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The Degree of Rational Approximation to Meromorphic Functions[1])

M. Freund[2])

Die Geschwindigkeit der besten rationalen Approximation meromorpher Funktionen auf kompakten Mengen wird durch das Wachstumsverhalten ihrer Nevanlinna-Charakteristik beschrieben. Die Ergebnisse sind in Form von $\mathcal{O}$-Abschätzungen der Approximationsgeschwindigkeit und beinhalten auch Abschätzungen von meromorphen Funktionen der Ordnung Null.

Описывается скорость наилучшей рациональной аппроксимации мероморфных функций на компактных множествах посредством характера роста их характеристики Неванлинны. Результаты имеют форму $\mathcal{O}$-оценок скорости аппроксимации и включают в себя оценки мероморфных функций порядка нуль.

The rate of best rational approximation of a meromorphic function on a compact set is described in terms of the growth of its Nevanlinna characteristic. The results are expressed in terms of $\mathcal{O}$-estimates of the rate of approximation and include estimates of meromorphic functions of zero order.

1. Introduction

Let f be meromorphic on the complex plane and analytic on a compact subset S of $\mathbf{C}$. Convergence theorems for rational approximants of such f have been proved e.g. by Walsh [9], Nuttall [5], Pommerenke [6], Wallin [8] and Karlsson [4]. Concerning the rate of convergence, there exist comparatively few general results. Denoting by

$$\mathcal{R}_{n,\nu}(S) = \{r_{n\nu};\, r_{n\nu} = p_n/q_\nu,\, p_n \in \mathcal{P}_n,\, q_\nu \in \mathcal{P}_\nu;\, q_\nu(z) \neq 0, z \in S\},$$

$n, \nu \in \mathbf{P} = \{0, 1, 2, \ldots\}$, the set of rational functions of type (n, ν), where $\mathcal{P}_n$ is the set of polynomials of degree $\leqq n$, we shall approximate f by $r_{nn} \in \mathcal{R}_{n,n}(S)$. Walsh [10, p. 222] has shown that for a function f, meromorphic in $\mathbf{C}$, there exists a sequence of rational functions of type (n, n) which converges to f as $n \to \infty$ uniformly on any compact set S containing no pole of f. In [9] he showed that for such functions

$$\lim_{n\to\infty} E_{nn}[f, A(S)]^{1/n} = 0. \tag{1.1}$$

Here $E_{n\nu}[f, A(S)] = \inf \|f - r_{n\nu}\|_{A(S)}$, where the infimum is taken over all $r_{n\nu} \in \mathcal{R}_{n,\nu}(S)$, and $A(S)$ denotes the space of continuous functions on S which are holomorphic in

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the interior of S, with maximum norm $\|\cdot\|_{A(S)}$. To improve this result, KARLSSON [4] employed the concept of order of a meromorphic function and showed that, if f is meromorphic of order $\leqq \varrho$, $0 < \varrho < \infty$, then for some compact disc U (depending on f) and for any $\alpha > \varrho$ we have

$$E_{nn}[f, A(U)]^{1/n} \leqq n^{-1/\alpha}, \tag{1.2}$$

for n sufficiently large.

The purpose of the present paper is to sharpen (1.2) and to admit also functions of zero order. For functions f, meromorphic in $\mathbf{C}$, with Nevanlinna characteristic $T(r, f) = \mathcal{O}(r^\varrho)$, $r \to \infty$, we will show under an additional condition upon the poles of f, that for any $\alpha > \varrho$

$$E_{nn}[f, A(S)] = \mathcal{O}(e^n d^{n+1} e^{g(n+1)} n^{-(n+1)/\alpha}), \qquad n \to \infty. \tag{1.3}$$

Here S is a compact set containing no pole of f, $d = \max\{|z|; z \in S\}$ and g is given by the assumptions on the poles. In contrast to Karlsson's result, (1.3) has the usual form of an approximation theorem, i.e. the degree of convergence is determined after fixing the compact set S on which f is to be approximated. In Section 3 we will give an extension of (1.3) admitting meromorphic functions of zero order, too.

2. Rational approximation of meromorphic functions of finite order

Let f be a meromorphic function in $\mathbf{C}$. The *Nevanlinna characteristic function* is defined by

$$T(r, f) = N(r, f) + m(r, f) \qquad (r > 0),$$

where

$$N(r, f) = \int_0^r \frac{n(t, f)}{t}\, dt, \qquad m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(r\, e^{i\theta})|\, d\theta,$$

and $n(t) = n(t, f)$ is the number of poles of f in $|z| < t$. For later use we collect some simple properties [3, pp. 5—7].

Lemma 1: a) *If $f_1, \ldots, f_p$ are meromorphic functions, one has*

$$T\left(r, \sum_{j=1}^p f_j\right) \leqq \sum_{j=1}^p T(r, f_j) + \log p, \qquad T\left(r, \prod_{j=1}^p f_j\right) \leqq \sum_{j=1}^r T(r, f_j).$$

b) *If $r_{mn} \in \mathcal{R}_{m,n}$, $m, n \in \mathbf{P}$, there exist $M, s_0 > 0$ such that*

$$T(r, r_{mn}) \leqq \max\{m, n\} \log r + M \qquad (r > s_0).$$

Let f be of growth

$$T(r, f) = \mathcal{O}(r^\varrho), \qquad r \to \infty, \tag{2.1}$$

for some $\varrho > 0$ and let the poles $z_1, \ldots, z_n, \ldots$ of f be numerated in such a way that $r_n \leqq r_{n+1}$, $n \in \mathbf{N}$, with $r_n = |z_n|$. We define for $R > 0$

$$\omega^{n(R)}(z) = \prod_{j=1}^{n(R)} (z - z_j), \tag{2.2}$$

$$g_R(z) = f(z)\, \omega^{n(R)}(z), \tag{2.3}$$

and consider functions f with property (2.1) for which g_R is analytic in $|z| < R$ and, for each $\alpha > \varrho$, there exist a function $g \in \Omega'$ and constants $M, r_0, R_0 > 0$ such that for each $R > R_0$ and $r_0 \leqq r < R$

$$M(r, g_R) = \max_{|z|=r} |g_R(z)| \leqq M\, e^{R^\alpha} \tilde{A}_R(r). \tag{2.4}$$

Here

$$\Omega' = \left\{g \in C^2(0, \infty);\, g(x) > 0,\, g''(x) < 0\ \forall x > 0,\ \lim_{x\to\infty} g'(x) = 0,\ \lim_{x\to\infty} g(x) = \infty\right\},$$

and, for $g \in \Omega'$ and $0 < r < R$,

$$\tilde{A}_R(r) = \exp\left\{g\left((g')^{-1}\left(\log\frac{R}{r}\right)\right) - (g')^{-1}\left(\log\frac{R}{r}\right)\log\frac{R}{r}\right\},$$

where $C^2(0, \infty)$ denotes the class of twice continuously differentiable functions on $(0, \infty)$.

Theorem 1: *Let f be meromorphic such that (2.1)–(2.4) hold for some $\varrho \in (0, \infty)$, and $g \in \Omega'$. Given a compact set S in the region of holomorphy of f and setting $d = \max\{|z|;\, z \in S\}$, it follows for each $\alpha > \varrho$ that*

$$E_{nn}[f, A(S)] = \mathcal{O}(e^n d^{n+1} e^{g(n+1)} n^{-(n+1)/\alpha}), \qquad n \to \infty.$$

Proof: For $r > 0$ we have

$$n(r) \log 2 \leqq \int_r^{2r} \frac{n(t)}{t}\, dt \leqq N(2r)$$

and, in view of (2.1), $N(r) = \mathcal{O}(r^\varrho)$, $r \to \infty$. Thus $n(r) = \mathcal{O}(r^\varrho)$, $r \to \infty$, and hence for $\alpha > \varrho$ there exists $s_1 > 0$ such that

$$n(r) \leqq r^\alpha \quad (r > s_1). \tag{2.5}$$

Let c_n^R denote the n-th Taylor coefficient of g_R for some $R > 0$. Using a slight modification of Lemma 1 in [2] (adding a factor e^{R^α} in the assumption and the assertion, so that the $\mathcal{O}$-constant becomes independent of R) it follows that (2.4) implies

$$|c_n^R| \leqq M\, e^{R^\alpha} \frac{e^{g(n)}}{R^n} \qquad (R > R_0,\, n > n_1)$$

for certain constants M, n_1. Using [2, Thm. 1] in the same manner, one obtains for some $n_2 \in \mathbb{N}$

$$E_n[g_R, A(D_d)] \leqq M\left(\frac{d}{R}\right)^{n+1} e^{g(n+1)}\, e^{R^\alpha} \qquad (R > R_1,\, n > n_2),$$

where $D_d = \{z;\, |z| \leqq d\}$ and $R_1 = \max\{d, R_0\}$. Denoting by $p_n^0 \in \mathcal{P}_n$ the polynomial of best approximation to g_R on D_d, it follows that

$$\left\|\omega^{n(R)}\left(f - \frac{p_n^0}{\omega^{n(R)}}\right)\right\|_{A(S)} \leqq M\left(\frac{d}{R}\right)^{n+1} e^{g(n+1)}\, e^{R^\alpha} \qquad (R > R_1,\, n > n_2), \tag{2.6}$$

noting that $S \subset D_d$ by assumption. Since the sequence of poles of f has no finite limit point, there exists a number $R_2 > 0$ such that

$$|z - z_j| \geqq 1 \qquad \big(z \in S, j \geqq n(R_2)\big). \tag{2.7}$$

Setting $k = \min\{|z - z_j|; z \in S, 1 \leqq j \leqq n(R_2)\}$ one obtains in view of (2.7) for $R > R_2$

$$|\omega^{n(R)}(z)| = |z - z_1| \dots |z - z_{n(R_2)}| \, |z - z_{n(R_2)+1}|$$
$$\dots |z - z_{n(R)}| \geqq k^{n(R_2)} \qquad (z \in S).$$

Inserting this into (2.6) and setting $R_3 = \max\{R_1, R_2\}$ yields

$$k^{n(R_2)} \left\| f - \frac{p_n^0}{\omega^{n(R)}} \right\|_{A(S)} \leqq \left\| \omega^{n(R)} \left(f - \frac{p_n^0}{\omega^{n(R)}} \right) \right\|_{A(S)}$$
$$\leqq M \left(\frac{d}{R} \right)^{n+1} e^{g(n+1)} e^{R^\alpha} \qquad (R > R_3, n > n_2),$$

and thus

$$E_{n,n(R)}[f, A(S)] \leqq M \left(\frac{d}{R} \right)^{n+1} e^{g(n+1)} e^{R^\alpha} \qquad (R > R_3, n > n_2), \tag{2.8}$$

where M is independent of R and n. Setting $R = n^{1/\alpha}$, there exists $n_3 \in \mathbf{N}$ such that $n^{1/\alpha} > R_4 = \max\{R_3, s_1\}$ for $n > n_3$. Then by (2.5) one has for these n and R $n(R) \leqq R^\alpha = n$, and it follows by definition that $E_{nn}[f, A(S)] \leqq E_{n,n(R)}[f, A(S)]$, so that (2.8) gives

$$E_{nn}[f, A(S)] \leqq M d^{n+1} e^n e^{g(n+1)} n^{-(n+1)/\alpha} \qquad (n > n_3),$$

which is the assertion ∎

Obviously, the statement of Theorem 1 is more precise than Karlsson's relation (1.2). We made the additional assumption (2.4) however, which is somewhat technical. Therefore (2.4) will be replaced by more natural sufficient conditions in Lemmas 2 and 4 below.

Lemma 2: *Let f be meromorphic and satisfy (2.1) for some $\varrho > 0$. If f has a finite number of poles, condition (2.4) holds with $g(x) = p \log x$ for any $p > 0$ and any $\alpha > \varrho$.*

Proof: Let z_j, $1 \leqq j \leqq \nu$, be the poles of f and $R_1 = \max\{|z_j|; 1 \leqq j \leqq \nu\}$. Then $n(R) = \nu$ and $g_R(z) = g_{\bar{R}}(z)$ for any $R, \bar{R} > R_1$. By (2.1) and Lemma 1 there is a number $t_2 > s_0$ such that

$$T(r, g_R) \leqq T(r, f) + T(r, \omega^\nu) \leqq M r^\varrho + \nu \log r + M \qquad (R > R_1, r > t_2),$$

where M is independent of r and R. Thus we have

$$\overline{\lim_{r \to \infty}} \frac{T(r, g_R)}{r^\varrho} \leqq M < \infty,$$

uniformly for all $R > R_1$, and [3, Thm. 1.7] yields

$$\overline{\lim_{r \to \infty}} \frac{\log^+ M(r, g_R)}{r^\varrho} \leqq M \qquad (R > R_1),$$

i.e. for any $\alpha > \varrho$ one can choose $t_3 > 0$ such that

$$M(r, g_R) \leqq e^{r^\alpha} \qquad (R > R_1, r > t_3).$$

Next we want to show that for any $p > 0$ and $R > r > \left(\frac{p}{\alpha}\right)^{1/\alpha}$ one has

$$e^{r^\alpha} \leqq e^{R^\alpha} \left(\log \frac{R}{r}\right)^{-p}. \tag{2.9}$$

The function $x^\alpha - p \log x$ is strictly increasing for $x > \left(\frac{p}{\alpha}\right)^{1/\alpha}$, whence

$$r^\alpha - p \log r - (R^\alpha - p \log R) \leqq p \qquad \left(\left(\frac{p}{\alpha}\right)^{1/\alpha} < r < R\right),$$

and, using $1 - 1/x \leqq \log x$, $x > 0$, one has

$$r^\alpha - R^\alpha \leqq p \left(1 - \log \frac{R}{r}\right) \leqq \log \left[\left(\log \frac{R}{r}\right)^{-p}\right].$$

Exponentiating we get (2.9), which in turn implies (2.4) with $r_0 = R_0 = \max\left\{\left(\frac{p}{\alpha}\right)^{1/\alpha}, t_3, R_1\right\}$, since $\tilde{A}_R(r) = \left(\frac{p}{e}\right)^p \left(\log \frac{R}{r}\right)^{-p}$ for $g(x) = p \log x$ where $g \in \Omega'$ ∎

Another class of meromorphic functions f with property (2.1) for which condition (2.4) holds is described by the following restriction on the poles. With the notation as in the proof of Theorem 1 and writing $a(R)$ for the number of poles of f on $|z| = R$, we assume that there exist $A, B > 0$ such that

$$\text{(i) } a(n) \leqq A \quad (n \in \mathbf{N}), \qquad \text{(ii) } |r_n - r_m| > B, \quad r_n \neq r_m \quad (n, m \in \mathbf{N}). \tag{2.10}$$

In order to show that (2.1), (2.10) imply (2.4) again, we need the following

Lemma 3: *Let* $\omega^{n(R)}(z) = \prod_{j=1}^{n(R)} (z - z_j)$, $R > 0$, *where* z_j *are the poles of a meromorphic function. Then,*

$$T\left(\frac{3}{2} R, \omega^{n(2R)}\right) = O\big(n(2R) \log 2R\big), \qquad R \to \infty.$$

Proof: We use Cartan's identity for a meromorphic function f (see e.g. [3, Thm. 1.3])

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} N\left(r, \frac{1}{f - e^{i\theta}}\right) d\theta + \log^+ |f(0)| \qquad (r > 0)$$

in case of $f = \omega^{n(2R)}$ and $r = (3/2) R$, $R > 0$. Assuming first that $|z_1| > 1$, we find $t_0 > 0$, independent of R, such that $n\big(t, 1/(\omega^{n(2R)} - a)\big) = 0$ for any $0 < t \leqq t_0$, $|a| = 1$, and so for each $(2/3)\, t_0 \leqq R$, $|a| = 1$ one has

$$N\left(\frac{3}{2} R, 1/(\omega^{n(2R)} - a)\right) \leqq n(2R) \log \frac{3}{2} R - n(2R) \log t_0.$$

By the unboundedness of the sequence $|z_j|$ for $j \to \infty$ one can choose $R_1 > 0$ so that $\log^+ |1/\omega^{n(2R)}(0)| = 0$, $R > R_1$, and thus $\log^+ |\omega^{n(2R)}(0)| = \log |\omega^{n(2R)}(0)|$, $R > R_1$.

Now

$$\log |\omega^{n(2R)}(0)| = \sum_{j=1}^{n(2R)} \log |z_j| \leqq n(2R) \log 2R \qquad (R > 0),$$

and setting $R_2 = \max \{R_1, (2/3)\, t_0\}$ one obtains for $R > R_2$ that

$$T\left(\frac{3}{2} R, \omega^{n(2R)}\right) \leqq n(2R) \log \frac{3}{2} R - n(2R) \log t_0 + n(2R) \log 2R,$$

as asserted. In the general case, let $\omega^{n(2R)}(z) = h(z)\, \tilde{\omega}^{n(2R)}(z)$, where

$$h(z) = \prod_{j=1}^{\nu} (z - z_j) \qquad \text{with } \nu \text{ such that } |z_j| > 1 \text{ for } j > \nu.$$

By Lemma 1 we have

$$T\left(\frac{3}{2} R, \omega^{n(2R)}\right) \leqq T\left(\frac{3}{2} R, h\right) + T\left(\frac{3}{2} R, \tilde{\omega}^{n(2R)}\right) = \mathcal{O}\big(n(2R) \log 2R\big),$$
$$R \to \infty,$$

and the proof is complete ∎

Lemma 4: *Let f be meromorphic with property* (2.1) *for some $\varrho > 0$. If f satisfies* (2.10), *then* (2.4) *holds with $g(x) = A \log x$ for any $\alpha > \varrho$.*

Proof: The function $g_{2R}(z) = f(z)\, \omega^{n(2R)}(z)$ is analytic in $|z| < 2R$, $R > 0$, so that [3, Thm. 1.6] yields for $0 < r < R$

$$\log^+ M(r, g_{2R}) \leqq \frac{\frac{3}{2} R + r}{\frac{3}{2} R - r}\, T\left(\frac{3}{2} R, g_{2R}\right).$$

By Lemmas 1a) and 3 one has for sufficiently large R

$$\begin{aligned} \log^+ M(r, g_{2R}) &\leqq 5T\left(\frac{3}{2} R, g_{2R}\right) \leqq 5\left(T\left(\frac{3}{2} R, f\right) + T\left(\frac{3}{2} R, \omega^{n(2R)}\right)\right) \\ &\leqq c\left(\left(\frac{3}{2} R\right)^{\varrho} + n(2R) \log 2R\right) \qquad (0 < r < R), \end{aligned}$$

where c is a constant. As in the proof of Theorem 1, for any $\alpha > \varrho$ one can choose $\varepsilon, R_1 > 0$ such that

$$n(2R) \leqq R^{\varrho + \varepsilon/2} \qquad (R > R_1) \tag{2.11}$$

and $\varrho + \varepsilon < \alpha$, and hence there exists $R_2 > R_1$ so that

$$M(r, g_{2R}) \leqq e^{R^{\varrho+\varepsilon}} \qquad (0 < r < R,\ R > R_2). \tag{2.12}$$

To transfer this estimate from g_{2R} to g_R we write

$$g_{2R}(z) = g_R(z)\, (z - z_{n(R)+1}) \ldots (z - z_{n(2R)})$$

and find a lower bound for the last $n(2R) - n(R)$ factors from which (2.4) follows by applying (2.12). Two cases are to be considered.

α) *At least one pole of f lies on $|z| = R$*: With $a(R)$ as in (2.10) we have

$$\prod_{j=1}^{a(R)} |z - z_{n(R)+j}| \geqq (R - r)^{a(R)} \qquad (|z| = r < R),$$

and, in view of assumption (2.10), $((R-r)/R)^{a(R)} \geqq ((R-r)/R)^A$, $0 < r < R$. Thus, for $R \geqq 1$,

$$|z - z_{n(R)+1}| \ldots |z - z_{n(R)+a(R)}| \geqq \left(\frac{R-r}{R}\right)^A. \tag{2.13}$$

For estimating the last factors in the representation of g_{2R} we assume without loss of generality that the constant B in (2.10) (ii) is less than 1, i.e., on $|z| = r$

$$\prod_{j=n(R)+a(R)+1}^{n(2R)} |z - z_j| \geqq B^{n(2R)-n(R)-a(R)} \geqq B^{n(2R)}. \tag{2.14}$$

Now it follows by (2.13) and (2.14) that

$$|g_{2R}(z)| \geqq |g_R(z)| \left(\frac{R-r}{R}\right)^A B^{n(2R)} \qquad (0 < r < R,\ R \geqq 1). \tag{2.15}$$

β) *There is no pole of f on $|z| = R$*: By assumption we have $R' > R$, where $R' = r_{n(R)+1}$. Now (2.15) can be obtained by using $a(R')$ instead of $a(R)$ in case α). The analog to (2.13) is then valid for the first $a(R')$ factors. Since $|z - z_{n(R)+a(R')+j}|$ is bounded below by B for $1 \leqq j \leqq n(2R) - n(R) - a(R')$ and $|z| = r < R$, we have (2.15) also in this case.

Thus, in both cases,

$$\max_{|z|=r} |g_R(z)| \leqq \max_{|z|=r} |g_{2R}(z)| \left(\frac{R}{R-r}\right)^A B^{-n(2R)} \qquad (0 < r < R,\ R \geqq 1)$$

and with (2.12) it follows for $R_3 = \max\{R_2, 1\}$ that

$$\max_{|z|=r} |g_R(z)| \leqq e^{R^{\varrho+\varepsilon}} \left(\frac{R}{R-r}\right)^A B^{-n(2R)} \qquad (0 < r < R,\ R > R_3).$$

Now $(R - r)^{-A} \leqq \left(\log \dfrac{R}{r}\right)^{-A}$, $1 \leqq r < R$, and so

$$(R - r)^{-A} \leqq M \tilde{A}_R(r) \qquad (1 \leqq r < R), \tag{2.16}$$

where M is some constant. If $R_4 > R_1$ is chosen such that $R^{\varrho+\varepsilon/2} \log(1/B) \leqq R^{\varrho+\varepsilon}$, $R > R_4$, we find with (2.11) that

$$B^{-n(2R)} \leqq e^{R^{\varrho+\varepsilon}} \qquad (R > R_4). \tag{2.17}$$

Since

$$e^{2R^{\varrho+\varepsilon}} R^A \leqq e^{R^\alpha} \qquad (R > R_5) \tag{2.18}$$

for some $R_5 > 0$, one obtains by inserting (2.16)—(2.18) in the last estimate of $g_R(z)$ that for any $1 \leqq r < R$ and $R > R_0$

$$M(r, g_R) \leqq e^{2R^{\varrho+\varepsilon}} R^A (R - r)^{-A} \leqq M\, e^{R^\alpha} \tilde{A}_R(r),$$

where $R_0 = \max\{R_3, R_4, R_5\}$, which is (2.4) ∎

Combining Lemmas 2 and 4 with Theorem 1 we have

Proposition 1: *Let f be meromorphic with* (2.1) *for some* $\varrho \in (0, \infty)$. *If f has a finite number of poles and S, d are given as in Theorem* 1, *it follows for any* $\alpha > \varrho$ *and* $p > 0$ *that*

$$E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1}\, e^n n^p n^{-(n+1)/\alpha}), \qquad n \to \infty.$$

Proposition 2: *Let f be meromorphic with* (2.10) *and* (2.1) *for some* $\varrho \in (0, \infty)$. *Then, for any* $\alpha > \varrho$,

$$E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1}\, e^n n^A n^{-(n+1)/\alpha}), \qquad n \to \infty,$$

where S, d are chosen as before.

Propositions 1 and 2 improve Karlsson's result (1.2) by providing a more precise bound as well as by admitting a free choice of the set S, where f is to be approximated, only observing that S must not contain any pole of f.

Remark: Concerning a converse of the results presented so far we notice that in [1] an example is given which demonstrates that the converse of (1.1) is not true, i.e. there exists a f with (1.1) which is not meromorphic in the plane. For entire functions a converse of (1.2) in case of polynomial approximation is valid: f is entire of order ϱ, $0 < \varrho < \infty$, if and only if $E_{n0}[f]^{1/n} \leqq n^{-1/\alpha}$, $\alpha > \varrho$, for n sufficiently large. But in case of rational approximation, the converse of Theorem 1 does not hold, as is shown by the counterexample $f(z) = e^z$. Here, $\log M(r, f) \leqq r$, thus in view of Lemma 2 f fulfills condition (2.4) with $g(x) = p \log x$ for any $p > 0$; $\alpha > 1$, but for large n we have $E_{nn}[e^z, A(S)]^{1/n} \leqq n^{-2+\varepsilon}$, $\varepsilon > 0$ (Trefethen [7], see also [4]).

In the next section it will be shown that the techniques used in proving Theorem 1 also yield a convergence theorem for certain meromorphic functions of zero order.

3. Extension to meromorphic functions of zero order

Apart from extending Theorem 1 to functions of zero order, Theorem 2 below uses a more refined assumption in place of (2.1), namely

$$T(r, f) = \mathcal{O}\big(B(r)\big), \qquad r \to \infty, \tag{3.1}$$

where for some $x_0 > 0$

$$B(r) = (h')^{-1}(\log r) \log r - h\big((h')^{-1}(\log r)\big) \qquad (r > x_0) \tag{3.2}$$

and h is an element of the following set

$$\Omega = \{h \in C^2[x_0, \infty);\, h''(x) > 0\ \forall x > x_0, \lim_{x\to\infty} h'(x) = \infty\}.$$

Writing $h_\varepsilon(x) = \varepsilon^{-1} h(x)$, $\varepsilon > 1$, we define $B_\varepsilon(r)$, $r > x_0$, $\varepsilon > 1$, corresponding to (3.2). For the proof of the next theorem we assume that for each $1 < \delta < \varepsilon$ and $C > 0$ there exist $t_0, n_0 > 0$ such that

$$\text{(i)}\ B_\delta(e^{h_\varepsilon'(n+1)}) \leqq n \quad (n > n_0), \qquad \text{(ii)}\ B(2r) \leqq C B_\delta(r) \quad (r > t_0): \tag{3.3}$$

Condition (2.4) now turns into the following: for f with (3.1) and each $\varepsilon > 1$ there exist $g \in \Omega'$, $M, r_0, R_0 > 0$ such that for each $R > R_0$ and $r_0 \leqq r < R$

$$M(r, g_R) \leqq M\, e^{B_\varepsilon(R)} \tilde{A}_R(r). \tag{3.4}$$

Theorem 2: *Let f be meromorphic with properties* (3.1) *for some $h \in \Omega$ and* (3.4). *Suppose that B_ε fulfills* (3.3). *With S, d as in Theorem* 1 *one has for each $\varepsilon > 1$*

$$E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1}\, e^{g(n+1)}\, e^{-h_\varepsilon(n+1)}), \qquad n \to \infty.$$

Proof: Let $1 < \delta < \varepsilon$. Using (3.1) and (3.3) (ii) one finds as in the proof of Theorem 1 that for $r > s_1$

$$n(r) \leqq \frac{1}{\log 2} N(2r) \leqq MB(2r) \leqq B_\delta(r), \tag{3.5}$$

which is analogous to (2.5). Assumption (3.4) implies, as in deducing (2.8) from (2.1), that one can choose R_2, $n_1 > 0$ so that

$$E_{n,n(R)}[f, A(S)] \leqq Md^{n+1}\, e^{g(n+1)}\, e^{B_\varepsilon(R)} R^{-(n+1)} \qquad (R > R_1, n > n_1).$$

Now let $n_2 > n_0$ such that $e^{h_\varepsilon'(n+1)} = R > s_1$, $n > n_2$. Thus

$$\frac{e^{B_\varepsilon(R)}}{R^{n+1}} = e^{-h_\varepsilon(n+1)} \qquad (n > n_2), \tag{3.6}$$

and one obtains in view of (3.5) and (3.3) (i)

$$n(R) \leqq B_\delta(R) = B_\delta(e^{h_\varepsilon'(n+1)}) \leqq n \qquad (n > n_3).$$

Together with (3.6) the last estimate implies the assertion ∎

Theorem 2 contains Theorem 1 as a special case, choosing $h(x) = (x/\varrho) \log x$, $\varrho > 0$, and $\alpha = \varepsilon\varrho$. Then condition (3.3) is fulfilled and (3.4) reduces to (2.4). We further note that Theorem 2 produces a better estimate than Theorem 1 since by the substitution (3.6) the factor e^n is avoided.

The following propositions are analogous to Propositions 1 and 2. The proofs are similar to those in the preceding section and are omitted.

Proposition 3: *Let f be meromorphic with* (3.1) *for some $h \in \Omega$. If f has a finite number of poles and if condition* (3.3) *holds for B_ε, $\varepsilon > 1$, it follows for any $\varepsilon > 1$ and $p > 0$ that*

$$E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1} n^p\, e^{-h_\varepsilon(n+1)}), \qquad n \to \infty,$$

with S, d as in Theorem 1.

For an extension of Lemma 4 we need a further technical restriction on B_ε which can be combined with (3.3) to the following sufficient condition: Let B_ε be defined corresponding to (3.2) for some h_ε with $h \in \Omega$, then for each $1 < \delta < \varepsilon$ we suppose that

$$\text{(i)} \lim_{r\to\infty} \frac{B_\delta'(r) \log r}{B_\varepsilon'(r)} = 0, \qquad \text{(ii)} \lim_{r\to\infty} \frac{B_\delta(r)}{rB_\varepsilon'(r)} = 0. \tag{3.7}$$

Proposition 4: *Let f be meromorphic with* (2.10) *and* (3.1) *for some $h \in \Omega$. If condition* (3.7) *holds for B_ε, $\varepsilon > 1$, one has for any $\varepsilon > 1$*

$$E_{nn}[f, A(S)] = \mathcal{O}(d^{n+1} n^A\, e^{-h_\varepsilon(n+1)}), \qquad n \to \infty,$$

where S, d are as in Theorem 1 *and A is defined by* (2.10) (i).

The following example ensures that the results of this section apply to functions of order $\varrho = 0$. Choosing $h(x) = x(\log x)^2 - 2x \log x + 2x$, $x > 1$, one has $h \in \Omega$ and

$$B(r) = 2e^{(\log r)^{1/2}}[(\log r)^{1/2} - 1] \qquad (r > 1). \tag{3.8}$$

By simple calculations one verifies that condition (3.7) is fulfilled in this case. Furthermore the definition of the order of a meromorphic function implies that all f with $T(r, f) = \mathcal{O}(B(r))$, $r \to \infty$, where B is given by (3.8), are of order zero.

REFERENCES

[1] AHANOROV, D., and J. L. WALSH: Some examples in degree of approximation by rational functions. Trans. Amer. Math. Soc. **159** (1971), 427–444.

[2] FREUND, M.: The degree of polynomial approximation and interpolation of analytic functions. Resultate Math. **13** (1988), 81–98

[3] HAYMAN, W. K.: Meromorphic Functions. Oxford: Clarendon Press 1964.

[4] KARSSON, J.: Rational interpolation and best rational approximants. J. Math. Anal. Appl. **53** (1976), 38–52.

[5] NUTTALL, J.: The convergence of Padé approximants of meromorphic functions. J. Math. Anal. Appl. **31** (1970), 147–153.

[6] POMMERENKE, C.: Padé approximants and convergence in capacity. J. Math. Anal. Appl. **41** (1973), 775–780.

[7] TREFETHEN, L. N.: The asymptotic accuracy of rational best approximation to e^z on a disk. J. Approx. Theory **40** (1984), 380–383.

[8] WALLIN, H.: The convergence of Padé approximants and the size of the power series coefficients. Applicable Anal. **4** (1974), 235–251.

[9] WALSH, J. L.: On the overconvergence of certain sequences of rational functions of best approximation. Acta Math. **57** (1931), 411–435.

[10] WALSH, J. L.: Interpolation and Approximation by Rational Functions in the Complex Domain, 2nd ed. Amer. Math. Soc. Colloq. Publ. 20, Providence 1956.

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VERFASSER:

Dr. MICHAEL FREUND
Lehrstuhl A für Mathematik
Rhein.-Westf. Technische Hochschule Aachen
Templergraben 55
D-5100 Aachen