Zeitschrift für Analysis
und ihre Anwendungen
Bd. 7 (3) 1988, S. 229–232

Clarkson's Inequalities, Besov Spaces and Triebel-Sobolev Spaces

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Es wird gezeigt, daß die Clarksonschen Ungleichungen auch in einigen Besov-Räumen $B^s_{p,q}$ und Triebel-Sobolev-Räumen $F^s_{p,q}$ gelten.

Доказывается, что неравенства Кларксона справедливы также в некоторых пространствах Бесова $B^s_{p,q}$ и пространствах Трибеля-Соболева $F^s_{p,q}$.

It is shown that Clarkson's inequalities also hold in some Besov spaces $B^s_{p,q}$ and Triebel-Sobolev spaces $F^s_{p,q}$.

1. Introduction. In 1936, J. A. CLARKSON [1] proved that the following inequalities hold in L_p:

$$\left(1/2\ \|f-g\|^{p'} + 1/2\ \|f+g\|^{p'}\right)^{1/p'} \leqq \left(\|f\|^p + \|g\|^p\right)^{1/p} \quad \text{if } 1 < p \leqq 2, \text{ with } 1/p + 1/p' = 1; \tag{1}$$

and

$$\left(1/2\ \|f-g\|^{p} + 1/2\ \|f+g\|^{p}\right)^{1/p} \leqq \left(\|f\|^{p'} + \|g\|^{p'}\right)^{1/p'} \quad \text{if } 2 \leqq p < \infty. \tag{2}$$

These generalise the parallelogram law in L_2, and show that L_p is uniformly convex if $1 < p < \infty$. Since the discovery of (1) and (2), many people have been interested in two problems which arise naturally: to find other Banach spaces in which these inequalities hold, and to find the most suitable way of proving them in such spaces. As examples we cite the paper by L. R. WILLIAMS and J. H. WELLS [9] for the case of L_p spaces, those by C. A. MCCARTHY [5] and C. E. CLEAVER [2] on Schatten p-classes, that by M. MILMAN [6] on Fourier-type spaces and the paper of the first-named author [3] concerning Sobolev spaces.

In this note we show that Clarkson's inequalities also hold in some Besov spaces $B^s_{p,q}$ and Triebel-Sobolev spaces $F^s_{p,q}$. To do this we compute the type or cotype constants for certain of these spaces.

It is known that the (Rademacher) type (resp. cotype) of $B^s_{p,q}$ and $F^s_{p,q}$ is min $(2,p,q)$ (resp. max $(2,p,q)$). Our procedure is to choose an appropriate L_r-norm in the definition of type (resp. cotype) in such a way that Clarkson's inequalities follow from the fact that these spaces have type (resp. cotype) constant equal to 1.

2. Preliminaries. Let N be a fixed positive integer, let $S(\mathbf{R}^n)$ and $S'(\mathbf{R}^n)$ be the Schwartz space of all infinitely differentiable rapidly decreasing (real- or complex-valued) functions on $\mathbf{R}^n$ and the collection of all (real- or complex-valued) tempered distributions on $\mathbf{R}^n$, respectively, and let F denote the Fourier transform on $S(\mathbf{R}^n)$. Let $(\phi_k)_{k\geq 0}$ be a sequence in $S(\mathbf{R}^n)$ with the following properties:

For all $\xi \in \mathbf{R}^n$ and all $k \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$, $\quad (F\phi_k)(\xi) \geqq 0$. (3)

$\operatorname{supp} F\phi_k \subset \{\xi \in \mathbf{R}^n : 2^{k-N} \leqq |\xi| \leqq 2^{k+N}\}$ for all $k \in \mathbf{N}$, (4)

and

$\operatorname{supp} F\phi_0 \subset \{\xi \in \mathbf{R}^n : |\xi| \leqq 2^N\}$.

There is a positive number c such that for all $\xi \in \mathbf{R}^n$, (5)

$$\sum_{k=0}^{\infty} (F\phi_k)(\xi) \geqq c.$$

Given any $\beta \in \mathbf{N}_0^n$, there is a positive number c_β such that for all (6)

$k \in \mathbf{N}$ and all $\xi \in \mathbf{R}^n \setminus \{0\}$, $|(D^\beta F\phi_k)(\xi)| \leqq c_\beta |\xi|^{-|\beta|}$.

The existence of such a sequence is established in Section 2.3.1 of [7].

Given any $s \in \mathbf{R}$, any $p \in (1, \infty)$ and any $q \in (1, \infty)$, we denote by $B^s_{p,q}$ the Besov space of all distributions $f \in S'(\mathbf{R}^n)$ having finite norm

$$\|f \mid B^s_{p,q}\| := \left[\sum_{k=0}^{\infty} \left(2^{sk} \|f * \phi_k \mid L_p(\mathbf{R}^n)\|\right)^q\right]^{1/q};$$

$F^s_{p,q}$ will stand for the Triebel-Sobolev space of all $f \in S'(\mathbf{R}^n)$ with finite norm

$$\|f \mid F^s_{p,q}\| := \left[\int_{\mathbf{R}^n} \left(\sum_{k=0}^{\infty} 2^{skq} |(f * \phi_k)(x)|^q\right)^{p/q} dx\right]^{1/p}.$$

Endowed with the corresponding norms, $B^s_{p,q}$ and $F^s_{p,q}$ are Banach spaces; they are independent of the particular choice of functions ϕ_k, changes of ϕ_k merely giving different, but equivalent, norms. The two scales of spaces, $B^s_{p,q}$ and $F^s_{p,q}$, cover many well-known classical spaces of functions and distributions on $\mathbf{R}^n$: for example, $F^s_{p,2}$ is the ordinary Sobolev space $W_p^s(\mathbf{R}^n)$ if $s \in \mathbf{N}$ and $p \in (1, \infty)$. Detailed information about these spaces can be found in [7] and [8].

Finally we turn to the notions of type and cotype. Given any $p \in (1, \infty)$ we denote by p' the number $p/(p-1)$. A Banach space X is said to be of (Rademacher) *type* p for some $p \in (1, 2]$ (resp. of *cotype* q for some $q \in [2, \infty)$) if there is a constant $C < \infty$ such that for every finite set of vectors $\{x_1, \ldots, x_m\}$ in X we have

$$\left(\int_0^1 \left\|\sum_{j=1}^m r_j(t)\, x_j\right\|^{p'} dt\right)^{1/p'} \leqq C \left(\sum_{j=1}^m \|x_j\|^p\right)^{1/p} \tag{7}$$

(respectively,

$$\left(\sum_{j=1}^m \|x_j\|^q\right)^{1/q} \leqq C \left(\int_0^1 \left\|\sum_{j=1}^m r_j(t)\, x_j\right\|^{q'} dt\right)^{1/q'},) \tag{8}$$

where r_j denotes the Rademacher function defined by $r_j(t) = \operatorname{sgn}(\sin(2^j\pi t))$ for $0 \leqq t \leqq 1$. In [4] these notions are introduced by using L_1 norms: by a result of J. P. Kahane (see [4], Theorem 1.e.13), that definition is equivalent to ours. Any constant C satisfying (7) (or (8)) is called a *type* p (or *cotype* q) *constant* of X. Note that if 1 is a type p (resp. cotype q) constant of X, then inequality (1) (respectively (2) with $p = q$) holds in X. This explains why we did not choose the L_1 norm in the definition of type and cotype. By way of illustration, we remark that 1 is a type p constant of L_p and l_p for all $p \in (1, 2]$ (see [6]). By a duality argument (see [4], Prop. 1.e.17), it is not hard to check that 1 is also a cotype q constant of L_q and l_q for all $q \in [2, \infty)$. This will be very useful in the next section.

3. Clarkson's inequalities. Now let us come to the results. For the *type* we have

Theorem 1: *Let $s \in \mathbf{R}$ and suppose that p, q are such that $1 < p \leqq 2$ and $p \leqq q \leqq p'$. Then 1 is a type p constant of $B^s_{p,q}$ and $F^s_{q,p}$.*

Proof: Let $m \in \mathbf{N}$ and let $f_1, \ldots, f_m \in B^s_{p,q}$. Then by the triangle inequality in $L_{p'/q}$, the fact that 1 is a type p constant of L_p, and the triangle inequality in $L_{q/p}$, we see that

$$\left(\int_0^1 \left\| \sum_{j=1}^m r_j(t)\, f_j \mid B^s_{p,q} \right\|^{p'} dt\right)^{1/p'}$$

$$= \left(\int_0^1 \left[\sum_{k=0}^\infty \left(2^{sk} \left\| \sum_{j=1}^m r_j(t)\, f_j * \phi_k \mid L_p(\mathbf{R}^n)\right\|\right)^q\right]^{p'/q} dt\right)^{1/p'}$$

$$\leqq \left(\sum_{k=0}^\infty 2^{skq} \left[\int_0^1 \left\| \sum_{j=1}^m r_j(t)\, f_j * \phi_k \mid L_p(\mathbf{R}^n)\right\|^{p'} dt\right]^{q/p'}\right)^{1/q}$$

$$\leqq \left(\sum_{k=0}^\infty 2^{skq} \left[\sum_{j=1}^m \|f_j * \phi_k \mid L_p(\mathbf{R}^n)\|^p\right]^{q/p}\right)^{1/q}$$

$$\leqq \left(\sum_{j=1}^m \left[\sum_{k=0}^\infty 2^{skq}\, \|f_j * \phi_k \mid L_p(\mathbf{R}^n)\|^q\right]^{p/q}\right)^{1/p} = \left(\sum_{j=1}^m \|f_j \mid B^s_{p,q}\|^p\right)^{1/p}.$$

This shows that 1 is a type p constant of $B^s_{p,q}$. For $F^s_{q,p}$ we proceed in a similar manner ∎

For the *cotype* we have

Theorem 2: *Let $s \in \mathbf{R}$ and suppose that p, q are such that $2 \leqq p < \infty$ and $p' \leqq q \leqq p$. Then 1 is a cotype p constant of $B^s_{p,q}$ and $F^s_{q,p}$.*

Proof: This time we give the details for $F^s_{q,p}$; the case of $B^s_{p,q}$ can be handled in the same way. Let $m \in \mathbf{N}$ and suppose that $f_1, \ldots, f_m \in F^s_{q,p}$. Then by the Minkowski inequality in $L_{q/p}$ (note that $q/p \leqq 1$ and so the usual Minkowski inequality is reversed), the fact that 1 is a cotype p constant of L_p and the generalised Minkowski inequality in $L_{q/p'}$, we have

$$\left(\sum_{j=1}^m \|f_j \mid F^s_{q,p}\|^p\right)^{1/p}$$

$$= \left(\sum_{j=1}^m \left[\int_{\mathbf{R}^n} \left(\sum_{k=0}^\infty 2^{skp}\, |(f_j * \phi_k)(x)|^p\right)^{q/p} dx\right]^{p/q}\right)^{1/p}$$

$$\leqq \left(\int_{\mathbf{R}^n} \left[\sum_{j=1}^m \left(\sum_{k=0}^\infty 2^{skp}\, |(f_j * \phi_k)(x)|^p\right)\right]^{q/p} dx\right)^{1/q}$$

$$\leqq \left(\int_{\mathbf{R}^n} \left[\int_0^1 \left(\sum_{k=0}^\infty 2^{skp} \left|\sum_{j=1}^m r_j(t)\, (f_j * \phi_k)(x)\right|^p\right)^{p'/p} dt\right]^{q/p'} dx\right)^{1/q}$$

$$\leqq \left(\int_0^1 \left[\int_{\mathbf{R}^n} \left(\sum_{k=0}^\infty 2^{skp} \left|\sum_{j=1}^m r_j(t)\, (f_j * \phi_k)(x)\right|^p\right)^{q/p} dx\right]^{p'/q} dt\right)^{1/p'}$$

$$= \left(\int_0^1 \left\| \sum_{j=1}^m r_j(t)\, f_j \mid F^s_{q,p}\right\|^{p'} dt\right)^{1/p'}.$$

The proof is complete ∎

Finally, as an immediate consequence of these theorems we obtain

Theorem 3: *Let $s \in \mathbf{R}$, $p \in (1, \infty)$, $q \in (1, \infty)$ and let X be $B^s_{p,q}$ or $F^s_{q,p}$. Then for all f and g in X we have, with $\|\cdot\|$ standing for the norm in X,*

$$(1/2\,\|f-g\|^{p'} + 1/2\,\|f+g\|^{p'})^{1/p'} \leqq (\|f\|^p + \|g\|^p)^{1/p}$$

if $1 < p \leqq 2$ *and* $p \leqq q \leqq p'$,

and

$$(1/2\,\|f-g\|^{p} + 1/2\,\|f+g\|^{p})^{1/p} \leqq (\|f\|^{p'} + \|g\|^{p'})^{1/p'}$$

if $2 \leqq p < \infty$ *and* $p' \leqq q \leqq p$.

Remark: Let Ω be an open subset of $\mathbf{R}^n$ and let $B^s_{p,q}(\Omega)$ (resp. $F^s_{p,q}(\Omega)$) be the restriction of $B^s_{p,q}$ (resp. $F^s_{p,q}$) to Ω, with norm

$$\|f \mid B^s_{p,q}(\Omega)\| := \inf\{\|g \mid B^s_{p,q}\| : g|_\Omega = f,\ g \in B^s_{p,q}\}$$

$$(\text{resp. } \|f \mid F^s_{p,q}(\Omega)\| := \inf\{\|g \mid F^s_{p,q}\| : g|_\Omega = f,\ g \in F^s_{p,q}\}).$$

Then it is clear that the inequalities of Theorem 3 hold when X is allowed to be $B^s_{p,q}(\Omega)$ or $F^s_{q,p}(\Omega)$.

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Manuskripteingang: 05. 08. 1987

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