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A Differential Equation for the Positive Zeros of the Function $\alpha J_\nu(z) + \gamma z J_\nu'(z)$

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Es wird eine Differentialgleichung für jede positive Nullstelle $\varrho(\nu)$ der Funktion $\alpha J_\nu(z) + \gamma z J_\nu'(z)$ angegeben, wobei J_ν die Besselfunktion erster Art der Ordnung $\nu > -1$, J_ν' die Ableitung von J_ν und α, γ reelle Zahlen sind. Es wird gezeigt:

(i) Die Funktion $\varrho(\nu)/(1+\nu)$ ist fallend mit $\nu > -1$ im Falle $\alpha \geqq 1$, und die Funktion $\varrho(\nu)/(\alpha+\nu)$ ist fallend mit $\nu > -\alpha$ im Falle $\alpha < 1$.

(ii) Die Nullstellen der Funktion $\alpha J_\nu(z) + z J_\nu'(z)$ wachsen mit $\nu > -1$ im Falle $\alpha \geqq 1$ und mit $\nu > -\alpha$ im Falle $\alpha < 1$. Das erste Resultat führt zu einer Anzahl oberer und unterer Schranken für die Nullstellen der Funktion $\alpha J_\nu(z) + z J_\nu'(z)$, die frühere bekannte Schranken vervollständigen und verbessern. Das zweite Resultat erweitert ein bekanntes Resultat.

Устанавливается дифференциальное уравнение для произвольного нуля $\varrho(\nu)$ функции $\alpha J_\nu(z) + \gamma z J_\nu'(z)$, где J_ν — функция Бесселя первого рода порядка $\nu > -1$, J_ν' — производная J_ν, а α, γ вещественные числа. Доказано:

(i) функция $\varrho(\nu)/(1+\nu)$ убывает при $\nu > -1$ в случае $\alpha \geqq 1$, а функция $\varrho(\nu)/(\alpha+\nu)$ убывает при $\nu > -\alpha$ в случае $\alpha < 1$.

(ii) Нули функции $\alpha J_\nu(z) + z J_\nu'(z)$ возрастают при $\nu > -1$ в случае $\alpha \geqq 1$ и при $\nu > -\alpha$ в случае $\alpha < 1$. Первый результат ведёт к набору вверхних и нижних границ для нулей функции $\alpha J_\nu(z) + z J_\nu'(z)$, что пополняет и улучшает более ранние известные границы. Второй результат улучшает один хорошо известный результат.

A differential equation for any positive zero $\varrho(\nu)$ of the function $\alpha J_\nu(z) + \gamma z J_\nu'(z)$ is found, where J_ν is the Bessel function of the first kind of order $\nu > -1$, J_ν' is the derivative of J_ν and α, γ are real numbers. It is proven that:

(i) The function $\varrho(\nu)/(1+\nu)$ decreases with $\nu > -1$ in the case $\alpha \geqq 1$, and the function $\varrho(\nu)/(\alpha+\nu)$ decreases with $\nu > -\alpha$ in the case $\alpha < 1$.

(ii) The zeros of the function $\alpha J_\nu(z) + z J_\nu'(z)$ increase with $\nu > -1$ in the case $\alpha \geqq 1$ and with $\nu > -\alpha$ in the case $\alpha < 1$. The first result leads to a number of lower and upper bounds for the zeros of the function $\alpha J_\nu(z) + z J_\nu'(z)$ which complete and improve previously known bounds. The second result improves a well-known result.

1. Introduction

The study of the behaviour of the zeros of the function $\alpha J_\nu(z) + \gamma z J_\nu'(z)$, where J_ν is the Bessel function of the first kind of order $\nu > -1$ and J_ν' the derivative of J_ν, dates back at least in 1884, as we can see from a work of H. LAMB concerning the induction of electric currents in a cylinder placed across the lines of magnetic force [6]. Special cases were also studied by Schwerd in 1835 and Rayleigh in 1873 (see the footnote in [11: p. 477]). Since then results concerning the interlacing of zeros, monotonicity of growth and upper or lower bounds for the zeros have been established by many authors. The reason for which the problem of the zeros of $\alpha J_\nu(z) + \gamma z J_\nu'(z)$ is very interesting is that these zeros arise in the solution of the wave equation in a sphere with mixed boundary conditions [1] and in other equations of physical interest (see the references in [10]). In this paper we follow a new approach presented in [2] and find a differential equation for any positive zero $\varrho(\nu)$ of $\alpha J_\nu(z) + \gamma z J_\nu'(z)$ with $\nu > -1$. This differential equation for $\gamma = 0$ is reduced to the differential equation for the k-positive zeros $j_{\nu,k}$ of J_ν found in [3]. For $\gamma = 1$ the differential equation has the form

$$\frac{d\varrho(\nu)}{d\nu} = \varrho(\nu) \frac{(L_\nu h, h) + 1}{(h, h) + \alpha + \nu}, \qquad \nu > \begin{cases} -1 & \text{for} \quad \alpha \geqq 1, \\ -\alpha & \text{for} \quad \alpha < 1, \end{cases} \tag{1.1}$$

where h is an element of an abstract Hilbert space $(H, (\cdot, \cdot))$ with orthonormal basis $e_1, e_2, \ldots$, and L_ν is the diagonal operator defined by $L_\nu e_n = e_n/(n + \nu)$, $n \neq -\nu$. For $\alpha = 0$ we find from (1.1) the differential equation for the k-positive zeros $j'_{\nu,k}$ of the derivative J_ν'

$$\frac{dj'_{\nu,k}}{d\nu} = j'_{\nu,k} \frac{(L_\nu h, h) + 1}{(h, h) + \nu}, \qquad \nu > 0. \tag{1.2}$$

Although we know nothing more about the element h we can easily prove the differential inequalities

$$\frac{d\varrho(\nu)}{d\nu} < \frac{\varrho(\nu)}{1 + \nu}, \qquad \nu > -1, \quad \text{for} \quad \alpha \geqq 1,$$

$$\frac{d\varrho(\nu)}{d\nu} < \frac{\varrho(\nu)}{\alpha + \nu}, \qquad \nu > -\alpha, \quad \text{for} \quad \alpha < 1.$$

The first inequality means that the function $\varrho(\nu)/(1 + \nu)$ decreases with $\nu > -1$ for $\alpha \geqq 1$ and the second one that $\varrho(\nu)/(\alpha + \nu)$ decreases with $\nu > -\alpha$ for $\alpha < 1$. These results lead to a number of lower and upper bounds for the k-positive zeros $\varrho_{\nu,k}$ ($k = 1, 2, \ldots$) of the function $\alpha J_\nu(z) + z J_\nu'(z)$, which complete and improve previously known bounds. For $\alpha = 0$ we obtain the well-known result that the function $j'_{\nu,k}/\nu$ decreases as $\nu > 0$ increases ([9: Theorem 5] and [7: Theorem 4.1]). Also from (1.2) we obtain the well-known result that $j_{\nu,k}$ increases as $\nu > 0$ increases [7, 11]. Finally, from (1.1) it follows immediately that the zeros $\varrho_{\nu,k}$ of the function $\alpha J_\nu(z) + z J_\nu'(z)$ increase for $\nu > -1$ in the case $\alpha \geqq 1$ and for $\nu > -\alpha$ in the case $\alpha < 1$. This improves the well-known result that each positive zero of $\alpha J_\nu(z) + z J_\nu'(z)$ increases with $\nu > 0$ [10: Lemma 4.1].

2. The differential equation

In this section we prove the following

Theorem 2.1: *Let $\nu > -1$ in the case $\alpha \geqq \gamma$ and $\nu > \max\{-\alpha/\gamma, -1\}$ in the case $\alpha < \gamma$. Then in each of the open intervals $(0, j_{\nu,1})$ and $(j_{\nu,k}, j_{\nu,k+1})$ ($k = 1, 2, \ldots$) there exists a unique zero $\varrho(\nu)$ of the function $\alpha J_\nu(z) + \gamma z J_\nu'(z)$. It satisfies the differential equation*

$$\frac{d\varrho(\nu)}{d\nu} = \varrho(\nu) \frac{2(L_\nu v, v) + \gamma^2}{2(v, v) + \alpha\gamma + \gamma^2 \nu}. \tag{2.1}$$

Here v is an element in an abstract Hilbert space $(H, (\cdot, \cdot))$ with the orthonormal basis $e_1, e_2, \ldots$, and L_ν is the diagonal operator defined by $L_\nu e_n = e_n/(n + \nu)$.

For the proof of Theorem 2.1 we use [2: Theorem 2.1], reducing the problem of the zeros of the more general function $(\alpha + \delta z) J_\nu(z) + (\beta + \gamma z) J_\nu'(z)$ to a special eigenvalue problem in the abstract Hilbert space $(H, (\cdot, \cdot))$. In fact, in the case $\beta = \delta = 0$, α, β and ν real numbers it is known that $\varrho(\nu)$ is a real zero of $\alpha J_\nu(z) + \gamma z J_\nu'(z)$ if and only if there exists an element $u(\nu) \neq 0$ in $(H, (\cdot, \cdot))$ such that

$$(C_0 + \nu)\, u(\nu) - \frac{\varrho(\nu)}{2}\, T_0 u(\nu) = -\frac{\gamma}{2}\, \varrho^2(\nu)\, e_1, \qquad (2.2)$$

$$\big(u(\nu), e_1\big) = -(\alpha + \gamma\nu),$$

where C_0 is the diagonal operator defined by $C_0 e_n = n e_n$, $T_0 = V + V^*$, V is the shift operator with respect to $\{e_n\}$ and V^* its adjoint. In the case $\nu > -1$ the problem (2.2) is equivalent to the following one:

$$S_\nu g(\nu) - \frac{2}{\varrho(\nu)}\, g(\nu) = \frac{\gamma \varrho(\nu)}{\sqrt{1 + \nu}}\, e_1, \qquad (2.3)$$

$$\big(g(\nu), e_1\big) = -\sqrt{1 + \nu}\, (\alpha + \gamma\nu), \qquad (2.4)$$

where

$$u(\nu) = L_\nu^{1/2} g(\nu) \qquad (2.5)$$

and S_ν is the compact and self-adjoint operator $S_\nu = L_\nu^{1/2} T_0 L_\nu^{1/2}$. The conditions stated in Theorem 2.1 exclude the case $\alpha + \gamma\nu_0 = 0$, for some real $\nu_0 > -1$, and essentially exclude the case that $\varrho(\nu_0)$ is a zero of J_{ν_0+1}. Also these conditions, in conformance with the results of [2], imply that there exists a unique zero of $\alpha J_\nu(z) + \gamma z J_\nu'(z)$ in each of the intervals $(0, j_{\nu,1})$ and $(j_{\nu,k}, j_{\nu,k+1})$ $(k = 1, 2, \ldots)$.

Moreover, for the proof of Theorem 2.1 we need the following

Lemma 2.1: *Let $\nu > -1$ in the case $\alpha \geqq \gamma$ and $\nu > \max\{-\alpha/\gamma, -1\}$ in the case $\alpha < \gamma$. Then the element $u(\nu) = \lambda(\nu)\, L_\nu^{1/2} A_\nu e_n$, where $\lambda(\nu) = -\gamma\varrho(\nu)/\sqrt{1 + \nu}$ and $A_\nu = \big(I - 2^{-1}\varrho(\nu)\, S_\nu\big)^{-1}$, as a function from $(-1, \infty)$ or $(\max\{-\alpha/\gamma, -1\}, \infty)$, respectively, into the abstract Hilbert space $(H, (\cdot, \cdot))$ is strongly continuous, i.e.,*

$$\mu \to \nu \quad \textit{implies} \quad \|u(\mu) - u(\nu)\| \to 0. \qquad (2.6)$$

Proof: The continuity of $u(\nu)$ follows by (2.5) from the continuity of $g(\nu)$ and this follows also if we prove that the operator A_ν is uniformly bounded on every compact subinterval of the intervals $(-1, \infty)$ or $(\max\{-\alpha/\gamma, -1\}, \infty)$, respectively. Let $\varrho(\nu)$ be a zero of $\alpha J_\nu(z) + \gamma z J_\nu(z)$ in $(0, j_{\nu,1})$. Then $\|2^{-1}\varrho(\nu)\, S_\nu\| = \varrho(\nu)/j_{\nu,1} < 1$ because $\|S_\nu\| = 2/j_{\nu,1}$ [4, 5]. So we find

$$\|A_\nu\| \leqq 1 + \frac{\varrho(\nu)}{j_{\nu,1}} + \frac{\varrho^2(\nu)}{j_{\nu,1}^2} + \cdots = \frac{j_{\nu,1}}{j_{\nu,1} - \varrho(\nu)}. \qquad (2.7)$$

$\varrho(\nu)$ and $j_{\nu,1}$ are continuous functions of ν as zeros of an analytic function with continuous coefficients [8]. To show uniform boundedness of A_ν on every compact subinterval I_0 of $(-1, \infty)$ in case $\alpha \geqq \gamma$ or of $(\max\{-\alpha/\gamma, -1\}, \infty)$ in case $\alpha < \gamma$, respectively, it is sufficient to show that $\inf\{j_{\nu,1} - \varrho(\nu) : \nu \in I_0\} \neq 0$. Let $2/\varrho(\nu) = 2/j_{\nu,1} + \varepsilon(\nu)$, $j_{\nu,1} - \varrho(\nu) \to 0$ as $\nu \to \nu_0$ for some $\nu_0 \in I_0$. Then $\varepsilon(\nu) \to 0$ as $\nu \to \nu_0$ and from (2.3) we find

$$\big(S_\nu g(\nu), e_1\big) - \frac{2}{j_{\nu,1}}\, \big(g(\nu), e_1\big) = \varepsilon(\nu)\, \big(g(\nu), e_1\big) + \frac{\gamma\varrho(\nu)}{\sqrt{1 + \nu}}. \qquad (2.8)$$

From (2.4) we see that

$$\lim_{\nu \to \nu_0} \big(g(\nu), e_1\big) = -\sqrt{1 + \nu_0}\, (\alpha + \gamma\nu_0) = \big(g(\nu_0), e_1\big) \qquad (2.9)$$

and from (2.3)

$$\lim_{\nu\to\nu_0} \left(S_\nu g(\nu), e_1\right) = \left(S_{\nu_0} g(\nu_0), e_1\right). \tag{2.10}$$

Now from (2.8)—(2.10) we obtain

$$\left(S_{\nu_0} g(\nu_0), e_1\right) - \frac{2}{j_{\nu_0,1}} \left(g(\nu_0), e_1\right) = \frac{\gamma\varrho(\nu_0)}{\sqrt{1+\nu_0}}.$$

This together with the relation

$$\left(S_{\nu_0} g(\nu_0), e_1\right) - \frac{2}{\varrho(\nu_0)} \left(g(\nu_0), e_1\right) = \frac{\gamma\varrho(\nu_0)}{\sqrt{1+\nu_0}}$$

implies $\left(2/j_{\nu_0,1} - 2/\varrho(\nu_0)\right)\left(g(\nu_0), e_1\right) = 0$. But $\left(g(\nu_0), e_1\right) = -\sqrt{1+\nu_0}\,(\alpha + \gamma\nu_0) \neq 0$. So we have $j_{\nu_0,1} = \varrho(\nu_0)$. This is impossible because the functions J_ν and $\alpha J_\nu(z) + \gamma z J_\nu'(z)$, $\gamma \neq 0$, cannot have common zeros [2: Cor. 2.4]. Thus $j_{\nu,1} - \varrho(\nu) \geqq \min \{j_{\nu,1} - \varrho(\nu) : \nu \in I_0\} = \tau_1 > 0$, and from (2.7) we have that $\|A_\nu\| \leqq \tau_1 \max \{j_{\nu,1} : \nu \in I_0\}$. This proves the boundedness of A_ν. Since

$$\begin{aligned} A_\mu - A_\nu &= A_\mu \left[\left(I - \frac{1}{2}\varrho(\nu)\, S_\nu\right) - \left(I - \frac{1}{2}\varrho(\mu)\, S_\mu\right)\right] A_\nu \\ &= \frac{1}{2} A_\mu [\varrho(\mu)\, S_\mu - \varrho(\nu)\, S_\nu]\, A_\nu, \end{aligned}$$

$\|S_\mu - S_\nu\| \to 0$ and $\varrho(\mu) \to \varrho(\nu)$ as $\mu \to \nu$, it follows that $\|A_\mu - A_\nu\| \to 0$ as $\mu \to \nu$. From this, (2.6) follows easily.

In the general case, i.e., in the case $\varrho(\nu) \in (j_{\nu,k}, j_{\nu,k+1})$ $(k = 1, 2, \ldots)$, the theorem follows in a similar way from the expansion of the operator A_ν in terms of the complete orthonormal system of the compact and self-adjoint operator S_ν [2: Theorem 4.1] ∎

Proof of Theorem 2.1: The existence and uniqueness are known from [2: Theorem 4.2]. Let

$$(C_0 + \mu)\, u(\mu) - \frac{1}{2}\varrho(\mu)\, T_0 u(\mu) = -\frac{1}{2}\gamma\varrho^2(\mu)\, e_1 \tag{2.11}$$

and

$$(C_0 + \nu)\, u(\nu) - \frac{1}{2}\varrho(\nu)\, T_0 u(\nu) = -\frac{1}{2}\gamma\varrho^2(\nu)\, e_1, \tag{2.12}$$

where $\left(u(\nu), e_1\right) = -(\alpha + \gamma\nu)$ and $\left(u(\mu), e_1\right) = -(\alpha + \gamma\mu)$. Then scalar product multiplication of (2.11) on the right by $u(\mu)$ and of (2.12) on the left by $u(\nu)$, subtraction and passage to the limit $\mu \to \nu$, using Lemma 2.1 and the continuity of $\varrho(\nu)$, lead easily to

$$\left(u(\nu), u(\nu)\right) + \frac{1}{2}\gamma^2\varrho^2(\nu) = \frac{d\varrho(\nu)}{d\nu}\left[\frac{1}{2}\left(T_0 u(\nu), u(\nu)\right) + \alpha\gamma\varrho(\nu) + \gamma^2\nu\varrho(\nu)\right]$$

or

$$\frac{d\varrho(\nu)}{d\nu} = \varrho(\nu)\, \frac{2\left(u(\nu), u(\nu)\right) + \gamma^2\varrho^2(\nu)}{2\left((C_0 + \nu)\, u(\nu), u(\nu)\right) + \varrho^2(\nu)\,(\alpha\gamma + \gamma^2\nu)}.$$

By setting $u(\nu) = L_\nu^{1/2} g(\nu)$ and $g(\nu) = \varrho(\nu)\, v(\nu)$ we find equation (2.1) ∎

Remark 2.1: For $\gamma = 0$ equation (2.1) is reduced to the differential equation for the k-positive zeros $j_{\nu,k}$ of J_ν found in [3]. Also for $\alpha = 0$ and $\gamma = \sqrt{2}$ we obtain from (2.1) the differential equation for the k-positive zeros $j'_{\nu,k}$ of J_ν'

$$\frac{dj'_{\nu,k}}{d\nu} = j'_{\nu,k} \frac{L_\nu v, v) + 1}{(v, v) + \nu}, \quad \nu > 0.$$

From this it follows that the zeros $j'_{\nu,k}$ of J_ν' increase as $\nu > 0$ increases. This result was proved first by Rayleigh [11: p. 510–511] for $\nu > 1/2$, with the use of arguments concerning transverse vibrations of a membrane in the form of a sector of a circle, and later with different methods by G. N. Watson [11: p. 510] and J. T. Lewis and M. E. Muldoon [7] for $\nu > 0$.

Corollary 2.1: *Let $\alpha\gamma > 0$. Then by* (2.1) *every positive zero of $\alpha J_\nu(z) + \gamma z J_\nu'(z)$ increases with ν in $(-1, \infty)$ in the case $\alpha \geqq \gamma$ and with ν in $(\max\{-\alpha/\gamma, -1\}, \infty)$ in the case $\alpha < \gamma$.*

For $\gamma = 1$, this corollary extends the range of validity of the order ν known from [10: Lemma 4.1].

In the following we set $\gamma = 1$ and $v(\nu) = h(\nu)/\sqrt{2}$ and find from (2.1) the equation

$$\frac{d\varrho(\nu)}{d\nu} = \varrho(\nu) \frac{(L_\nu h, h) + 1}{(h, h) + \alpha + \nu}, \quad g(\nu) = \frac{\varrho(\nu)}{\sqrt{2}} h(\nu). \tag{2.13}$$

Remark 2.2: The conditions stated in Theorem 2.1 imply the existence of a unique zero of $\alpha J_\nu(z) + \gamma z J_\nu'(z)$ in the interval $(0, j_{\nu,1})$ [2]. Note that for every real α and $\nu > -1$ there exists a unique zero of $\alpha J_\nu(z) + z J_\nu'(z)$ in each of the intervals $(j_{\nu,k}, j_{\nu,k+1})$ $(k = 1, 2, \ldots)$. For these zeros we do not need the above conditions. In fact, Lemma 2.1 holds for every $\nu \neq -\alpha/\gamma$ and the differential equation (2.13) holds also for every $\nu > -1$ and $\nu \neq -\alpha$. We can therefore state the following

Theorem 2.2: *For $\nu > -1$ and $\nu \neq -\alpha$ any positive zero of $\alpha J_\nu(z) + z J_\nu'(z)$ satisfies the differential equation* (2.13).

3. Differential inequalities

In this section we prove the following

Theorem 3.1: *Every positive zero $\varrho(\nu)$ of the function $\alpha J_\nu(z) + z J_\nu'(z)$ satisfies the differential inequality*

$$\frac{d\varrho(\nu)}{d\nu} < \frac{\varrho(\nu)}{1 + \nu}, \quad \nu > -1, \quad \textit{for} \quad \alpha \geqq 1,$$

$$\frac{d\varrho(\nu)}{d\nu} < \frac{\varrho(\nu)}{\alpha + \nu}, \quad \nu > -\alpha, \quad \textit{for} \quad \alpha < 1.$$

Proof: Since $\|L_\nu\| = 1/(1 + \nu)$ we obtain from (2.13)

$$\frac{d\varrho(\nu)}{d\nu} < \frac{\varrho(\nu)}{1 + \nu} \frac{(h, h) + 1 + \nu}{(h, h) + \alpha + \nu} \leqq \frac{\varrho(\nu)}{1 + \nu} \quad \text{for} \quad \alpha \geqq 1.$$

In the case $\alpha < 1$ we find for $\nu > -\alpha$

$$\frac{d\varrho(\nu)}{d\nu} < \frac{\varrho(\nu)}{1+\nu} \frac{(h, h) + 1 + \nu}{(h, h) + \alpha + \nu}$$

$$= \frac{\varrho(\nu)}{1+\nu} \left(1 + \frac{1-\alpha}{(h, h) + \alpha + \nu}\right) < \frac{\varrho(\nu)}{1+\nu} \left(1 + \frac{1-\alpha}{\alpha+\nu}\right) = \frac{\varrho(\nu)}{\alpha+\nu} \quad \blacksquare$$

Corollary 3.1: *Every positive zero $\varrho(\nu)$ of $\alpha J_\nu(z) + zJ_\nu'(z)$ satisfies the inequality*

$$\frac{\varrho(\nu)}{1+\nu} < \frac{\varrho(\mu)}{1+\mu}, \quad \nu > \mu > -1, \quad \text{in case} \quad \alpha \geqq 1, \tag{3.1}$$

$$\frac{\varrho(\nu)}{\alpha+\nu} < \frac{\varrho(\mu)}{\alpha+\mu}, \quad \nu > \mu > -\alpha, \quad \text{in case} \quad \alpha < 1, \tag{3.2}$$

or

$$\frac{\varrho(\nu)}{1+\nu} > \frac{\varrho(\mu)}{1+\mu}, \quad -1 < \nu < \mu, \quad \text{in case} \quad \alpha \geqq 1, \tag{3.3}$$

$$\frac{\varrho(\nu)}{\alpha+\nu} > \frac{\varrho(\mu)}{\alpha+\mu}, \quad -\alpha < \nu < \mu, \quad \text{in case} \quad \alpha < 1. \tag{3.4}$$

Proof: From Theorem 3.1 it follows easily that

$$\left.\begin{aligned} &\frac{d}{d\nu}\left(\frac{\varrho(\nu)}{1+\nu}\right) < 0, \quad \nu > -1, \quad \text{for} \quad \alpha \geqq 1, \\ &\frac{d}{d\nu}\left(\frac{\varrho(\nu)}{\alpha+\nu}\right) < 0, \quad \nu > -\alpha, \quad \text{for} \quad \alpha < 1. \end{aligned}\right\} \tag{3.5}$$

This means that the functions $\varrho(\nu)/(1+\nu)$ and $\varrho(\nu)/(\alpha+\nu)$ decrease as ν increases, which proves the desired relations ∎

Corollary 3.2: *Let $j'_{\nu,k}$ be the k-th positive zero of the derivative J_ν'. Then, from (3.5) for $\alpha = 0$, the function $j'_{\nu,k}/\nu$ decreases as $\nu > 0$ increases.*

Remark 3.1: Corollary 3.2 was proved independently and with different methods by R. C. McCann [9: Theorem 5] and by J. T. Lewis and M. E. Muldoon [7: Theorem 4.1].

4. Lower and upper bounds

From Corollary 3.1 we can easily find a number of lower and upper bounds for the zeros of $\alpha J_\nu(z) + zJ_\nu'(z)$. We denote by $\varrho_{\nu,k}$ the k-th positive zero of $\alpha J_\nu(z) + zJ_\nu'(z)$ and we recall that $\varrho_{\nu,k} \in (j_{\nu,k}, j_{\nu,k+1})$ $(k = 1, 2, \ldots)$ for $\nu > -1$ [2: Theorem 4.2]. Also from the relation $\alpha J_\alpha(z) + zJ_\alpha'(z) = zJ_{\alpha-1}(z)$ we see that

$$\varrho_{\alpha,k} = j_{\alpha-1,k}. \tag{4.1}$$

1. For $\mu = \alpha$ in (3.1) we obtain $\varrho_{\nu,k} < (1+\nu)\,\varrho_{\alpha,k}/(1+\alpha)$, which together with (4.1) gives the upper bound

$$\varrho_{\nu,k} < \frac{1+\nu}{1+\alpha} j_{\alpha-1,k}, \quad \nu > \alpha, \quad \text{in case} \quad \alpha \geqq 1. \tag{4.2}$$

In the same way for $\mu = \alpha$ in (3.3) we find the lower bound

$$\varrho_{\nu,k} > \frac{1+\nu}{1+\alpha} j_{\alpha-1,k}, \quad -1 < \nu < \alpha, \quad \text{in case} \quad \alpha \geqq 1. \tag{4.3}$$

2. For $\mu = \alpha$ in (3.2) and (3.4) we find

$$\varrho_{\nu,k} < \frac{\alpha+\nu}{2\alpha} j_{\alpha-1,k}, \quad \nu > \alpha, \quad \text{in case} \quad \alpha < 1, \tag{4.4}$$

$$\varrho_{\nu,k} > \frac{\alpha+\nu}{2\alpha} j_{\alpha-1,k}, \quad -\alpha < \nu < \alpha, \quad \text{in case} \quad \alpha < 1. \tag{4.5}$$

3. For $\mu = 0$ in (3.1), (3.3), (3.2) and (3.4) we find, respectively,

$$\varrho_{\nu,k} < (1+\nu)\, \varrho_{0,k}, \quad \nu > 0, \quad \text{in case} \quad \alpha \geqq 1, \tag{4.6}$$

$$\varrho_{\nu,k} > (1+\nu)\, \varrho_{0,k}, \quad -1 < \nu < 0, \quad \text{in case} \quad \alpha \geqq 1, \tag{4.7}$$

$$\varrho_{\nu,k} < \frac{\alpha+\nu}{\alpha} \varrho_{0,k}, \quad \nu > 0, \quad \text{in case} \quad \alpha < 1, \tag{4.8}$$

$$\varrho_{\nu,k} > \frac{\alpha+\nu}{\alpha} \varrho_{0,k}, \quad -\alpha < \nu < 0, \quad \text{in case} \quad \alpha < 1. \tag{4.9}$$

Remark 4.1: The bounds (4.9), (4.5) complete, with respect to the range of validity of the order ν [2: Cor. 4.5]. The bounds (4.7), (4.9) complete, with respect to the range of validity of the parameter α [2: Cor. 4.4]. The bounds (4.3), (4.5) for $k = 1$ complete, with respect to the range of validity of the order ν [2: (3.13)]. The bounds (4.6), (4.8) complete with respect to the range of validity of the parameter α [2: Cor. 4.2]. At last the bound (4.7) completes with respect to the range of validity of the order ν [2: Cor. 4.3] for $\alpha \geqq 1$.

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