(1.2)

Uniqueness of the Solution of an Inverse Problem for a Quasilinear Parabolic Equation in Divergence Form

S. DÜMMEL and S. HANDROCK-MEYER

Es wird die quasilineare parabolische Differentialgleichung $u_t = \text{div}(a(u) \text{ grad } u)$ mit einer konstanten Anfangsbedingung und einer Randbedingung 1. Art betrachtet, wobei a eine positive analytische Funktion von u ist. Unter gewissen Zusatzvoraussetzungen bezüglich u wird ein Eindeutigkeitssatz für die Bestimmung des Koeffizienten a bewiesen.

Рассматривается квазилинейное параболическое дифференциальное уравнение \ddot{u} $=$ div $(a(u)$ grad u) с постоянным начальным условием и граничным условием первого рода, причем а есть положительная аналитическая функция от *и*. Под некоторыми дополнительными предположениями относительно и доказывается теорема единственности для определения коэффициента а.

The quasilinear parabolic differential equation $u_t =$ div ($a(u)$ grad u) with a constant initial condition and a boundary condition of the first kind is considered where a is a positive analytic function of u . Under some additional assumptions on \tilde{u} a uniqueness theorem for the determination of the coefficient a is proved.

1. Introduction

We use the following notations. D is a bounded region of the n -dimensional Euclidean space \mathbb{R}^n with a sufficiently smooth boundary ∂D , T a positive number, $Z_T = D$ \times (0, T), $\Gamma_T = \partial D \times [0, T)$. By \overline{M} we denote the closure of a set $M \subseteq \mathbb{R}^n$ or $M \subseteq \mathbb{R}^{n+1}$. Points of \mathbb{R}^n are denoted by $x = (x_1, x_2, ..., x_n)$, t is a real variable (time) with $0 \le t \le T$. For points $P = (x, t)$, $P' = (x', t') \in Z_T$ we introduce the distance $d(P, P') = (||x - x'||_{\mathbb{R}^n}^2 + |t - t'|)^{1/2}$. Using this metric we denote the space of all real functions which are uniformly Hölder continuous with the exponent α ($0 < \alpha \leq 1$) in Z_T by $\overline{C}_a(Z_T)$. By $\overline{C}_{2+a}(Z_T)$ we denote the space of all functions possessing uniformly Hölder continuous derivatives (exponent α) up to the order 2 with respect to x_1, \ldots, x_n and up to the order 1 with respect to t in Z_T . For the precise definitions see [3: p. 61].

We consider the boundary value problem

$$
u_t(x, t) = \text{div} \left(a(u(x, t)) \text{ grad } u(x, t) \right) \text{ in } Z_T
$$

\n
$$
u(x, 0) = d \quad (d \text{ constant}) \qquad \text{in } \overline{D}
$$

\n
$$
u(x, t) = v(x, t) \qquad \text{on } \Gamma_T'
$$
\n(1.1)

with ψ a given real function satisfying the conditions

$$
\psi \in C(\overline{\Gamma}_T), \qquad \psi(x, 0) = d \quad (x \in \partial D).
$$

Now we define

$$
v_0 = \min \left\{ d, \min_{(x,t \in \overline{\Gamma}_T)} \psi(x,t) \right\}, \quad v_1 = \max \left\{ d, \max_{(x,t) \in \overline{\Gamma}_T} \psi(x,t) \right\}
$$

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We assume that $v_0 < v_1$ and that $a = a(u)$ is a real function of the real variable u
which is analytic in an interval $[v_0^*, v_1^*]$ with $v_0^* < v_0$ and $v_1 < v_1^*$. Further let
 $a(u) > 0$ for all $u \in [v_0, v_1]$. In this cas which is analytic in an interval $[v_0^*, v_1^*]$ with $v_0^* < v_0$ and $v_1 < v_1^*$. Further let $a(u) > 0$ for all $u \in [v_0, v_1]$. In this case we say that a is of the class A or $a \in A$. Let the real function *u* satisfy the conditions

$$
u \in C(\bar{Z}_T) \cap \bar{C}_{2+\alpha}(Z_T), \quad u_{x_i} \in C(\bar{Z}_T) \quad (i = 1, ..., n), \qquad \Delta u \in C(\bar{Z}_T).
$$
 (1.3)

Lastly we assume that, under the stated assumptions, for every $a \in A$ there exists a unique solution u of the boundary value problem (1.1) which fulfils the condition (1.3). We denote this solution by $u = u(a, x, t)$. Using a maximum principle one obtains that $v_0 \leq u(a, x, t) \leq v_1$ for all $(x, t) \in Z_T$ and $a \in A$.

In this paper we consider the inverse problem of determining the coefficient a if $\hat{u}(a, x_0, t)$ is known in an interior point $x_0 \in D$ for all $t \in [0, T]$, and we prove a uniqueness theorem for this problem. The paper is closely related to that of S. Dünnel [2] where the mentioned inverse problem is considered for the more simple equation $u_i = a(u) \Delta u$. The special case where $a(u)$ is constant can be found in S. Dümmen [1]. The equation $u_t = \text{div}(a(u) \text{ grad } u)$ is investigated by S. Meyer [5, 6] for $a(u) = b(u - v_0)^t + c$ (b, c constant, *l* natural number). The last equation with additional information about *u* on the boundary of *D* can be found in some other papers, e.g. in N. V. MUZYLEV [7]. For further references see [2].

2. Some lemmas

In this section we shall state some lemmas which will be needed in the proof of the uniqueness theorem. Let $a_1 \in A$ and $u_1, u_2 \in \overline{C}_{2+a}(Z_T)$. We define functions f_1 and f_2 by

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Substituting the values of the boundary of
$$
D
$$
 can be found to be considered.

\nSome lemmas

\nthis section we shall state some lemmas which will be needed in the *i*queness theorem. Let $a_1 \in A$ and $u_1, u_2 \in \overline{C}_{2+a}(Z_T)$. We define fun-
$$
f_1(x, t) = \begin{cases} a_1(u_1(x, t)) - a_1(u_2(x, t)) & \text{if } u_1(x, t) = u_2(x, t) \\ a_1'(u_2(x, t)) & \text{if } u_1(x, t) = u_2(x, t), \\ a_1'(u_2(x, t)) & \text{if } u_1(x, t) = u_2(x, t), \end{cases}
$$
\n
$$
f_2(x, t) = \begin{cases} a_1'(u_1(x, t)) - a_1'(u_2(x, t)) & \text{if } u_1(x, t) = u_2(x, t) \\ a_1''(u_2(x, t)) & \text{if } u_1(x, t) = u_2(x, t). \end{cases}
$$

\nLemma 1: One has $f_1, f_2 \in \overline{C}_a(Z_T)$.

\nis lemma can be proved as in [7].

\nWe introduce the following notations:

$$
I_1(x, t) = \begin{cases} u_1(x, t) - u_2(x, t) & \text{if } u_1(x, t) = u_2(x, t), \\ a_1'(u_2(x, t)) & \text{if } u_1(x, t) = u_2(x, t), \end{cases}
$$

\n
$$
f_2(x, t) = \begin{cases} \frac{a_1'(u_1(x, t)) - a_1'(u_2(x, t))}{u_1(x, t) - u_2(x, t)} & \text{if } u_1(x, t) = u_2(x, t), \\ a_1''(u_2(x, t)) & \text{if } u_1(x, t) = u_2(x, t). \end{cases}
$$

\nand 1: One has $f_1, f_2 \in \overline{C}_a(Z_T)$.
\nand can-be proved as in [7].
\nproduce the following notations:
\n
$$
k_- = \begin{cases} k/2 & \text{for even } k \\ (k - 1)/2 & \text{for odd } k, \end{cases} \qquad k_+ = \begin{cases} k/2 & \text{for even } k \\ (k + 1)/2 & \text{for odd } k, \end{cases}
$$

\nbe real analytic functions in an interval [c., c_a], By $W(u)$ with

This lemma can-be proved as in [7].

We introduce the following notations:

$$
\begin{aligned}\n\left[a_1''(u_2(x,t))\right] &\text{if } u_1(x,t) = u_2(x,t). \\
\text{and } 1: \text{ One has } f_1, f_2 \in \overline{C}_a(Z_T). \\
\text{ma can be proved as in [7].} \\
\text{roduce the following notations:} \\
k_- &= \begin{cases} k/2 & \text{for even } k \\ (k-1)/2 & \text{for odd } k \end{cases}, \qquad k_+ = \begin{cases} k/2 & \text{for even } k \\ (k+1)/2 & \text{for odd } k \end{cases} \\
\text{be real analytic functions in an interval } [c_1, c_2]. By } W(u) \text{ we determine} \\
W(u) &= \begin{vmatrix} g_1(u) & g_2(u) \\ g_1'(u) & g_2'(u) \end{vmatrix}.\n\end{aligned}
$$

Let g_1, g_2 be real analytic functions in an interval $[c_1, c_2]$. By $W(u)$ we denote Wronski's determinant

$$
\begin{aligned}\n\left| \begin{array}{cc} u_1(x, y) & u_2(x, y) \\ a_1''(u_2(x, t)) & \text{if } u_1(x, t) = u_2(x, t).\n\end{array} \right.\n\end{aligned}
$$
\na 1: One has $f_1, f_2 \in \overline{C}_a(Z_T)$.

\nma can be proved as in [7].

\nroduce the following notations:

\n
$$
k_{-} = \begin{cases} k/2 & \text{for even } k \\ (k - 1)/2 & \text{for odd } k, \end{cases} \qquad k_{+} = \begin{cases} k/2 & \text{for even } k \\ (k + 1)/2^{\setminus} \text{for odd } k. \end{cases}
$$
\nbe real analytic functions in an interval $[c_1, c_2]$. By $W(u)$ we denote Wron-
\n**min**

\n
$$
W(u) = \begin{vmatrix} g_1(u) & g_2(u) \\ g_1'(u) & g_2'(u) \end{vmatrix}.
$$
\n
$$
(2.1)
$$

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Lemma 2: The derivatives $W^{(k)}$ $(k = 1, 2, ...)$ of W are given by

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\nLemma 2: The derivatives
$$
W^{(k)}
$$
 $(k = 1, 2, \ldots)$ of W are given by

\n
$$
W^{(k)} = \begin{vmatrix} g_1 & g_2 \\ g_1^{(k+1)} & g_2^{(k+1)} \end{vmatrix} + b_1 \begin{vmatrix} g_1' & g_2' \\ g_1^{(k)} & g_2^{(k)} \end{vmatrix} + \cdots + b_k \begin{vmatrix} g_1^{(k-1)} & g_2^{(k-1)} \\ g_1^{(k+2)} & g_2^{(k+2)} \end{vmatrix}
$$
\nwhere $b_1 = k - 1$ and b_2, \ldots, b_k are real numbers.

\nThe proof can be made by mathematical induction.

\n3. A uniqueness theorem

\nUnder, the stated assumptions and some additional conditions we obtain that the inverse problem of the determination of the coefficient a has at most one solution,

where $b_1 = k - 1$ and $b_2, ..., b_k$ are real numbers.
The proof can be made by mathematical induction.

Under, the stated assumptions and some additional conditions we obtain that the inverse problem of the determination of the coefficient a has at most one solution, if we suppose that *a(d)* is known. ^u*¹* (a, x, *t) 0. ((x, I) € ZT; a* € *A) (3.1)*

Theorem: For all $a \in A$ *let* $u(a, \cdot, \cdot)$ *, be the solution of (1.1) satisfying the conditions (1.3) and in addition the condition*

$$
u_t(a, x, t) \geq 0 \qquad ((x, t) \in Z_T; a \in A). \tag{3.1}
$$

where the function w fulfils the condition (1.2). Suppose that $x_0 \in D$ and $h \in C^1([0, T])$.
 Furthermore let there exist a positive number t₀ such that $h'(t) > 0$ *for all* $t \in (0, t_0]$ *.

<i>Lastly let* a_0 *be a positiv Furthermore let there exist a positive number* t_0 *such that* $h'(t) > 0$ *for all* $t \in (0, t_0]$ *. Lastly let* a_0 *be a positive number. Then there is at most one* $a \in A$ *with* $a(d) = a_0$ *such that* $u(a, x_0, t) = h(t)$ *for all* $t \in [0, T]$. Under, the stated assumptions and some additional converse problem of the determination of the coefficient

if we suppose that $a(d)$ is known.

Theorem: For all $a \in A$ let $u(a, \cdot, \cdot)$ be the solution of
 (1.3) and in ad **i** *roblem* of the determination of the coefficient *a* has at most one solution,
 pose that $a(d)$ is known.
 em: *For all* $a \in A$ *let* $u(a, \cdot, \cdot)$, *be the solution of* (1.1) *satisfying the conditions*
 u₁(*a,*

Proof: We suppose that there are two functions $a_1, a_2 \in A$ with $a_1(d) = a_2(d) = a_0$ such that

$$
u(a_1, x_0, t) = u(a_2, x_0, t) = h(t) \text{ for all } t \in [0, T].
$$
\n(3.2)

For brevity we set $u_i(x, t) = u(a_i, x, t)$ $(i = 1, 2)$ and $u_{12} = u_1 - u_2, a_{12} = a_1 - a_2.$ From $u(a_1, x_0, t) = u(a_2, x_0, t) = h(t)$ for all $t \in [0, T]$.

(ity we set $u_i(x, t) = u(a_i, x, t)$ $(i = 1, 2)$ and $u_{12} = u_1 - u_2, a_{12} = a_1 - (u_i)_t = \text{div}(a_i(u_i) \text{ grad } u_i) = a_i(u_i) du_i + a_i'(u_i) (\text{grad } u_i)^2$ $(i = 1, 2)$

(i a linear parabolic differential equ

$$
(u_i)_i = \text{div} (a_i(u_i) \text{ grad } u_i) = a_i(u_i) \, \Delta u_i + a_i'(u_i) \, (\text{grad } u_i)^2 \qquad (i = 1, 2)
$$

we obtain a linear parabolic differential equation of second order for the function u_{12}
by elementary computations:

such that
\n
$$
u(a_1, x_0, t) = u(a_2, x_0, t) = h(t) \text{ for all } t \in [0, T].
$$
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\nFor brevity we set $u_i(x, t) = u(a_i, x, t)$ $(i = 1, 2)$ and $u_{12} = u_1 - u_2, a_{12} = a_1 - a_2$.
\nFrom
\n $(u_i)_t = \text{div } (a_i(u_i) \text{ grad } u_i) = a_i(u_i) \Delta u_i + a_i'(u_i) (\text{grad } u_i)^2$ $(i = 1, 2)$
\nwe obtain a linear parabolic differential equation of second order for the function u_{12}
\nby elementary computations:
\n $(u_{12})_t - a_1(u_1) \Delta u_{12} - a_1'(u_1) \sum_{j=1}^n ((u_1)_{z_j} + (u_2)_{z_j}) (u_{12})_{z_j}$
\n $\qquad - [f_1 \Delta u_2 + f_2(\text{grad } u_2)^2] u_{12} = \text{div } (a_{12}(u_2) \text{ grad } u_2) \quad ((x, t) \in Z_T).$ (3.3)
\nMoreover u_{12} satisfies the conditions
\n $u_{12}(x, 0) = 0 \quad (x \in \overline{D})$ and $u_{12}(x, t) = 0 \quad ((x, t) \in \Gamma_T).$ (3.4)
\nBecause of $a \in A$, (1.3) and Lemma 1 the assumptions of [4: § 4, Theorem 3] are
\nfulfilled. Using this theorem we obtain that there exists a unique solution of the

$$
u_{12}(x, 0) = 0 \quad (x \in \overline{D}) \quad \text{and} \quad u_{12}(x, t) = 0 \quad ((x, t) \in \Gamma_T). \tag{3.4}
$$

Because of $a \in A$, (1.3) and Lemma 1 the assumptions of [4: §4, Theorem 3] are because of $u \in A$, (1.3) and Lemma 1 the assumptions of [4: §4, Theorem 3] are fulfilled. Using this theorem we obtain that there exists a unique solution of the initial boundary value problem (3.3), (3.4). This solution initial boundary value problem (3.3), (3.4). This solution can be represented by Green's function G of the operator of the differential equation (3.3). Consequently we have we obtain a linear parabolic differential equation of second order for the t
by elementary computations:
 $(u_{12})_t - a_1(u_1) du_{12} - a_1'(u_1) \sum_{j=1}^n ((u_1)_{z_j} + (u_2)_{z_j}) (u_{12})_{z_j}$
 $- [f_1 du_2 + f_2(\text{grad } u_2)^2] u_{12} = \text{div } (a_{12}(u_2) \$ Moreover u_{12} satisfies the conditions
 $u_{12}(x, 0) = 0$ $(x \in \overline{D})$ and $u_{12}(x, t) = 0$ ((a

Because of $a \in A$, (1.3) and Lemma 1 the assumptions c

fulfilled. Using this theorem we obtain that there exists

initial bou

$$
u_{12}(x, t) = \int\limits_{0}^{t} \int\limits_{D} (G(x, t, x', t') \operatorname{div} (a_{12}(u_2(x', t') \operatorname{grad} u_2(x', t'))) dx' dt'.
$$
 (3.5)

Because of (3.2) for the left-hand side of (3.5) **there** holds

$$
u_{12}(x_0, t) = u_1(x_0, t) - u_2(x_0, t) = 0 \quad \text{for all} \quad t \in [0, T]. \tag{3.6}
$$

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of (3.2) for the left-hand side of (3.5) there holds
 $u_{12}(x_0, t) = u_1(x_0, t) - u_2(x_0, t) = 0$ for all $t \in [0, T]$.

netion a_{12} is analytic in the interval $[v_0^*, v_1^*]$. From the identity
 The function a_{12} is analytic in the interval $[v_0^*, v_1^*]$. From the identity theorem for analytic functions it follows that either $a_{12}(u) = 0$ for all $u \in [v_0, v_1]$, and we have the uniqueness of the function a , or a_{12} has at most finitely many zeros in have the uniqueness of the function *a*, or a_{12} has at most finitely many zeros in $[v_0, v_1]$. We investigate the second case. Since $\partial u_2/\partial t \ge 0$ in \bar{Z}_T , $u_2(x, \cdot)$ is monotone increasing for fixed $x \in \bar{D}$. increasing for fixed $x \in \overline{D}$. Thus $d = v_0$, and we have $a_1(v_0) = a_2(v_0)$. Hence v_0 is a zero of the function a_{12} . By w we denote the smallest of these zeros which is greater than v_0 , if such a zero exists. If such a zero does not exist we set $w = v_1$. Then, for all *u* with $v_0 < u < w$, either $a'_{12}(u) > 0$ or $a_{12}(u) < 0$. We consider the first case.
The second case can be treated analogously. In [2] it was proved that there exists $aT_0 > 0$ such that $a_{12}(u_2(x, t))$ does not change the sign for all $x \in \overline{D}$ and all $t \in (0, T_0)$. Now we define $T_0^* = \min \{t_0, T_0\}$. Using $h'(t) > 0$ $(t \in (0, t_0])$ we can see as in [2] that there exists a neighborhood $S_{\epsilon}(x_0)$ of x_0 and an interval $(t_1, t_2) \subset [0, T_0^*]$ such increasing for

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a $T_0 > 0$ suce

Now we define that

that (u_2) *M* case can be treated analogously. In [2] it was proved that there exists such that $a_{12}(u_2(x, t))$ does not change the sign for all $x \in \overline{D}$ and all $t \in (0, T_0)$.

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cond case. Since $\partial u_2/\partial t \ge 0$ in \bar{Z}_T , $u_2(x, \cdot)$ is n
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If such a zero do *y*. In [2] it was proved that there exists

ge the sign for all $x \in \overline{D}$ and all $t \in (0, T_0)$.
 $\langle t \rangle > 0$ ($t \in (0, t_0]$) we can see as in [2]
 $\langle t_0, t_0 \rangle$ and an interval $(t_1, t_2) \subset [0, T_0^*]$ such
 $\langle (x, t) \in B_{T_0$ 244 S. DOWNET and S. HANDROOK-Mix're

Becquise of (3.2) for the left-hand side of (3.6) there holds
 $u_1(x_0, t) = u_1(x_0, t) - u_2(x_0, t) = 0$ for all $t \in [0, T]$. (3.6)

The function x_0 , it and the interview $\left[u_0^2, t\right] = 0$

$$
(u_2)_t(x,t) > 0, \qquad u_2(x,t) > v_0, \qquad \big((x,t) \in B_{T_0^*} = S_{\epsilon}(x_0) \times (t_1,t_2)\big).
$$
 (3.7)

From this we obtain that)

$$
a_{12}(u_2) > 0
$$
 and $a_2(u_2) \Delta u_2 + a_2'(u_2)$ (grad u_2)² = $(u_2)_t > 0$

for all $(x, t) \in B_{T_0^*}$. Let $W(u)$ be the determinant (2.1) with $g_1 = a_2$ and $g_2 = a_{12}$. Then we obtain

$$
(u_2)_t (u, v) > 0, \t u_2(u, v) > v_0, \t (u, v) \in D_{T_0} = S_t(u_0) \wedge (v_1, v_2). \t (5.7)
$$
\nso we obtain that

\n
$$
a_{12}(u_2) \geq 0 \quad \text{and} \quad a_2(u_2) \Delta u_2 + a_2'(u_2) \text{ (grad } u_2)^2 = (u_2)_t > 0
$$
\n, $t \in B_{T_0}.$ Let $W(u)$ be the determinant (2.1) with $g_1 = a_2$ and $g_2 = a_{12}$.

\nobtain

\n
$$
\text{div } (a_{12}(u_2) \text{ grad } u_2) = a_{12}(u_2)' \Delta u_2 + a_{12}'(u_2) \text{ (grad } u_2)^2
$$
\n
$$
> \frac{W(u_2)}{a_2(u_2)} \text{ (grad } u_2)^2 \qquad \text{((x, t) < B_{T_0}.)}
$$
\n(3.8)

\nwe prove that there exists a real number $w_1 > 0$ such that $W(u) > 0$ for all $0 + w_1$. The function $a_{12} = a_{12}(u)$ is analytic in a neighborhood of v_0 and 0 . Either the derivatives $a_{12}^{(k)}(v_0) = 0$ for every k, and we have $a_1(u) = a_2(u)$ is the nontrivial number m such that

\n
$$
a_{12}^{(k)}(v_0) = 0 \qquad (k \in \{0, 1, ..., m\}) \quad \text{and} \quad a_{12}^{(m+1)}(v_0) \neq 0.
$$
\n(3.9)

\nso u such that $v_0 < u < w$. Then by Taylor's theorem we obtain, for some

\n
$$
w = \frac{1}{2} \sum_{k=1}^{m} \frac{1}{k!} \sum_{k=1}^{n} \frac{1}{k!} \sum_{k=1}^{n} \frac{1}{k!} \sum_{k=1}^{n} \frac{1}{k!} \sum_{k=1}^{n} \frac{1}{k!} \sum_{k=
$$

Now we prove that there exists a real number $w_1 > 0$ such that $W(u) > 0$ for all $u \in (v_0, v_0 + w_1)$. The function $a_{12} = a_{12}(u)$ is analytic in a neighborhood of v_0 and *a*₁₂(*v*₀, *v*₀ + *w*₁). The function *a*₁₂ = *a*₁₂(*u*) is analytic in a neighborhood of *v*₀ and $a_{12}(v_0) = 0$. Either the derivatives $a_{12}^{(i)}(v_0) = 0$ for every *k*, and we have $a_1(u) = a_2(u)$ in a ne in a neighborhood of v_0 , or there exists a natural number m such that $a_0 + w_1$). The function $a_{12}^{(k)}(v_0) = 0$
 *a*₁₂ $(v_0) = 0$
 *a*₁₂ $(v_0) = 0$
 *a*₁₂ $(u) = \frac{a_{12}^{(m+1)}}{2}$
 *a*₁₂ $(u) = \frac{a_{12}^{(m+1)}}{2}$ $a_{12}(u_2)$ (gra
 a_2)² ((*x*,
 $\sec w_1 > 0$

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 $a_{12}^{(m+1)}$

Taylor's the l
 $\tan(\xi) = \frac{(n+1)(\xi)}{2}$ $\frac{W(u_2)}{a_2(u_2)}$ (grad

Now we prove that there exists a real nun
 $u \in (v_0, v_0 + w_1)$. The function $a_{12} = a_{12}(u)$ is
 $a_{12}(v_0) = 0$. Either the derivatives $a_{12}^{(k)}(v_0) = 0$

in a neighborhood of v_0 , or there ex

$$
a_{12}^{(k)}(v_0) = 0 \qquad (k \in \{0, 1, ..., m\}) \quad \text{and} \quad a_{12}^{(m+1)}(v_0) \neq 0. \tag{3.9}
$$

 $\xi \in (v_0, u)$,

$$
(\frac{1}{2}u_{11} + u_{12}) = \frac{a_{12}(u_{11} - u_{12}(u_{12}))}{(u_{12} - u_{12}(u_{12}))} = 0 \text{ for every } k, \text{ and we have } a_{1}(u) = a_{2}(u)
$$

oborhood of v_{0} , or there exists a natural number m such that
 $a_{12}^{(k)}(v_{0}) = 0$ $(k \in \{0, 1, ..., m\})$ and $a_{12}^{(m+1)}(v_{0}) \neq 0$. (3.9)
we u such that $v_{0} < u < \psi$. Then by Taylor's theorem we obtain, for some
 $a_{12}(u) = \frac{a_{12}^{(m+1)}(\xi) (u - v_{0})^{m+1}}{(m+1)!}, \quad a_{12}^{(m+1)}(\xi) = \frac{(m+1)! a_{12}(u)}{(u - v_{0})^{m+1}} > 0$
use of (3.9),
 $a_{12}^{(m+1)}(v_{0}) = \lim_{u_{1} \to v_{0}} a_{12}^{(m+1)}(\xi) > 0.$ (3.10)
with u is also an analytic function and from (2.2) and (3.10) we obtain for

$$
W^{(k)}(v_{0}) = 0 \quad (k \in \{0, 1, ..., m - 1\}),
$$

$$
W^{(m)}(v_{0}) = a_{2}(v_{0}) a_{12}^{(m+1)}(v_{0}) > 0.
$$
(3.11)
0, there holds $W^{(0)}(v_{0}) = W(v_{0}) > 0$. Again using Taylor's theorem, (3.7),

and, because of (3.9) ,

- -

$$
a_{12}^{(m+1)}(v_0) = \lim_{u_1 \to v_0} a_{12}^{(m+1)}(\xi) > 0. \tag{3.10}
$$

 $W = W(u)$ is also an analytic function and from (2.2) and (3.10) we obtain for and, beca
 $W = W$
 $W \ge 1$

$$
a_{12}(u) = \frac{a_{12}^2 - (\xi)(u - v_0)^2}{(m+1)!}, \quad a_{12}^{(m+1)}(\xi) = \frac{(m+1)! \cdot a_{12}(u)}{(u - v_0)^{m+1}} > 0
$$

and, because of (3.9),

$$
a_{12}^{(m+1)}(v_0) = \lim_{u_1 \to v_0} a_{12}^{(m+1)}(\xi) > 0.
$$
(3.10)

$$
W = W(u)
$$
 is also an analytic function and from (2.2) and (3.10) we obtain for

$$
m \ge 1
$$

$$
W^{(k)}(v_0) = 0 \quad (k \in \{0, 1, ..., m - 1\}),
$$

$$
W^{(m)}(v_0) = a_2(v_0) a_{12}^{(m+1)}(v_0) > 0.
$$

For $m = 0$ there holds $W^{(0)}(v_0) = W(v_0) > 0$. Again using Taylor's theorem, (3.7),
(3.11) and the continuity of $W^{(m)}(u)$ in $[v_0, v_1]$ one can easily see that there exists

a $w_1 > v_0$ such that

. Uniqueness of the Solution of an Inverse Problem 245
\nwhich that
\n
$$
W(u_2) = \frac{W^{(m)}(\xi) (u_2 - v_0)^m}{m!} > 0 \qquad (u_2 \in (v_0, v_0 + w_1)).
$$
\n(3.12)
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\n

best of the Solution of an Inverse Problem 245
 >0 $(u_2 \in (v_0, v_0 + w_1)).$ (3.12)

ere exists a $T_1 > 0$ such that $W(u_2(x, t))$ does
 \overline{z}_{T_1} . We define $T_1^* = \min \{T_0^*, T_1\}$ and $B_{T_1^*}$ As in [2] it can be proved that there exists a $T_1 > 0$ such that $W(u_2(x, t))$ does not change the sign for all $(x, t) \in \overline{Z}_{T_1}$. We define $T_1^* = \min \{T_0^*, T_1\}$ and $B_{T_1^*}$, similarly as the set $B_{T_1^*}$. Both $B_{T_2^*}$ and $B_{T_1^*}$ can be chosen in such way that $B_{T_1^*}$. • Uniqueness of the Solution of
 $w_1 > v_0$ such that
 $W(u_2) = \frac{W^{(m)}(\xi) (u_2 - v_0)^m}{m!} > 0$ $(u_2 \in (v_0, v_0$

As in [2] it can be proved that there exists a $T_1 > 0$

not change the sign for all $(x, t) \in \overline{Z}_{T_1}$. We defi ch that $W(u_2)$
min $\{T_0^*, T_1\}$
n in such way such that
 $W(u_2) = \frac{W^{(m)}(\xi) (u_2 - v_0)^m}{m!} > 0$ $(u_2 \in (v_0, v_0 + w_1))$ (3.12)

2] it can be proved that there exists a $T_1 > 0$ such that $W(u_2(x, t))$ does

ge the sign for all $(x, t) \in \overline{Z}_{T_1}$. We define $T_1^* = \min \{T_0^*, T$ $W(u_2) = \frac{W^{(m)}(\xi) (u_2 - v_0)^m}{m!} > 0$ (u_2

As in [2] it can be proved that there exists a

t change the sign for all $(x, t) \in \overline{Z}_{T_1}$. We def

nilarly as the set B_{T_1} . Both B_{T_2} , and B_{T_1} , can
 B_{T_2} . T = min { T_0^*, T_1 } and $B_{T_1^*}$,
sen in such way that $B_{T_1^*}$
((x, *t*) $\in B_{T_1^*}$). (3.13)
nd $W(u_2) \ge 0$. Then if we
((x, *t*) $\in Z_{T_1^*}$). (3.14) nere exists a $T_1 > 0$ such that $W(u_2(x, t))$ does \overline{Z}_{T_1} . We define $T_1^* = \min \{T_0^*, T_1\}$ and $B_{T_1^*}$
and $B_{T_1^*}$ can be chosen in such way that $B_{T_1^*}$
we obtain
 $\frac{2}{2}$ (grad $u_2)^2 \ge 0$ $\qquad (x, t) \in B_{T_$ **u** *l* **W**<sub>(u_2) = $\frac{W^{(m)}(\xi)}{m!}$ *(u₂* $(u_2 - v_0)^m$ > 0 $\{u_2 \in (v_0, v_0 + w_1)\}$

As in [2] it can be proved that there exists a $T_1 > 0$ such that

not change the sign for all $(x, t) \in \mathbb{Z}_{T_1}$. We define T_1^*

$$
\subseteq B_{T_{\bullet}^*}.
$$
 Then from (3.8) and (3.12) we obtain
\n
$$
\text{div}\left(a_{12}(u_2)\,\text{grad}\,u_2\right) > \frac{W(u_2)}{a_2(u_2)}\left(\text{grad}\,u_2\right)^2 \geq 0 \quad \left((x,t) \in B_{T_{\bullet}^*}\right). \tag{3.13}
$$
\nMoreover for all $(x,t) \in Z_{T_{\bullet}^*}$ we have $a_{12}(u_2(x,t)) \geq 0$ and $W(u_2) \geq 0$. Then if we

repeat the above consideration it ,follows from (3.1) that

for all
$$
(x, t) \in Z_T
$$
, we have $a_{12}(u_2(x, t)) \ge 0$ and $W(u_2) \ge 0$. Then if we
\ne above consideration it follows from (3.1) that
\ndiv $(a_{12}(u_2)$ grad $u_2) \ge \frac{W(u_2)}{a_2(u_2)}$ (grad u_2)² ≥ 0 $((x, t) \in Z_T$.) (3.14)
\ne have (see [3: p. 83])
\n $G(x_0, T_1^*, x, t) > 0$ $((x, t) \in Z_T$.) (3.15)
\n $m.(3.13) - (3.15)$ we obtain

Lastly we have (see $[3: p. 83]$)

$$
G(x_0, T_1^*, x, t) > 0 \qquad ((x, t) \in Z_{T_1^*}). \qquad (3.15)
$$

Now from $(3.13) - (3.15)$ we obtain

for all
$$
(x, t) \in Z_{T_1}
$$
, we have $a_{12}(u_2(x, t)) \geq 0$ and $W(u_2) \geq 0$. The above consideration it follows from (3.1) that $\text{div}(a_{12}(u_2) \text{ grad } u_2) \geq \frac{W(u_2)}{a_2(u_2)} (\text{grad } u_2)^2 \geq 0$ $((x, t) \in Z_{T_1})$. Anve (see [3: p. 83]) $G(x_0, T_1^*, x, t) > 0$ $((x, t) \in Z_{T_1^*})$. $G(3.13) - (3.15)$ we obtain T_1^* $\iint_G G(x_0, T_1^*, x', t') \, \text{div}(a_{12}(u_2(x', t')) \text{ grad } u_2(x', t')) \, dx' \, dt' > 0$. is a contradiction to (3.5) and (3.6), and thus $a_1(u) = a_2(u_2(u_2(u_2(x', t'))))$.

But this is a contradiction to (3.5) and (3.6), and thus $a_1(u) = a_2(u)$ for all $u \in [v_0, v_1]$

Remark: If we consider the class of coefficients a with the property: a is an analytic function in the interval $[v_0, v_1]$.

Remark: If we consider the class of coefficients a with the property: a is an analytic function analytic function in the interval $[v_0^*, v_1^*], a(u) > 0$ and $a'(u) < 0$ in the interval $[v_0, v_1]$, then the proof of the uniqueness theorem is very simple. $G(x_0, T_1^*, x, t) > 0$ $((x, t) \in Z_{T_1^*})$. (3.15)

Now from (3.13)-(3.15) we obtain
 $\int_{0}^{T_1^*} G(x_0, T_1^*, x', t') \operatorname{div} (a_{12}(u_2(x', t')) \operatorname{grad} u_2(x', t')) dx'dt' > 0$.

But this is a contradiction to (3.5) and (3.6), and thus $a_1(u) = a_2(u)$ *(1.1)* $G(x_0, T_1^*, x, t)$ div $(a_{12}(u_0, u_1))$

But this is a contradiction to (3.5)
 $u \in [v_0, v_1]$ **R**

Remark: If we consider the class

analytic function in the interval $[v_0^*, v_1]$, then the proof of the uniquenes

In **Formal Remark:** If we consider

analytic function in the int
 $[v_0, v_1]$, then the proof of the

In the uniqueness theore

proposition: Let $a \in A$

(1.1) satisfying the conditions
 $u_t \in C(\overline{Z}_T) \cap \overline{C}_{2+\alpha}(Z)$

Suppose

In the uniqueness theorem we have used the supposition (3.1). In the following

Pr op o sit ion: *Let a € A and u be the unique solution of, the boundary value problem* (1.1) satisfying the conditions

uniqueness theorem we have used the supposition (3.1). In the following
on we shall give sufficient conditions which imply this relation.
sition: Let
$$
a \in A
$$
 and u be the unique solution of the boundary value problem
glying the conditions
 $u_t \in C(\bar{Z}_T) \cap \bar{C}_{2+\alpha}(Z_T)$, $u_{x_t} \in C(\bar{Z}_T)$ $(i = 1, ..., n)$, $\Delta u \in C(\bar{Z}_T)$.
(3.16)

• Suppose that the boundary junction fulfils (1.2) and in addition (x, . € C'([O, T]), $u_t \in C(\bar{Z}_T) \cap \bar{C}_{2+\alpha}(Z_T)$, $u_{x_t} \in C(\bar{Z}_T)$ $(i = 1, ..., n)$, $\Delta u \in C(\bar{Z}_T)$.

(3.16)

ose that the boundary function ψ fulfils (1.2) and in addition $\psi(x, \cdot) \in C^1([0, T])$,

(0) = 0 for all $x \in \partial D$, $\psi_t(x, t) \ge 0$ for all $\frac{\psi}{\geq}$ **Example 6.** $U(Z_T) \cap U_{2+a}(Z_T)$, $u_{x_i} \in C(Z_T)$ $(i = 1, ..., n)$, $Z(u \in C(Z_T)$.

(3.16)

Suppose that the boundary function ψ fulfils (1.2) and in addition $\psi(x, \cdot) \in C^1([0, T])$,
 $\psi_t(x, 0) = 0$ for all $x \in \partial D$, $\psi_t(x, t) \ge 0$ for a proposition with
 $\begin{aligned} \text{Proposition, w:} \ \Omega(1.1) \text{ satisfying} \ u_t &\in \ \text{Suppose that } t \\ \text{suppose that } (x, 0) &= 0 \ \text{for all } (x, t) \in \ \text{Proof: Set} \\ \text{have} \qquad \qquad w_t &= \ \end{aligned}$ valisfying the conditions
 $u_t \in C(\bar{Z}_T) \cap \bar{C}_{2+\alpha}(Z_T)$, $u_{x_t} \in C(\bar{Z}_T)$ $(i = 1, ..., n)$, $\Delta u \in C(\bar{Z}_T)$.

(3.16)

(3.16)

(3.16)

(2.16)

(2.16)

(4.18)

(4.18)

(4.19)

(4.19)

(4.19)

(4.19)

(4.19)

(4.19)

(4.19)

(4.1

have
\n
$$
w_t = a(u) \Delta w + 2a'(u) \text{ grad } u \text{ grad } w
$$
\n
$$
+ [a'(u) \Delta u + a''(u) (\text{grad } u)^2] w \text{ in } Z_T,
$$
\n
$$
w(x, 0) = 0 \text{ in } \overline{D}, w(x, t) = \psi_t(x, t) \text{ on } \Gamma_T. \text{ The functions } a(u), a'(u), a''(u), u_{x_1}, \dots, u_{x_n}
$$
\n(3.17)

and Δu are bounded in Z_T . Hence the coefficients in the differential equation (3.17) are also bounded in Z_T . On the boundary $\Gamma_T \cup \overline{D}$ we have $w(x, t) \geq 0$. Using a 'maximum principle [4: p. 8] we obtain $u_i(x, t) = w(x, t) \ge 0$ for all $(x, t) \in \bar{Z}_T$ **I**

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Manuskripteingang: 30.01.1987

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