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Inequalities between Entropy and Approximation Numbers of Compact Maps

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Für eine kompakte lineare Abbildung T mit unendlichdimensionalem Wertebereich von einem Hilbert-Raum in einen anderen wird gezeigt, daß für beliebig vorgegebenes $N \in \mathbf{N}$ gilt $e_n(T) \leqq 2a_{N+1}(T)$ für alle n, die einer gewissen Ungleichung genügen. Es werden dann Anwendungen dieses Resultates angegeben.

Для компактного линейного отображения T с бесконечномерной областью значений от одного Гильбертова пространства в другое показывается, что для произвольно заданного $N \in \mathbf{N}$ выполнено $e_n(T) \leqq 2a_{N+1}(T)$ для всех n удовлетворяющих некоторому неравенству. Даются применения этого результата.

It is shown that if T is a compact linear map, with infinite-dimensional range, from one Hilbert space to another, then given any $N \in \mathbf{N}$, $e_n(T) \leqq 2a_{N+1}(T)$ for any n satisfying a certain inequality. Applications of this are given.

1. Introduction. The object of this paper is to obtain upper and lower estimates of the approximation numbers of a compact map acting between Hilbert spaces in terms of its entropy numbers. This result is applied to show that, in various circumstances, upper and lower bounds for the approximation numbers give rise to upper and lower bounds of the same type for the entropy numbers. In particular, it is shown that if the n^{th} approximation number is bounded above and below by constant multiples of $n^{-\alpha}$, then so is the n^{th} entropy number. This recovers, in a Hilbert space setting, a result of Carl [2].

2. The result. Given any Banach spaces X and Y, denote the closed unit ball in X by B_X and let $B(X, Y)$ stand for the space of all bounded linear maps from X to Y. The space of all compact linear maps from X to Y will be denoted by $K(X, Y)$, and $K(X, X)$ will be abbreviated to $K(X)$. For any $T \in B(X, Y)$ and any $n \in \mathbf{N}$, the n^{th} *entropy number* of T, $e_n(T)$, is defined by

$$e_n(T) = \inf \{\varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{n-1} \text{ balls of radius } \varepsilon\}$$

and the n^{th} *approximation number* of T, $a_n(T)$, by

$$a_n(T) = \inf \{\|T - F\| : F \in B(X, Y), \operatorname{rank} F < n\}.$$

Throughout this paper H, H_1 and H_2 will denote Hilbert spaces; and, for definiteness, we shall assume that all Hilbert spaces occurring here are complex.

We can now give the following theorem.

Theorem 1: *Let $T \in K(H)$ have infinite-dimensional range. Then given any $N \in \mathbf{N}$,*

$$e_n(T) \leqq 2a_{N+1}(T) \leqq 2\sqrt{2}\, e_{N+2}(T) \tag{1}$$

for any $n \in \mathbf{N}$ satisfying the inequality

$$(n-1)\log 2 \geqq 2\log\left\{\prod_{j=1}^{N}\frac{3a_j(T)}{a_{N+1}(T)}\right\}. \tag{2}$$

Proof: Since $\dim T(H) = \infty$, it follows that $a_m(T) > 0$ for all $m \in \mathbf{N}$. Let $u \in H$; then (cf. Theorem 1.4 of Simon [6])

$$Tu = \sum_{m=1}^{\infty} a_m(T)\,(u, \phi_m)\,\Psi_m$$

where ϕ_m is a normalised eigenvector of $|T|$, the non-negative square root of T^*T, corresponding to the eigenvalue $a_m(T)$ of $|T|$ and $\Psi_m = \big(1/a_m(T)\big)\,T\phi_m$.

Let $N \in \mathbf{N}$ and define $P_N \in K(H)$ by

$$P_N u = \sum_{m=1}^{\infty} a_m(T)\,(u, \phi_m)\,\Psi_m \qquad (u \in H).$$

Clearly, rank $P_N = N$ and $\|T - P_N\| = a_{N+1}(T)$. Since $P_N(B_H)$ can be identified with an ellipsoid E in $\mathbf{R}^{2N}$ with semi-axes $a_1(T), a_1(T), \ldots, a_N(T), a_N(T)$, we claim that given any $\varepsilon \in (0, a_N(T)]$, $P_N(B_H)$ can be covered by N_ε balls of radius ε, where N_ε = integer part of

$$\left(\prod_{j=1}^{N}\frac{3a_j(T)}{\varepsilon}\right)^2.$$

Accepting this for the moment, it follows that $T(B_H)$ can be covered by N_ε balls of radius $\varepsilon + a_{N+1}(T)$. The choice of $\varepsilon = a_{N+1}(T) := \varepsilon'$ shows that $T(B_H)$ can be covered by $N_{\varepsilon'}$ balls of radius $2a_{N+1}(T)$. Consideration of the inequality $N_{\varepsilon'} \leqq 2^{n-1}$ shows now that $e_n(T) \leqq 2a_{N+1}(T)$, provided that n satisfies inequality (2). This completes the proof of the left-hand inequality in (1).

The right-hand inequality follows since from Carl [1] we have $a_{N+1}(T) \leqq \sqrt{2} \times e_{N+2}(|T|)$, and by Edmunds and Edmunds [4], $e_{N+2}(|T|) = e_{N+2}(T)$.

It remains to establish the claim made above. To do this, let $S = \{x^{(j)}: j = 1, \ldots, M\}$ be a maximal family of points in E such that $|x^{(j)} - x^{(k)}| \geqq \varepsilon$ whenever $j \neq k$. Since every ball with centre $x^{(j)}$ and radius $\varepsilon/2$ is contained in $3/2E$, a volume argument shows that

$$M \leqq \left(\prod_{j=1}^{N}\frac{3a_j(T)}{\varepsilon}\right)^2.$$

Moreover, the balls of radius ε with centres $x^{(1)}, \ldots, x^{(M)}$ cover E, for otherwise the maximality of S would be contradicted. This completes the proof of the theorem ∎

We now extend Theorem 1 so that it applies to maps which act from one Hilbert space to another. To do this it is convenient to establish the following result due to H. Triebel and pointed out to us by him.

Theorem 2: *Let $T \in B(H_1, H_2)$. Then for all $n \in \mathbf{N}$, $e_n(T) = e_n(T^*) = e_n(|T|)$.*

Proof (Triebel): By the polar decomposition theorem [5], there exists a partial isometry $U \in B(H_1, H_2)$ from $(\ker T)^\perp$ to $\overline{\operatorname{im} T}$ such that $T = U\,|T|$ and $|T| = U^*T$. It follows that for all $n \in \mathbf{N}$,

$$e_n(T) \leqq e_n(|T|)\,\|U\| = e_n(|T|), \qquad e_n(|T|) \leqq e_n(T)\,\|U^*\| = e_n(T),$$

and hence $e_n(T) = e_n(|T|)$. Use of the facts that $T^* = |T|\,U^*$ and $|T| = T^*U$ shows that $e_n(T^*) = e_n(|T|)$, and the proof is complete ∎

Corollary 3: *Let $T \in K(H_1, H_2)$ have infinite-dimensional range. Then the conclusion of Theorem 1 holds.*

Proof: By Theorem 2, $e_j(|T|) = e_j(T)$; also $a_j(|T|) = a_j(T)$. These results, together with Theorem 1, give the corollary ∎

3. Examples. To illustrate the usefulness of these results, we give the following examples.

Example A: *Let $T \in K(H_1, H_2)$ and suppose that for some $\alpha > 0$, $a_n(T) \asymp n^{-\alpha}$; that is, there are positive constants c_1 and c_2 such that $c_1 n^{-\alpha} \leqq a_n(T) \leqq c_2 n^{-\alpha}$ for all $n \in \mathbf{N}$. Then $e_n(T) \asymp n^{-\alpha}$, a result obtained by* Carl ([2], Cor. 2) *when H_1 and H_2 are Banach spaces with H_1 and H_2^* of type 2.*

To show this, let $N \in \mathbf{N}$ and observe that

$$\begin{aligned}\prod_{j=1}^{N} \frac{3a_j(T)}{a_{N+1}(T)} &\leqq \prod_{j=1}^{N} \frac{3c_2 j^{-\alpha}}{c(N+1)^{-\alpha}} = \left(\frac{3c_2}{c_1}\right)^N (N+1)^{\alpha N} (N!)^{-\alpha}\\ &\leqq \left(\frac{3c_2}{c_1}\right)^N (N+1)^{\alpha N} \{c_3 N^N (N+1)^{1/2}\, \mathrm{e}^{-N}\}^{-\alpha}\\ &= \left(\frac{3c_2}{c_1}\right)^N c_3^{-\alpha} \left\{\left(\frac{N+1}{N}\right)^N (N+1)^{-1/2}\, \mathrm{e}^N\right\}^{\alpha}\\ &\leqq c_3^{-\alpha}\, \mathrm{e}^{\alpha(N+1)} \left(\frac{3c_2}{c_1}\right)^N.\end{aligned}$$

Thus

$$\log\left\{\prod_{j=1}^{N} \frac{3a_j(T)}{a_{N+1}(T)}\right\} \leqq KN + K_1$$

and hence inequality (2) will be satisfied by any n which is a suitably large multiple of N. From (1) it now follows that $e_n(T) = O(n^{-\alpha})$. The lower bound for $e_n(T)$ is an immediate consequence of the right-hand inequality of (1) ∎

Example B: *Let $T \in K(H_1, H_2)$ and suppose that $a_n(T) \asymp 1/\log(n+1)$. Then $e_n(T) \asymp 1/\log(n+1)$.*

To establish this let $N \geqq 3$. For some constant $K \geqq 1$,

$$\log\left(3^N \prod_{j=1}^{N} \frac{a_j(T)}{a_{N+1}(T)}\right) \leqq N \log K + N \log\log(N+2) - \sum_{j=1}^{N} \log\log(j+1),$$

and since

$$\begin{aligned}\sum_{j=3}^{N} \log\log(j+1) &\geqq \int_3^N \log\log x\, dx\\ &\geqq N\log\log N - 3\log\log 3 - \int_{\log 3}^{\log N} \frac{1}{y}\, \mathrm{e}^y\, dy\\ &\geqq N\log\log N - 3\log\log 3 - \int_{\log 3}^{\log N} \mathrm{e}^y\, dy\\ &= N(\log\log N - 1) - 3(\log\log 3 - 1),\end{aligned}$$

it follows that for some $K_1 > 0$,

$$\log\left(3^N \prod_{j=1}^{N} \frac{a_j(T)}{a_{N+1}(T)}\right) \leqq N(1 + \log K) + K_1 + N \log\left(\frac{\log (N+2)}{\log N}\right).$$

Use of the Mean-Value Theorem shows that

$$\log\log (N+2) - \log\log N = O(N^{-1}),$$

and so for some positive constants K_2, K_3,

$$\log\left(3^N \prod_{j=1}^{N} \frac{a_j(T)}{a_{N+1}(T)}\right) \leqq K_2 N + K_3 \quad \text{for all} \quad N \geqq 3.$$

The result now follows in the same way as Example A ▮

Example C: *Let $T \in K(H_1, H_2)$ and suppose that for some $\alpha > 0$, $a_n(T) \asymp \left(\frac{\log n}{n}\right)^\alpha$. Then $e_n(T) \asymp \left(\frac{\log n}{n}\right)^\alpha$.*

This is established by using techniques similar to those in Examples A and B ▮

Example D: *Let $p, q \in \mathbf{N}$, $n = p + q$, $\sigma > 0$, $\sigma' > 0$. Denote points of $\mathbf{R}^n$ by (x, y), where $x = (x_i) \in \mathbf{R}^p$, $y = (y_j) \in \mathbf{R}^q$, and let $V: \mathbf{R}^n \to \mathbf{R}$ be defined by*

$$V(x, y) = |x|^\sigma \, |y|^{\sigma'}, \quad where \quad |x| = \sum_{i=1}^{p} |x_i|, \quad |y| = \sum_{j=1}^{q} |y_j|.$$

Let

$$H^{1,n}_{\sigma,\sigma'} = \{u \in W^{1,2}(\mathbf{R}^n) : Vu \in L^2(\mathbf{R}^n)\}.$$

Endowed with the norm

$$\|u \mid H^{1,n}_{\sigma,\sigma'}\| := [\|u \mid W^{1,2}(\mathbf{R}^n)\|^2 + \|Vu \mid L^2(\mathbf{R}^n)\|^2]^{1/2},$$

$H^{1,n}_{\sigma,\sigma'}$ is a Hilbert space. DESPLANCHES *has proved in* [3] *that the k^{th} approximation number $a_k(I)$ of the embedding $I: H^{1,n} \to L^2(\mathbf{R}^n)$ has the following properties as $k \to \infty$:*

$$a_k(I) \asymp \begin{cases} k^{-\sigma/p(1+\sigma+\sigma')} & \text{if } q\sigma < p\sigma' \\ (k^{-1}\log k)^{\sigma/p(1+\sigma+\sigma')} & \text{if } q\sigma = p\sigma' \\ k^{-\sigma'/q(1+\sigma+\sigma')} & \text{if } q\sigma > p\sigma'. \end{cases}$$

It follows immediately from Examples A *and* C *that the entropy numbers $e_k(I)$ have exactly the same properties.*

Other inequalities connecting entropy numbers and approximation numbers may also be obtained. For example, an obvious adaptation of Proposition 2.d.3 of KÖNIG [5] shows that the following holds.

Theorem 4: *Let $T \in K(H)$ be self-adjoint. Then for all $k \in \mathbf{N}$,*

$$\mu_k \leqq e_{k+1}(T) \leqq 6\mu_k$$

where

$$\mu_k = \sup\left\{2^{-k/2n}\left(\prod_{j=1}^{n} a_j(T)\right)^{1/n} : n \in \mathbf{N}\right\}.$$

Corollary 5: *Let $T \in K(H_1, H_2)$. Then the conclusion of Theorem 4 holds.*

Proof: Simply combine Theorems 2 and 4 ∎

Corollary 6: *Let $T \in K(H_1, H_2)$. Then $(a_n(T)) \in l^p$ if, and only if, $(e_n(T)) \in l^p$.*

Proof: Similar to that of Corollary 2.d.3 of König [5] ∎

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REFERENCES

[1] Carl, B.: Entropy numbers, s-numbers, and eigenvalue problems. J. Funct. Anal. **41** (1981), 290–306.
[2] Carl, B.: Inequalities of Bernstein-Jackson-type and the degree of compactness of operators in Banach spaces. Ann. Inst. Fourier **35** (1985), 79–118.
[3] Desplanches, R.: $k^{\text{ième}}$ diamètre de classes d'espaces de Sobolev sur $\mathbf{R}^n$ associés à des opérateurs de type "Schrödinger" (to appear).
[4] Edmunds, D. E., and R. M. Edmunds: Entropy numbers of compact operators. Bull. London Math. Soc. 18 (1986), 392–394.
[5] König, H.: Eigenvalue distribution of compact operators. Basel: Birkhäuser Verlag 1986.
[6] Simon, B.: Trace ideals and their applications (London Math. Soc. Lect. Note Series 35). Cambridge: University Press 1979.

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