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Uniqueness of the Solution of an Inverse Problem for a Quasilinear Parabolic Equation in Divergence Form

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Es wird die quasilineare parabolische Differentialgleichung $u_t = \operatorname{div}(a(u) \operatorname{grad} u)$ mit einer konstanten Anfangsbedingung und einer Randbedingung 1. Art betrachtet, wobei a eine positive analytische Funktion von u ist. Unter gewissen Zusatzvoraussetzungen bezüglich u wird ein Eindeutigkeitssatz für die Bestimmung des Koeffizienten a bewiesen.

Рассматривается квазилинейное параболическое дифференциальное уравнение $u_t = \operatorname{div}(a(u) \operatorname{grad} u)$ с постоянным начальным условием и граничным условием первого рода, причем a есть положительная аналитическая функция от u. Под некоторыми дополнительными предположениями относительно u доказывается теорема единственности для определения коэффициента a.

The quasilinear parabolic differential equation $u_t = \operatorname{div}(a(u) \operatorname{grad} u)$ with a constant initial condition and a boundary condition of the first kind is considered where a is a positive analytic function of u. Under some additional assumptions on u a uniqueness theorem for the determination of the coefficient a is proved.

1. Introduction

We use the following notations. D is a bounded region of the n-dimensional Euclidean space $\mathbf{R}^n$ with a sufficiently smooth boundary ∂D, T a positive number, $Z_T = D \times (0, T)$, $\Gamma_T = \partial D \times [0, T)$. By $\overline{M}$ we denote the closure of a set $M \subseteq \mathbf{R}^n$ or $M \subseteq \mathbf{R}^{n+1}$. Points of $\mathbf{R}^n$ are denoted by $x = (x_1, x_2, \ldots, x_n)$, t is a real variable (time) with $0 \leqq t \leqq T$. For points $P = (x, t)$, $P' = (x', t') \in Z_T$ we introduce the distance $d(P, P') = (\|x - x'\|^2_{\mathbf{R}^n} + |t - t'|)^{1/2}$. Using this metric we denote the space of all real functions which are uniformly Hölder continuous with the exponent α $(0 < \alpha \leqq 1)$ in Z_T by $\overline{C}_\alpha(Z_T)$. By $\overline{C}_{2+\alpha}(Z_T)$ we denote the space of all functions possessing uniformly Hölder continuous derivatives (exponent α) up to the order 2 with respect to $x_1, \ldots, x_n$ and up to the order 1 with respect to t in Z_T. For the precise definitions see [3: p. 61].

We consider the boundary value problem

$$\left.\begin{array}{lr} u_t(x, t) = \operatorname{div}\big(a(u(x, t)) \operatorname{grad} u(x, t)\big) & \text{in } Z_T \\ u(x, 0) = d \quad (d \text{ constant}) & \text{in } \overline{D} \\ u(x, t) = \psi(x, t) & \text{on } \Gamma_T \end{array}\right\} \tag{1.1}$$

with ψ a given real function satisfying the conditions

$$\psi \in C(\overline{\Gamma}_T), \quad \psi(x, 0) = d \quad (x \in \partial D). \tag{1.2}$$

Now we define

$$v_0 = \min \left\{d, \min_{(x,t) \in \overline{\Gamma}_T} \psi(x, t)\right\}, \quad v_1 = \max \left\{d, \max_{(x,t) \in \overline{\Gamma}_T} \psi(x, t)\right\}.$$

We assume that $v_0 < v_1$ and that $a = a(u)$ is a real function of the real variable u which is analytic in an interval $[v_0^*, v_1^*]$ with $v_0^* < v_0$ and $v_1 < v_1^*$. Further let $a(u) > 0$ for all $u \in [v_0, v_1]$. In this case we say that a is of the class A or $a \in A$. Let the real function u satisfy the conditions

$$u \in C(\bar{Z}_T) \cap \bar{C}_{2+\alpha}(Z_T), \quad u_{x_i} \in C(\bar{Z}_T) \quad (i = 1, \ldots, n); \qquad \Delta u \in C(\bar{Z}_T). \tag{1.3}$$

Lastly we assume that, under the stated assumptions, for every $a \in A$ there exists a unique solution u of the boundary value problem (1.1) which fulfils the condition (1.3). We denote this solution by $u = u(a, x, t)$. Using a maximum principle one obtains that $v_0 \leqq u(a, x, t) \leqq v_1$ for all $(x, t) \in Z_T$ and $a \in A$.

In this paper we consider the inverse problem of determining the coefficient a if $u(a, x_0, t)$ is known in an interior point $x_0 \in D$ for all $t \in [0, T]$, and we prove a uniqueness theorem for this problem. The paper is closely related to that of S. DÜMMEL [2] where the mentioned inverse problem is considered for the more simple equation $u_t = a(u)\,\Delta u$. The special case where $a(u)$ is constant can be found in S. DÜMMEL [1]. The equation $u_t = \operatorname{div}\big(a(u)\operatorname{grad} u\big)$ is investigated by S. MEYER [5, 6] for $a(u) = b(u - v_0)^l + c$ (b, c constant, l natural number). The last equation with additional information about u on the boundary of D can be found in some other papers, e.g. in N. V. MUZYLEV [7]. For further references see [2].

2. Some lemmas

In this section we shall state some lemmas which will be needed in the proof of the uniqueness theorem. Let $a_1 \in A$ and $u_1, u_2 \in \bar{C}_{2+\alpha}(Z_T)$. We define functions f_1 and f_2 by

$$f_1(x, t) = \begin{cases} \dfrac{a_1\big(u_1(x, t)\big) - a_1\big(u_2(x, t)\big)}{u_1(x, t) - u_2(x, t)} & \text{if } u_1(x, t) \neq u_2(x, t) \\ a_1'\big(u_2(x, t)\big) & \text{if } u_1(x, t) = u_2(x, t), \end{cases}$$

$$f_2(x, t) = \begin{cases} \dfrac{a_1'\big(u_1(x, t)\big) - a_1'\big(u_2(x, t)\big)}{u_1(x, t) - u_2(x, t)} & \text{if } u_1(x, t) \neq u_2(x, t) \\ a_1''\big(u_2(x, t)\big) & \text{if } u_1(x, t) = u_2(x, t). \end{cases}$$

Lemma 1: *One has* $f_1, f_2 \in \bar{C}_\alpha(Z_T)$.
This lemma can be proved as in [7].

We introduce the following notations:

$$k_- = \begin{cases} k/2 & \text{for even } k \\ (k-1)/2 & \text{for odd } k, \end{cases} \qquad k_+ = \begin{cases} k/2 & \text{for even } k \\ (k+1)/2 & \text{for odd } k. \end{cases}$$

Let g_1, g_2 be real analytic functions in an interval $[c_1, c_2]$. By $W(u)$ we denote Wronski's determinant

$$W(u) = \begin{vmatrix} g_1(u) & g_2(u) \\ g_1'(u) & g_2'(u) \end{vmatrix}. \tag{2.1}$$

Lemma 2: *The derivatives* $W^{(k)}$ $(k = 1, 2, \ldots)$ *of* W *are given by*

$$W^{(k)} = \begin{vmatrix} g_1 & g_2 \\ g_1^{(k+1)} & g_2^{(k+1)} \end{vmatrix} + b_1 \begin{vmatrix} g_1' & g_2' \\ g_1^{(k)} & g_2^{(k)} \end{vmatrix} + b_2 \begin{vmatrix} g_1'' & g_2'' \\ g_1^{(k-1)} & g_2^{(k-1)} \end{vmatrix} + \cdots + b_{k_-} \begin{vmatrix} g_1^{(k_-)} & g_2^{(k_-)} \\ g_1^{((k+2)_+)} & g_2^{((k+2)_+)} \end{vmatrix} \tag{2.2}$$

where $b_1 = k - 1$ *and* $b_2, \ldots, b_{k_-}$ *are real numbers.*

The proof can be made by mathematical induction.

3. A uniqueness theorem

Under the stated assumptions and some additional conditions we obtain that the inverse problem of the determination of the coefficient a has at most one solution, if we suppose that $a(d)$ is known.

Theorem: *For all* $a \in A$ *let* $u(a, \cdot, \cdot)$ *be the solution of* (1.1) *satisfying the conditions* (1.3) *and in addition the condition*

$$u_t(a, x, t) \geqq 0 \qquad \big((x, t) \in Z_T;\ a \in A\big), \tag{3.1}$$

where the function ψ *fulfils the condition* (1.2). *Suppose that* $x_0 \in D$ *and* $h \in C^1([0, T])$. *Furthermore let there exist a positive number* t_0 *such that* $h'(t) > 0$ *for all* $t \in (0, t_0]$. *Lastly let* a_0 *be a positive number. Then there is at most one* $a \in A$ *with* $a(d) = a_0$ *such that* $u(a, x_0, t) = h(t)$ *for all* $t \in [0, T]$.

Proof: We suppose that there are two functions $a_1, a_2 \in A$ with $a_1(d) = a_2(d) = a_0$ such that

$$u(a_1, x_0, t) = u(a_2, x_0, t) = h(t) \quad \text{for all } t \in [0, T]. \tag{3.2}$$

For brevity we set $u_i(x, t) = u(a_i, x, t)$ $(i = 1, 2)$ and $u_{12} = u_1 - u_2$, $a_{12} = a_1 - a_2$. From

$$(u_i)_t = \operatorname{div}\big(a_i(u_i) \operatorname{grad} u_i\big) = a_i(u_i)\, \Delta u_i + a_i'(u_i) (\operatorname{grad} u_i)^2 \qquad (i = 1, 2)$$

we obtain a linear parabolic differential equation of second order for the function u_{12} by elementary computations:

$$(u_{12})_t - a_1(u_1)\, \Delta u_{12} - a_1'(u_1) \sum_{j=1}^{n} \big((u_1)_{x_j} + (u_2)_{x_j}\big) (u_{12})_{x_j}$$
$$- [f_1 \Delta u_2 + f_2 (\operatorname{grad} u_2)^2]\, u_{12} = \operatorname{div}\big(a_{12}(u_2) \operatorname{grad} u_2\big) \quad \big((x, t) \in Z_T\big). \tag{3.3}$$

Moreover u_{12} satisfies the conditions

$$u_{12}(x, 0) = 0 \quad (x \in \bar{D}) \quad \text{and} \quad u_{12}(x, t) = 0 \quad \big((x, t) \in \Gamma_T\big). \tag{3.4}$$

Because of $a \in A$, (1.3) and Lemma 1 the assumptions of [4: § 4, Theorem 3] are fulfilled. Using this theorem we obtain that there exists a unique solution of the initial boundary value problem (3.3), (3.4). This solution can be represented by Green's function G of the operator of the differential equation (3.3). Consequently we have

$$u_{12}(x, t) = \int_0^t \int_D G(x, t, x', t') \operatorname{div}\big(a_{12}(u_2(x', t') \operatorname{grad} u_2(x', t'))\big)\, dx'\, dt'. \tag{3.5}$$

Because of (3.2) for the left-hand side of (3.5) there holds

$$u_{12}(x_0, t) = u_1(x_0, t) - u_2(x_0, t) = 0 \quad \text{for all} \quad t \in [0, T]. \tag{3.6}$$

The function a_{12} is analytic in the interval $[v_0^*, v_1^*]$. From the identity theorem for analytic functions it follows that either $a_{12}(u) = 0$ for all $u \in [v_0, v_1]$, and we have the uniqueness of the function a, or a_{12} has at most finitely many zeros in $[v_0, v_1]$. We investigate the second case. Since $\partial u_2/\partial t \geqq 0$ in $\bar{Z}_T$, $u_2(x, \cdot)$ is monotone increasing for fixed $x \in \bar{D}$. Thus $d = v_0$, and we have $a_1(v_0) = a_2(v_0)$. Hence v_0 is a zero of the function a_{12}. By w we denote the smallest of these zeros which is greater than v_0, if such a zero exists. If such a zero does not exist we set $w = v_1$. Then, for all u with $v_0 < u < w$, either $a_{12}(u) > 0$ or $a_{12}(u) < 0$. We consider the first case. The second case can be treated analogously. In [2] it was proved that there exists a $T_0 > 0$ such that $a_{12}(u_2(x, t))$ does not change the sign for all $x \in \bar{D}$ and all $t \in (0, T_0)$. Now we define $T_0^* = \min \{t_0, T_0\}$. Using $h'(t) > 0$ $(t \in (0, t_0])$ we can see as in [2] that there exists a neighborhood $S_\varepsilon(x_0)$ of x_0 and an interval $(t_1, t_2) \subset [0, T_0^*]$ such that

$$(u_2)_t (x, t) > 0, \quad u_2(x, t) > v_0 \quad \left((x, t) \in B_{T_0^*} = S_\varepsilon(x_0) \times (t_1, t_2)\right). \tag{3.7}$$

From this we obtain that

$$a_{12}(u_2) > 0 \quad \text{and} \quad a_2(u_2) \Delta u_2 + a_2'(u_2) (\operatorname{grad} u_2)^2 = (u_2)_t > 0$$

for all $(x, t) \in B_{T_0^*}$. Let $W(u)$ be the determinant (2.1) with $g_1 = a_2$ and $g_2 = a_{12}$. Then we obtain

$$\begin{aligned} \operatorname{div} \left(a_{12}(u_2) \operatorname{grad} u_2\right) &= a_{12}(u_2) \Delta u_2 + a_{12}'(u_2) (\operatorname{grad} u_2)^2 \\ &> \frac{W(u_2)}{a_2(u_2)} (\operatorname{grad} u_2)^2 \quad \left((x, t) \in B_{T_0^*}\right). \end{aligned} \tag{3.8}$$

Now we prove that there exists a real number $w_1 > 0$ such that $W(u) > 0$ for all $u \in (v_0, v_0 + w_1)$. The function $a_{12} = a_{12}(u)$ is analytic in a neighborhood of v_0 and $a_{12}(v_0) = 0$. Either the derivatives $a_{12}^{(k)}(v_0) = 0$ for every k, and we have $a_1(u) = a_2(u)$ in a neighborhood of v_0, or there exists a natural number m such that

$$a_{12}^{(k)}(v_0) = 0 \quad (k \in \{0, 1, \ldots, m\}) \quad \text{and} \quad a_{12}^{(m+1)}(v_0) \neq 0. \tag{3.9}$$

We choose u such that $v_0 < u < w$. Then by Taylor's theorem we obtain, for some $\xi \in (v_0, u)$,

$$a_{12}(u) = \frac{a_{12}^{(m+1)}(\xi) (u - v_0)^{m+1}}{(m + 1)!}, \quad a_{12}^{(m+1)}(\xi) = \frac{(m + 1)!\, a_{12}(u)}{(u - v_0)^{m+1}} > 0$$

and, because of (3.9),

$$a_{12}^{(m+1)}(v_0) = \lim_{u_1 \to v_0} a_{12}^{(m+1)}(\xi) > 0. \tag{3.10}$$

$W = W(u)$ is also an analytic function and from (2.2) and (3.10) we obtain for $m \geqq 1$

$$\begin{aligned} &W^{(k)}(v_0) = 0 \quad (k \in \{0, 1, \ldots, m - 1\}), \\ &W^{(m)}(v_0) = a_2(v_0)\, a_{12}^{(m+1)}(v_0) > 0. \end{aligned} \tag{3.11}$$

For $m = 0$ there holds $W^{(0)}(v_0) = W(v_0) > 0$. Again using Taylor's theorem, (3.7), (3.11) and the continuity of $W^{(m)}(u)$ in $[v_0, v_1]$ one can easily see that there exists a $w_1 > v_0$ such that

$$W(u_2) = \frac{W^{(m)}(\xi)\,(u_2 - v_0)^m}{m!} > 0 \qquad \big(u_2 \in (v_0, v_0 + w_1)\big). \tag{3.12}$$

As in [2] it can be proved that there exists a $T_1 > 0$ such that $W\big(u_2(x, t)\big)$ does not change the sign for all $(x, t) \in \bar{Z}_{T_1}$. We define $T_1{}^* = \min\{T_0{}^*, T_1\}$ and $B_{T_1{}^*}$ similarly as the set $B_{T_0{}^*}$. Both $B_{T_0{}^*}$ and $B_{T_1{}^*}$ can be chosen in such way that $B_{T_1{}^*} \subseteqq B_{T_0{}^*}$. Then from (3.8) and (3.12) we obtain

$$\operatorname{div}\big(a_{12}(u_2)\operatorname{grad} u_2\big) > \frac{W(u_2)}{a_2(u_2)}(\operatorname{grad} u_2)^2 \geqq 0 \qquad \big((x, t) \in B_{T_1{}^*}\big). \tag{3.13}$$

Moreover for all $(x, t) \in Z_{T_1{}^*}$ we have $a_{12}\big(u_2(x, t)\big) \geqq 0$ and $W(u_2) \geqq 0$. Then if we repeat the above consideration it follows from (3.1) that

$$\operatorname{div}\big(a_{12}(u_2)\operatorname{grad} u_2\big) \geqq \frac{W(u_2)}{a_2(u_2)}(\operatorname{grad} u_2)^2 \geqq 0 \qquad \big((x, t) \in Z_{T_1{}^*}\big). \tag{3.14}$$

Lastly we have (see [3: p. 83])

$$G(x_0, T_1{}^*, x, t) > 0 \qquad \big((x, t) \in Z_{T_1{}^*}\big). \tag{3.15}$$

Now from (3.13)—(3.15) we obtain

$$\int_0^{T_1{}^*}\int_D G(x_0, T_1{}^*, x', t')\operatorname{div}\big(a_{12}(u_2(x', t'))\operatorname{grad} u_2(x', t')\big)\,dx'\,dt' > 0.$$

But this is a contradiction to (3.5) and (3.6), and thus $a_1(u) = a_2(u)$ for all $u \in [v_0, v_1]$ ∎

Remark: If we consider the class of coefficients a with the property: a is an analytic function in the interval $[v_0{}^*, v_1{}^*]$, $a(u) > 0$ and $a'(u) < 0$ in the interval $[v_0, v_1]$, then the proof of the uniqueness theorem is very simple.

In the uniqueness theorem we have used the supposition (3.1). In the following proposition we shall give sufficient conditions which imply this relation.

Proposition: *Let $a \in A$ and u be the unique solution of the boundary value problem (1.1) satisfying the conditions*

$$u_t \in C(\bar{Z}_T) \cap \bar{C}_{2+\alpha}(Z_T), \quad u_{x_i} \in C(\bar{Z}_T) \quad (i = 1, \ldots, n), \quad \Delta u \in C(\bar{Z}_T). \tag{3.16}$$

Suppose that the boundary function ψ fulfils (1.2) and in addition $\psi(x, \cdot) \in C^1([0, T])$, $\psi_t(x, 0) = 0$ for all $x \in \partial D$, $\psi_t(x, t) \geqq 0$ for all $(x, t) \in \Gamma_T$. Then we have $u_t(x, t) \geqq 0$ for all $(x, t) \in \bar{Z}_T$.

Proof: Set $u_t = w$. From (3.16) it follows that $w \in C(\bar{Z}_T) \cap \bar{C}_{2+\alpha}(Z_T)$, and we have

$$w_t = a(u)\,\Delta w + 2a'(u)\operatorname{grad} u\operatorname{grad} w + [a'(u)\,\Delta u + a''(u)(\operatorname{grad} u)^2]\,w \quad \text{in } Z_T, \tag{3.17}$$

$w(x, 0) = 0$ in $\bar{D}$, $w(x, t) = \psi_t(x, t)$ on Γ_T. The functions $a(u)$, $a'(u)$, $a''(u)$, $u_{x_1}, \ldots, u_{x_n}$ and Δu are bounded in Z_T. Hence the coefficients in the differential equation (3.17) are also bounded in Z_T. On the boundary $\Gamma_T \cup \bar{D}$ we have $w(x, t) \geqq 0$. Using a maximum principle [4: p. 8] we obtain $u_t(x, t) = w(x, t) \geqq 0$ for all $(x, t) \in \bar{Z}_T$ ∎

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