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An Explicit Expression for the Korteweg-de Vries Hierarchy

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Für die Korteweg-de Vries-Hierarchie von solitonischen partiellen Differentialgleichungen wird eine bisher nicht bekannte explizite Darstellung hergeleitet.

Выводится явное и до сих пор неизвестное представление для иерархии Кортевега-де Фриза солитоновых дифференциальных уравнений.

An explicit representation for the Korteweg-de Vries hierarchy of solitonic partial differential equations is derived which has not been known before.

Introduction

A nonlinear partial differential equation is called *solitonic* if it admits

- particle-like solutions, the so-called solitons;
- Bäcklund transformations;
- a Lax and/or prolongation representation;
- application of the inverse scattering method;
- infinitely many conservation laws.

The typical solitonic equations named after Korteweg-de Vries, Burgers, Boussinesq, Kadomtsev-Petviashvili, ..., have been discovered as physical models. The deeper mathematical reasons for their highly peculiar behaviour were difficult to recognize. Some progress has been achieved by extending a "seed equation" to a "hierarchy", that means to an infinite sequence of related solitonic equations of increasing order. The hierarchies considerably enlarge the reservoir of differential equations in soliton theory.

P. D. Lax [8] proposed to extend the Korteweg-de Vries equation

$$u_t = 6uu_x + u_{xxx}$$

for a function $u = u(x, t)$ of position x and time t to a hierarchy

$$u_t = u_3 + 6uu_1,$$
$$u_t = u_5 + 10uu_3 + 20u_1u_2 + 30u^2u_1,$$
$$u_t = u_7 + 14uu_5 + \cdots,$$
$$\vdots$$

The general law for this

$$u_t = \frac{\partial}{\partial x} G_n[u] \qquad (n = 2, 3, \ldots)$$

usually is described by recursion relations for the differential polynomials $G_n = G_n[u]$ ($n = 1, 2, \ldots$). (A differential polynomial in u is a polynomial in $u, u_1, u_2, \ldots$ with constant coefficients and without an absolute term.) I. M. Gel'fand and L. A. Dikij [16] presented formulas for the G_n which become explicit when certain multiple integrals or generating functions, respectively, are evaluated. The purpose of this paper is to derive a fully explicit representation for the right-hand sides of the Korteweg-de Vries hierarchy, namely

$$\begin{aligned} & n! \left[(2n-1)!\right]^{-1} G_n[u] \\ & = \sum \left[(q_2 + n - 1)(q_3 + n - 2) \ldots (q_n + 1)\right]^{-1} \\ & \quad \times c(0, q_2)\, c(q_2, q_3) \ldots c(q_{n-1}, q_n)\, u_{-q_2} u_{q_2 - q_3} \ldots u_{q_{n-1} - q_n} u_{q_n}. \end{aligned}$$

We use the notations

$$u_t = \frac{\partial u}{\partial t}, \qquad u_1 = u_x = \frac{\partial u}{\partial x}, \qquad u_p = u_{xx\ldots x} = \frac{\partial^p u}{\partial x^p} \qquad (p \geqq 2)$$

and also formally

$$u_0 = u, \qquad u_{-1} = 0, \qquad u_{-2} = 1.$$

The numerical coefficients above are given by

$$c(p, q) = \binom{p}{q} + \delta_q^{\,p+2} \quad \text{for integers } p, q \geqq 0$$

and the sum runs over all integers $q_2, q_3, \ldots, q_n$ such that

$$0 \leqq q_2 \leqq 2,\; 0 \leqq q_3 \leqq q_2 + 2, \ldots, 0 \leqq q_n \leqq q_{n-1} + 2.$$

In § 1 we develop the definition of the sequence $(G_n) = (G_n[u])_{n \geqq 0}$ and introduce the Korteweg-de Vries hierarchy through Lax pairs. The calculation behind this has been done essentially by J. L. Burchnall and T. W. Chaundy [2] as early as in 1922. Their priority is an interesting historical fact we would like to emphasize. The § 2 is devoted to the proof of our formula. To be precise, a somewhat more general formula concerning Hadamard's coefficients to the one-dimensional Schrödinger operator is established. Use of the Minakshisundaram-Pleijel asymptotic expansion of the fundamental solution to the heat equation is made. In § 3 we present, for the sake of completeness, additional properties of the $G_1, G_2, \ldots$ and of differential polynomials which are closely related to the $G_1, G_2, \ldots$

§ 1 Definition of the Korteweg-de Vries hierarchy

The construction starts with the one-dimensional Schrödinger equation (sometimes also called Sturm-Liouville equation)

$$(L + k^2)\, y = y'' + (u + k^2)\, y = 0. \tag{1.1}$$

Here $L = D^2 + u$ is the one-dimensional Schrödinger operator; $D = \partial/\partial x$; the energy constant k is assumed to be real and nonzero; the potential $u = u(x)$ is assumed to be real-valued and smooth in some interval. (An additional dependence on a time t is introduced later.) Using standard arguments from the literature (cf., e.g., [17, 18]) one can construct a formal solution to (1.1) of the form

$$y = e^{ikx} \sum_{n=0}^{\infty} z_n (2ik)^{-n} \tag{1.2}$$

with real coefficients $z_0 = 1, z_1, z_2, \ldots$ From it we build up

$$G := k^{-1} y \bar{y} =: 4 \sum_{n=0}^{\infty} G_n (-1)^n (2k)^{-2n-1}. \tag{1.3}$$

(The power series may converge or not; in fact they serve as generating functions for the sequences of their coefficients.) Here the G_n are again real and $G_0 = 1/2$. Considering the first integral

$$y' \bar{y} - y \bar{y}' = \text{const.} = 2ik,$$

we obtain from (1.1) the nonlinear second-order equation

$$2GG'' - G'^2 + 4(u + k^2)\, G^2 = 4.$$

Differentiation with respect to x produces the linear third-order equation

$$G''' + 4(u + k^2)\, G' + 2u'G = 0. \tag{1.4}$$

Equivalent to this, the sequence $(G_n)_{n \geq 0}$ obeys a differential-recursion equation system which has been named after A. Lenard [3, 9, 10]:

$$G'_{n+1} = G_n''' + 4uG_n' + 2u'G_n, \qquad G_0 := 1/2. \tag{1.5}$$

The next step will be the construction of certain linear differential operators A_n with respect to x. Note that we do not notationally distinguish between a function and the multiplication operator (or linear differential operator of zero order) defined by it.

Proposition 1.1: *The operator-valued formal power series A in $(2k)^{-1}$ defined by*

$$4A = (G' - 2GD)\,(L + k^2)^{-1} \tag{1.6}$$

satisfies

$$AL - LA \equiv [A, L] = G'. \tag{1.7}$$

Proof: We calculate

$$\begin{aligned} 4[A, L]\,(L + k^2) &= 4[A, L + k^2]\,(L + k^2) = [G' - 2GD, L + k^2] \\ &= [G' - 2GD, L] = \cdots = 4G'D^2 - (G'' + 2u'G) = 4G'(D^2 + u + k^2) \\ &= 4G'(L + k^2). \end{aligned}$$

The factor $(L + k^2)$ in the first and in the last expression can be cancelled because it possesses an inverse in the ring of formal power series in $(2k)^{-1}$ ∎

Proposition 1.2: *The linear differential operators A_n defined by*

$$A = 4 \sum_{n=0}^{\infty} A_n (-1)^n (2k)^{-2n-1} \tag{1.8}$$

admit the recursive representation

$$A_{n+1} = 4A_n L + 2G_n D - G_n', \qquad A_0 = 0, \qquad A_1 = D \tag{1.9}$$

as well as the explicit representation

$$A_{n+1} = \sum_{m=0}^{n} (2G_m D - G_m') (4L)^{n-m}. \tag{1.10}$$

Each operator A_n is formally anti-selfadjoint, i.e. $A_n^* = -A_n$. (Here $*$ means the formal adjoint of a linear differential operator.) Further, there holds

$$A_n L - L A_n \equiv [A_n, L] = G_n'. \tag{1.11}$$

Proof: The equations (1.9), (1.11) for the sequences (G_n), (A_n) are equivalent to the equations (1.6), (1.7) for their generating functions G, A, respectively. Then (1.10) and $A_n + A_n^* = 0$ follow from (1.9) by mathematical induction with respect to n. Considering $L^* = L$ and $D^* = -D$ we obtain in the induction step $A_{n+1} + A_{n+1}^* = \cdots = 4[A_n, L] - 4G_n' = 0$ ∎

Our introduction of the differential operators A_n is a standard one; cf., e.g., O. I. BOGOJAVLENSKIJ [15]. A considerably older construction is due to J. L. BURCHNALL and T. W. CHAUNDY [2]; these authors directly worked with the sequences (G_n), (A_n).

Now is the time to introduce the Korteweg-de Vries hierarchy.

Definition: Let $u = u(x, t)$ and all objects constructed from u depend on the time t as a parameter. The partial differential equation for $u = u(x, t)$ in the Lax representation

$$\frac{\partial L}{\partial t} = [A_n, L] \quad (n = 2, 3, \ldots) \tag{1.12}$$

or in the equivalent function representation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} G_n[u] \quad (n = 2, 3, \ldots) \tag{1.13}$$

is called *n-th Korteweg-de Vries equation*. The sequence of these equations is called *Korteweg-de Vries hierarchy*.

Let us recall that two linear differential operators A, L depending on t as a parameter form a *Lax pair* if

$$\frac{\partial L}{\partial t} = [A, L] \equiv AL - LA.$$

Here $\partial L/\partial t$ is defined by the Leibniz rule

$$\frac{\partial}{\partial t}(Ly) =: \frac{\partial L}{\partial t} y + L \frac{\partial y}{\partial t}.$$

Particularly, for $L = D^2 + u$ the operator $\partial L/\partial t$ is the multiplication by $\partial u/\partial t$; hence (1.12) and (1.13) are equivalent.

Example: The sequence (G_n) begins as

$$G_0 = 1/2, \qquad G_1 = u, \qquad G_2 = u_2 + 3u^2,$$
$$G_3 = u_4 + 10uu_2 + 5u_1^2 + 10u^3,$$
$$G_4 = u_6 + 14uu_4 + 28u_1u_3 + 21u_2^2 + 70u^2u_2 + 70uu_1^2 + 35u^4.$$

These expressions are given in [4, 11, 14] and in other papers. The sequence of differential operators (A_n) begins as

$$A_1 = D, \qquad A_2 = 4D^3 + D3u + 3uD,$$
$$A_3 = 16D^5 + D^3 20u + 20uD^3 + D5(3u^2 - u_2) + 5(3u^2 - u_2)\, D.$$

These are given in [17] and in other papers.

§ 2 Derivation of the explicit expression

The time s used in this paragraph has nothing to do with the time t in the Korteweg-de Vries hierarchy and is, therefore, notationally distinguished.

Definition: The *fundamental solution* $K = K(x, x_0, s)$ *of the heat equation*

$$\frac{\partial K}{\partial s} = LK = D^2 K + uK \tag{2.1}$$

or, shortly, the *heat kernel* is defined for $s > 0$ and $x, x_0 \in I$, where $I \subsetneqq R$ is some open interval. Moreover, the function

$$H = H(x, x_0, s) = (4\pi s)^{1/2} \exp\left(\frac{(x - x_0)^2}{4s}\right) K(x, x_0, s)$$

is required to admit a smooth extension to $s \geqq 0$ such that $H(x, x_0, 0) = 1$.

Proposition 2.1: *If $u = u(x)$ is defined and smooth in some interval, then there exists a subinterval I and a heat kernel $K = K(x, x_0, s)$ for $x, x_0 \in I$ and $s > 0$. This is uniquely determined, smooth and symmetric in its arguments x, x_0, that means*

$$K(x, x_0, s) = K(x_0, x, s).$$

The corresponding function $H = H(x, x_0, s)$ satisfies

$$\left[(x - x_0)\, D + s\,\frac{\partial}{\partial s}\right] H = sLH. \tag{2.2}$$

For the proof of existence, unicity, and of further properties of K we refer to the literature [1, 12]. In fact, the construction of the heat kernel works for quite more general second-order differential operators L. In our case, the formula (2.2) is equivalent to the heat equation (2.1).

The following proposition is essentially due to H. P. McKean and P. van Moerbeke [11]. Our proof is a simplified version of that in [11].

Proposition 2.2: *The "diagonal values" $H(x, s) := H(x, x, s)$ obey*

$$2\left(2s\,\frac{\partial}{\partial s} - 1\right) DH(x, s) = s(D^3 + 2Du + 2uD)\, H(x, s). \tag{2.3}$$

Proof: We apply the operator $D + 3D_0$ to

$$\left[(x - x_0)\, D + s\,\frac{\partial}{\partial s}\right] H = sLH$$

and, analogously, the operator $D_0 + 3D$ to

$$\left[(x_0 - x) D_0 + s \frac{\partial}{\partial s}\right] H = sL_0 H,$$

where

$$D = \frac{\partial}{\partial x}, \qquad D_0 = \frac{\partial}{\partial x_0}, \qquad L = D^2 + u(x), \qquad L_0 = D_0{}^2 + u(x_0).$$

The second equation for $H = H(x, x_0, s)$ follows from the first and from the symmetry in x, x_0. We add the results and restrict then to the diagonal $x = x_0$ of $I \times I$. The identities

$$(D + 3D_0)(x - x_0) D + (D_0 + 3D)(x_0 - x) D_0$$
$$= -2(D + D_0) + (x - x_0)(D^2 - D_0{}^2),$$
$$(D + 3D_0) D^2 + (D_0 + 3D) D_0{}^2 = (D + D_0)^3$$

are taken into consideration as well as the rules

$$[(D + D_0) y](x, x) = d[y(x, x)], \qquad [(D + D_0)^3 y](x, x) = d^3[y(x, x)],$$

for two-point functions $y = y(x, x_0)$, where $d = d/dx$ denotes the total derivative with respect to x. We arrive at

$$2\left(2s \frac{\partial}{\partial s} - 1\right) dH(x, x, s) = s(d^3 + 4ud + 2u_1) H(x, x, s),$$

which is equivalent to (2.3) ∎

The following classical construction is connected with the names S. MINAKSHISUNDARAM and A. PLEIJEL [12].

Proposition 2.3: *Let a sequence of two-point quantities* $H_n = H_n(x, x_0)$ $(n = 0, 1, \ldots)$ *be defined by*

$$[(x - x_0) D + n] H_n = LH_{n-1} \quad \textit{for} \ n \geqq 1, \qquad H_0 = 1. \tag{2.4}$$

With this there holds the asymptotic expansion

$$H(x, x_0, s) \sim \sum_{n=0}^{\infty} H_n(x, x_0) s^n \quad \textit{for} \ s \to +0.$$

For a proof cf. [1, 11, 12].

In fact, the differential-recursion equation system (2.4) has a unique smooth solution $H_n = H_n(x, x_0)$ $(n = 0, 1, \ldots)$ in $I \times I$, I here denoting the domain of definition of u, which is recursively given by

$$H_n(x, x_0) = \int_0^1 \lambda^{n-1}(LH_{n-1})\left(\lambda x + (1 - \lambda) x_0\right) d\lambda \quad \text{for } n \geqq 1.$$

This can be directly verified or already follows from classical considerations by J. HADAMARD [6] for more general second-order differential operators L. We will call, following [5], the $H_n = H_n(x, x_0)$ $(n = 0, 1, \ldots)$ "Hadamard's coefficients".

Proposition 2.4: *The one-point quantities*

$$H_n{}^p = H_n{}^p(x) := (D^p H_n)(x, x) \quad \textit{for} \ p, n \geqq 0$$

derived from the two-point quantities $H_n = H_n(x, x_0)$ *obey the algebraic recursion system*

$$(p + n) H_n{}^p = H_{n-1}^{p+2} + \sum_{q=0}^{p} \binom{p}{q} u_{p-q} H_{n-1}^q \quad \text{for} \quad n \geqq 1, \quad H_0{}^p = \delta_0{}^p. \qquad (2.5)$$

As a consequence, each $H_n{}^p$ *for* $n \geqq 1$ *is a differential polynomial in* u, *that means a polynomial in* $u, u_1, u_2, \ldots$ *with constant coefficients and without an absolute term.*

Proof: We apply $D^p = (\partial/\partial x)^p$ to both sides of the equation (2.4) and restrict then to the diagonal $x = x_0$ of $I \times I$. Throughout the paper $\delta_q{}^p$ denotes the Kronecker symbol ∎

Proposition 2.5: *The diagonal values of Hadamard's coefficients are proportional to the quantities* G_n *of* § 1, *more precisely*

$$2(n!)\, G_n[u]\,(x) = (2n)!\, H_n(x, x) \quad \text{for} \quad n \geqq 0. \qquad (2.6)$$

Proof: The equation (2.3) for

$$H(x, s) = \sum_{n=0}^{\infty} H_n(x, x)\, s^n$$

implies the differential-recursion system

$$2(2n - 1)\, DH_{n+1}^0 = (D^3 + 2uD + 2Du)\, H_n{}^0, \qquad H_0{}^0 = 1.$$

The comparison with

$$DG_{n+1} = (D^3 + 2uD + 2Du)\, G_n, \qquad G_0 = 1/2$$

shows (2.6) by mathematical induction. The sequences $(H_n{}^0)$, (G_n) are uniquely determined by the above recursions and by the additional property of being differential polynomials in u ∎

In the following, we use the notations

$$u_0 = u, \qquad u_{-1} = 0, \qquad u_{-2} = 1, \qquad c(p, q) = \binom{p}{q} + \delta_q{}^{p+2}$$

for integers $p, q \geqq 0$.

Theorem: *There holds for* $n \geqq 2$

$$\begin{aligned} H_n{}^p = \sum_{q_2, q_3, \ldots, q_n} & [(q_1 + n)(q_2 + n - 1) \ldots (q_n + 1)]^{-1} \\ & \times c(q_1, q_2)\, c(q_2, q_3) \ldots c(q_{n-1}, q_n) \\ & \times u_{q_1 - q_2} u_{q_2 - q_3} \ldots u_{q_{n-1} - q_n} u_{q_n}, \end{aligned} \qquad (2.7)$$

where $q_1 = p$ *and where the sum runs according to*

$$0 \leqq q_2 \leqq q_1 + 2,\ 0 \leqq q_3 \leqq q_2 + 2, \ldots, 0 \leqq q_n \leqq q_{n-1} + 2. \qquad (2.8)$$

Corollary 1: *There holds for* $n \geqq 1$

$$H_n{}^p = (n!)^{-1} \binom{2n + p - 1}{n}^{-1} u_{2n+p-2} + \cdots \qquad (2.9)$$

where the points ... indicate terms in u of lower order and higher degree. Further, there holds for $n \geqq 1$

$$H_n^p = \sum [(r_1 + 1)(r_1 + r_2 + 2) \dots (r_1 + \dots + r_n + n)]^{-1} \times \binom{p}{r_1, r_2, \dots, r_n} u_{r_1} u_{r_2} \dots u_{r_n} + \cdots \qquad (2.10)$$

where the sum runs over all integers $r_1, r_2, \dots, r_n \geqq 0$ *such that* $r_1 + r_2 + \dots + r_n = p$ *and where the points ... indicate terms of lower degree. The symbol* $\binom{p}{r_1, \dots, r_n} = p!/(r_1! \dots r_n!)$ *is the usual polynomial coefficient.*

Corollary 2: *There holds for* $n \geqq 1$

$$nH_n^0 = \sum [(q_2 + n - 1)(q_3 + n - 2) \dots (q_n + 1)]^{-1} \times c(0, q_2)\, c(q_2, q_3) \dots c(q_{n-1}, q_n)\, u_{-q_2} u_{q_2 - q_3} \dots u_{q_{n-1} - q_n} u_{q_n}, \qquad (2.11)$$

where the sum runs over all integers $q_2, q_3, \dots, q_n$ *according to* (2.8).

Proofs: With our special notations the recursion system (2.5) can be formally simplified to

$$H_n^p = \sum_{q=0}^{p+2} (p + n)^{-1} c(p, q)\, u_{p-q} H_{n-1}^q \quad \text{for } n = 2,$$

$$H_0^p = \delta_0^p, \qquad H_1^p = (p + 1)^{-1} u_p.$$

Hence (2.7) follows by mathematical induction with respect to n. The first assertion (2.3) of Corollary 1 is better shown by a mathematical induction directly applied to (2.5). The terms of maximal degree in H_n^p appear for

$$r_n := q_1 - q_2 \geqq 0, \dots, r_2 := q_{n-1} - q_n \geqq 0, r_1 := q_n \geqq 0,$$

$$c(q_1, q_2) = \binom{q_1}{q_2}, \quad c(q_2, q_3) = \binom{q_2}{q_3}, \dots$$

$$c(q_1, q_2)\, c(q_2, q_3) \dots c(q_{n-1}, q_n) = \binom{q_1}{r_1, r_2, \dots, r_n}.$$

This gives the second assertion (2.10) of Corollary 1. In order to obtain Corollary 2 we have to insert into (2.7) the particular value $p = q_1 = 0$. Our formulas (2.6), (2.11) together give the explicit expression for G_n quoted in the introduction ∎

Example: There holds

$$(p + 1) H_1^p = u_p,$$

$$(p + 1)(p + 2)(p + 3) H_2^p = (p + 1) u_{p+2} + (p + 3) \sum_{q=0}^{p} \binom{p + 1}{q + 1} u_q u_{p-q}.$$

J. E. LAGNESE [7] explicitly calculated H_3^p too in this fashion; we omit here this lengthy expression.

§ 3 Other properties of the differential polynomials

The following notions have been introduced in [13] and are useful to describe differential polynomials associated to the Korteweg-de Vries hierarchy.

Definition: A (differential) monomial

$$cu^{k_0}u_1^{k_1}\dots u_p^{k_p}$$

with integers $p \geqq 0$, $k_0 \geqq 0, \dots, k_{p-1} \geqq 0$, $k_p \geqq 1$ has the *order* p, the *degree*

$$d := k_0 + k_1 + \dots + k_p,$$

and the *weight*

$$w := 2k_0 + 3k_1 + \dots + (p+2)\,k_p.$$

A differential polynomial has as its *order, degree, weight* the maximum of the orders, degrees, weights, respectively, of its monomials. A differential polynomial is called *homogeneous* if all its monomials have the same weight.

Proposition 3.1: *The differential polynomial H_n^p for $n \geqq 1$ is homogeneous of weight $2n+p$, has the degree n and the order $2n+p-2$. The differential polynomial G_n for $n \geqq 1$ is homogeneous of weight $2n$, has the degree n and the order $2n-2$.*

The proof is done by mathematical induction with respect to n ∎

The dependence of our differential polynomials on the variable $u = u_0$ is fully known.

Proposition 3.2: *There holds for $n \geqq 1$*

$$G_n = \sum_{m=0}^{n} \binom{n-1/2}{n-m} (4u)^{n-m}\,[G_m]_{u=0}, \tag{3.1}$$

$$H_n^p = \sum_{m=0}^{n} [(n-m)!]^{-1}\,u^{n-m}[H_m^p]_{u=0}. \tag{3.2}$$

Proof: We define

$$\tilde u = \tilde u(x, x_0) = u(x) - u(x_0), \qquad \tilde k = k\big(1 + k^{-2}u(x_0)\big)^{1/2}$$

and indicate objects belonging to $\tilde u$, $\tilde k$ by $\sim$. There holds $\tilde G = G$ because this generating function depends only on $\tilde u + \tilde k^2 = u + k^2$. Inserting

$$(2\tilde k)^{-2n-1} = \sum_{m=0}^{\infty} \binom{n+m-1/2}{m} \big(-4u(x_0)\big)^m (2k)^{-2(n+m)-1}$$

into

$$\sum_{n=0}^{\infty} \tilde G_n(-1)^n (2\tilde k)^{-2n-1} = \sum_{n=0}^{\infty} G_n(-1)^n (2k)^{-2n-1}$$

we obtain

$$G_n = \sum_{m=0}^{n} \binom{n-1/2}{n-m} \big(4u(x_0)\big)^{n-m} \tilde G_m,$$

which is equivalent to (3.1). The heat kernel transforms as

$$K(x, x_0, s) = e^{u(x_0)s}\tilde K(x, x_0, s).$$

Herefrom follows, step by step,

$$H(x, x_0, s) = e^{u(x_0)s} \tilde{H}(x, x_0, s),$$

$$H_n(x, x_0) = \sum_{m=0}^{n} [(n-m)!]^{-1} u(x_0)^{n-m} \tilde{H}_m(x, x_0),$$

$$H_n{}^p(x_0) = \sum_{m=0}^{n} [(n-m)!]^{-1} u(x_0)^{n-m} \tilde{H}_m{}^p(x_0).$$

The last formula is equivalent to (3.2) ∎

Proposition 3.3: *There holds for $n \geqq 1$*

$$G_n = u_{2n-2} + \sum_{p+q=2n-4} \left\{ \binom{2n-2}{p+1} + (-1)^p \right\} u_p u_q + \cdots, \tag{3.3}$$

where the points ... indicate for $n \geqq 3$ terms of degree greater than 2 and of order less than $2n-5$. As a consequence, we have for the right-hand sides of the Korteweg-de Vries hierarchy

$$\frac{\partial}{\partial x} G_n = u_{2n-1} + \sum_{p+q=2n-3} \binom{2n-1}{p+1} u_p u_q + \cdots. \tag{3.4}$$

The sums in (3.3), (3.4) begin with $p = 0$ and end with $q = 0$. Further, there holds for $n \geqq 6$

$$\begin{aligned} 60 \binom{2n}{n}^{-1} G_n = {} & 30u^n + 10 \binom{n}{2} u^{n-2} u_2 + 3 \binom{n}{3} u^{n-3}(u_4 + 5u_1^2) \\ & + 6 \binom{n}{4} u^{n-4}(4u_1u_3 + 3u_2^2) + 110 \binom{n}{5} u^{n-5} u_1^2 u_2 \\ & + 75 \binom{n}{6} u^{n-6} u_1^4 + \cdots, \end{aligned} \tag{3.5}$$

where the points ... indicate terms of degree less than $n-2$. This expression is also valid for $n \geqq 0$ if terms with negative powers of u are formally omitted. As a consequence, we have for the right-hand sides of the Korteweg-de Vries hierarchy

$$\begin{aligned} 60 \binom{2n}{n}^{-1} \frac{\partial}{\partial x} G_n = {} & 30nu^{n-1}u_1 + 10 \binom{n}{2} u^{n-2} u_3 \\ & + 3 \binom{n}{3} u^{n-3}(u_5 + 20u_1u_2 + 5u_1^3) \\ & + 12 \binom{n}{4} u^{n-4}(3u_1u_4 + 5u_2u_3) \\ & + 10 \binom{n}{5} u^{n-5}(23u_1^2u_3 + 31u_1u_2^2) \\ & + 960 \binom{n}{6} u^{n-6} u_1^3 u_2 + \cdots. \end{aligned}$$

Proof: Let L_n denote the linear part of G_n and Q_n the quadratic part, respectively. These obey the differential-recursion system

$$L_{n+1} = D^2 L_n, \qquad L_1 = u,$$
$$DQ_{n+1} = D^3 Q_n + 2(Du + uD) L_n, \qquad Q_2 = 3u^2.$$

Mathematical induction shows

$$L_n = u_{2n-2}, \qquad Q_n = \sum_{p+q=2n-4} \left\{ \binom{2n-2}{p+1} + (-1)^p \right\} u_p u_q .$$

All other terms in G_n must have an order less than $2n - 5$. Considering

$$4^{n-m} \binom{n-1/2}{n-m} = \binom{2n}{n} \binom{n}{m} \binom{2m}{m}^{-1}$$

we rewrite (3.1) as

$$\binom{2n}{n}^{-1} G_n = \sum_{m=0}^{n} \binom{2m}{m}^{-1} \binom{n}{m} u^{n-m} [G_m]_{u=0}$$

and insert herein

$$G_0 = 1/2, \qquad G_1 = u, \qquad G_2 = u_2 + 3u^2,$$
$$[G_3]_{u=0} = u_4 + 5u_1^2, \qquad [G_4]_{u=0} = 28u_1u_3 + 21u_2^2 + \cdots,$$
$$[G_5]_{u=0} = 462u_1^2u_2 + \cdots, \qquad [G_6]_{u=0} = 1155u_1^4 + \cdots,$$

where the points indicate terms which do not contribute to (3.5). The result (3.3) is due to P. B. Gilkey [4], who established it by a quite other method, not being aware of the relation to the Korteweg-de Vries hierarchy ∎

More properties of the sequence (G_n) can be found in the literature [3, 8–10, 16, 17]. In [16] it is shown that

$$\frac{\partial}{\partial u} G_{n+1} = \frac{\delta}{\delta u} G_{n+1} = 2(2n+1) G_n,$$

where

$$\frac{\delta}{\delta u} := \sum_{p \geqq 0} (-D)^p \frac{\partial}{\partial u_p}$$

denotes the variational or Euler-Lagrange derivative. For each couple of integers $m, n \geqq 1$ the quantity G_m is a conserved density of the n-th Korteweg-de Vries equation $u_t = G_n[u]_x$ [3, 9, 10, 16].

We will finish with a formula which merely follows from the symmetry property $H_n(x, x_0) = H_n(x_0, x)$. The proof runs along the lines of [7] and will, therefore, be omitted.

Proposition 3.4: *There holds for integers $n \geqq 1$, $k \geqq 0$*

$$2H_n^{2k+1} = D \sum_{q=0}^{2k} \binom{2k+1}{q+1} (-D)^q H_n^{2k-q}.$$

As a consequence, H_n^{2k+1} can be expressed as a linear differential expression in H_n^0, $H_n^2, \ldots, H_n^{2k}$.

The relevance of the quantities G_n $(n = 1, 2, \ldots)$ for the theory of Huygens' principle is discussed in the paper [14]. If the wave-like operator in $2m + 2$ dimensions

$$L - \Delta \equiv \frac{\partial^2}{\partial x^2} + u(x) - \sum_{i=2}^{2m+2} \frac{\partial^2}{\partial x_i^2}$$

is a Huygens-type one, then $G_n[u] = 0$ for $n \geqq m$.

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