function represents a holomorphic function in G with (Hölder) continuous bound-
\nvalues on Γ if $K = nd$, $n \in \mathbb{N}$, i.e.
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W(z) = nd \frac{1 + Cz^n}{1 - Cz^n}
$$
\n
$$
h |C| < 1 \text{ (cp. [2]). Putting } C = -\varrho e^{-ins} \text{ with } 0 < \varrho < 1, \text{ a arbitrary real, from}
$$
\nwe obtain
\n
$$
u(s) = \text{Re } W(e^{is}) = nd \frac{1 - \varrho^2}{1 + \varrho^2 + 2\varrho \cos n(s - \alpha)}
$$
\n
$$
u(s) = nd \frac{\sinh \varphi}{\cos n(s - \alpha) + \cosh \varphi}, \qquad \varphi > 0,
$$
\nare the parameter φ is introduced by
\n
$$
1 + \varrho^2
$$
\n
$$
1 - \varrho^2
$$

with $|C|$ < 1 (cp. [2]). Putting $C = -\rho e^{-in\pi}$ with $0 < \rho < 1$, a arbitrary real, from (13) we obtain

$$
\begin{aligned}\n\text{as on } & \varGamma \text{ if } K = nd, \, n \in \mathbb{N}, \text{ i.e.} \\
& W(z) = nd \, \frac{1 + Cz^n}{1 - Cz^n} \\
&< 1 \text{ (cp. [2]).} \text{ Putting } C = -\varrho \, \text{e}^{-\text{i} n z} \text{ with } 0 < \varrho \\
\text{btain} \\
& u(s) = \text{Re } W(\text{e}^{\text{i} s}) = nd \, \frac{1 - \varrho^2}{1 + \varrho^2 + 2\varrho \cos n(s - \alpha)} \\
& u(s) = nd \, \frac{\sinh \varphi}{\cos n(s - \alpha) + \cosh \varphi}, \qquad \varphi > 0, \\
\text{e parameter } \varphi \text{ is introduced by}\n\end{aligned}
$$

or

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$$
u(s) = nd \frac{\sinh \varphi}{\cos n(s - \alpha) + \cosh \varphi}, \qquad \varphi > 0,
$$

where the parameter φ is introduced by

es on
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\Gamma
$$
 if $K = nd, n \in \mathbb{N}$, i.e.
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$$
W(z) = nd \frac{1 + Cz^{n}}{1 - Cz^{n}}
$$
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 (cp. [2]). Putting $C = -\rho e^{-in\pi}$ with $0 < \rho$ that
\n
$$
u(s) = \text{Re } W(e^{is}) = nd \frac{1 - \rho^{2}}{1 + \rho^{2} + 2\rho \cos n(s - \alpha)}
$$
\n
$$
u(s) = nd \frac{\sinh \varphi}{\cos n(s - \alpha) + \cosh \varphi}, \qquad \varphi > 0,
$$
\ne parameter φ is introduced by
\n
$$
\cosh \varphi = \frac{1 + \rho^{2}}{2\rho} \qquad \text{or} \qquad \sinh \varphi = \frac{1 - \rho^{2}}{2\rho}.
$$
\ne (with $\alpha = 0$) again the solutions of J. Sarsum.
\n $L = 2\pi$.
\n
$$
u(s) u_{s} = (Hu \cdot u)_{s} = 0
$$
\n
$$
\text{special (positive) } 2\pi \text{-periodic functions } d. \text{ If the}
$$
\n
$$
u_{s} = \lambda(s) Hu
$$

These are (with $\alpha = 0$) again the solutions of J. SATSUMA and M. MEMURA [5] for the period $2L = 2\pi$. $u(s) = nd \frac{1}{\cos n(s)}$
where the parameter φ is
 $\cosh \varphi = \frac{1 + \varrho^2}{2\varrho}$
These are (with $\alpha = 0$) ag
period $2L = 2\pi$. $u(s) = nd \frac{\sinh \varphi}{\cos n(s - \alpha) + \cosh \varphi}, \qquad \varphi > 0,$

where the parameter φ is introduced by
 $\cosh \varphi = \frac{1 + \varrho^2}{2\varrho} \qquad \text{or} \qquad \sinh \varphi = \frac{1 - \varrho^2}{2\varrho}.$

These are (with $\alpha = 0$) again the solutions of J. SATSUMA and M. I

per

$$
(d(s) u_s)_s - (Hu \cdot u)_s = 0 \qquad \qquad \bullet
$$
 (15)

for some special (positive) 2π -periodic functions *d*. If the eigenvalue problem

$$
u_s = \lambda(s) H u
$$

where the parameter φ is introduced by
 • $\cos n(s - \alpha) + \cosh \varphi'$
 • $\cos n(s - \alpha) + \cosh \varphi'$
 $\cos h \varphi = \frac{1 + e^2}{2g}$ or $\sinh \varphi = \frac{1 - e^2}{2g}$.
 These are (with $\alpha = 0$ **) again the solutions of J. SATSUMA and M. MIMURA [5] fo** for a positive sufficiently smooth function λ has a 2π -periodic (positive) solution u_0 , then obviously u_0 is a steady state solution of the equation (15) with the diffusiou coefficient $d = u_0/\lambda$. The problem (16) is equivalent to the *Steklov problem* These are (with $\alpha = 0$) again the solutions of J. SATSUMA and M. MIMURA [5] for the

period $2L = 2\pi$.

4. Variable diffusion coefficient. Finally, we consider the equation
 $\{d(s) u_s\}_s - \{Hu \cdot u_s\}_s = 0$ (15)

for some spec

$$
\frac{\partial V}{\partial r} - \lambda(s) V = 0 \quad \text{on } \Gamma, \quad V(0) = 0, \quad (17)
$$

for the harmonic function *V* in the unit disk $G: r < 1$, which is the conjugate function to the Poisson integral *U* of *u*. For instance, for $\lambda = k \in \mathbb{N}$ we have the eigensolutions r^k cos *ks*, r^k sin *ks* of (17) and therefore the solution $u_0 = a_0 + a_1 \cos ks$ $+a_2$ sin *ks* with arbitrary constants a_0 , a_1 , a_2 of equation (15) for $d = d_0 + d_1$ cos *ks* $+ d_2 \sin ks$, where $a_i = kd_i$, $j = 0, 1, 2$. In the particular case $d = d_0 + d_1 \cos s$ $+$ d_{2} sin s with . $u_s = A(s) Hu$ (10)
positive sufficiently smooth function λ has a 2π -periodic (positive) solution u_0 ,
bobviously u_0 is a steady state solution of the equation (15) with the diffusion
cient $d = u_0/\lambda$. The problem (1

$$
d_0 > \sqrt{d_1^2 + d_2^2} > 0
$$

 $\frac{1}{r}$

(insuring $d > 0$) equation (15) as above leads to the complex differential equation Stead
 A(z) W' = *W²* - *K*²
 A(z) W' = *W²* - *K²*
 A(z) W' = *W²* - *K²*
 A
 A(x) W in *G*, where *K* is a complement of *A₁ A I A₂ A I I J*_{*d₂ A I I J*_{*d₂*}} ren
.

$$
A(z) W' = W^2 - K^2
$$

for the holomorphic function W in G , where K is a complex constant and

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\n(insuring
$$
d > 0
$$
) equation (15) as above leads to the complex differential equation
\n $A(z) W' = W^2 - K^2$
\nfor the holomorphic function W in G, where K is a complex constant and
\n
$$
A(z) = 2z \left[d_0 + \frac{d_1}{2} \left(z + \frac{1}{z} \right) + \frac{d_2}{2i} \left(z - \frac{1}{z} \right) \right]
$$
\nis a holomorphic function in G. We write
\n
$$
A(z) = \overline{D}z^2 + 2d_0z + D = \overline{D}(z - z_1) (z - z_2)
$$
\nwith $D = d_1 + id_2$ and

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\n
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$$
\begin{aligned}\n\text{norphic function in } G. \text{ We write} \\
A(z) &= \overline{D}z^2 + 2d_0z + D = \overline{D}(z - z_1)(z - z_2) \\
&= d_1 + \mathrm{id}_2 \text{ and} \\
z_1 &= \frac{1}{\overline{D}} \left[-d_0 + \Delta \right] \in G, \qquad z_2 = \frac{1}{\overline{D}} \left[-d_0 - \Delta \right] \notin \overline{G},\n\end{aligned}
$$

 $\frac{|D|^2}{|D|^2} > 0$ by (18). That is, W $\frac{1}{2}$

Steady State Solutions ... 169
\n(insuring
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d > 0
$$
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\n $z_1 = \frac{1}{\overline{D}} \left[-d_0 + \Delta \right] \in G$, $z_2 = \frac{1}{\overline{D}} \left[-d_0 - \Delta \right] \notin \overline{G}$,
\nwhere $A = \sqrt{d_0^2 - |\overline{D}|^2} > 0$ by (18). That is, W satisfies the differential equation
\n $\frac{W'}{W^2 - K^2} = \frac{1}{\overline{D}(z - z_1) (z - z_2)}$ in G.
\nAs can be easily seen for $K = 0$ no non-trivial regular solutions to (19) in G exist.
\nFor $K = 0$ we have the general solution
\n $W(z) = K \frac{1 + C\xi^{K/d}}{1 - C\xi^{K/d}}$, $\zeta = \frac{z - z_1}{z - z_2}$, C a complex number.
\nThis function represents a holomorphic function in G with (Hölder) continuous
\nboundary values on Γ if $K = n\Delta$, $n \in N$, i.e. putting $C = C_0^n$.
\n $W(z) = n\Delta \frac{(z - z_2)^n + C_0^n(z - z_1)^n}{(z - z_2)^n - C_0^n(z - z_1)^n}$ (20)
\nwith the restriction on C_0 that $(z - z_2)^n + C_0^n(z - z_1)^n$

where
 \cdot
 \cdot can be easily seen for $K = 0$ no non-trivial regular solutions to (19) in G exist.
 $W(z) = K \frac{1 + C\zeta^{K/4}}{1 - C\zeta^{K/4}}$, $\zeta = \frac{z - z_1}{z - z_2}$, C a complex number. As can be easily s

For $K = 0$ we have
 $W(z) = K$

$$
W(z) = K \frac{1 + C\zeta^{K/d}}{1 - C\zeta^{K/d}}, \qquad \zeta = \frac{z - z_1}{z - z_2}, \quad C \text{ a complex number.}
$$

This function represents a holomorphic function in G with (Hölder) continuous this function represents a notomorphic function in G with
boundary values on Γ if $K = n\lambda$, $n \in \mathbb{N}$, i.e. putting $C = C_0$ ⁿ. r if K
 $(z - z_2)$

As can be easily seen for
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K = 0
$$
 no non-trivial regular solutions to (19) in G exist.
\nFor $K + 0$ we have the *q*eneral solution
\n
$$
W(z) = K \frac{1 + C\zeta^{K/d}}{1 - C\zeta^{K/d}}, \qquad \zeta = \frac{z - z_1}{z - z_2}, \quad C \text{ a complex number.}
$$
\nThis function represents a holomorphic function in G with (Hölder) continuous
\nboundary values on Γ if $K = n\Delta, \, \hat{n} \in \mathbb{N}$, i.e. putting $C = C_0^n$.
\n
$$
W(z) = n\Delta \frac{(z - z_2)^n + C_0^n(z - z_1)^n}{(z - z_2)^n - C_0^n(z - z_1)^n}
$$
\nwith the restriction on C_0 that $(z - z_2)^n + C_0^n(z - z_1)^n$ for $|z| \le 1$, i.e.
\n
$$
\left| z_2 - C_0 e^{\frac{i2k\pi}{n}} \right| > |1 - C_0 e^{\frac{i2k\pi}{n}} z_1 |, \qquad 0 \le k \le n - 1.
$$
\n(21)
\nSince $|z_2| > 1$, the condition (21) is fulfilled for sufficiently small $|C_0|$. From (20) the
\nsteady state solutions u of (15) are given by
\n
$$
u(s) = n\Delta \frac{|e^{is} - z_2|^{2n} - |C_0|^{2n}|e^{is} - z_1|^{2n}}{|(e^{is} - z_2)^n - C_0^n(e^{is} - z_1)^n|^2}, \qquad n \in \mathbb{N}.
$$
\n(22)
\nEspecially for $n = 1$ with $C_0 = 1$ we retrieve the above solution $u_0 = d$.

with the restriction on
$$
C_0
$$
 that $(z - z_2)^n - C_0^n(z - z_1)^n$
\nwith the restriction on C_0 that $(z - z_2)^n + C_0^n(z - z_1)^n$ for $|z| \le 1$, i.e.
\n
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$$
u(s) = n\Delta \frac{|e^{is} - z_2|^{2n} - |C_0|^{2n} |e^{is} - z_1|^{2n}}{|(e^{is} - z_2)^n - C_0^n (e^{is} - z_1)^n|^2}, \qquad n \in \mathbb{N}.
$$
 (22)
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