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On the Spectrum of Schrödinger Operators at the Half Space with a Certain Class of Boundary Conditions

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Es wird das Spektrum von freien Schrödinger-Operatoren auf dem Halbraum mit Randbedingung $\varphi_n - Q\varphi = 0$ (φ_n ist die Normalableitung, Q ein selbstadjungierter Operator auf dem Rand) untersucht und ein Zusammenhang zwischen seinem negativen Teil und dem Spektrum einer Familie von Operatoren vom Klein-Gordon-Typ hergestellt.

Рассматривается спектр свободных шрёдингеровских операторов на полупространстве с граничным условием $\varphi_n - Q\varphi = 0$ (здесь φ_n обозначает нормальную производную, а Q — некоторый самосопряженный оператор на границе) и устанавливается связь между его отрицательной частью и спектром семейства операторов типа Клейна-Гордона.

The spectrum of free Schrödinger operators at the half space with boundary condition $\varphi_n - Q\varphi = 0$ (φ_n being the normal derivative, Q a self-adjoint operator at the boundary) is investigated and a connection between its negative part and the spectrum of a family of Klein-Gordon type operators is stated.

0. Introduction

For the understanding of surface effects it is useful to consider the motion of particles in domains with position-dependent boundary conditions. In the one-dimensional case it has been shown that the operator $H_q = -d^2/dx^2$ with boundary conditions $\varphi'(0) = q\varphi(0)$, $q \in \mathbf{R}$, is the norm resolvent limit of $-d^2/dx^2 + nV(nx)$ with Neumann boundary conditions for $n \to \infty$, where V is an L_1-function satisfying $\int_0^\infty V(x)\,dx = q$ (see [1]). One can conjecture that an analogous property holds in the multidimensional case, when $H_Q = -\Delta$ in $L_2(\mathbf{R}^{n-1} \times \mathbf{R}_+)$ with boundary conditions $\partial\varphi/\partial x_n|_{x_n=0} = Q\varphi|_{x_n=0}$, where Q is a multiplication operator representing the action of boundary forces. However, in this paper we will not restrict ourselves to multiplication operators. Thus our results may be applied to the case of non-local boundary forces, too. A detailed analysis of H_Q with convolution-type operators Q will be provided in a forthcoming paper [7]. Our main result, the statement of a connection between the spectra of H_Q and $K_{Q,E} = (-\Delta - E)^{1/2} + Q$ in $L_2(\mathbf{R}^{n-1})$, $E < 0$, is formulated and proved in Section 1. This connection enables us to make use of the theory of pseudodifferential and, particularly, Klein-Gordon operators, which took a rapid development in the recent years (see, e.g., [4, 5, 10–12]). Section 2 contains two propositions on the applicability of our Theorem for certain classes of functions Q. The last Section 3 is devoted to the proof of some technical lemmata. Applications of the results of the present paper will be published in [2, 3, 8].

1. The main result

In the following we will use the notations

$\mathbf{R}_+^n = \mathbf{R}^{n-1} \times \mathbf{R}_+$,

$\|\cdot\|$, $\|\cdot\|'$ for the norms of $L_2(\mathbf{R}_+^n)$, $L_2(\mathbf{R}^{n-1})$, resp.,

Δ, Δ' for the Laplacians in $\mathbf{R}_+^n$, $\mathbf{R}^{n-1}$, resp.,

$$H_Q = -\Delta,$$

$$\begin{aligned} D(H_Q) = \{\varphi \in L_2(\mathbf{R}_+^n)\colon \Delta\varphi \in L_2(\mathbf{R}_+^n),\\ \operatorname*{l.i.m.}_{h\searrow 0} (\partial/\partial x_n)\, \varphi(x_1, \ldots, x_{n-1}, h) \in L_2(\mathbf{R}^{n-1}),\\ \operatorname*{l.i.m.}_{h\searrow 0} Q\varphi(x_1, \ldots, x_{n-1}, h) \in L_2(\mathbf{R}^{n-1}),\\ \operatorname*{l.i.m.}_{h\searrow 0} (\partial\varphi/\partial x_n - Q\varphi)|_{x_n=h} = 0\}, \end{aligned} \tag{1}$$

$$K_{Q,E} = (-\Delta' - E)^{1/2} + Q. \tag{2}$$

$$\mathring{\varphi}(p, x) = \frac{1}{(2\pi)^{(n-1)/2}} \int\limits_{\mathbf{R}^{n-1}} e^{-i(p,y)} \varphi(y, x)\, d^{n-1}y \tag{3}$$

for the Fourier transform over the first $n - 1$ variables.

Now we can state

Theorem 1: *Let $Q = Q^*$ be $K_{0,0}$-bounded with a relative bound less than 1. Then*
(i) *H_Q is self-adjoint,*
(ii) $[0, \infty) \subset \sigma(H_Q)$,
(iii) $0 > E \in \sigma(H_Q)$ *iff* $0 \in \sigma(K_{Q,E})$,
(iv) $0 > E \in \sigma_{\mathrm{pp}}(H_Q)$ *iff* $0 \in \sigma_{\mathrm{pp}}(K_{Q,E})$,
(v) $0 > E \in \sigma_{\mathrm{ess}}(H_Q)$ *iff* $0 \in \sigma_{\mathrm{ess}}(K_{Q,E})$,
where σ, σ_{pp} and *σ_{ess} denote the spectrum, the pure point spectrum and the essential spectrum, respectively.*

Proof: (i) $D(H_Q)$ is dense in $L_2(\mathbf{R}_+^n)$, since $C_0^\infty(\mathbf{R}_+^n) \subset D(H_Q)$. Equipped with the norm

$$\|\varphi\|_Q = \|\varphi\| + \|\Delta\varphi\| + \lim_{h\searrow 0} (\|Q\varphi|_{x_n=h}\|' + \|\partial\varphi/\partial x_n|_{x_n=h}\|'),$$

$C_0^\infty(\mathbf{R}_+^n)$ can be completed to a Banach space M_Q which contains $D(H_Q)$. Since $D(H_Q)$ is the kernel of a continuous map from M_Q into $L_2(\mathbf{R}^{n-1})$, it is closed, and so is H_Q. From Gauss' Theorem (cf. [6]) it follows that H_Q is symmetric. On the other hand, it is well known that $\overline{(H_Q \pm i)\, C_0^\infty(\mathbf{R}_+^n)} = L_2(\mathbf{R}_+^n)$, which implies (i).

(ii) can be verified by taking C_0^∞-test functions, which approximate plane waves.

(iii) Here we need some technical lemmata, which we will prove in Section 3:

Lemma 1: *Let $\varphi \in D(H_Q)$ and $\phi = \varphi|_{x_n=0}$. Then, for $E < 0$,*

$$\|(H_Q - E)\, \varphi\| \geqq 2^{1/2}\, \|K_{0,E}^{1/2} K_{Q,E} \phi\|' \tag{4}$$

and

$$\left| \|\varphi\| - \frac{1}{2^{1/2}}\, \|K_{0,E}^{-1/2} \phi\|' \right| \leqq \frac{1}{-E}\, \|(H_Q - E)\, \varphi\|. \tag{5}$$

Lemma 2: *Let $\phi \in D(K_{0,E}^{3/2})$, $E < 0$. Then there exists a function $\varphi \in D(H_Q)$, with $\varphi|_{x_n=0} = \phi$, satisfying*

$$\|(H_Q - E)\,\varphi\| = 2^{1/2}\,\|K_{0,E}^{1/2} K_{Q,E}\phi\|' \tag{6}$$

and

$$\left|\|\varphi\| - \frac{1}{2^{1/2}}\,\|K_{0,E}^{1/2}\phi\|'\right| \leqq \frac{1}{2(-E)^{3/4}}\,\|K_{Q,E}\phi\|'. \tag{7}$$

Lemma 3: *Let $A = A^*$ and $0 \in \sigma(A)$. Then for all B_1, B_2, with*

a) *B_1 and B_2^* relatively bounded with respect to A,*

b) *B_2^{-1} bounded,*

it holds that

$$0 \in \sigma(B_1 A B_2). \tag{8}$$

Now we continue the proof of Theorem 1/(iii).

1. Let $0 > E \in \sigma(H_Q)$. Then there exists a sequence $(\varphi_k) \subset D(H_Q)$ with $\|\varphi_k\| = 1$ and $\|(H_Q - E)\,\varphi_k\| \to 0$. Let $\phi_k = \varphi_k|_{x_n=0}$ and $\psi_k = 2^{-1/2} K_{0,E}^{-1/2}\phi_k$. From (4) it follows that

$$\|K_{0,E}^{1/2} K_{Q,E} K_{0,E}^{1/2}\psi_k\|' \leqq \frac{1}{2}\,\|(H_Q - E)\,\varphi_k\| \to 0,$$

while $\|\psi_k\|' \to 1$ due to (5). Thus $0 \in \sigma(K_{0,E}^{1/2} K_{Q,E} K_{0,E}^{1/2})$ and hence (since $K_{0,E}^{1/2} \geqq (-E)^{1/4} > 0$), $0 \in \sigma(K_{Q,E} K_{0,E}^{1/2})$. Since the assumption on Q implies $D(K_{Q,E}) = D(K_{0,E})$, and thus $D(K_{Q,E} K_{0,E}^{1/2}) = D(K_{0,E}^{3/2})$, there exists a sequence $(\psi_k') \subset D(K_{0,E}^{3/2})$, $\|\psi_k\|' = 1$, such that $\|K_{Q,E} K_{0,E}^{1/2}\psi_k'\|' \to 0$. Now let $\phi_k' = K_{0,E}^{1/2}\psi_k'$. We obtain

$$\|K_{Q,E}\phi_k'\|'/\|\phi_k'\|' = \|K_{Q,E} K_{0,E}^{1/2}\psi_k'\|'/\|K_{0,E}^{1/2}\psi_k'\|' \leqq \frac{1}{(-E)^{1/4}}\,\|K_{Q,E} K_{0,E}^{1/2}\psi_k'\|' \to 0,$$

and therefore $0 \in \sigma(K_{Q,E})$.

2. Suppose $0 \in \sigma(K_{Q,E})$, $E < 0$. By reason of the Closed Graph Theorem, the assumption on Q yields that $K_{0,E}$ and, all the more, $K_{0,E}^{1/2}$ are $K_{Q,E}$-bounded. Thus Lemma 3 implies $0 \in \sigma(K_{0,E}^{1/2} K_{Q,E} K_{0,E}^{1/2})$. Hence there exists a sequence $(\psi_k) \subset L_2(\mathbf{R}^{n-1})$ such that $\|\psi_k\|' = 1$ and

$$\|K_{0,E}^{1/2} K_{Q,E} K_{0,E}^{1/2}\psi_k\|' \to 0. \tag{9}$$

Now we set $\phi_k = 2^{1/2} K_{0,E}^{1/2}\psi_k$. According to Lemma 2, there exists a sequence $(\varphi_k) \subset D(H_Q)$ with $\varphi_k|_{x_n=0} = \phi_k$. We get $\|\varphi_k\| \to 1$ (it follows from (7)) and $\|(H_Q - E)\,\varphi_k\| \to 0$, which proves $E \in \sigma(H_Q)$.

(iv) 1. Let $\varphi \in D(H_Q)$, $H_Q\varphi = E\varphi$, $E < 0$. Thus $\mathring{\varphi}$ satisfies the differential equation $-\partial^2\mathring{\varphi}(p, x)/\partial x^2 + |p|^2\,\mathring{\varphi}(p, x) = E\mathring{\varphi}(p, x)$. Therefore

$$\mathring{\varphi}(p, x) = \exp\left(-(|p|^2 - E)^{1/2}\,x\right)\mathring{\varphi}(p, 0).$$

Define $\mathring{\psi}(p) = 2^{-1/2}(|p|^2 - E)^{-1/4}\,\mathring{\varphi}(p, 0)$. Then $\psi \in L_2(\mathbf{R}^{n-1})$, $\|\psi\|' = \|\varphi\|$ and, according to (1), $K_{Q,E} K_{0,E}^{1/2}\psi = 0$. Thus $K_{0,E}^{1/2}\psi \in D(K_{Q,E}) = D(K_{0,E})$, hence $\phi = 2^{1/2} K_{0,E}^{1/2}\psi = \varphi|_{x_n=0} \in L_2(\mathbf{R}^{n-1})$, and $K_{Q,E}\phi = 0$.

2. Suppose $\phi \in L_2(\mathbf{R}^{n-1})$, $K_{Q,E}\phi = 0$ and $\mathring{\varphi}(p, x) = \mathring{\phi}(p)\exp\left(-|p|^2 - E\right)^{1/2} x)$. Then $\varphi \in D(H_Q)$ and $H_Q\varphi = E\varphi$, with $\|\varphi\| \leqq 2^{-1/2}(-E)^{-1/4}\,\|\phi\|'$.

(v) By choosing the ψ_k in (9) orthonormal one proves $E \in \sigma_{\mathrm{ess}}(H_Q)$ for $0 \in \sigma_{\mathrm{ess}}(K_{Q,E})$. The analyticity of $K_{Q,E}$ and (iv) imply $E \in \sigma_{\mathrm{disc}}(H_Q)$ for $0 \in \sigma_{\mathrm{disc}}(K_{Q,E})$ ∎

Remarks: 1. One could expect that H_Q will be self-adjoint for all such Q, which define a self-adjoint operator $K_{Q,E}$, but, however, this still remains to be verified generally.

2. It is remarkable that, in the case of a multiplication operator Q, the boundary values of the negative energy wave functions of H_Q describe relativistic particles moving along the boundary in a potential field Q with a rest mass corresponding to the binding energy $-E$.

2. Examples

In this section we state some classes of functions Q which are $K_{0,0}$-bounded (as multiplication operators) with a relative bound less than 1.

Theorem 2: *Let $Q \in L_p(\mathbf{R}^m) + L_\infty(\mathbf{R}^m)$, with $p = 2$ for $m = 1$ and $p > m$ for $m \geqq 2$. Then Q is infinitesimally small with respect to $K_{0,0}$.*

Proof: Let $\phi \in C_0^\infty(\mathbf{R}^m)$, $2 \leqq q < 2m/(m-2)$ for $m > 2$ and $2 \leqq q < \infty$ otherwise, $1/q + 1/s = 1$ (hence $s > 2m/(2+m)$). The Hausdorff-Young inequality yields

$$\begin{aligned}\|\phi\|_q^s &\leqq c_1 \|\mathring{\phi}\|_s^s = c_1 \int |\mathring{\phi}(u)|^s\, d^m u \\ &= c_1 \int (|u|^2+1)^{-s/2} |(|u|^2+1)^{1/2} \mathring{\phi}(u)|^s\, d^m u \\ &\leqq c_1 \|(|u|^2+1)^{-s/2}\|_{2/(2-s)} \|(|u|^2+1)^{1/2} \mathring{\phi}(u)\|_2^s \\ &= c_2 \|(|u|^2+1)^{1/2} \mathring{\phi}\|_2^s,\end{aligned}$$

and hence $\|\phi\|_q \leqq c_3 \|(|u|^2+1)^{1/2}\mathring{\phi}\|_2$. Let $\mathring{\phi}_r(u) = r^{m/s}\mathring{\phi}(ru)$. Then we get

$$\begin{aligned}\|\mathring{\phi}\|_s &= \|\mathring{\phi}_r\|_s \leqq c_4 \|(|u|^2+1)^{1/2} \mathring{\phi}_r\|_2 \\ &= c_4 \Big(\int (|u|^2+1)\, r^{2m/s} |\mathring{\phi}(ru)|^2\, d^m u\Big)^{1/2} \\ &= c_4 \Big(r^{2m/s-m} \int (|u|^2 r^{-2}+1)\, |\mathring{\phi}(u)|^2\, d^m u\Big)^{1/2} \\ &\leqq c_4 r^{m/s-m/2-1} \|K_{0,0}\phi\|_2 + c_4 r^{m/s-m/2} \|\phi\|_2.\end{aligned}$$

The first term tends to zero when r tends to infinity. Thus, for all positive a, one finds a positive b such that

$$\|\phi\|_q \leqq a \|K_{0,0}\phi\|_2 + b \|\phi\|_2. \tag{10}$$

Now we suppose $Q = Q_1 + Q_2$, where $Q_1 \in L_p(\mathbf{R}^m)$ $(p > m)$, $Q_2 \in L_\infty(\mathbf{R}^m)$, $\phi \in D(Q) \cap D(K_{0,0})$. Then $\|Q\phi\|_2 \leqq \|Q_1\phi\|_2 + \|Q_2\|_\infty \|\phi\|_2$. On the other hand,

$$\|Q_1\phi\|_2 = \Big(\int |Q_1\phi|^2\, d^m x\Big)^{1/2} \leqq (\|Q_1^2\|_{p/2} \|\phi^2\|_t)^{1/2},$$

where $2/p + 1/t = 1$; hence $\|Q_1\phi\|_2 \leqq \|Q_1\|_p \|\phi\|_{2t}$. Since $p > m$, we obtain $2t < 2m/(m-2)$. (If $m = 1$, we set $p = 2$ and $t = \infty$.) Finally, we get $\|Q\phi\|_2 \leqq \|Q_1\|_p \times (a \|K_{0,0}\phi\|_2 + b \|\phi\|_2) + \|Q_2\|_\infty \|\phi\|_2$ ∎

Theorem 3 [4, 11]: *Let $Q = |x|^{-\alpha}$, $x \in \mathbf{R}^m$, with $0 < \alpha \leqq 1$. Then*
(i) *if $\alpha < 1$, then Q is infinitesimally small with respect to $K_{0,0}$, and*
(ii) *for $\alpha = 1$, Q is $K_{0,0}$-bounded with a relative bound $C_m = 1/(m/2 - 1)$.*
Consequently, Theorem 1 will be applicable when $Q = -C\,|x|^{-1}$, with $C < (n-3)/2$.

In analogy with [7: Theorem XIII.96], one can get the following criterion.

Theorem 4: *Let Q be a uniformly local $L_p(\mathbf{R}^m)$-function with $p = 2$ for $m = 1$ and $p > m$ for $m \geqq 2$. Then Q is infinitesimally small with respect to $K_{0,0}$.*

Proof: Given a real number $r \geqq 1$ and a measurable set $C < \mathbf{R}^m$, denote $\|\cdot\|_{r,C} = \|\cdot\|_{L_r(C)}$. Let C_k, C_k' ($k \in \mathbf{Z}^m$) be the cube of that $x \in \mathbf{R}^m$ for which $|x_i - k_i| \leqq 1/2$ (3/2, resp.), $i = 1, \dots, m$. For Q as in the assumption, we define $|||Q||| = \sup\{\|Q\|_{p,C_k}: k \in \mathbf{Z}^m\}$. We note that, for $\psi \in D(K_{0,0})$, the Parseval identity implies

$$\|K_{0,0}\psi\|_2 = \|\nabla\psi\|_2. \tag{11}$$

Now, let $\eta \in C_0^\infty(C_k')$, with $\eta(x) \leqq 1$ and $\eta(x) = 1$ for all $x \in C_k$, and let $q = 2p/(p-2)$. Using (10) and (11) we obtain

$$\begin{aligned}\|\phi\|_{q,C_k} &\leqq \|\eta\phi\|_q \leqq a\,\|K_{0,0}(\eta\phi)\|_2 + b\,\|\eta\phi\|_2 \\ &= a\,\|\nabla(\eta\phi)\|_2 + b\,\|\eta\phi\|_2 \leqq a\,\|\eta\,\nabla\phi\|_2 + a\,\|\phi\,\nabla\eta\|_2 + b\,\|\eta\phi\|_2 \\ &\leqq a\,\|\nabla\phi\|_{2,C_k'} + (a\,\|\nabla\eta\|_\infty + b)\,\|\phi\|_{2,C_k'} = a\,\|\nabla\phi\|_{2,C_k'} + b'\,\|\phi\|_{2,C_k'}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\|Q\phi\|_2^2 &= \sum_k \|Q\phi\|_{2,C_k}^2 \leqq \sum_k \|Q\|_{2,C_k}^2\,\|\phi\|_{2,C_k}^2 \\ &\leqq |||Q|||^2 \sum_k (a\,\|\nabla\phi\|_{2,C_k'} + b'\,\|\phi\|_{2,C_k'})^2 \\ &\leqq 2\,|||Q|||^2 \sum_k (a^2\,\|\nabla\phi\|_{2,C_k'}^2 + b'^2\,\|\phi\|_{2,C_k'}^2) \\ &= 2\cdot 3^m\,|||Q|||^2\,(a^2\,\|\nabla\phi\|_2^2 + b'^2\,\|\phi\|_2^2) \\ &= 2\cdot 3^m\,|||Q|||^2\,(a^2\,\|K_{0,0}\phi\|_2^2 + b'^2\,\|\phi\|_2^2).\end{aligned}$$

The first step follows from the Hölder inequality, the second uses the elementary inequality $(x + y)^2 \leqq 2(x^2 + y^2)$, and the third one is a consequence of the fact that any interior point of C_k is contained in 3^m cubes C_n', $|k - n| \leqq 1$. Since a can be choosen arbitrarily small, the statement follows ∎

3. Appendix

Here we recover the proofs of the three Lemmata used in the proof of Theorem 1/(iii).

Proof of Lemma 1: Let $(H_Q - E)\,\varphi = f$. Then $\mathring{\varphi}$ satisfies the differential equation

$$-\frac{\partial^2}{\partial x^2}\,\mathring{\varphi}(p, x) + (p^2 - E)\,\mathring{\varphi}(p, x) = \mathring{f}(p, x). \tag{12}$$

We represent the solution of (12) in the form

$$\mathring{\varphi}(p, x) = \mathring{c}(p)\exp\left(-(p^2 - E)^{1/2}\,x\right) + \frac{1}{2}\,(p^2 - E)^{-1/2}\int_0^\infty \exp\left(-(p^2 - E)^{1/2}\,|x - z|\right)\mathring{f}(p, z)\,dz. \tag{13}$$

Denoting $\mathring{\phi}(p) = \mathring{\varphi}(p, 0)$ and using the boundary condition we get the equation

$$(p^2 - E)^{1/2}\,\mathring{\phi}(p) + (Q\phi)^0\,(p) = \int\limits_0^\infty \frac{\mathring{f}(p, z)}{\exp\left((p^2 - E)^{1/2} z\right)}\, dz \equiv \mathring{F}(p).$$

Standard estimates imply $\|2^{1/2}(p^2 - E)^{1/4}\,\mathring{F}(p)\|' \leqq \|\mathring{f}\|$. Thus (4) is verified. From (13) we get

$$\|2^{-1/2}(p^2 - E)^{-1/4}\,\mathring{c}(p)\|' = \|\mathring{c}(p) \exp\left((p^2 - E)^{1/2} x\right)\|$$

$$= \|\mathring{\varphi}\| + \left(\frac{1}{4}\int \frac{d^{n-1}p}{p^2 - E}\int\limits_0^\infty dx \left|\int\limits_0^\infty \exp\left(-\frac{|x - z|}{(p^2 - E)^{1/2}}\right) \mathring{f}(p, z)\, dz\right|^2\right)^{1/2}$$

$$\leqq \|\mathring{\varphi}\| + \frac{3}{4}\,\|(p^2 - E)^{-1}\,\mathring{f}(p, x)\|$$

and

$$\|2^{-1/2}(p^2 - E)^{-1/4}\,\mathring{\phi}\|' \leqq \|2^{-1/2}(p^2 - E)^{-1/4}\,\mathring{c}\|'$$

$$+ \left\|2^{-3/2}(p^2 - E)^{-3/4}\int\limits_0^\infty \exp\left(-(p^2 - E)^{1/2} z\right) \mathring{f}(p, z)\, dz\right\|'$$

$$\leqq \|2^{-1/2}(p^2 - E)^{-1/4}\,\mathring{c}\|' + \frac{1}{4}\,\|(p^2 - E)^{-1}\,\mathring{f}\| \leqq \|\mathring{\varphi}\| + \|(p^2 - E)^{-1}\,\mathring{f}\|.$$

Using the opposite direction of the triangle inequality and the boundedness of $(p^2 - E)^{-1}$ we get (5) ∎

Proof of Lemma 2: Let $\phi \in D(K_{0,E}^{3/2})$, $(p^2 - E)^{1/2}\,\mathring{\phi}(p) + (Q\phi)^0_\cdot(p) = \hat{F}(p)$, and set

$$\mathring{f}(p, x) = 2(p^2 - E)^{1/2}\,\hat{F}(p) \exp\left(-(p^2 - E)^{1/2} x\right).$$

Obviously, $\|\mathring{f}(p, x)\| = 2^{1/2}\,\|K_{0,E}^{1/2} F\|'$. Now we define a function $\mathring{\varphi}(p, x)$ as the unique solution in $L_2(\mathbf{R}_+{}^n)$ of the differential equation

$$\frac{\partial^2}{\partial x^2}\,\mathring{\varphi}(p, x) + (p^2 - E)\,\mathring{\varphi}(p, x) = \mathring{f}(p, x), \qquad \mathring{\varphi}(p, 0) = \mathring{\phi}(p). \tag{14}$$

We represent $\mathring{\varphi}(p, x)$ as in (13) by setting, in correspondence to the initial condition of (14), $\mathring{c}(p) = \mathring{\phi}(p) - 1/2(p^2 - E)^{-1/2}\,\mathring{F}(p)$, and get

$$\mathring{\varphi}(p, x) = \mathring{\phi}(p) \exp\left(-(p^2 - E)^{1/2} x\right) + \mathring{F}(p)\, x \exp\left(-(p^2 - E)^{1/2} x\right).$$

Using again the triangle inequality in both directions, we obtain (7) ∎

Proof of Lemma 3: Suppose that there are positive real numbers a_1, a_2, b_1, b_2 such that for all φ

$$\|B_1\varphi\| \leqq a_1\,\|A\varphi\| + b_1\,\|\varphi\| \quad \text{and} \quad \|B_2\varphi\| \leqq a_2\,\|A\varphi\| + b_2\,\|\varphi\|,$$

and that $A = \int_{-\infty}^{\infty} tP(dt)$, where P is the family of spectral projectors of A. We denote $\mathcal{H}_m = B_2^{-1} \operatorname{Ran}\left(P\left(-\frac{1}{m}, \frac{1}{m}\right)\right)$, $m \in \mathbb{N}$. For $\varphi \in \mathcal{H}_m$, $\|\varphi\| = 1$, we get

$$\begin{aligned} \|B_2\varphi\|^2 &= (B_2^*B_2\varphi, \varphi) \leqq \|B_2^*B_2\varphi\| \\ &\leqq a_2 \|AB_2\varphi\| + b_2 \|B_2\varphi\| \leqq \left(\frac{a_2}{m} + b_2\right) \|B_2\varphi\|. \end{aligned}$$

Thus $\|B_2\varphi\| \leqq a_2/m + b_2$. Now we choose a sequence (φ_m) such that $\varphi_m \in \mathcal{H}_m$, $\|\varphi_m\| = 1$. Then

$$\begin{aligned} \|B_1AB_2\varphi_m\| &\leqq a_1 \|A^2B_2\varphi_m\| + b_1 \|AB_2\varphi_m\| \\ &\leqq \left(\frac{a_1}{m^2} + \frac{b_1}{m}\right) \|B_2\varphi_m\| \leqq \left(\frac{a_1}{m^2} + \frac{b_1}{m}\right)\left(\frac{a_2}{m} + b_2\right) \to 0 \quad \blacksquare \end{aligned}$$

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