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On the Analysis of a Particular Volterra-Stieltjes Convolution Integral Equation

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Es wird eine spezielle Volterra-Stieltjessche Faltungsgleichung, welche bei der mathematischen Modellierung von Aquifer-Untergrundspeichern von Interesse ist, aus der Sicht der Korrektheit entstehender Aufgaben untersucht. Ein direktes und zwei inverse Probleme können formuliert und bezüglich Existenz, Eindeutigkeit und Stabilität von Lösungen analysiert werden.

Исследуется частное интегральное уравнение типа Вольтерра-Стилтьеса, которое возникает в математическом моделлировании подземных хранилищ. Формулируются одна прямая и две обратные задачи. Для таких задач анализируются существование, единственность и устойчивость решений.

A particular Volterra-Stieltjes convolution integral equation arising in the mathematical modelling of aquifers is investigated. One direct and two inverse problems are formulated and analysed with respect to the uniqueness, existence and stability of associated solutions.

1. Introduction

Let $C[0, 1]$ be the space of all real continuous functions on $[0, 1]$, $\|\cdot\|$ designate the associated maximum norm and also the norm in the space of all bounded linear operators in $C[0, 1]$. Analogously, let $C^1[0, 1] \subset C[0, 1]$ be the space of all continuously differentiable real functions on $[0, 1]$ and $\|\cdot\|_1$ with $\|g\|_1 = \|g\| + \|g'\|$ designate the associated norm (g' denotes the first derivative of g) as well as the norm of all bounded linear operators from $C^1[0, 1]$ into $C[0, 1]$. Moreover, p_0, v_0, $v_{\min}$ and $v_{\max}$ are assumed to be fixed positive values throughout this paper. Finally, we denote by

$$\mathcal{P} = \{p \in C[0, 1]: p(t) > 0 \ (0 \leqq t \leqq 1), p(0) = p_0\},$$

$$\mathcal{V} = \{v \in C[0, 1]: 0 < v_{\min} \leqq v(t) \leqq v_{\max} \ (0 \leqq t \leqq 1), v(0) = v_0\},$$

$$\mathcal{X} = \{x \in C^1[0, 1]: x(t) > 0, x'(t) \leqq 0 \ (0 \leqq t \leqq 1)\}$$

subsets which are under consideration in the sequel.

Now we are going to deal with triples

$$(p, v, x) \in \mathcal{P} \times \mathcal{V} \times \mathcal{X} \subset C[0, 1] \times C[0, 1] \times C^1[0, 1]$$

satisfying the Volterra-Stieltjes convolution integral equation

$$\int_0^t x(t - \tau) \, d\Omega(\tau) = p(t) - p_0, \qquad \Omega(t) = \frac{v(t)}{p(t)} \quad (0 \leqq t \leqq 1), \tag{1}$$

for which the following three problems are of interest:

(**P1**) *Find* $p \in \mathcal{P}$ *if* $p_0 > 0$, $v \in \mathcal{V}$ *and* $x \in \mathcal{X}$ *are given!*
(**P2**) *Find* $v \in \mathcal{V}$ *if* $v_0 > 0$, $p \in \mathcal{P}$ *and* $x \in \mathcal{X}$ *are given!*
(**P3**) *Find* $x \in \mathcal{X}$ *if* $p \in \mathcal{P}$ *and* $v \in \mathcal{V}$ *are given!*

The purpose of the present paper is to make statements regarding the existence, uniqueness and stability of solutions to these three problems. Thus, we are going to decide whether (P1), (P2) and (P3) are well-posed or ill-posed in the sense of Hadamard (cf. e.g. [7, p. 16]).

Remark 1: Triples (p, v, x) satisfying (1) arise in the mathematical modelling of aquifers by means of the influence function method (see e.g. [2]). In this context, the time-dependent functions p and v represent reservoir pressure and volume of gas, i.e., the field data of an aquifer. The pore volume of the aquifer is reflected by the continuous function Ω. Finally, the monotonic nonincreasing smooth function x has memory character. It is a material function expressing the response to field data changes caused by the special geometry and by the geological properties of the aquifer.

Remark 2: It is evident that changes of the function values $p(t)$ for growing time t are always caused by the behaviour of $v(\tau)$ $(0 \leqq \tau \leqq t)$. The conditions of this causality are given by the function x. Therefore, (P1) is a direct problem, whereas (P2) and (P3) are both of inverse nature (as for inverse problems cf. [4]). From another point of view (P1) is a prediction problem, since p is to be predicted when v is prescribed. Then, (P2) gets a control problem: How to choose v in order to obtain the desired function p. Finally, (P3) may be considered to be a problem of parameter identification (cf. e.g. [6]).

2. Intrinsic properties of the occurring operators

For a given triple $(p, v, x) \in \mathcal{P} \times \mathcal{V} \times \mathcal{X}$, the function F defined by

$$F(p, v, x)(t) = \int_0^t x(t - \tau)\, d\Omega(\tau) - p(t) + p_0 \qquad (0 \leqq t \leqq 1)$$

is continuous (as for Ω cf. (1)). Namely, the integral

$$\int_0^t x(t - \tau)\, d\Omega(\tau)$$

exists whenever

$$\int_0^t \Omega(\tau)\, dx(t - \tau) = -\int_0^t x'(t - \tau)\, \Omega(\tau)\, d\tau$$

exists. Thus, we can express equation (1) in the form of an operator equation

$$F(p, v, x) = 0, \qquad F: \mathcal{P} \times \mathcal{V} \times \mathcal{X} \to C[0, 1].$$

From the partial integration formula we easily derive that the operator F is continuous with respect to all three variables p, v and x. Moreover, the equation (1) may be rewritten in the form

$$\int_0^t x'(t-\tau)\frac{v(\tau)}{p(\tau)}\,d\tau + x(0)\,\frac{v(t)}{p(t)} = x(t)\,\Omega_0 + p(t) - p_0 \quad (0 \leqq t \leqq 1), \tag{2}$$

$$\Omega_0 = \frac{v_0}{p_0}.$$

Note that, for given p and v, equation (1) is a linear Volterra-Stieltjes convolution integral equation of the first kind with respect to x. Then problem (P3) may be expressed by the operator equation

$$Cx = 0, \qquad C\cdot = F(p, v, \cdot), \qquad C: \mathcal{X} \to C[0, 1].$$

Here, C is an inhomogeneous linear continuous operator. On the other hand, for given v_0, p and x, equation (2) attains the form of a linear Volterra integral equation of the second kind with respect to v. Therefore, the operator equation

$$Bv = 0, \qquad B\cdot = F(p, \cdot, x), \qquad B: \mathcal{V} \to C[0, 1]$$

expressing (P2) is also associated with an inhomogeneous linear continuous operator B. Finally, for given p_0, v and x, equation (2) represents a nonlinear Volterra integral equation with respect to p. Hence, (P1) may be written as an operator equation

$$Ap = 0, \qquad A\cdot = F(\cdot, v, x), \qquad A: \mathcal{P} \to C[0, 1]$$

with nonlinear operator A. Due to (2) it can be shown that there exists a Fréchet derivative $A'(\hat{p}): C[0, 1] \to C[0, 1]$ of the operator A at any point $\hat{p} \in \mathcal{P}$ such that, for $(p, v, x) \in C[0, 1] \times C[0, 1] \times \mathcal{X}$, we have

$$\big(A'(\hat{p})\,p\big)(t) = -\int_0^t \frac{x'(t-\tau)\,v(\tau)}{\hat{p}^2(\tau)}\,p(\tau)\,d\tau - \left(\frac{x(0)\,v(t)}{\hat{p}^2(t)} + 1\right)p(t)$$

$$(0 \leqq t \leqq 1).$$

Lemma 1: *The inverse $\big(A'(\hat{p})\big)^{-1}$ of the operator $A'(\hat{p})$ exists and both operators are uniformly bounded with respect to all $p \in \mathcal{P}$ and $v \in \mathcal{V}$, where the inequalities*

$$\|A'(\hat{p})\| \leqq 2 + 2x(0)\,\|v\|\,\|\hat{p}^{-2}\| \qquad (\hat{p} \in \mathcal{P},\ v \in \mathcal{V},\ x \in \mathcal{X}) \tag{3}$$

and

$$\big\|\big(A'(\hat{p})\big)^{-1}\big\| \leqq \frac{x(0)}{x(1)} \qquad (\hat{p} \in \mathcal{P},\ x \in \mathcal{X}) \tag{4}$$

hold.

Proof: Let us factorize $A'(\hat{p})\,p = (A_2 + I)\,A_1 p$, where I denotes the unity operator and

$$q(t) = (A_1 p)(t) = -\left(\frac{x(0)\,v(t)}{\hat{p}^2(t)} + 1\right)p(t), \qquad (A_2 q)(t) = \int_0^t k(t, x)\,q(\tau)\,d\tau,$$

$$0 \geqq k(t, \tau) = \frac{x'(t-\tau)\,v(\tau)}{x(0)\,v(\tau) + \hat{p}^2(\tau)} \geqq \frac{x'(t-\tau)}{x(0)}.$$

Here, A_1 and A_2 are linear bounded operators in $C[0, 1]$. We have

$$\|A_1\| = \max_{0\le t\le 1} \left|\frac{q(t)}{p(t)}\right| = \max_{0\le t\le 1} \left(\frac{x(0)\, v(t)}{\hat{p}^2(t)} + 1\right) \leqq 1 + x(0)\, \|v\|\, \|\hat{p}^{-2}\|,$$

$$\|A_1^{-1}\| = \max_{0\le t\le 1} \left|\frac{p(t)}{q(t)}\right| < 1,$$

$$\|A_2\| = \max_{0\le t\le 1} \int_0^t |\hat{k}(t, \tau)|\, d\tau = -\max_{0\le t\le 1} \int_0^t \frac{x'(t-\tau)}{x(0)}\, d\tau = \frac{x(0) - x(1)}{x(0)} < 1.$$

Therefore, $(A'(\hat{p}))^{-1}$ exists and due to $\|(A_2 + I)^{-1}\| \leqq 1/(1 - \|A_2\|)$ (see e.g. [5, p. 140]) we obtain

$$\|(A'(\hat{p}))^{-1}\| \leqq \frac{\|A_1^{-1}\|}{1 - \|A_2\|} \leqq \left(1 - \frac{x(0) - x(1)}{x(0)}\right)^{-1} = \frac{x(0)}{x(1)}$$

and thus the inequality (4). Moreover,

$$\|A'(\hat{p})\| \leqq \|A_1\|\, (\|A_2\| + 1) \leqq 2 + 2x(0)\, \|v\|\, \|\hat{p}^{-2}\|$$

provides formula (3) ∎

The factorization technique used above also helps to investigate the operator B and its Fréchet derivative $B'\colon C[0, 1] \to C[0, 1]$ defined as

$$(B'v)(t) = \int_0^t \frac{x'(t-\tau)}{p(\tau)}\, v(\tau)\, d\tau + \frac{x(0)}{p(t)}\, v(t) \qquad (0 \leqq t \leqq 1)$$

for $(p, v, x) \in \mathcal{P} \times C[0, 1] \times \mathcal{X}$. Exploiting the ideas of Lemma 1 again we obtain

Lemma 2: *For all $p \in \mathcal{P}$ and $x \in \mathcal{X}$, the inverse $(B')^{-1}$ of the operator B exists and we have*

$$\|B'\| \leqq 2x(0)\, \|p^{-1}\| \tag{5}$$

as well as

$$\|(B')^{-1}\| \leqq \|p\|/x(1). \tag{6}$$

Finally, let us consider the Fréchet derivative $C'\colon C^1[0, 1] \to C[0, 1]$ given by

$$(C'x)(t) = \int_0^t x'(t-\tau)\, \Omega(\tau)\, d\tau + x(0)\, \Omega(t) - x(t)\, \Omega_0 \qquad (0 \leqq t \leqq 1),$$

where $x \in C^1[0, 1]$ and $\Omega \in C[0, 1]$ with $\Omega(t) > 0$ $(0 \leqq t \leqq 1)$ and $\Omega(0) = \Omega_0$ (for the definition of Ω and Ω_0 cf. (1) and (2)). If considering the continuous function

$$\Sigma(t) = \Omega(t) - \Omega_0 \qquad (0 \leqq t \leqq 1) \tag{7}$$

we can also write

$$(C'x)(t) = \int_0^t x'(t-\tau)\, \Sigma(\tau)\, d\tau + x(0)\, \Sigma(t). \tag{8}$$

This is due to

$$x(t)\, \Omega_0 = x(0)\, \Omega_0 + \int_0^t x'(t-\tau)\, \Omega_0\, d\tau.$$

Lemma 3: *For all $\Omega \in C[0, 1]$ with $\Omega(t) > 0$ $(0 \leqq t \leqq 1)$, $\Omega(0) = \Omega_0$, C': $C^1[0, 1] \to C[0, 1]$ is a linear bounded and compact operator satisfying the inequality*

$$\|C'\|_1 \leqq 2\,\|\Omega\|. \tag{9}$$

Consequently, C' is not surjective. This operator is injective if and only if there is no real number $\varepsilon > 0$ such that $\Omega(t) = \Omega_0$ $(0 \leqq t \leqq \varepsilon)$. In the injective case, the inverse operator $(C')^{-1}$ is unbounded.

Proof: The inequality (9) follows from

$$\|C'x\| \leqq \max_{0 \leq t \leq 1} \int_0^t |x'(t-\tau)|\,\Omega(\tau)\,d\tau + \max_{0 \leq t \leq 1} \{\Omega(t)\,x(0) + \Omega_0 x(t)\}$$
$$\leqq (\|x'\| + 2\,\|x\|)\,\|\Omega\| \leqq 2\,\|\Omega\|\,\|x\|_1.$$

On the other hand, the compactness of C' is a consequence of formula (8). Define the operators

$$C_1: C[0, 1] \to C[0, 1] \quad \text{by} \quad (C_1 g)(t) = \int_0^t \Sigma(t-\tau)\,g(\tau)\,d\tau$$

(cf. (7)) and

$$C_2: C[0, 1] \to C[0, 1] \quad \text{by} \quad (C_2 h)(t) = h(0)\,\Sigma(t) \qquad (0 \leqq t \leqq 1).$$

Then C_1 and C_2 are both compact. As for C_1 we refer e.g. to [3, p. 247]. The compactness of C_2 is immediately caused by Arzela's theorem (cf. e.g. [5, p. 20]). Thus any bounded subset of $\mathcal{X} \subset C^1[0, 1]$ is transformed into a compact subset of $C[0, 1]$ by applying the operator C'. This provides the compactness of C'. However, no compact operator of linear type is surjective and if it is injective, then its inverse is unbounded (see e.g. [4, p. 23]). If $\Omega(t) = \Omega_0$ and $\Sigma(t) = 0$ $(0 \leqq t \leqq \varepsilon)$, then the values of $x(\tau)$ and $x'(\tau)$ $(1 - \varepsilon \leqq \tau \leqq 1)$ do not influence $C'x$. Therefore, C' cannot be injective whenever such an $\varepsilon > 0$ exists. Finally, the stated sufficient condition for the injectivity of C' comes from Lemma 4 proved below ∎

Lemma 4: *Let $x \in C^1[0, 1]$ and $y \in C[0, 1]$ be functions so that*

$$\int_0^t x(t-\tau)\,dy(\tau) = 0 \qquad (0 \leqq t \leqq 1). \tag{10}$$

Then $x(t) = 0$ $(0 \leqq t \leqq 1)$ whenever there is no $\varepsilon > 0$ such that $y(t) = \text{const}$ $(0 \leqq t \leqq \varepsilon)$.

Proof: As already discussed above we can write

$$\int_0^t x'(t-\tau)\,z(\tau)\,d\tau + x(0)\,z(t) = 0, \qquad z(t) = y(t) - y(0) \qquad (0 \leqq t \leqq 1)$$

instead of (10). For $x(0) \neq 0$, this would contradict the well-known fact that Volterra integral operators with continuous kernels cannot have nonzero eigenvalues (cf. e.g. [5, p. 435]). Therefore, (10) may be expressed by the couple of equations

$$x(0) = 0 \quad \text{and} \quad \int_0^t x'(t-\tau)\,z(\tau)\,d\tau = 0 \qquad (0 \leqq t \leqq 1).$$

When applying Tichmarsh's theorem in the form of [1, p. 138] we obtain $x'(t) = 0$ $(0 \leqq t \leqq 1)$ and thus $x(t) = 0$ $(0 \leqq t \leqq 1)$ ∎

At the end of this section we still remark that A', B', C' are the partial Fréchet derivatives

$$\partial_p F(\cdot, v, x) = A'(\cdot), \qquad \partial_v F(p, \cdot, x) = B'(\cdot), \qquad \partial_x F(p, v, \cdot) = C'(\cdot)$$

of the operator F with respect to p, v and x, respectively.

3. A particular maximum principle

Now we return to triples $(p, v, x) \in \mathcal{P} \times \mathcal{V} \times \mathcal{X}$ satisfying equation (1). As we will show, all values $p(t)$ $(0 \leqq t \leqq 1)$ associated with such triples are uniformly bounded below and above by a couple of positive values $p_{\min}$ and $p_{\max}$. These upper and lower bounds depend on the given values p_0, v_0, $v_{\min}$ and $v_{\max}$. However, they are completely independent of the function $x \in \mathcal{X}$. Such a behaviour reminds us of the maximum principle established for classes of heat equation problems. Thus, problem (P1) is closely related to initial-boundary value problems in parabolic partial differential equations. Furthermore, the particular maximum principle stated below helps studying the correctness of inverse problems (P2) and (P3). On the other hand, it is of particular interest for the theory of nonlinear Volterra integral equations.

Theorem 1: *Let $(p, v, x) \in \mathcal{P} \times \mathcal{V} \times \mathcal{X}$ satisfying* (1). *Then we have*

$$\Omega_{\min} = \frac{v_{\min}}{v_0} \Omega_0 \leqq \Omega(t) \leqq \Omega_{\max} = \frac{v_{\max}}{v_0} \Omega_0 \qquad (0 \leqq t \leqq 1) \tag{11}$$

and

$$0 < p_{\min} = \frac{v_{\min}}{v_{\max}} p_0 \leqq p(t) \leqq p_{\max} = \frac{v_{\max}}{v_{\min}} p_0 \qquad (0 \leqq t \leqq 1). \tag{12}$$

Proof: Choose $\hat{t} \in [0, 1]$ so that $\Omega(t) \leqq \Omega(\hat{t})$ $(0 \leqq t \leqq 1)$. Owing to $x'(t) \leqq 0$ $(0 \leqq t \leqq 1)$ we have

$$\begin{aligned} p(\hat{t}) &= p_0 + x(0)\, \Omega(\hat{t}) - x(\hat{t})\, \Omega_0 + \int_0^{\hat{t}} x'(\hat{t} - \tau)\, \Omega(\tau)\, d\tau \\ &\geqq p_0 + x(0)\, \Omega(\hat{t}) - x(\hat{t})\, \Omega_0 + \Omega(\hat{t}) \int_0^{\hat{t}} x'(\hat{t} - \tau)\, d\tau \\ &= p_0 + x(0)\, \Omega(\hat{t}) - x(\hat{t})\, \Omega_0 + \Omega(\hat{t}) \big(x(\hat{t}) - x(0)\big) \\ &= p_0 + x(\hat{t}) \big(\Omega(\hat{t}) - \Omega_0\big) \geqq p_0 . \end{aligned}$$

Consequently, we get the right-hand side inequality of (11)

$$\Omega(t) \leqq \frac{v(\hat{t})}{p(\hat{t})} \leqq \frac{v_{\max}}{p_0} = \frac{v_{\max}}{v_0} \Omega_0 \qquad (0 \leqq t \leqq 1).$$

This implies

$$p(t) = \frac{v(t)}{\Omega(t)} \geqq \frac{v_{\min}}{v_{\max}} p_0 \qquad (0 \leqq t \leqq 1)$$

and thus the left-hand side inequality of (12). By choosing $\hat{t}$ so that $\Omega(\hat{t}) \leqq \Omega(t)$ $(0 \leqq t \leqq 1)$ the other couple of inequalities is derived in an analogous way ∎

Thus we are able to confine our considerations in the sequel to the subset of $\mathcal{P}$ (cf. (12))

$$\tilde{\mathcal{P}} = \{p \in \mathcal{P} : 0 < p_{\min} \leqq p(t) \leqq p_{\max} \quad (0 \leqq t \leqq 1)\}.$$

We can also summarize that, for fixed $x \in \mathcal{X}$, the operators $A'(\hat{p})$, $(A'(\hat{p}))^{-1}$, B', $(B')^{-1}$ and C' are uniformly bounded with respect to all $\hat{p} \in \tilde{\mathcal{P}}$, $p \in \tilde{\mathcal{P}}$ and $v \in \mathcal{V}$. Moreover, if we consider a neighbourhood $\mathcal{S} \subset \mathcal{X}$ of an element $\hat{x} \in \mathcal{X}$ such that $x(0) \leqq x_{\max}$ and $x(1) \geqq x_{\min} > 0$ for all $x \in \mathcal{S}$, then these operators are also uniformly bounded with respect to all $x \in \mathcal{S}$.

4. On the well-posedness of problem (P1)

First we are going to prove an existence and uniqueness theorem.

Theorem 2: *For any given $p_0 > 0$, $v \in \mathcal{V}$ and $x \in \mathcal{X}$, the problem* (P1) *is uniquely solvable with respect to $p \in \mathcal{P}$.*

Proof: For given $x \in \mathcal{X}$ and $v \in \mathcal{V}$ we consider the operator equation

$$L(p, \vartheta) = 0, \qquad p \in C[0,1], \quad \vartheta \in [0,1], \tag{13}$$

where the continuous nonlinear operator $L: C[0,1] \times [0,1] \to C[0,1]$ is defined by the formulae

$$L(p, \vartheta)(t) = \int_0^t x(t-\tau)\, d\, \frac{v_\vartheta(\tau)}{p_\varepsilon(\tau)} - p_\varepsilon(t) + p_0$$

$$(0 \leqq t \leqq 1,\ 0 \leqq \vartheta \leqq 1,\ 0 < \varepsilon < p_{\min}),$$

$$v_\vartheta(t) = v_0 + \vartheta\big(v(t) - v_0\big), \qquad p_\varepsilon(t) = \max\big(\varepsilon, p(t)\big).$$

As we have learned from Theorem 1, for any solution $(p, \vartheta) \in C[0,1] \times [0,1]$ of (13) we have $p \in \tilde{\mathcal{P}}$. If we introduce a set

$$\tilde{\mathcal{P}}_\delta = \{p \in \mathcal{P} : 0 < \varepsilon < p_{\min} - \delta \leqq p(t) \leqq p_{\max} + \delta \ (0 \leqq t \leqq 1)\}$$

according to a small positive number δ, then in view of Section 2 we can state that L and its partial Fréchet derivative $\partial_p L$ are uniformly continuous with respect to $(p, \vartheta) \in \tilde{\mathcal{P}}_\delta \times [0,1]$. Moreover, $(\partial_p L(p, \vartheta))^{-1}$ exists and is uniformly bounded with respect to all $(p, \vartheta) \in \tilde{\mathcal{P}}_\delta \times [0,1]$. Namely, we have $\|(\partial_p L(p, \vartheta))^{-1}\| \leqq x(0)/x(1)$ (cf. (4)). Finally, we a priori know that (13) has a unique solution with respect to p if $\vartheta = 0$. If we set $v_{\min} = v_0 = v_{\max}$, then it follows from Theorem 1 that $p_{\min} = p_0 = p_{\max}$. Thus, the constant function $p(t) = p_0$ $(0 \leqq t \leqq 1)$ is a solution of (1) for $v(t) = v_0$ $(0 \leqq t \leqq 1)$. Then under the conditions derived above a well-known corollary of the implicit-function theorem (see e.g. [8, p. 63]) yields that $L(p, \vartheta) = 0$ is also uniquely solvable with respect to $p \in C[0,1]$ if $\vartheta = 1$. Therefore, (P1) is uniquely solvable with respect to $p \in \mathcal{P}$. In view of Theorem 1 this uniquely determined solution belongs to the subset $\tilde{\mathcal{P}}$ of $\mathcal{P}$ ∎.

In order to complete the proof of well-posedness for problem (P1), we still have to show the stability of solutions with respect to small changes in v and x. For given $x \in \mathcal{X}$, let $p = Pv$, $P: \mathcal{V} \to \tilde{\mathcal{P}}$, represent the dependence of solutions p to equation (1) upon the element v. On the other hand, let $p = Qx$, $Q: \mathcal{X} \to \tilde{\mathcal{P}}$, denote the dependence between p and x in (1) for given $v \in \mathcal{V}$. As Theorem 2 indicates, both operators P and Q are uniquely determined.

Under the assumptions stated above the implicit-function theorem (cf. e.g. [8, p. 51]) applies. Thus, the operators P and Q are continuous and even Fréchet-differentiable. We obtain

$$P'(\hat{v}) = \left(A'(P\hat{v})\right)^{-1} B' \quad (\hat{v} \in \mathcal{V}) \quad \text{and} \quad Q'(\hat{x}) = \left(A'(Q\hat{x})\right)^{-1} C' \quad (\hat{x} \in \mathcal{X}).$$

Due to the mean-value theorem (cf. e.g. [5, p. 535]) these two formulae imply Lipschitz conditions stated as follows.

Theorem 3: *Let, for given $p_0 > 0$ and $x \in X$, $\hat{p} = P\hat{v}$ and $\hat{\hat{p}} = P\hat{\hat{v}}$, where $\hat{v}$ and $\hat{\hat{v}}$ are arbitrary elements of $\mathcal{V}$. Then,*

$$\|\hat{p} - \hat{\hat{p}}\| \leqq \frac{2\left(x(0)\right)^2}{x(1)} p_{\min}^{-1} \|\hat{v} - \hat{\hat{v}}\|. \tag{14}$$

On the other hand, let $\tilde{p} = Q\tilde{x}$ and $\tilde{\tilde{p}} = Q\tilde{\tilde{x}}$ for given $p_0 > 0$ and $v \in \mathcal{V}$, where $\tilde{x}$ and $\tilde{\tilde{x}}$ are arbitrary elements of $\mathcal{X}$. Then,

$$\|\tilde{p} - \tilde{\tilde{p}}\| \leqq \frac{2v_{\max}}{p_0} \frac{\max\left(\tilde{x}(0), \tilde{\tilde{x}}(0)\right)}{\min\left(\tilde{x}(1), \tilde{\tilde{x}}(1)\right)} \|\tilde{x} - \tilde{\tilde{x}}\|_1. \tag{15}$$

The results of this section show that (P1) is well-posed in the sense of Hadamard. In addition, the formulae (14) and (15) yield measures for the sensitivity of solutions with respect to perturbations in the data.

5. A uniqueness and stability theorem regarding problem (P2)

As we know, problem (P2) is of inverse nature. Thus, we suspect that it is ill-posed. Indeed, if $p \in \mathcal{P}$ but $p \notin \tilde{\mathcal{P}}$, then as a consequence of Theorem 1 there is, independently of the choice of $x \in \mathcal{X}$, no element $v \in \mathcal{V}$ such that equation (1) may be satisfied. Thus, the existence requirement of Hadamard's well-posedness definition is injured. However, it is known that we have at least one element of $\mathcal{P}$, namely the constant function $p(t) = p_0$ $(0 \leqq t \leqq 1)$, that possesses a solution $v \in \mathcal{V}$ to problem (P2) (see the proof of Theorem 2). From the example formulated below we will learn that the obviously closed union set of solutions $v \in \mathcal{V}$ to (P2) over all elements $p \in \tilde{\mathcal{P}}$ is not necessarily convex.

Example: Let us now consider the extremal case $x(t) = c > 0$ $(0 \leqq t \leqq 1)$. Then equation (1) attains the form

$$c\left(\Omega(t) - \Omega_0\right) = p(t) - p_0 \quad (0 \leqq t \leqq 1). \tag{16}$$

The values

$$p(t) = \frac{p_0}{2} - \frac{c\Omega_0}{2} + \left\{\left[\frac{p_0}{2} - \frac{c\Omega_0}{2}\right]^2 + cv(t)\right\}^{1/2} \quad (0 \leqq t \leqq 1) \tag{17}$$

verified from (16) form a continuous function $p \in \tilde{\mathcal{P}}$. As formula (17) shows, $p(t)$ does not depend on $v(\tau)$ $(0 < \tau < t)$. On the other hand, for given $p(t) > 0$, we derive the uniquely determined value

$$v(t) = \frac{p^2(t) + (c\Omega_0 - p_0)\, p(t)}{c} = p(t)\left(\Omega_0 + \frac{p(t) - p_0}{c}\right) \quad (0 \leqq t \leqq 1). \tag{18}$$

Thus, the function v may take on even negative values whenever $0 < p(t) < p_0$ and $c > 0$ gets sufficiently small. Now let for fixed $t \in [0, 1]$ the pairs $(p_1(t), v_1(t))$ and $(p_2(t), v_2(t))$ both belong to the rectangle $[p_{\min}, p_{\max}] \times [v_{\min}, v_{\max}]$ and together with a pair $(p_3(t), v_3(t))$ $(p_3(t) = (p_1(t) + p_2(t))/2)$ satisfy a condition (18) for the dependence between $v_i(t)$ and $p_i(t)$, $i = 1, 2, 3$. Then, it may occur that $v_3(t) < v_{\min}$ since $v(t)$ is a quadratic function of $p(t)$.

Now we are going to formulate a uniqueness and stability theorem for problem (P2).

Theorem 4: *For any given $v_0 > 0$, $p \in \mathcal{P}$ and $x \in \mathcal{X}$, there is a uniquely determined function $v \in C[0, 1]$, $v(0) = v_0$, such that the equation (1) holds. Consequently, if the problem (P2) is solvable with respect to $v \in \mathcal{V}$, then this solution is uniquely determined. Moreover, for given $v_0 > 0$ and $x \in \mathcal{X}$, we have*

$$\|\hat{v} - \hat{\hat{v}}\| \leqq \frac{x(0)\,\Omega_0 + 2p_{\max} - p_0}{x(1)} \|\hat{p} - \hat{\hat{p}}\| \tag{19}$$

whenever $\hat{v} \in \mathcal{V}$ and $\hat{\hat{v}} \in \mathcal{V}$ are solutions of (P2) corresponding to $\hat{p} \in \tilde{\mathcal{P}}$ and $\hat{\hat{p}} \in \tilde{\mathcal{P}}$, respectively. On the other hand, for given $v_0 > 0$ and $p \in \tilde{\mathcal{P}}$, we obtain

$$\|\tilde{v} - \tilde{\tilde{v}}\| \leqq \frac{\tilde{x}(0)\,\Omega_0 + p_{\max} - p_0}{\tilde{x}(1)\,\tilde{\tilde{x}}(1)} \|p\|\,\|\tilde{x} - \tilde{\tilde{x}}\|_1 \tag{20}$$

whenever $\tilde{v} \in \mathcal{V}$, $\tilde{\tilde{v}} \in \mathcal{V}$ are solutions of (P2) corresponding to $\tilde{x} \in \mathcal{X}$ and $\tilde{\tilde{x}} \in \mathcal{X}$, respectively.

Proof: For given $v_0 > 0$, $x \in \mathcal{X}$ and $p \in \mathcal{P}$, the problem (P2) corresponds to the equation

$$(B'v)(t) = x(t)\,\Omega_0 + p(t) - p_0 \qquad (0 \leqq t \leqq 1).$$

Due to Lemma 2 the linear operator B' is injective. Therefore, (P2) is uniquely solvable if the requirement $v \in \mathcal{V}$ is weakened to $v \in C[0, 1]$. Now consider formula (2) as a linear equation with respect to $\Omega \in C[0, 1]$:

$$\int_0^t x'(t - \tau)\,\Omega(\tau)\,d\tau + x(0)\,\Omega(t) = x(t)\,\Omega_0 + p(t) - p_0 \qquad (0 \leqq t \leqq 1). \tag{21}$$

For given $x \in \mathcal{X}$, let $\hat{\Omega}, \hat{\hat{\Omega}}$ denote the solutions of (21) according to $p = \hat{p}$ and $p = \hat{\hat{p}}$, respectively. Then we have (see Theorem 1 and the proof of Lemma 1)

$$\|\hat{\Omega} - \hat{\hat{\Omega}}\| \leqq \frac{\|\hat{p} - \hat{\hat{p}}\|}{x(1)} \quad \text{and} \quad \|\hat{\Omega}\| \leqq \frac{x(0)\,\Omega_0 + p_{\max} - p_0}{x(1)},$$

$$\|\hat{v} - \hat{\hat{v}}\| \leqq \|\hat{\Omega}\|\,\|\hat{p} - \hat{\hat{p}}\| + \|\hat{\hat{p}}\|\,\|\hat{\Omega} - \hat{\hat{\Omega}}\| \leqq \frac{x(0)\,\Omega_0 + 2p_{\max} - p_0}{x(1)} \|\hat{p} - \hat{\hat{p}}\|.$$

Now, for given $p \in \tilde{\mathcal{P}}$, let $\tilde{\Omega}, \tilde{\tilde{\Omega}}$ denote the solutions to (21) according to $\tilde{x}$ and $\tilde{\tilde{x}}$, respectively. Then

$$\|\tilde{\Omega} - \tilde{\tilde{\Omega}}\| \leqq \frac{\Omega_0\,\|\tilde{x} - \tilde{\tilde{x}}\|}{\tilde{\tilde{x}}(1)} + \frac{\tilde{x}(0)\,\Omega_0 + (p_{\max} - p_0)\,\|\tilde{x}' - \tilde{\tilde{x}}'\|}{\tilde{x}(1)\,\tilde{\tilde{x}}(1)}$$

and finally

$$\|\tilde{v} - \tilde{\tilde{v}}\| \leqq \|\tilde{\Omega} - \tilde{\tilde{\Omega}}\|\,\|p\| \leqq \frac{\tilde{x}(0)\,\Omega_0 + p_{\max} - p_0}{\tilde{x}(1)\,\tilde{\tilde{x}}(1)} \|p\|\,\|\tilde{x} - \tilde{\tilde{x}}\|_1 \quad \blacksquare$$

If a solution $v \in \mathcal{V}$ of problem (P2) according to $p \in \mathcal{P}$ and $x \in \mathcal{X}$ satisfies the inequalities

$$v_{\min} < \min_{0 \le t \le 1} v(t), \qquad v_{\max} > \max_{0 \le t \le 1} v(t),$$

then the estimations (19) and (20) show that (P2) is also solvable with respect to $v \in \mathcal{V}$ when the elements p and x change by a sufficiently small amount. Thus, at least the uniqueness and stability requirements of Hadamard's well-posedness definition are satisfied for problem (P2).

6. Notes on the ill-posedness of problem (P3)

Provided $p \in \mathcal{P}$ and $v \in \mathcal{V}$ are given the identification problem (P3) is expressed by the linear equation

$$(C'x)(t) = p(t) - p_0 \qquad (0 \leqq t \leqq 1). \tag{22}$$

Therefore, the properties of solutions to problem (P3) may be derived from the assertions of Lemma 3. It is evident that the existence requirement as well as the stability requirement of Hadamard's well-posedness definition get injured for problem (P3) since C' is a compact linear operator of $C^1[0, 1]$ into $C[0, 1]$, but $\mathcal{X}$ fails to be a compact subset of $C^1[0, 1]$. As a consequence of the non-closed range of C' the equation (22) can be inconsistent. Moreover, small perturbations of p and v may lead to significant changes in the solution x, because the inverse $(C')^{-1}$ is unbounded whenever it exists. Thus, problem (P3) becomes an ill-posed one. However, we can formulate a necessary and sufficient condition for the unique solvability of (P3) in the consistent case.

Theorem 5: *For given $p \in \tilde{\mathcal{P}}$ and $v \in \mathcal{V}$, let exist a solution $x \in \mathcal{X}$ of problem* (P3). *Then, this solution is uniquely determined if and only if there is no real number $\varepsilon > 0$ such that $v(t) = v_0$ $(0 \leqq t \leqq \varepsilon)$.*

Proof: If $(p, v, x) \in \tilde{\mathcal{P}} \times \mathcal{V} \times \mathcal{X}$ satisfy the equation (1), then $v(t) = v_0$ $(0 \leqq t \leqq \varepsilon)$ is equivalent to $\Omega(t) = \Omega_0$ $(0 \leqq t \leqq \varepsilon)$. This is due to the maximum principle established in Theorem 1. Thus, Theorem 5 immediately follows from Lemma 3 ∎

In order to identify the function $x \in \mathcal{X}$ in a unique manner, it suffices to require a non-steady state of v for an arbitrarily small initial interval $t \in [0, \varepsilon]$. However, in the computational identification of x based on observation data of p and v substantial difficulties arise from the instability of (P3) outlined above.

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