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The Degree of Rational Approximation to Meromorphic Functions¹)

M. FREUND²)

Die Geschwindigkeit der besten rationalen Approximation meromorpher Funktionen auf kompakten Mengen wird durch das Wachstumsverhalten ihrer Nevanhinna-Charakteristik beschrieben. Die Ergebnisse sind in Form von O-Abschätzungen der Approximationsgeschwindigkeit und beinhalten auch Abschiitzungen von meromorphen Funktionen der Ordnung Null.

Описывается скорость наилучшей рациональной аппроксимации мероморфных функций на компактных множествах посредством характера роста их характеристики Неванлинны. Результаты имеют форму *О*-оценок скорости аппроксимации и включают в себя оценки мероморфных функций порядка нуль.

The rate of best rational approximation of a meromorphic function on a compact set is described in terms of the growth of its Nevanlinna characteristic. The results are expressed in terms of 0-estimates of the rate of approximation and include estimates of meromorphic 'functions of zero order.

1. Introduction

Let f be meromorphic on the complex plane and analytic on a compact subset S of C. Convergence theorems for rational approximants of such */* have been proved e.g. by WALSH [9], NUTTALL [5], POMMERENKE [6], WALLIN [8] and KARLSSON [4].
Concerning the rate of convergence, there exist comparatively few general results.
Denoting by
 $\mathcal{R}_{n,\nu}(S) = \{r_n, r_n = p_n/q, p_n \in \mathcal{P}_n, q, \in \mathcal{P}_n$ Concerning the rate of convergence, there exist comparatively few general results. Denoting by

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$$

 $n, \nu \in \mathbf{P} = \{0, 1, 2, \ldots\}$, the set of rational functions of type (n, ν) , where \mathcal{P}_n is the set of polynomials of degree $\leq n$, we shall approximate *f* by $r_{nn} \in \mathcal{R}_{n,n}(S)$. WALSH [10, p. 222]. has shown that for a function *f,* meromorphie in C, there exists a se-' quence of rational functions of type (n, n) which converges to *f* as $n \to \infty$ uniformly quence or rational functions of type (n, n) which converges to f as $n \to \infty$ uniformly
on any compact set *S* containing no pole of *f*. In [9] he showed that for such func-
 $\lim_{n \to \infty} E_{nn}[f, A(S)]^{1/n} = 0.$ (1.1) tions

$$
\lim_{n \to \infty} E_{nn}[f, A(S)]^{1/n} = 0.
$$
\n(1.1)

Here $E_{n} [f, A(S)] = \inf ||f - r_{n}||_{A(S)}$, where the infimum is taken over all $r_{n} \in \mathcal{R}_{n}$, (S), and $A(S)$ denotes the space of continuous functions on *S* which are holomorphic in lim $E_{nn}[f, A(S)]^{1/n} = 0$.

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1) AMS Subject Classification Numbers (1980): Primary 3

²) The author was supported by the Deutsche Forschungsgemeinschaft under grant No. 261/ 7-1.

¹) AMS Subject Classification Numbers (1980): Primary 30 E 10, 41 A 20; Secondary 41 A 25, *30* D

Key words and phrases: Best uniform approximation, rational functions, degree of approximation, $growth$ of meromorphic functions.

the interior of S, with maximum norm $\|\cdot\|_{A(S)}$. To improve this result, KARLSSON [4] *-* employed the concept of order of a merornorphic function and showed that, if *f* is meromorphic of *S*, with maximum norm $\|\cdot\|_{A(S)}$. To improve this result, KARLSSON [4]
employed the concept of order of a meromorphic function and showed that, if *f* is
meromorphic of order $\leq \varrho$, $0 < \varrho < \infty$, then f on *f*) and for any $\alpha > \varrho$ we have **94** • **M. FREUND**
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employed the concept of order of a meromorphic function and showed that, if *f* is

meromorphic of order $\leq \varrho$,

$$
E_{nn}[f, A(U)]^{1/n} \leq n^{-1/\alpha}
$$
,

The purpose of the present paper is to sharpen (1.2) and to admit also functions of zero order. For functions f , meromorphic in C , with Nevanlinna characteristic $T(r, f) = \mathcal{O}(r^e)$, $r \to \infty$, we will show under an additional condition upon the poles of *f*, that for any $\alpha > \rho$ **of** \mathbb{R}^n **.** *of* S *, with maximum r*
 of any *of S***, with maximum** *a***

of meromorphic** of order $\leq \varrho$, $0 < \varrho$
 on *f*) and for any $\alpha > \varrho$ we have
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for *n* sufficiently large.

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(1.2)

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onal condition upon the poles
 $n \to \infty$. (1.3)

max $\{|z|; z \in$

$$
E_{nn}[f, A(S)] = \mathcal{O}(e^n d^{n+1} e^{g(n+1)} n^{-(n+1)/3}), \qquad n \to \infty.
$$
 (1.3)

Here S is a compact set containing no pole of f, $d = \max \{|z|; z \in S\}$ and g is given by the assumptions on the poles. In contrast to Karlsson's result, (1.3) has the usual form of an approximation theorem, i.e. the degree of convergence is determined after fixing the compact set S on which */* is to be approximated. In Section 3 we will give an extension of (1.3) admitting meromorphic functions of zero order, too. of *f*, that for any $\alpha > \varrho$
 $E_{nn}[f, A(S)] = \mathcal{O}(e^n d^{n+1} e^{g(n+1)} n^{-(n+1)/s})$, $n \to \infty$.

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2. Rational approximation of i
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tension of (1.3) admitting meromorphic functions of zero order,

imation of meromorphic functions of finite order,

orphic function in C. The *Neva*

2. Rational approximation of meromorphic functions of finite order,

Let f be a meromorphic function in C. The *Nevanlinna characteristic function* is defined by

$$
T(r, f) = N(r, f) + m(r, f) \qquad (r > 0),
$$

where

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Intudinal approximation of the following functions of finite order,
\n
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f \quad b \quad a \quad meromorphic function in C. The Nevanlinna characteristic function is\n
$$
T(r, f) = N(r, f) + m(r, f) \qquad (r > 0),
$$
\nhere\n
$$
N(r, f) = \int_{0}^{r} \frac{n(t, f)}{t} dt, \qquad m(r, f) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |f(r e^{i\theta})| d\theta,
$$
\n
$$
d n(t) = n(t, f) \text{ is the number of poles of } f \text{ in } |z| < t. \text{ For later use we collect some\nnple properties [3, pp. 5-7].\n
$$
\text{Lemma 1: a) If } f_1, \ldots, f_p \text{ are meromorphic functions, one has}
$$
\n
$$
T\left(r, \sum_{j=1}^p f_j\right) \leq \sum_{j=1}^p T(r, f_j) + \log p, \qquad T\left(r, \prod_{j=1}^p f_j\right) \leq \sum_{j=1}^r T(r, f_j).
$$
\n
$$
b) If r_{mn} \in \mathcal{A}_{m,n}, m, n \in \mathbf{P}, there exist M, s_0 > 0 such that
$$
\n
$$
T(r, r_{mn}) \leq \max \{m, n\} \log r + M \qquad (r > s_0).
$$
\nLet f be of growth
$$
$$

and $n(t) = n(t, f)$ is the number of poles of f in $|z| < t$. For later use we collect some $h(t) = n(t, f)$ is the number of poles of f
simple properties [3, pp. 5–7].

$$
N(r, f) = \int_{0}^{\infty} \frac{r}{t} dt, \quad m(r, f) = \frac{1}{2\pi} \int_{0}^{\infty} \log^{+} |f(r e^{i\theta})| d\theta,
$$

and $n(t) = n(t, f)$ is the number of poles of f in $|z| < t$. For later use we collect
simple properties [3, pp. 5-7].
Lemma 1: a) If $f_1, ..., f_p$ are meromorphic functions, one has

$$
T\left(r, \sum_{j=1}^{p} f_j\right) \leq \sum_{j=1}^{p} T(r, f_j) + \log p, \quad T\left(r, \prod_{j=1}^{p} f_j\right) \leq \sum_{j=1}^{r} T(r, f_j).
$$

b) If $r_{mn} \in \mathcal{R}_{m,n}$, $m, n \in \mathbf{P}$, there exist $M, s_0 > 0$ such that
 $T(r, r_{mn}) \leq \max \{m, n\} \log r + M \quad (r > s_0).$
Let f be of growth

$$
T(r, f) = \mathcal{O}(r^e), \quad r \to \infty,
$$

for some $\varrho > 0$ and let the poles $z_1, ..., z_n, ..., \varphi_n$ and f be numerical in such a
 $r_n \leq r_{n+1}$, $n \in \mathbf{N}$, with $r_n = |z_n|$. We define for $R > 0$

 $(r > s_0)$.

Let */* be of growth

$$
T(r, f) = \mathcal{O}(r^e), \qquad r \to \infty,
$$
\n(2.1)

 $\begin{aligned} \n\text{simple prime} \quad \text{if } r_n \text{ is } \n\text{b) } &\text{if } r_n \text{ is } t \text{ is } \n\text{for some} \n\end{aligned}$ Let f be of growth
 $T(r, f) = \mathcal{O}(r^e), \quad r \to \infty,$ (2.1)

for some $\rho > 0$ and let the poles $z_1, ..., z_n,$ of f be numerated in such a way that
 $r_n \le r_{n+1}, n \in \mathbb{N},$ with $r_n = |z_n|$. We define for $R > 0$ $T(r, r_{mn}) \leq \max(m, n) \log r + M$ $(r > n)$

Let *f* be of growth
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 $\omega^{n(R)}(z) = \prod_{j=1$ some
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Lemma 1: a) If
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T(r, \sum_{j=1}^p f_j) \leq \sum_{j=1}^p T(r, f_j) + \log p, \qquad T(r, \prod_{j=1}^p f_j) \leq \sum_{j=1}^r T(r, f_j).
$$
\nb) If $r_{mn} \in \mathcal{R}_{m,n}$, $m, n \in \mathbf{P}$, there exist $M, s_0 > 0$ such that
\n $T(r, r_{mn}) \leq \max \{m, n\} \log r + M \qquad (r > s_0).$
\nLet f be of growth
\n $T(r, f) = \mathcal{O}(r^e), \qquad r \to \infty,$
\nfor some $\rho > 0$ and let the poles $z_1, ..., z_n, ...$ of f be numerical in such a way that
\n $r_n \leq r_{n+1}, n \in \mathbf{N}$, with $r_n = |z_n|$. We define for $R > 0$
\n $\omega^{n(R)}(z) = \prod_{j=1}^{n(R)} (z - z_j),$
\n $g_R(z) = f(z) \omega^{n(R)}(z),$ (2.2)

Rational Approximation to Meromorphic Functions 195 and consider functions f with property (2.1) for which g_R is analytic in $|z| < R$ and,

for each $\alpha > \varrho$, there exist a function $g \in \mathcal{Q}'$ and constants *M*, r_0 , $R_0 > 0$ such that for each $R > R_0$ and $r_0 \le r < R$ **Mational Approximation to Mero

ider functions f with property (2.1) for which** g_R **is
** $\alpha > g$ **, there exist a function** $g \in \Omega'$ **and constant
** $R > R_0$ **and** $r_0 \le r < R$ **
** $M(r, g_R) = \max_{|z|=r} |g_R(z)| \le M e^{R^a} \tilde{A}_R(r)$ **.
 \in C^2(0, \in** *v*- Functions
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$$
M(r, g_R) = \max_{|z|=r} |g_R(z)| \leq M e^{R^a} \tilde{A}_R(r).
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Rational Approximation to Meromorphic Functions
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\nfor each $R > R_0$ and $r_0 \le r < R$
\n
$$
M(r, g_R) = \max_{|z|=r} |g_R(z)| \le M e^{R^a} \tilde{A}_R(r).
$$
\nHere
\n
$$
Q' = \left\{ g \in C^2(0, \infty); g(x) > 0, g''(x) < 0 \forall x > 0, \lim_{x \to \infty} g'(x) = 0, \lim_{x \to \infty} g(x) = \infty \right\},
$$
\nand, for $g \in Q'$ and $0 < r < R$,
\n
$$
\tilde{A}_R(r) = \exp \left\{ g \left((g')^{-1} \left(\log \frac{R}{r} \right) \right) - (g')^{-1} \left(\log \frac{R}{r} \right) \log \frac{R}{r} \right\},
$$
\nwhere $C^2(0, \infty)$ denotes the class of twice continuously differentiable functions
\n $(0, \infty)$.
\nTherefore, 1: Let f be meromorphic such that (2.1)–(2.4) hold for some $\varrho \in (0, \infty)$ and $g \in Q'$. Given a compact set S in the region of holomorphy of f and set
\n $d = \max |z|$: $z \in S$, it follows to each set S in the region of holomorphy of f and set

$$
\epsilon \ C^{2}(0, \infty); g(x) > 0, g''(x) < 0 \ \forall x > 0, \lim_{x \to \infty} g'(x) = 0, \lim_{x \to \infty} g
$$
\n
$$
g \in \Omega' \text{ and } 0 < r < R,
$$
\n
$$
\tilde{A}_{R}(r) = \exp \left\{ g \left((g')^{-1} \left(\log \frac{R}{r} \right) \right) - (g')^{-1} \left(\log \frac{R}{r} \right) \log \frac{R}{r} \right\},\
$$

where $C^2(0, \infty)$ denotes the class of twice continuously differentiable functions on $(0, \infty)$.

Theorem 1: Let *f* be meromorphic such that $(2.1) - (2.4)$ hold for some $\rho \in (0, \infty)$, and $g \in \Omega'$. Given a compact set S in the region of holomorphy of *f* and setting $d = \max \{|z|; z \in S\}$, *it follows for each* $\alpha > \varrho$ that For 1: Let f be meromorphic such that $(2.1) - (2.4)$ hold for some $\rho \in (0, \infty)$
 n. Given a compact set *S* in the region of holomorphy of f and setting
 $\{ |z|, z \in S \}$, it follows for each $\alpha > \rho$ that
 $E_{nn}[f, A(S)] = \mathcal{$

$$
E_{nn}[f, A(S)] = \mathcal{O}(e^n d^{n+1} e^{g(n+1)} n^{-(n+1)/\alpha}), \qquad n \to \infty
$$

Proof: For $r > 0$ we have

$$
n(r) \log 2 \leq \int_{r}^{2r} \frac{n(t)}{t} dt \leq N(2r)
$$

and, in view of (2.1), $N(r) = \mathcal{O}(r^e)$, $r \to \infty$. Thus $n(r) = \mathcal{O}(r^e)$, $r \to \infty$, and hence for $\alpha > \varrho$ there exists $s_1 > 0$ such that

$$
n(r) \leq r^2 \quad (r > s_1). \tag{2.5}
$$

Let c_n^R denote the *n*-th Taylor coefficient of g_R for some $R > 0$. Using a slight modification of Lemma 1 in [2] (adding a factor e^{Re} in the assumption and the assertion, so that the \mathcal{O} -constant becomes independent of *R*) it follows that (2.4) implies *•* Let c_n^R denote the *n*-th Taylor coefficient of g_R for some R >
modification of Lemma 1 in [2] (adding a factor e^{R^a} in the a
assertion, so that the O -constant becomes independent of R) is
implies
 $|c_n^R| \le$

$$
|c_n^R| \leq M e^{R^{\alpha}} \frac{e^{\varrho(n)}}{R^n} \qquad (R > R_0, n > n_1)
$$

for certain constants *M*, *n*₁. Using [2, Thm, 1] in the same manner, one obtains for some $n_2 \in \mathbb{N}$
 $E_n[g_R, A(D_d)] \leq M \left(\frac{d}{R}\right)^{n+1} e^{\rho(n+1)} e^{R^a}$ $(R > R_1, n > n_2)$, where $D_n = \{x : |x| \leq d\}$ and \overline{R}_n and $\overline{R}_$ some $n_2 \in N$

$$
E_n[g_R, A(D_d)] \leq M \left(\frac{d}{R}\right)^{n+1} e^{\rho(n+1)} e^{R^a} \qquad (R > R_1, n > n_2),
$$

where $D_d = \{z; |z| \leq d\}$ and $R_1 = \max \{d, R_0\}$. Denoting by $p_n^0 \in \mathcal{P}_n$ the polynomial

$$
\lim_{\epsilon \to \infty} \text{constants } M, n_1. \text{ Using [2, Thm, 1] in the same manner, one obtains for } \epsilon \text{ N}
$$
\n
$$
E_n[g_R, A(D_d)] \leq M \left(\frac{d}{R}\right)^{n+1} e^{q(n+1)} e^{R^a} \qquad (R > R_1, n > n_2),
$$
\n
$$
h_d = \{z \mid |z| \leq d\} \text{ and } R_1 = \max \{d, R_0\}. \text{ Denoting by } p_n^0 \in \mathcal{P}_n \text{ the polynomial approximation to } g_R \text{ on } D_d, \text{ it follows that}
$$
\n
$$
\left\| \omega^{n(R)} \left(f - \frac{p_n^0}{\omega^{n(R)}} \right) \right\|_{A(S)} \leq M \left(\frac{d}{R}\right)^{n+1} e^{q(n+1)} e^{R^a} \qquad (R > R_1, n > n_2),
$$
\n
$$
(2.6)
$$

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(2.4)

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noting that $S \subset D_d$ by assumption. Since the sequence of poles of *f* has no finite limit point, there exists a number $R_2 > 0$ such that

$$
|z - z_j| \geq 1 \qquad (z \in S, j \geq n(R_2)). \tag{2.7}
$$

• (zES,jn(R2)). ' (2.7)' Setting $k = \min\{|z - z_j|| \ge 1$ $(z \in S, j \ge n(R_2))\}$.

Setting $k = \min\{|z - z_j|; z \in S, 1 \le j \le n(R_2)\}$ one obtains in view of (2.7) for $R > R_2$ $\begin{aligned} z &= \min \left\{ |z - z_j|; z \in S, 1 \leq j \leq n(R_2) \right\} \text{ one of } \\ |\omega^{n(R)}(z)| &= |z - z_1| \dots |z - z_{n(R_1)}| \left| z - z_{n(R_1)+1} \right| \end{aligned}$

$$
|z - z_j| \ge 1 \qquad (z \in S, j \ge n(R_2)).
$$

\n
$$
k = \min \{|z - z_j|; z \in S, 1 \le j \le n(R_2)\} \text{ one of}
$$

\n
$$
|\omega^{n(R)}(z)| = |z - z_1| \dots |z - z_{n(R_1)}| |z - z_{n(R_1) + 1}|
$$

\n
$$
\dots |z - z_{n(R)}| \ge k^{n(R_1)} \qquad (z \in S).
$$

\nthis into (2.6) and setting $R_3 = \max \{R_1, R_2\}$.

Inserting this into (2.6) and setting $R_3 = \max \{R_1, R_2\}$ yields θ

 a)n(1:) Q - J?....) *o A(S) A(S) M (-j-)* e1 *(R > R3 , n > n2),* e' eR4 *(R > B31 n >n2), (2.8)* where ill is independent of B and *n.* Sett. iiig *R* = *n' ¹ ,* there exists *n3* € N, such that (2.8) gives - ^S Proof: **Let** *z,* **1** v, be the poles of */* and B1 = max {I z I; **1** *j v}.* **Then**

and thus

$$
\leq M \left(\frac{d}{R}\right)^{n+1} e^{g(n+1)} e^{R^{\alpha}} \qquad (R > R_3, n > n_2),
$$

$$
E_{n, n(R)}[f, A(S)] \leq M \left(\frac{d}{R}\right)^{n+1} e^{g(n+1)} e^{R^{\alpha}} \qquad (R > R_3, n > n_2),
$$
 (2.8)

where *M* is independent of *R* and *n*. Setting $R = n^{1/a}$, there exists $n_3 \in N$ such that $n^{1/\alpha} > R_4 = \max \{R_3, s_1\}$ for $n > n_3$. Then by (2.5) one has for these *n* and *R* $n(R) \leq R^* = n$, and it follows by definition that $E_{nn}(f, A(S)) \leq E_{n,n(R)}[f, A(S)]$, so and thus
 $E_{n,n(R)}[f, A(S)] \leq M \left(\frac{d}{R}\right)^{n+1} e^{g(n+1)} e^{R^a}$ $(R > R_3, n > n_2),$

where *M* is independent of *R* and *n*. Setting $R = n^{1/a}$, there exists $n_3 \in \mathbb{N}$ such

that $n^{1/a} > R_4 = \max\{R_3, s_1\}$ for $n > n_3$. Then by (2 *•* $\alpha = ||V - \omega^{n(R)}||_{A(S)} \equiv ||\omega - \sqrt{2}(\omega^{n(R)})||_{B(S)}$ *
 i* $\leq M \left(\frac{d}{R}\right)^{n+1} e^{g(n+1)} e^{\lambda h}$ *
 iii* $E_{n,n(R)}[f, A(S)] \leq M \left(\frac{d}{R}\right)^{n+1} e^{g(n+1)} e^{\lambda h}$ *

where <i>M* is independent of *R* and *n*. Setting *R* = that $n^{1/a} > R_4 = \max\{R_$ and thus
 $E_{n,n(R)}[f, A(S)] \leq M \left(\frac{d}{R}\right)^{n+1} e^{g(n+1)} e^{R^a}$ $(R > R_3, n > n_2),$

where *M* is independent of *R* and *n*. Setting $R = n^{1/s}$, there exists $n_3 \in$

that $n^{1/s} > R_4 = \max \{R_3, s_1\}$ for $n > n_3$. Then by (2.5) one has where *M* is

that $n^{1/s} > 1$
 $n(R) \leq R^s =$

that (2.8) giv
 *E*_{nn}

which is the

Obviously

(1.2). We ma

Therefore (2

and 4 below.

Lemma 2

number of p

$$
E_{nn}[f, A(S)] \leq M d^{n+1} e^{n} e^{g(n+1)} n^{-(n+1)/\alpha} \qquad (n > n_3)
$$

Obviously, the statement of Theorem 1 is more precise than Karisson's relation (1.2). We made the additional assumption (2.4) however, which is somewhat technical.
Therefore (2.4) will be replaced by more natural sufficient conditions in Lemmas 2 where $m^2 > R_4 = \max\{R_3, s_1\}$ for $n > n_3$. Then by (2.5) one has for these *n* and *R* $n(R) \le R_4 = n$, and it follows by definition that $E_{nn}[f, A(S)] \le E_{n,n(R)}[f, A(S)]$, so that (2.8) gives
that (2.8) gives
 $E_{nn}[f, A(S)] \leq M d^{n+1} e^n$ where M is independent of R and n. Setting $R = n^{1/4}$,

that $n^{1/4} > R_4 = \max \{R_3, s_1\}$ for $n > n_3$. Then by $(2.5) \alpha_R(R) \leq R^* = n$, and it follows by definition that $E_{nn}[f, A$

that (2.8) gives
 $E_{nn}[f, A(S)] \leq M d^{n+1} e^n e^{g$ *E*_{nn}[*f*, *A*(*S*)] $\leq M d^{n+1} e^n e^{g(n+1)} n^{-(n+1)/s}$ $(n > n_3)$,
which is the assertion **that**
Obviously, the statement of Theorem 1 is more precise than Karlssor
(1.2). We made the additional assumption (2.4) however, whi Obviously, the statement of Theorem 1 is more precise tha.

(1.2). We made the additional assumption (2.4) however, which is

Therefore (2.4) will be replaced by more natural sufficient con

and 4 below.

Lemma 2: Let f

Lemma 2: Let *f* be meromorphic and satisfy (2.1) for some $\rho > 0.1$ *If f has a finite number of poles, condition (2.4) holds with* $g(x) = p \log x$ *for any* $p > 0$ *and any*

 $n(R) = v$ and $g_R(z) = g_{\overline{R}}(z)$ for any $R, \overline{R} > R_1$. By (2.1) and Lemma 1 there is a *Figure 1.1, i.e.* \vec{r} *Figure 1.1, i.e.* \vec{r} *Figure 1.1, i.e.* \vec{r} *****Figure 1.1, i.e. Figure 1.1, i.e. Figure 1.1, i.e.* \vec{r} *<i>Figure 1.1, i.e. Figure 1.1, i.e.* \vec{r} *<i>z e. P.fold* $f(z_j, 1 \leq j \leq r,$
 $g_R(z) = g_R(z)$
 g_0 such that
 g_R) $\leq T(r, f) +$

dependent of r
 $\frac{T(r, g_R)}{r^e} \leq M$

$$
T(r, g_R) \leq T(r, f) + T(r, \omega^{\prime}) \leq Mr^{\varrho} + \nu \log r + M \qquad (R > R_1, r > t_2),
$$

where M is independent of r and R . Thus we have

$$
\overline{\lim_{r\to\infty}}\,\frac{T(r,\,g_R)}{r^e}\leqq M<\infty,
$$

uniformly for all $R>R_{1}$, and $[3,$ Thm. 1.7] yields

$$
\overline{\lim}_{r \to \infty} \frac{T(r, g_R)}{r^{\varrho}} \leq M < \infty,
$$
\nwhere

\n
$$
y \text{ for all } R > R_1 \text{, and } [3, \text{ Thm. 1.7}] \text{ yield}
$$
\n
$$
\overline{\lim}_{r \to \infty} \frac{\log^+ M(r, g_R)}{r^{\varrho}} \leq M \qquad (R > R_1),
$$

i.e. for any $\alpha > \varrho$ one can choose $t_3 > 0$ such that

$$
M(r, g_R) \leq e^{r^{\alpha}} \qquad (R > R_1, r > t_3).
$$

Mathematical Approximation to Meromorphic Function
 M(r, g_R) $\leq e^{r^{\alpha}}$ ($R > R_1$, $r > t_3$).

Next we want to show that for any $p > 0$ and $R > r > \left(\frac{p}{\alpha}\right)^{1/\alpha}$ one has

$$
y \alpha > \varrho
$$
 one can cl
\n
$$
M(r, g_R) \leq e^{r^{\alpha}}
$$
\n
$$
y \alpha \beta \beta \beta
$$
\n
$$
y \alpha \beta
$$

Rational Approximation to Meromorphic 1

an choose $t_3 > 0$ such that
 $(R > R_1, r > t_3)$.

that for any $p > 0$ and $R > r > \left(\frac{p}{\alpha}\right)^{1/\alpha}$ on
 $\frac{R}{r}$ $\Big)^{-p}$.
 \therefore x is strictly increasing for $x > \left(\frac{p}{\alpha}\right)^{1/\alpha}$,

Rational Approximation to Meromorphic Functions
\ni.e. for any
$$
\alpha > \varrho
$$
 one can choose $t_3 > 0$ such that
\n
$$
M(r, g_R) \leq e^{r\alpha} \qquad (R > R_1, r > t_3).
$$
\nNext we want to show that for any $p > 0$ and $R > r > \left(\frac{p}{\alpha}\right)^{1/\alpha}$ one has
\n
$$
e^{r\alpha} \leq e^{R^{\alpha}} \left(\log \frac{R}{r}\right)^{-p}.
$$
\nThe function $x^{\alpha} - p \log x$ is strictly increasing for $x > \left(\frac{p}{\alpha}\right)^{1/\alpha}$, whence
\n $r^{\alpha} - p \log r - (R^{\alpha} - p \log R) \leq p \qquad \left(\left(\frac{p}{\alpha}\right)^{1/\alpha} < r < R\right),$
\nand, using $1 - 1/x \leq \log x, x > 0$, one has
\n $r^{\alpha} - R^{\alpha} \leq p \left(1 - \log \frac{R}{r}\right) \leq \log \left[\left(\log \frac{R}{r}\right)^{-p}\right].$

and, using $1 - 1/x \leqq \log x, x > 0$, one has

$$
e^{r^a} \leq e^{R^a} \left(\log \frac{\pi}{r} \right)^{\nu}.
$$

tion $x^a - p \log x$ is strictly increasing for $x > r^a - p \log r - (R^a - p \log R) \leq p \qquad \left(\left(\frac{p}{\alpha} \right)^{1/2} \right)$
g $1 - 1/x \leq \log x, x > 0$, one has
 $r^a - R^a \leq p \left(1 - \log \frac{R}{r} \right) \leq \log \left[\left(\log \frac{R}{r} \right)^{-p} \right].$
tiating we get (2.9) which in turn implies (2.4)

Exponentiating we get (2.9), which in turn implies (2.4) with $r_0 = R_0 = \max \Big\{ \Big(\frac{p}{r} \Big) \Big\}$ R_1 , since $\tilde{A}_R(r)$ $\log \frac{R}{r}$)^{-p}.
 (a)
 $\log x$ is strictly increasing for $x > \left(\frac{p}{\alpha}\right)^{1/2}$, whence
 $r - (R^2 - p \log R) \leq p$ $\left(\left(\frac{p}{\alpha}\right)^{1/\alpha} < r < R\right)$,
 $\leq \log x, x > 0$, one has
 $p\left(1 - \log \frac{R}{r}\right) \leq \log \left[\left(\log \frac{R}{r}\right)^{-p}\right]$.
 $\det (2.9)$ $R \setminus p$ for $g(x) = p \log x$ where $g \in \Omega'$ **i**

Another class of meromorphic functions f with property (2.1) for which condition (2.4) holds is described by the following restriction on the poles. With the notation as in the proof of Theorem 1 and writing $a(R)$ for the number of poles of f on $|z| = R$, we assume that there exist $A, B > 0$ such that Exponentiating we get (2.9), which in turn implies (2.4) with $r_0 = R_0 = \max \left\{ \left(\frac{p}{\alpha} \right)^{1/a}, \, t_j, R_1 \right\}$, since $\overline{A}_R(r) = \left(\frac{p}{e} \right)^p \left(\log \frac{R}{r} \right)^{-p}$ for $g(x) = p \log x$ where $g \in \Omega'$ **a**
Another class of meromorph

(i)
$$
a(n) \leq A
$$
 $(n \in \mathbb{N}),$ (ii) $|r_n - r_m| > B,$ $r_n \neq r_m$ $(n, m \in \mathbb{N}).$ (2.10)

Lemma 3: Let $\omega^{n(R)}(z) = \prod_{i=1}^{n(R)} (z - z_i)$, $R > 0$, where z_i are the poles of a mero*morphic function. Then,* to show that (2.1), (2.10) imply (2.4) again, we ne

ia 3: Let $\omega^{n(R)}(z) = \prod_{j=1}^{n(R)} (z - z_j)$, $R > 0$, where z_j

inntion. Then,
 $T\left(\frac{3}{2}R, \omega^{n(2R)}\right) = \mathcal{O}(n(2R) \log 2R)$, $R \to \infty$.

$$
T\left(\frac{3}{2}|R,\,\omega^{n(2R)}\right)=\mathcal{O}\big(n(2R)\log 2R\big),\qquad R\to\infty.
$$

Proof: We use Cartan's identity for a meromorphic function f (see e.g. [3, Thm, 1.3])

$$
T\left(\frac{3}{2} R, \omega^{n(2R)}\right) = O(n(2R) \log 2R), \qquad R \to \infty.
$$

f: We use Cartan's identity for a meromorphic function f (\sqrt{r})

$$
T(r, f) = \frac{1}{2\pi} \int_{0}^{2\pi} N\left(r, \frac{1}{f - e^{i\theta}}\right) d\theta + \log^+ |f(0)| \qquad (r > 0)
$$

in case of $f = \omega^{n(2R)}$ and $r = (3/2) R$, $R > 0$. Assuming first that $|z_1| > 1$, we find in case of $f = \omega^{n(2R)}$ and $r = (3/2) R$, $R > 0$. Assuming first that $|z_1| > 1$, we find $t_0 > 0$, independent of R , such that $n(t, 1/(\omega^{n(2R)} - a)) = 0$ for any $0 < t \le t_0$, $|a| = 1$, and so for each $(2/3) t_0 \le R$, $|a| = 1$ $|a| = 1$, and so for each $(2/3) t_0 \leq R$, $|a| = 1$ one has

$$
N\left(\frac{3}{2} R, 1/(\omega^{n(2R)}-a)\right) \leq n(2R) \log \frac{3}{2} R - n(2R) \log t_0.
$$

By the unboundedness of the sequence $|z_i|$ for $j \rightarrow \infty$ one can choose $R_1 > 0$ so that $\log^+ |1/\omega^{n(2R)}(0)| = 0, R > R_1$, and thus $\log^+ |\omega^{n(2R)}(0)| = \log |\omega^{n(2R)}(0)|, R > R_1$.

(2.9)

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Now

•

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\nNow
\n
$$
\log |\omega^{n(2R)}(0)| = \sum_{j=1}^{n(2R)} \log |z_j| \leq n(2R) \log 2R \qquad (R > 0),
$$
\n
$$
\log |\omega^{n(2R)}(0)| = \sum_{j=1}^{n(2R)} \log |z_j| \leq n(2R) \log 2R \qquad (R > 0),
$$

and setting $R_2 = \max \{R_1, (2/3) t_0\}$ one obtains for $R > R_2$ that $\frac{198}{198}$ \ Now
 and setting and setting a sample of the set of

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\n
$$
\log |\omega^{n(2R)}(0)| = \sum_{j=1}^{n(2R)} \log |z_j| \leq n(2R) \log 2R \qquad (R > 0),
$$
\n
$$
\log R_2 = \max \{R_1, (2/3) t_0\} \text{ one obtains for } R > R_2 \text{ that}
$$
\n
$$
T\left(\frac{3}{2} R, \omega^{n(2R)}\right) \leq n(2R) \log \frac{3}{2} R - n(2R) \log t_0 + n(2R) \log 2R,
$$
\n
$$
\text{ed. In the general case, let } \omega^{n(2R)}(z) = h(z) \tilde{\omega}^{n(2R)}(z), \text{ where}
$$

as asserted. In the general-case, let $\omega^{n(2R)}(z) = h(z) \bar{\omega}^{n(2R)}(z)$, where

$$
h(z) = \prod_{j=1}^{r} (z - z_j) \quad \text{with } v \text{ such that } |z_j| > 1 \text{ for } j > v
$$

• By Lemma 1 we have

$$
T\left(\frac{3}{2}R,\omega^{n(2R)}\right) \leq n(2R)\log\frac{3}{2}R - n(2R)\log t_0 + n(2R)\log 2R,
$$

as asserted. In the general case, let $\omega^{n(2R)}(z) = h(z)\omega^{n(2R)}(z)$, where

$$
h(z) = \prod_{j=1}^{r} (z - z_j) \quad \text{with } r \text{ such that } |z_j| > 1 \text{ for } j > r.
$$

By Lemma 1 we have

$$
T\left(\frac{3}{2}R,\omega^{n(2R)}\right) \leq T\left(\frac{3}{2}R,h\right) + T\left(\frac{3}{2}R,\tilde{\omega}^{n(2R)}\right) = \mathcal{O}(n(2R)\log 2R),
$$

$$
R \to \infty,
$$

and the proof is complete
Lemma 4: Let f be meromorphic with property (2.1) for some $\rho > 0$. If f sa
(2.10), then (2.4) holds with $g(x) = A \log x$ for any $\alpha > \rho$.
Proof: The function $g_{2R}(z) = f(z) \omega^{n(2R)}(z)$ is analytic in $|z| < 2R, R > 0$, so
[3, Thm. 1.6] yields for $0 < r < R$

$$
\frac{3}{2}R + r
$$

$$
\log^+ M(r, g_{2R}) \leq \frac{\frac{3}{2}R + r}{2}T\left(\frac{3}{2}R, g_{2R}\right).
$$

and the proof is complete \blacksquare

Lemma 4 : Let *f* be meromorphic with property (2.1) for some $\rho > 0$. If *f* satisfies (2.10) , then (2.4) holds with $g(x) = A \log x$ for any $\alpha > 0$.

Proof: The function $g_{2R}(z) = f(z) \omega^{n(2R)}(z)$ is analytic in $|z| < 2R$, $R > 0$, so that [3, Thm. 1.6] yields for $0 < r < R$

By Lemma 1 we have
\n
$$
T\left(\frac{3}{2} R, \omega^{n(2R)}\right) \leq T\left(\frac{3}{2} R, h\right) + T\left(\frac{3}{2} R, \tilde{\omega}^{n(2R)}\right) =
$$
\n $R \to \infty$,
\nand the proof is complete
\n
$$
\text{Lemma 4: Let } f \text{ be meromorphic with property (2.1) for set } g(2.10), \text{ then } (2.4) \text{ holds with } g(x) = A \log x \text{ for any } x > \varrho.
$$
\n
$$
\text{Proof: The function } g_{2R}(z) = f(z) \omega^{n(2R)}(z) \text{ is analytic in } |z|
$$
\n
$$
[3, \text{ Thm. 1.6}] \text{ yields for } 0 < r < R
$$
\n
$$
\frac{3}{2} R + r
$$
\n
$$
\log^+ M(r, g_{2R}) \leq \frac{3}{2} R - r
$$
\n
$$
\frac{3}{2} R - r
$$
\nBy Lemmas 1a) and 3 one has for sufficiently large R

By Lemmas 1a) and 3 one has for sufficiently large R

$$
T\left(\frac{1}{2}R, \omega^{n\alpha N}\right) \leq n(2R) \log \frac{1}{2}R - n(2R) \log t_0 + n(2R) \log 2R,
$$

as asserted. In the general case, let $\omega^{n(2R)}(z) = h(z) \omega^{n(2R)}(z)$, where

$$
h(z) = \prod_{j=1}^{r} (z - z_j) \quad \text{with } r \text{ such that } |z_j| > 1 \text{ for } j > r.
$$

By Lemma 1 we have

$$
T\left(\frac{3}{2}R, \omega^{n(2R)}\right) \leq T\left(\frac{3}{2}R, h\right) + T\left(\frac{3}{2}R, \tilde{\omega}^{n(2R)}\right) = \mathcal{O}\left(n(2R) \log 2R\right),
$$
 $R \to \infty,$
and the proof is complete
Lemma 4: Let *f* be meromorphic with property (2.1) for some $\rho > 0$. If *f* satisfies
(2.10), then (2.4) holds with $g(x) = A \log x$ for any $\alpha > \rho$.
Proof: The function $g_{2R}(z) = f(z) \omega^{n(2R)}(z)$ is analytic in $|z| < 2R, R > 0$, so that
(3, Thn. 1.6] yields for $0 < r < R$

$$
\frac{3}{2}R + r
$$

$$
\log^+ M(r, g_{2R}) \leq \frac{2}{3} \frac{3}{2}R - r
$$

$$
\frac{3}{2}R - r
$$

$$
\log^+ M(r, g_{2R}) \leq 5T\left(\frac{3}{2}R, g_{2R}\right) \leq 5\left(T\left(\frac{3}{2}R, f\right) + T\left(\frac{3}{2}R, \omega^{n(2R)}\right)\right)
$$

$$
\leq c\left(\left(\frac{3}{2}R\right)^{\rho} + n(2R) \log 2R\right) \qquad (0 < r < R),
$$

where *c* is a constant. As in the proof of Theorem 1, for any $\alpha > \rho$ one can choose
 $\varepsilon, R_1 > 0$ such that
 $n(r, g_{2R}) \leq R^{e + \varepsilon/2}$

where *c* is a constant. As in the proof of Theorem 1, for any $\alpha > \varrho$ one can choose $\frac{1}{\sqrt{2}}$
one can $\begin{bmatrix} 1 \end{bmatrix}$ $\varepsilon, R_1 > 0$ such that $n(2R) \leq R^{e+\epsilon/2}$ $(R > R_1)$
and $\varrho + \varepsilon < \alpha$, and hence there exists $R_2 > R_1$ so that $\begin{aligned} \text{here } c \text{ is a consta} \ \textit{R}_{1} &> 0 \text{ such that} \ \textit{n}(2R) &\leq 1 \ \text{and } \textit{q} + \textit{\epsilon} &< \alpha \text{, and} \ \textit{M}(r, g_{2R}) \ \text{to transfer this est} \end{aligned}$ where c is a constant. As in the proof of Theorem 1, for any $\alpha > \varrho$ o
 $\epsilon, R_1 > 0$ such that
 $n(2R) \leq R^{e+\epsilon/2}$ $(R > R_1)$.

and $\varrho + \epsilon < \alpha$, and hence there exists $R_2 > R_1$ so that
 $M(r, g_{2R}) \leq e^{R^{\varrho + \epsilon}}$ $(0 < r < R, R > R$

$$
n(2R) \leq R^{e+\epsilon/2} \qquad (R > R_1) \tag{2.11}
$$

$$
M(r, g_{2R}) \leq e^{R^{\theta+\epsilon}} \qquad (0 < r < R, R > R_2). \tag{2.12}
$$

To transfer this estimate from g_{2R} to g_R we write

$$
g_{2R}(z) = g_R(z) (z - z_{n(R)+1}) \ldots (z - z_{n(2R)})
$$

and find a lower bound for the last $n(2R) - n(R)$ factors from which (2.4) follows by applying (2.12). Two cases are to be considered. **/**

a) At least one pole of *f* lies on $|z| = R$: With $a(R)$ as in (2.10) we have

Rational Approximation to Meromorphic Functions
\n*least one pole of f lies on*
$$
|z| = R
$$
: With $a(R)$ as in (2.10) we have
\n
$$
\prod_{j=1}^{a(R)} |z - z_{n(R)+j}| \geq (R - r)^{a(R)} \qquad (|z| = r < R),
$$
\n
$$
a(2.2) \cdot (4R - 1)^{a(R)} \geq (4R - 1)^{a(R)}
$$

imation to Meromorphic F(

Vith $a(R)$ as in (2.10) we
 $(|z| = r < R),$
 $R)^{a(R)} \geq ((R - r)/R)^{A}, 0$ and, in view of assumption (2.10), $((R - r)R)^{a(R)} \geq ((R - r)R)^{a(R)}$ and, in view of assumption (2.10), $((R - r)R)^{a(R)} \geq ((R - r)R)^{a}, 0 < r < R$. Thus, $\text{for } R \geq 1$, for $R\geqq 1$,

least one pole of *f* lies on
$$
|z| = R
$$
: With $a(R)$ as in (2.10) we have
\n
$$
\prod_{j=1}^{a(R)} |z - z_{n(R)+j}| \ge (R - r)^{a(R)} \qquad (|z| = r < R),
$$
\n
$$
\text{new of assumption (2.10), } ((R - r)/R)^{a(R)} \ge ((R - r)/R)^{k}, 0 < r < R.
$$
 Thus,
\n1,
\n
$$
|z - z_{n(R)+1}| \dots |z - z_{n(R)+a(R)}| \ge \left(\frac{R - r}{R}\right)^{A}.
$$
\n
$$
\text{mating the last factors in the representation of } g_{2R} \text{ we assume without loss}
$$
\n
$$
\text{ality that the constant } B \text{ in (2.10) (ii) is less than 1, i.e., on } |z| = r
$$
\n
$$
\prod_{j=n(R)+a(R)+1}^{n(2R)} |z - z_j| \ge B^{n(2R)-n(R)-a(R)} \ge B^{n(2R)}.
$$
\n(2.14)

For estimating the last factors in the representation of g_{2R} we assume without loss of generality that the constant *B* in (2.10) (ii) is less than 1, i.e., on $|z| = r$

$$
R
$$

hatting the last factors in the representation of g_{2R} we assume without loss
lity that the constant *B* in (2.10) (ii) is less than 1, i.e., on $|z| = r$

$$
\prod_{j=n(R)+a(R)+1}^{\lfloor n(2R) \rfloor} |z-z_j| \geq B^{n(2R)-n(R)-a(R)} \geq B^{n(2R)}.
$$
 (2.14)

$$
\iint_{j=1}^{a(R)} |z - z_{n(R)+j}| \ge (R - r)^{a(R)} \quad (|z| = r < R),
$$
\nand, in view of assumption (2.10),
$$
((R - r)/R)^{a(R)} \ge ((R - r)/R)^k, 0 < r < R.
$$
\nThus, for $R \ge 1$,
\n
$$
|z - z_{n(R)+1}| \dots |z - z_{n(R)+a(R)}| \ge \left(\frac{R - r}{R}\right)^A.
$$
\nFor estimating the last factors in the representation of g_{2R} we assume without loss of generality that the constant B in (2.10) (ii) is less than 1, i.e., on $|z| = r$
\n
$$
\iint_{j=n(R)+a(R)+1} |z - z_j| \ge B^{n(2R)-n(R)-a(R)} \ge B^{n(2R)}.
$$
\n(2.14)
\nNow it follows by (2.13) and (2.14) that
\n
$$
|g_{2R}(z)| \ge |g_R(z)| \left(\frac{R - r}{R}\right)^A B^{n(2R)} \quad (0 < r < R, R \ge 1):
$$
\n(2.15)
\n
$$
\beta \quad \text{There is no pole of } f \text{ on } |z| = R.
$$
\nBy assumption we have $R' > R$, where R'
\n
$$
= r_{\alpha(R)+1} \quad \text{Now (2.15) can be obtained by using } a(R') \text{ instead of } a(R) \text{ in case } \alpha.
$$

) *There is no pole of <i>f* on $|z| = R$: By assumption we have $R' > R$, where R' $=r_{n(R)+1}$. Now (2.15) can be obtained by using $a(R')$ instead of $a(R)$ in case α). p) There is no pole of f on $|z| = R$: By assumption we have $R' > R$, where $R' = r_{n(R)+1}$. Now (2.15) can be obtained by using $a(R')$ instead of $a(R)$ in case α).
The analog to (2.13) is then valid for the first $a(R')$ fac have (2.15) also in this case. *max ignth* $\begin{cases} R & \text{if } |R| \leq R \leq R \end{cases}$. Now (2.15) can be obtained by \log to (2.13) is then valid for the f ed below by *B* for $1 \leq j \leq n(2R)$.
 $\begin{cases} \text{if } |R| \leq R \leq R \end{cases}$.
 $\begin{cases} \text{if } |R| \leq R \leq R \end{cases}$.
 $\begin{cases} \text$ assumption we have $R > R$, which
 σ $\alpha(R)$ in client $a(R')$ factors. Since $|z - z_{n(R)}|$
 $\beta(p) - n(R) - a(R')$ and $|z| = r <$
 β
 The analog to (2.13) is then valid for the first $a(R')$ factors. Since $|z - z_{n(R)+o(R')+j}|$ is bounded below by *B* for $1 \leq j \leq n(2R) - n(R) - a(R')$ and $|z| = r < R$, we

Thus, in both cases,

$$
\max_{|z|=r} |g_R(z)| \leq \max_{|z|=r} |g_{2R}(z)| \left(\frac{R}{R-r}\right)^A B^{-n(2R)} \qquad (0 < r < R, R \geq 1) \quad .
$$

and with (2.12) it follows for $R_3 = \max \{R_2, 1\}$ that

The language of (2.16) is bounded below by B for
$$
1 \leq j \leq n(2R) - n(R) - a(R')
$$
 and $|z| = r < R$, we have (2.15) also in this case.
\nThus, in both cases,
\n
$$
\max |g_R(z)| \leq \max_{|z|=r} |g_{2R}(z)| \left(\frac{R}{R-r}\right)^A B^{-n(2R)} \qquad (0 < r < R, R \geq 1)
$$
\nand with (2.12) it follows for $R_3 = \max \{R_2, 1\}$ that
\n
$$
\max |g_R(z)| \leq e^{R^{\theta+\epsilon}} \left(\frac{R}{R-r}\right)^A B^{-n(2R)} \qquad (0 < r < R, R > R_3)
$$
\nNow $(R-r)^{-A} \leq \left(\log \frac{R}{r}\right)^{-A}$, $1 \leq r < R$, and so
\n $(R-r)^{-A} \leq M \tilde{A}_R(r) \qquad (1 \leq r < R)$, (2.16)
\nwhere M is some constant. If $R_4 > R_1$ is chosen such that $R^{\theta+\epsilon/2} \log (1/B) \leq R^{\theta+\epsilon}$,
\n $R > R_4$, we find with (2.11) that
\n $B^{-n(2R)} \leq e^{R^{\theta+\epsilon}} \qquad (R > R_4)$:
\nSince
\n $e^{2R^{\theta+\epsilon}} R^A \leq e^{R^{\alpha}} \qquad (R > R_5)$ (2.18)
\nfor some $R_5 > 0$, one obtains by inserting (2.16)–(2.18) in the last estimate of
\n $g_R(z)$ that for any $1 \leq r < R$ and $R > R_0$

 $\left(\log \frac{R}{a}\right)^{-A}$ and with (2.12) i

nax $|g_R$
 $\lim_{|z|=r}$

Now $(R-r)^{-A}$:
 $(R-r)$

where *M* is some
 $R > R_4$, we find
 $B^{-n(2R)}$

Since
 $e^{2R^{\theta+\epsilon}}R^{\epsilon}$

for some $R_5 > 0$

$$
(R-r)^{-A} \leqq M \tilde{A}_R(r) \qquad (1 \leqq r < R),
$$

where *M* is some constant. If $R_4 > R_1$ is chosen such that $R^{2+\epsilon/2} \log (1/B)$ $R > R_4$, we find with (2.11) that Now $(R - r)^{-A} \leq (\log \frac{R}{r})^{-A}$, $1 \leq r < R$, and so
 $(R - r)^{-A} \leq M \tilde{A}_R(r)$ $(1 \leq r < R)$,

where *M* is some constant. If $R_4 > R_1$ is chosen such that $R^{p+\ell/2}$
 $R > R_4$, we find with (2.11) that
 $B^{-n(2R)} \leq e^{R^{\ell+\epsilon}}$ $(R > R_$

$$
B^{-n(2R)} \leq e^{R^{\theta+\epsilon}} \qquad (R > R_4): \tag{2.17}
$$

$$
e^{2R^{e^{+e}}}R^A \leq e^{R^a} \qquad (R > R_5) \tag{2.18}
$$

for some $R_5 > 0$, one obtains by inserting $(2.16) - (2.18)$ in the last estimate of R_5)
 $\text{inserting} \ \ (2.16) - (R > R_0)$
 $r)^{-A} \leq M e^{R^a} \tilde{A}_R(r),$ where *M* is some constant. If $R_4 > R_1$ is chosen such that $R^{o+\epsilon/2} \log R$, R_4 , we find with (2.11) that
 $B^{-n(2R)} \leq e^{R^{o+\epsilon}}$ $(R > R_4)$:

Since
 $e^{2R^{o+\epsilon}}R^4 \leq e^{R^a}$ $(R > R_5)$

for some $R_5 > 0$, one obtains by inser

$$
M(r, g_R) \leq \tilde{e}^{2R^{2+\epsilon}} R^A(R-r)^{-A} \leq M e^{R^a} \tilde{A}_R(r)
$$

Combining Lemmas 2 and 4 with Theorem 1 we have

Proposition 1: Let *f* be meromorphic with (2.1) for some $\rho \in (0, \infty)$. If *f* has a *finite number of poles and S, d are given as in Theorem 1, it follows for any* α > and $p > 0$ that
 $E_{nn}[f, A(S)] = O(d^{n+1} e^n n^p n^{-(n+1)/\alpha})$, $n \to \infty$. *and* $p>0$ *that Example 1 Eq. 1 Eq. 1 Eq. 1 Eq. 1 i C example in the interpedient* $\frac{1}{2}$ *. <i>Let f be meromorphic with* $\frac{1}{2}$.
 Chat $E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1} e^{n} n^p n^{-(n+1)/\alpha}),$ 200 M. FREUN

Combining Len

Proposition 1

finite number of 1

and $p > 0$ that

E_{nn}[f, A(

Proposition 2

Then, for any $\alpha > E_{nn}[f, A(\alpha)]$

where S, d are chos **200 M. FREUND**
 Combining Lemmas 2 and 4 with Theorem 1 we have
 Proposition 1: Let f be meromorphic with (2.1) for some $\rho \in (0, \infty)$. If f has a
 finite number of poles and S, d are given as in Theorem 1, **Combining Lemmas 2 and 4 with Theoren**
 Proposition 1: Let f be meromorphic with

finite number of poles and S , d are given as

and $p > 0$ that
 $E_{nn}[f, A(S)] = O(d^{n+1} e^n n^p n^{-(n+1)/\alpha})$
 Proposition 2: Let f be mer

 \cdot

$$
E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1} e^{n} n^{p} n^{-(n+1)/\alpha}), \qquad n \to \infty.
$$

Proposition 2: Let *f* be meromorphic with (2.10) and (2.1) for some $\rho \in (0, \infty)$. $E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1} e^n n^p n^{-(n+1)/a}), \qquad n \to \infty.$

ssition 2: Let f be meromorphic with (2.10) and

any $\alpha > \varrho$,
 $E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1} e^n n^4 n^{-(n+1)/a}), \qquad n \to \infty,$

$$
E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1} e^n n^4 n^{-(n+1)/\alpha}), \qquad n \to \infty.
$$

Propositions 1 and 2 improve Karlsson's result (1.2) by providing a more precise bound as well as by admitting a free choice of the set S , where f is to be approximated, only observing that S must not contain any pole of *f*.

Remark: Concerning a converse of the results presented so far we notice that in $[1]$ an example is given which demonstrates that the converse of (1.1) is not true, i.e. there exists a f with (1.1) which is not meromorphic in the plane. For entire functions a converse of (1.2) in case of polynomial approximation is valid: f is entire of order ρ , $0 < \rho < \infty$, if and only if $E_{n0}[f]^{1/n} \leq n^{-1/\alpha}$, $\alpha > \rho$, for *n* sufficiently large: But in case of rational approximation, the converse of Theorem 1 does not hold, as is shown by the counterexample $f(z) = e^z$. Here, $\log M(r, f) \leq r$, thus in view of Lemma 2 *f* fulfills condition (2.4) with $g(x) = p \log x$ for any $p > 0$; $\alpha > 1$, but for large *n* we have $E_{nn}[e^z, A(S)]^{1/n} \leq n^{-2+\epsilon}$, $\epsilon > 0$ (TREFETHEN [7], see also mated, only observing that *S* must not contain any pole of *f*.

Remark: Concerning a converse of the results presented so far we notice that in [1] an example is given which demonstrates that the converse of (1.1) is no [4]). is shown by the counterexample $f(z)$ =
Lemma 2 f fulfills condition (2.4) with $g(x)$
arge *n* we have $E_{nn}[e^z, A(S)]^{1/n} \leq n^{-2+\epsilon}$
next section it will be shown that the tech
l a convergence theorem for certain meror
is a

In the next section it will be shown that the techniques used in proving Theorem 1. also yield a convergence theorem for certain meromorphic functions of zero order.

3. Extension to nieromorphic functions of zero order

Apart from extending Theorem 1 to functions of zero order, Theorem 2 below-uses. a more refined assumption in place of (2.1), namely **But the sum of the sum of the sum of the section is of the section of a convergence theorem for certain meromorphic functions of zero order.

Blue theorem 1 Blue theorem 1 convergence of the sum of the section of ze**

$$
T(r, f) = \mathcal{O}(B(r)), \qquad r \to \infty, \tag{3.1}
$$

where for some $x_0 > 0$

$$
B(r) = (h')^{-1} (\log r) \log r - h((h')^{-1} (\log r)) \qquad (r > x_0)
$$
 (3.2)

and *h* is an element of the following set

$$
\Omega = \{h \in C^2[x_0, \infty); h''(x) > 0 \,\forall x > x_0, \lim_{x \to \infty} h'(x) = \infty\}.
$$

Writing $h_{\epsilon}(x) = \epsilon^{-1}h(x)$, $\epsilon > 1$, we define $B_{\epsilon}(r)$, $r^{\circ} > x_0$, $\epsilon > 1$, corresponding to (3.2). For the proof of the next theorem we assume that for each $1 < \delta < \varepsilon$ and Apart from extending Theorem 1 to functions of zero order,

a more refined assumption in place of (2.1), namely
 $T(r, f) = \mathcal{O}(B(r))$, $r \to \infty$,

where for some $x_0 > 0$
 $B(r) = (h')^{-1} (\log r) \log r - h((h')^{-1} (\log r))$ $(r > \infty)$

and h is an $\Omega = \{h \in C^2[x_0, \infty); h''(x) > 0 \,\forall x > x_0, \lim_{x \to \infty} h'(x) = \infty\}.$
 $h_t(x) = \varepsilon^{-1}h(x), \varepsilon > 1, \text{ we define } B_t(r), r^2 > x_0, \varepsilon > 1, \text{ corresponding to } r \text{ the proof of the next theorem we assume that for each } 1 < \delta < \varepsilon \text{ and } r \text{ are exist } t_0, n_0 > 0 \text{ such that}$

(i) $B_\delta(e^{h_t(n+1)}) \leq n \quad (n > n_0), \qquad \text{(ii)} \ B(2r) \leq C B$

(i)
$$
B_{\delta}(e^{h_{\epsilon}'(n+1)}) \leq n \quad (n > n_0),
$$
 (ii) $B(2r) \leq CB_{\delta}(r) \quad (r > t_0).$ (3.3)

Condition (2.4) now turns into the following: for *f* with (3.1) and each $\varepsilon > 1$ there exist $g \in \Omega'$, *M*, r_0 , $R_0 > 0$ such that for each $R > R_0$ and $r_0 \le r < R$

$$
M(r, g_R) \leq M e^{B_{\epsilon}(R)} \tilde{A}_R(r). \tag{3.4}
$$

Fational Approximation to Meromorphic Functions 201
 1 (2.4) now turns into the following: for f with (3.1) and each $\varepsilon > 1$ there
 Ω' , M , r_0 , $R_0 > 0$ such that for each $R > R_0$ and $r_0 \le r < R$
 $M(r, g_R) \leq M e^{$ *Theorem 2: Let* f *be meromorphic with properties (3.1) for some* $h \in \Omega$ *and (3.4). Suppose that B_s fulfills (3.3). With S, d as in Theorem 1 one has for each* $\epsilon > 1$ $M(r, g_R) \leq M e^{B_t(R)} \tilde{A}_R(r)$.
 $e \text{ m } 2$: Let *f* be meromorphic with properties (3.1),
 hat B_t *fulfills* (3.3). With *S*, *d* as in Theorem 1 of
 $E_{nn}[f, A(S)] = O(d^{n+1} e^{g(n+1)} e^{-h_t(n+1)})$, $n \to$

$$
E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1} e^{g(n+1)} e^{-h_{\varepsilon}(n+1)}), \qquad n \to \infty.
$$

Proof: Let $1 < \delta < \varepsilon$. Using (3.1) and (3.3) (ii) one finds as in the proof of Theorem 1 that for $r > s_i$. Suppose that B_{ϵ} fulfills (3.3). With S , d as in Theorem 1 one has for each
 $E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1} e^{g(n+1)} e^{-h_{\epsilon}(n+1)}), \qquad n \to \infty.$

Proof: Let $1 < \delta < \epsilon$. Using (3.1) and (3.3) (ii) one finds as in the 1

rem 1 tha

Rational Approximation to Meromorphic Functions
\n
$$
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$$
\n
$$
1 (2.4) \text{ now turns into the following: for } f \text{ with } (3.1) \text{ and each } \varepsilon > 1 \text{ there}
$$
\n
$$
2f, M, r_0, R_0 > 0 \text{ such that for each } R > R_0 \text{ and } r_0 \leq r < R
$$
\n
$$
M(r, g_R) \leq M e^{B_{\varepsilon}(R)} \tilde{A}_R(r).
$$
\n
$$
1 (3.4) \cdot \text{rem } 2: \text{ Let } f \text{ be meromorphic with properties } (3.1) \text{ for some } h \in \Omega \text{ and } (3.4).
$$
\n
$$
2f \text{ then } R \text{ is the } \frac{1}{2} \text{ for all } R \text{ is the } \frac{1}{2} \text
$$

which is analogous to (2.5). Assumption (3.4) implies, as in deducing (2.8) from (2.1), that one can choose R_2 , $n_1 > 0$ so that

$$
E_{n,n(R)}[f, A(S)] \leq M d^{n+1} e^{g(n+1)} e^{B_{\ell}(R)} R^{-(n+1)} \qquad (R > R_1, n > n_1).
$$

Now let $n_2 > n_0$ such that $e^{h_{\varepsilon}'(n+1)} = R > s_1$, $n > n_2$. Thus

$$
\frac{e^{B_{\epsilon}(R)}}{R^{n+1}} = e^{-h_{\epsilon}(n+1)} \qquad (n > n_2),
$$
\n(3.6)

and one obtains in view of (3.5) and (3.3) (i)

/

$$
n(R) \leq B_{\delta}(R) = B_{\delta}(\mathrm{e}^{h_{\delta}(n+1)}) \leq n \qquad (n > n_{\delta}).
$$

Together with (3.6) the last estimate implies the assertion

Theorem 2 contains Theorem 1 as a special case, choosing $h(x) = (x/g) \log x, \rho > 0$, and $\alpha = \epsilon_0$. Then condition (3.3) is fulfilled and (3.4) reduces to (2.4). We further note that Theorem 2 produces a better estimate than Theorem 1 since by the substitution (3.6) the factor $eⁿ$ is avoided.

The following propositions are analogous to Propositions **I** and 2. The proofs are similar to those in the preceding section and are omitted.

Proposition 3: Let *f* be meromorphic with (3.1) for some $h \in \Omega$. If *f* has a finite *number of poles and if condition* (3.3) holds for B_{ϵ} , $\epsilon > 1$, it follows for any $\epsilon > 1$ The following presimilar to those in the Proposition 3:
 and p > 0 that
 and p > 0 that Efference in the preceding section and are omitted.
 Efference in the preceding section and are omitted.
 *E*_{nn}[*f, A(S)]* = $\mathcal{O}(d^{n+1}n^p e^{-h_{\epsilon}(n+1)})$, $n \to \infty$,
 A as in Theorem 1

$$
E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1}n^p e^{-h_{\epsilon}(n+1)}), \qquad n \to \infty,
$$

with 5, d as in Theorem, **1.**

For an extension of Lemma **4** we need a, further technical restriction on *B,* which can be combined with (3.3) to the following sufficient condition: Let B_c be defined with *S*, *d as* in Theorem 1.
For an extension of Lemma 4 we need a further technical restriction on B_{ϵ} which
can be combined with (3.3) to the following sufficient condition: Let B_{ϵ} be defined
corresponding note that Theorem 2
stitution (3.6) the factor interference in the factor of poles in the
Proposition 3: in
number of poles and
and $p > 0$ that
 $E_{nn}[f, A(S)]$
with S, d as in Theor
For an extension can be combined with
corre *a as in* Theorem 1.
 n extension of Lemma 4 we need a further to

combined with (3.3) to the following sufficial

blowing sufficial
 i $\lim_{r\to\infty} \frac{B_{\delta}(r)}{B_{\delta}(r)} = 0$, (ii) $\lim_{r\to\infty} \frac{B_{\delta}(r)}{rB_{\delta}(r)}$

(ii) ion 3: Let *f* be meromorphic with (3.1) for some $h \in \Omega$. If *f* has a finite
les and if condition (3.3) holds for $B_{\epsilon}, \epsilon > 1$, it follows for any $\epsilon > 1$
 $f, A(S)$] = $\mathcal{O}(d^{n+1}n^p e^{-h_{\epsilon}(n+1)})$, $n \to \infty$,
 n *Theorem*

(i)
$$
\lim_{r \to \infty} \frac{B_{\delta}'(r) \log r}{B_{\epsilon}'(r)} = 0
$$
, (ii) $\lim_{r \to \infty} \frac{B_{\delta}(r)}{r B_{\epsilon}'(r)} = 0$. (3.7)

Proposition 4: *Let f be meromorphic with* (2.10) *and* (3.1) *for some h* \in *Q. If condition* (3.7) *holds for B_t*, ε > 1, *one has for any* ε > 1
 $E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1}n^4 e^{-h_t(n+1)})$, $n \to \infty$, *condition (3.7) holds for* B_i , $\varepsilon > 1$, one has for any $\varepsilon > 1$ (i) $\lim_{r \to \infty} \frac{B_{\delta}(r) \log r}{B_{\epsilon}(r)} = 0$, (ii) $\lim_{r \to \infty}$

Proposition 4: Let *f* be meromorphic *n*

condition (3.7) holds for $B_{\epsilon}, \epsilon > 1$, one has *f*
 $E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1}n^A e^{-h_{\epsilon}(n+1)})$,

there *S d* are *as*

$$
E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1}n^A e^{-h_{\varepsilon}(n+1)}), \qquad n \to \infty,
$$

202 M. FREUND
The following example ensures that the
of order $a = 0$. Choosing, $h(x) = x(\log x)$ The following example ensures that the results of this setion apply to functions **of** order $\theta = 0$. Choosing $h(x) = x(\log x)^2 - 2x \log x + 2x$, $x > 1$, one has $h \in \Omega$
and $B(x) = 2 \cdot \frac{(\log x)^{1/2} \Gamma(\log x)^{1/2}}{2}$, $A = \frac{1}{2} \left(\log x \right)^{1/2}$, $A = \frac{1}{2} \left(\log x \right)^{1/2}$, $A = \frac{1}{2} \left(\log x \right)^{1/2}$, $A = \frac{1}{2} \left(\log x \right)^{$ 202 M. FREUND

The following example ensures that the results of this section apply to fur

of order $\rho = 0$. Choosing $h(x) = x(\log x)^2 - 2x \log x + 2x$, $x > 1$, one has

and
 $B(r) = 2e^{(\log r)^{1/2}}[(\log r)^{1/2} - 1]$ $(r > 1)$.

By simple c **B** *B* **EXECTS**
 B *B* **EXECTS B EXECTS E** 202 M. FREUND
 *The following example ensures that the results of this section apply to func

of order* $\rho = 0$ *. Choosing* $h(x) = x(\log x)^2 - 2x \log x + 2x, x > 1$ *, one has <i>l*

and
 $B(r) = 2e^{(\log r)^{1/2}}[(\log r)^{1/2} - 1]$ $(r > 1)$.

By simpl

$$
B(r) = 2e^{(\log r)^{1/2}}[(\log r)^{1/2} - 1] \qquad (r > 1). \tag{3.8}
$$

By simple calculations one verifies that condition (3.7) is fulfilled in this case. Furthermore the definition of the order *of* a meromorphic function implies that all / with $B(r) = 2e^{(\log r)^{1/2}} - 1$ $(r > 1)$.

imple calculations one verifies that condition (3.7) is fulfilled in this case.

if the definition of the order of a meromorphic function implies that a
 $f) = \mathcal{O}(B(r))$, $r \to \infty$, where B

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