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## On the Analysis of a Particular Volterra-Stieltjes Convolution **Integral Equation**

## **B. HOFMANN**

Es wird eine spezielle Volterra-Stieltjessche Faltungsgleichung, welche bei der mathematischen Modellierung von Aquifer-Untergrundspeichern von Interesse ist, aus der Sicht der Korrektheit entstehender Aufgaben untersucht. Ein direktes und zwei inverse Probleme können formuliert und bezüglich Existenz, Eindeutigkeit und Stabilität von Lösungen analysiert werden.

Исследуется частное интегральное уравнение типа Вольтерра-Стилтьеса, которое возникает в математическом моделлировании подземных хранилищ. Формулируются одна. прямая и две обратные задачи. Для таких задач анализируются существование, единственность и устойчивость решений.

A particular Volterra-Stieltjes convolution integral equation arising in the mathematical modelling of aquifers is investigated. One direct and two inverse problems are formulated and analysed with respect to the uniqueness, existence and stability of associated solutions.

### 1. Introduction

Let  $C[0, 1]$  be the space of all real continuous functions on [0, 1], || || designate the associated maximum norm and also the norm in the space of all bounded linear operators in C[0, 1]. Analogously, let  $C^{1}[0, 1] \subset C[0, 1]$  be the space of all continuously differentiable real functions on [0, 1] and  $||\cdot||_1$  with  $||g||_1 = ||g|| + ||g'||$  designate the associated norm (g' denotes the first derivative of g) as well as the norm. of all bounded linear operators from  $C^1[0, 1]$  into  $C[0, 1]$ . Moreover,  $p_0$ ,  $v_0$ ,  $v_{\min}$  and  $v_{\text{max}}$  are assumed to be fixed positive values throughout this paper. Finally, we denote by

$$
\mathcal{P} = \{p \in C[0, 1]: p(t) > 0 \ (0 \le t \le 1), p(0) = p_0\},\
$$
  

$$
\mathcal{V} = \{v \in C[0, 1]: 0 < v_{\min} \le v(t) \le v_{\max} \ (0 \le t \le 1), v(0) = v_0\}
$$
  

$$
\mathcal{I} = \{x \in C^1[0, 1]: x(t) > 0, x'(t) \le 0 \ (0 \le t \le 1)\}
$$

subsets which are under consideration in the sequel.

Now we are going to deal with triples

$$
(p,v,x)\in \mathscr{P}\times \mathscr{V}\times \mathscr{X}\subset C[0,1]\times C[0,1]\times C^1[0,1]
$$

satisfying the Volterra-Stieltjes convolution integral equation

$$
\int_{0}^{t} x(t-\tau) d\Omega(\tau) = p(t) - p_0, \qquad \Omega(t) = \frac{v(t)}{p(t)} \quad (0 \leq t \leq 1), \tag{1}
$$

for which the following three problems are of interest:

**(P1)** *Find*  $p \in \mathcal{P}$  *if*  $p_0 > 0$ ,  $v \in \mathcal{V}$  and  $x \in \mathcal{X}$  are given! **(P2)** *Find*  $v \in \mathcal{V}$  *if*  $v_0 > 0$ ,  $p \in \mathcal{P}$  and  $x \in \mathcal{X}$  are given!

(P3) *Find*  $x \in \mathcal{X}$  *if*  $p \in \mathcal{P}$  *and*  $v \in \mathcal{V}$  *are given!* 

**248 B. HOFMANN**<br> **for which the following three problems are of interest:**<br> **(P1)** *Find*  $p \in \mathcal{P}$  *if*  $p_0 > 0$ ,  $v \in \mathcal{V}$  and  $x \in \mathcal{X}$  are given!<br> **(P2)** *Find*  $v \in \mathcal{V}$  *if*  $v_0 > 0$ ,  $p \in \mathcal{P}$  and The purpose of the present paper is to make statements regarding the existence, uniqueness and stability of solutions to these three problems. Thus, we are going to decide whether  $(P1)$ ,  $(P2)$  and  $(P3)$  are well-posed or ill-posed in the sense of Hadamard (of. è.g: [7, p. 16]).

Remark 1: Triples  $(p, v, x)$  satisfying (1) arise in the mathematical modelling of aquifers by means of the influence function method (see e.g.  $[2]$ ). In this context, the time-dependent functions  $p$  and  $v$  represent reservoir pressure and volume of gas, i.e., the field data of an aquifer. The pore volume of the aquifer is reflected by the continuous function  $\Omega$ . Finally, the monotonic nonincreasing smooth function  $x$  has memory character. It is a material function expressing the response to field data changes caused by the special geometry and by the geological properties of the aquifer.

Remark 2: It is evident that changes of the function values  $p(t)$  for growing time *t* are always caused by the behaviour of  $v(\tau)$  ( $0 \leq \tau \leq t$ ). The conditions of this causality are given by the function  $x$ . Therefore,  $(P1)$  is a direct-problem, whereas (P2) and (P3) are both of inverse nature (as for inverse problems cf. [4]). From another point of view.  $(P1)$  is a prediction problem, since  $p$  is to be predicted when *v* is prescribed. Then, (P2) gets a control problem: How to choose *v* in order to o is prescribed. Then,  $(F2)$  gets a control problem: How to choose v in order to obtain the desired function p. Finally,  $(P3)$  may be considered to be a problem of parameter identification (cf. e.g.  $[6]$ ). gas, i.e., the field data of an aquifer. The pore vol<br>by the continuous function  $\Omega$ . Finally, the monoto<br>tion x has memory character. It is a material funct<br>field data changes caused by the special geometry a<br>of the aqu

For a given triple  $(p,v,x)\in \mathscr{P}\times\mathscr{V}\times\mathscr{X}$ , the function  $F$  defined by

isic properties of the occurring operators.

\nven triple 
$$
(p, v, x) \in \mathcal{P} \times \mathcal{V} \times \mathcal{X}
$$
, the function  $F$  defined by

\n $F(p, v, x) \, (t) = \int_0^t x(t - \tau) \, d\Omega(\tau) - p(t) + p_0 \qquad (0 \leq t \leq 1)$ 

\nuous (as for  $\Omega$  cf. (1)). Namely, the integral

\n $\int_0^t x(t - \tau) \, d\Omega(\tau)$ 

\nwhenever

is continuous (as for  $\Omega$  cf. (1)). Namely, the integral

$$
\int x(t-\tau)\,d\Omega(\tau)
$$

exists whenever

$$
\int_{0}^{t} \Omega(\tau) \, dx(t-\tau) = -\int_{0}^{t} x'(t-\tau) \, \Omega(\tau) \, d\tau
$$

exists. Thus, we can express equation (1) in the form of an operator equation

$$
F(p, v, x) = 0, \qquad F: \mathcal{P} \times \mathcal{V} \times \mathcal{X} \to C[0, 1].
$$

 $F(p, v, x)(t) = \int_{0}^{t} x(t - \tau) d\Omega(\tau) - p(t) + p_0$  ( $0 \le t \le 1$ )<br>
ntinuous (as for  $\Omega$  cf. (1)). Namely, the integral<br>  $\int_{0}^{t} x(t - \tau) d\Omega(\tau)$ <br>
s whenever<br>  $\int_{0}^{t} \Omega(\tau) dx(t - \tau) = -\int_{0}^{t} x'(t - \tau) \Omega(\tau) d\tau$ <br>
s. Thus, we can express equa From the partial integration formula we easily derive that the operator  $F$  is continuous with respect to all three variables  $p, v$  and x. Moreover, the equation. (1)

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# A Particular Volterra-Stieltjes Convolution Integral Equation 249

may be rewritten in the form

A Particular Volterra-Stieltjes Convolution Integral Equation 249  
rewritten in the form  

$$
\int_{0}^{t} x'(t-\tau) \frac{v(\tau)}{p(\tau)} d\tau + x(0) \frac{v(t)}{p(t)} = x(t) \Omega_{0} + p(t) - p_{0} \quad (0 \le t \le 1),
$$
(2)  

$$
\Omega_{0} = \frac{v_{0}}{p_{0}}.
$$
At, for given p and v, equation (1) is a linear Volterra-Stieltjes convolution  
equation of the first kind with respect to x. Then problem (P3) may be  
do by the operator equation  
 $Cx = 0$ ,  $C \cdot = F(p, v, \cdot)$ ,  $C \cdot \mathcal{I} \rightarrow C[0, 1].$   
is an inhomogeneous linear continuous operator. On the other hand, for  
p and x, equation (2) attains the form of a linear Volterra integral equation  
cond kind with respect to v. Therefore, the operator equation

Note that, for given p and *v,* equation (1) is a linear Volterra-Stieltjés convolution integral equation of the first kind with respect to,x. Then problem (P3) may be expressed by the operator equation

$$
Cx = 0, \qquad C \cdot = F(p, v, \cdot), \qquad C \colon \mathcal{X} \to C[0, 1].
$$

Here,  $C$  is an inhomogeneous linear continuous operator. On the other hand, for given  $v_0$ , p and x, equation (2) attains the form of a linear Volterra integral equation of the second kind with respect to *v.* Therefore, the operator equation  $Cx = 0,$   $C \cdot F(p, v, \cdot),$   $C \cdot \mathcal{X} \rightarrow C[0, 1]$ <br>is an inhomogeneous linear continuous operator<br>*p* and *x*, equation (2) attains the form of a linear<br>cond kind with respect to *v*. Therefore, the operator<br> $Bv = 0,$   $B \cdot F(p, \cdot, x),$ 

$$
Bv = 0, \qquad B \cdot = F(p, \cdot, x), \qquad B \colon \mathcal{V} \to C[0, 1]
$$

expressing (P2) is also associated with an inhomogeneous linear continuous operator *B*. Finally, for given  $p_0$ ,  $v$  and  $x$ , equation (2) represents a nonlinear Volterra integral equation with respect to  $p$ . Hence, (P1) may be written as an operator equation % of the sequence of the sequence  $B$ . Findegral equation  $\begin{aligned} \text{integral} \quad \text{expresses} \ \text{Here,} \ C \ \text{given } v_0, \ \text{of the see} \ \text{expression} \ \text{for} \ B. \ \text{F} \quad \text{integral} \ \text{equation} \ \text{with non-derivative} \ \text{for} \ (p, v, \end{aligned}$ *Ap* = 0,  $A \cdot P(x, y) = F(x, y)$ ,  $A \cdot P(x, y) = F(x, y)$ ,  $A \cdot P(x, y) = F(x, y) + F(x, y)$ ,  $A \cdot P(x, y) = F(x, y, x)$ ,  $A \cdot P(x, y) = F(x, y) + F(x, y) + F(x, y)$ ,  $A \cdot P(x, y) = F(x, y) + F(x, y) + F(x, y) + F(x, y)$ ,  $A \cdot P(x, y) = F(x, y) + F(x, y) + F(x, y) + F(x, y) + F(x, y)$ Here, *C* is an inhomogeneous linear continuous opera<br>given  $v_0$ , *p* and *x*, equation (2) attains the form of a line<br>of the second kind with respect to *v*. Therefore, the ope<br> $Bv = 0$ ,  $B = F(p_j \cdot, x)$ ,  $B: \mathcal{V} \to C[0, 1]$ 

$$
Ap = 0, \qquad A \cdot = F(\cdot, v, x), \qquad A : \mathcal{P} \to C[0, 1]
$$

with nonlinear operator *A*. Due to (2) it can be shown that there exists a Fréchet derivative  $A'(\hat{p})$ :  $C[0, 1] \rightarrow C[0, 1]$  of the operator. A at any point  $\hat{p} \in \mathcal{P}$  such that,

equation  
\n
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$$
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\nfor  $(p, v, x) \in C[0, 1] \times C[0, 1] \times \mathcal{F}$ , we have  
\n
$$
(A'(\hat{p}) p) (t) = -\int_{0}^{t} \frac{x'(t - \tau) v(\tau)}{\hat{p}^2(\tau)} p(\tau) d\tau - \left(\frac{x(0) v(t)}{\hat{p}^2(t)} + 1\right) p(t)
$$
\n
$$
(0 \le t \le 1).
$$
\nLemma 1: The inverse  $(A'(\hat{p}))^{-1}$  of the operator  $A'(\hat{p})$  exists and both operators  
\nare uniformly bounded with respect to all  $\hat{p} \in \mathcal{P}$  and  $v \in V$ , where the inequalities  
\n $||A'(\hat{p})|| \le 2 + 2x(0) ||v|| ||\hat{p}^{-2}|| \quad (\hat{p} \in \mathcal{P}, v \in V, x \in \mathcal{X})$ \n(3)  
\nand  
\n $||(A'(\hat{p}))^{-1}|| \le \frac{x(0)}{x(1)}$   $(\hat{p} \in \mathcal{P}, x \in \mathcal{X})$ \n(d)  
\nProof: Let us factorize  $A'(\hat{p}) p = (A_2 + I) A_1 p$ , where I denotes the unity  
\noperator and  
\n $q(t) = (A_1 p) (t) = -\left(\frac{x(0) v(t)}{\hat{p}^2(t)} + 1\right) p(t), \quad (A_2 q) (t) = \int_{0}^{t} k(t, x) q(\tau) d\tau$ ,

 $I = L$ emma 1: The inverse  $(A'(\hat{p}))^{-1}$  of the operator  $A'(\hat{p})$  exists and both operators Lemma 1: *1*<br>
are uniformly bo<br>  $||A'(\hat{p})||$ <br>
and<br>  $||(A'(\hat{p})$ <br>
hold.<br>
Proof: Let

$$
(0 \leq t \leq 1).
$$
  
\n
$$
\text{max 1: } The \text{ inverse } (A'(\hat{p}))^{-1} \text{ of the operator } A'(\hat{p}) \text{ exists and both operators}
$$
  
\n
$$
\text{formly bounded with respect to all } p \in \mathcal{P} \text{ and } v \in \mathcal{V}, \text{ where the inequalities}
$$
  
\n
$$
||A'(\hat{p})|| \leq 2 + 2x(0) ||v|| ||\hat{p}^{-2}|| \qquad (\hat{p} \in \mathcal{P}, v \in \mathcal{V}, x \in \mathcal{X})
$$
  
\n
$$
||[(A'(\hat{p}))^{-1}|| \leq \frac{x(0)}{x(1)} \qquad (\hat{p} \in \mathcal{P}, x \in \mathcal{X})
$$
\n
$$
(4)
$$

• 

$$
||(A'(\hat{p}))^{-1}|| \leq \frac{x(0)}{x(1)} \quad (\hat{p} \in \mathcal{P}, x \in \mathcal{X})
$$

are uniformly bounded with respect to all 
$$
p \in \mathcal{P}
$$
 and  $v \in \mathcal{V}$ , where the inequalities  
\n
$$
||A'(\hat{p})|| \leq 2 + 2x(0) ||v|| ||\hat{p}^{-2}|| \qquad (\hat{p} \in \mathcal{P}, v \in \mathcal{V}, x \in \mathcal{X})
$$
\n(3)  
\nand  
\n
$$
|| (A'(\hat{p}))^{-1} || \leq \frac{x(0)}{x(1)} \qquad (\hat{p} \in \mathcal{P}, x \in \mathcal{X})
$$
\n(4)  
\nhold.  
\nProof: Let us factorize  $A'(\hat{p}) p = (A_2 + I) A_1 p$ , where *I* denotes the unity  
\noperator and  
\n
$$
q(t) = (A_1 p) (t) = -(\frac{x(0) v(t)}{\hat{p}^2(t)} + 1) p(t), \quad (\hat{A}_2 q) (t) = \int_0^t k(t, x) q(\tau) d\tau,
$$
\n
$$
0 \geq k(t, \tau) = \frac{x'(t - \tau) v(\tau)}{x(0) v(\tau) + \hat{p}^2(\tau)} \geq \frac{x'(t - \tau)}{x(0)}.
$$

Here,  $A_1$  and  $A_2$  are linear bounded operators in  $C[0, 1]$ . We have

250 B. HormANS  
\nHere, 
$$
A_1
$$
 and  $A_2$  are linear bounded operators in  $C[0, 1]$ . We have  
\n
$$
||A_1|| = \max_{0 \le t \le 1} \left| \frac{q(t)}{p(t)} \right| = \max_{0 \le t \le 1} \left( \frac{x(0) v(t)}{p^2(t)} + 1 \right) \le 1 + x(0) ||v|| ||p^{-2}||,
$$
\n
$$
||A_1^{-1}|| = \max_{0 \le t \le 1} \left| \frac{p(t)}{q(t)} \right| < 1,
$$
\n
$$
||A_2|| = \max_{0 \le t \le 1} \left| \frac{h(t, \tau)}{q(t)} \right| \le 1,
$$
\nTherefore,  $(A'(\hat{p}))^{-1}$  exists and due to  $||(A_2 + I)^{-1}|| \le 1/(1 - ||A_2||)$  (see e.g. [5, p. 140]) we obtain  
\n
$$
||(A'(\hat{p}))^{-1}|| \le \frac{||A_1^{-1}||}{1 - ||A_2||} \le \left(1 - \frac{x(0) - x(1)}{x(0)}\right)^{-1} = \frac{x(0)}{x(1)}
$$
\nand thus the inequality (4). Moreover,  
\n
$$
||A'(\hat{p})|| \le ||A_1|| (||A_2|| + 1) \le 2 + 2x(0) ||v|| ||\hat{p}^{-2}||
$$
\nprovides formula (3)

p. 140]) we obtain *2* exists and *1*  $\leq \frac{||A_1^{-1}||}{1 - ||A_2||}$ 

Therefore, 
$$
(A'(\hat{p}))^{-1}
$$
 exists and due to  $||(A_2 + I)^{-1}|| \leq 1/(1 - ||A_2||)$ . Therefore,  $(A'(\hat{p}))^{-1}$  exists and due to  $||(A_2 + I)^{-1}|| \leq 1/(1 - ||A_2||)$ .  
\n $||(A'(\hat{p}))^{-1}|| \leq \frac{||A_1^{-1}||}{1 - ||A_2||} \leq \left(1 - \frac{x(0) - x(1)}{x(0)}\right)^{-1} = \frac{x(0)}{x(1)}$ .  
\nand thus the inequality (4). Moreover,  $||A'(\hat{p})|| \leq ||A_1|| (||A_2|| + 1) \leq 2 + 2x(0) ||v|| ||\hat{p}^{-2}||$ .  
\nprovides formula (3)  $\blacksquare$   
\nThe factorization technique used above also helps to investigate and its Fréchet derivative  $B': C[0, 1] \rightarrow C[0, 1]$  defined as  $(B'v)(t) = \int_0^t \frac{x'(t - \tau)}{t^2} v(\tau) d\tau + \frac{x(0)}{t^2} v(t) \quad (0 \leq t \leq 1)$ .

$$
||A'(\hat{p})|| \leq ||A_1|| (||A_2|| + 1) \leq 2 + 2x(0) ||v|| ||\hat{p}^{-2}||
$$

provides formula (3) **<sup>I</sup>**

The factorization technique used above also helps to investigate the operator *B •* 

Therefore, 
$$
(A (p))
$$
 exists and due to  $||(A_2 + 1)|| = \pm 1/(1 - ||A_2||)(3$ ,  
\n $||A'(\hat{p})|^{-1}|| \le \frac{||A_1^{-1}||}{1 - ||A_2||} \le (1 - \frac{x(0) - x(1)}{x(0)})^{-1} = \frac{x(0)}{x(1)}$   
\nand thus the inequality (4). Moreover,  
\n $||A'(\hat{p})|| \le ||A_1|| (||A_2|| + 1) \le 2 + 2x(0) ||v|| ||\hat{p}^{-2}||$   
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\nThe factorization technique used above also helps to investigate the c  
\nand its Fréchet derivative  $B': C[0, 1] \rightarrow C[0, 1]$  defined as  
\n $(B'v)(t) = \int_0^t \frac{x'(t - \tau)}{p(\tau)} v(\tau) d\tau + \frac{x(0)}{p(t)} v(t)$   $(0 \le t \le 1)$   
\nfor  $(p, v, x) \in \mathcal{P} \times C[0, 1] \times \mathcal{X}$ . Exploiting the ideas of Lemma 1 again we  
\nLemma 2: For all  $p \in \mathcal{P}$  and  $x \in \mathcal{X}$ , the inverse  $(B')^{-1}$  of the operator  $B$   
\nwe have  
\n $||B'|| \le 2x(0) ||p^{-1}||'$   
\nas well as  
\n $||(B')^{-1}|| \le ||p||/x(1)$ .  
\nFinally, let us consider the Fréchet derivative  $C': C^1[0, 1] \rightarrow C[0, 1]$  give  
\n $(C'x)(t) = \int_0^t \frac{x'(t - x)}{x'(t - x)} Q(\tau) d\tau + x'(0) Q(t) - x'(t) Q(\tau) d\tau + x'(0) Q(\tau) d\tau$ 

for  $(p, v, x) \in \mathcal{P} \times C[0, 1] \times \mathcal{X}$ . Exploiting the ideas of Lemma 1 again we obtain

*Lemma 2: For all*  $p \in \mathcal{P}$  *and*  $x \in \mathcal{X}$ *, the inverse*  $(B')^{-1}$  *of the operator B exists and we have*  provides formula<br>
The factorizat<br>
and its Fréchet c<br>  $(B'v)$  (*t*)<br>
for  $(p, v, x) \in \mathcal{P}$ <br>
Lemma 2: Fc<br>
we have<br>  $||B'|| \le$ <br>
as well as<br>  $||(B')^{-1}||$  $\frac{1}{16}$  ideas of Lemma 1 again we obtain<br>
nverse  $(B')^{-1}$  of the operator B exists and<br>
(5)

$$
||B'|| \leq 2x(0) ||p^{-1}|| \tag{5}
$$

-

(6)

 $\frac{1}{2}$ 

$$
|| (B')^{-1} || \leq ||p||/x(1).
$$

Finally, let us consider the Fréchet derivative  $C'$ :  $C^1[0, 1] \rightarrow C[0, 1]$  given by<br>  $\therefore$ <br>
Finally, let us consider the Fréchet derivative  $C'$ :  $C^1[0, 1] \rightarrow C[0, 1]$  given by

Lemma 2: For all 
$$
p \in \mathcal{P}
$$
 and  $x \in \mathcal{X}$ , the inverse  $(B')^{-1}$  of the operator  $B$  exists and  
\nwe have  
\n
$$
||B'|| \leq 2x(0) ||p^{-1}||
$$
\n(5)  
\nas well as  
\n
$$
||(B')^{-1}|| \leq ||p||/x(1).
$$
\n(6)  
\nFinally, let us consider the Fréchet derivative  $C': C^1[0, 1] \rightarrow C[0, 1]$  given by  
\n
$$
(C'x) (t) = \int_0^t x'(t - \tau) \Omega(\tau) d\tau + x(0) \Omega(t) - x(t) \Omega_0 \qquad (0 \leq t \leq 1),
$$
\nwhere  $x \in C^1[0, 1]$  and  $\Omega \in C[0, 1]$  with  $\Omega(t) > 0$  ( $0 \leq t \leq 1$ ) and  $\Omega(0) = \Omega_0$  (for  
\nthe definition of  $\Omega$  and  $\Omega_0$  of. (1) and (2)). If considering the continuous function  
\n
$$
\Sigma(t) = \Omega(t) - \Omega_0 \qquad (0 \leq t \leq 1)
$$
\nwe can also write  
\n
$$
(C'x) (t) = \int_0^t x'(t - \tau) \Sigma(\tau) d\tau + x(0) \Sigma(t).
$$
\n(8)  
\nThis is due to  
\n
$$
x(t) \Omega_0 = x(0) \Omega_0 + \int_0^t x'(t - \tau) \Omega_0 d\tau.
$$

where  $x \in C^1[0, 1]$  and  $\Omega \in C[0, 1]$  with  $\Omega(t) > 0$   $(0 \le t \le 1)$  and  $\Omega(0) = \Omega_0$  (for the definition of  $\Omega$  and  $\Omega_0$  cf. (1) and (2)). If considering the continuous function  $||(B')^{-1}|| \leq ||p||/x(1).$ <br>Finally, let us consider the<br> $(C'x) (t) = \int_0^t x'(t - t)$ <br>where  $x \in C^1[0, 1]$  and  $\Omega \in C$ <br>the definition of  $\Omega$  and  $\Omega_0$  cf.<br> $\varSigma(t) = \Omega(t) - \Omega_0$ <br>we can also write<br> $(C'x) (t) = \int_0^t x'(t - t)$ <br>This is due to<br> $x$ 

$$
\Sigma(t) = \Omega(t) - \Omega_0 \qquad (0 \le t \le 1)
$$

we can also write

$$
(C'x)(t)=\int\limits_0^t x'(t-\tau)\ \Sigma(\tau)\,d\tau\ +\ x(0)\ \Sigma(t).
$$

$$
x(t) \Omega_0 = x(0) \Omega_0 + \int_0^t x'(t'-\tau) \Omega_0 d\tau.
$$

 

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A Particular Volterra<sub>i</sub>Stieltjes Convolution Integral Equation 251<br>
Lemma 3: *For all*  $\Omega \in C[0, 1]$  *with*  $\Omega(t) > 0$   $(0 \le t \le 1)$ ,  $\Omega(0) = \Omega_0$ ,  $C'$ :  $C^1[0, 1]$  $\rightarrow$  C[0, 1] *is a linear bounded and compact operator satisfying the inequality* 

$$
\|C'\|_1 \leq 2 \|\Omega\|.
$$

*A* Particular Volterra<sub>i</sub>Stieltjes Convolution Integral Equation 251<br> *J*<br> *J J <i>Z I is a linear bounded and compact operator satisfying the inequality<br>*  $||C'||_1 \leq 2 ||Q||$ *.<br>
<i>J I*<sup>*I*</sup> *I is not surjective. This ./*- *Consequently, C' is not surjective. This operator is injectie if and only if there is no*   $\rightarrow$  C[0, 1] is a unear bounded and compact operator satisfying the inequality<br>  $||C'||_1 \leq 2 ||\Omega||$ .<br>
Consequently, C' is not surjective. This operator is injective if and only if there is no<br>
real number  $\varepsilon > 0$  such that *operator*  $(C')^{-1}$  *is unbounded.* urjective. This operator is injective if and or<br>
that  $\Omega(t) = \Omega_0 \ (0 \le t \le \varepsilon)$ . In the injective<br>
ded.<br>  $\begin{aligned} \n\Omega(t) &= \Omega_0 \ (0 \le t \le \varepsilon) \ \Omega(t) &= \Omega_0 \ (0 \le t \le \varepsilon) \ \Omega(t) &= \Omega_0 \ \Omega_0 \end{aligned}$ (9)<br> *is injective if and only if there is no*<br>  $\sum_{i=1}^{n} E_i$ . In the injective case, the inverse<br>  $\max \{ \Omega(t) x(0) + \Omega_0 x(t) \}$ <br>  $\therefore$ <br>
consequence of formula (8). Define<br>  $\int_{t}^{t} \Sigma(t-\tau) g(\tau) d\tau$ <br>  $h(0) \Sigma(t)$   $(0 \le t \le 1).$ 

Proof: The inequality (9) follows **from** 

operator 
$$
(C')^{-1}
$$
 is unbounded.

\nProof: The inequality (9) follows from

\n
$$
||C'x|| \leq \max_{0 \leq t \leq 1} \int_{0}^{t} |x'(t - \tau)| \Omega(\tau) d\tau + \max_{0 \leq t \leq 1} \Omega(t) x(0) + \Omega_0 x(t)|
$$
\n
$$
\leq (||x'|| + 2'||x||) ||\Omega|| \leq 2 ||\Omega|| ||x||.
$$
\nOn the other hand, the compactness of  $C'$  is a consequence of formula (8). Define

the operators

$$
\geq ||x|| + 2 ||x|| + 2 ||x|| \leq 2 ||x|| + 2 ||x||.
$$
\nOn the other hand, the compactness of C' is a consequence of formula (8). Define the operators\n
$$
C_1: C[0, 1] \rightarrow C[0, 1] \text{ by } (C_1g)(t) = \int_0^t \Sigma(t - \tau) g(\tau) d\tau
$$
\n(cf. (7)) and\n
$$
C_2: C[0, 1] \rightarrow C[0, 1] \text{ by } (C_2h)(t) = h(0) \Sigma(t) \quad (0 \leq t \leq 1).
$$
\nThen  $C_1$  and  $C_2$  are both compact. As for  $C_1$  we refer e.g. to [3, p. 247]. The compact-

*(cf. (7))* and

 

$$
C_2: C[0, 1] \to C[0, 1]
$$
 by  $(C_2 h)(t) = h(0) \Sigma(t)$   $(0 \le t \le 1).$ 

ness of  $C_2$  is immediately caused by Arzela's theorem (cf. e.g. [5, p. 20]). Thus any bounded subset of  $\mathcal{X} \subset C^1[0, 1]$  is transformed into a compact subset of  $C[0, 1]$ by applying the operator *C'.* This provides the compactness of *C'.* However, no compact operator of linear type is surjective and if it is injective, then its inverse<br>is unbounded (see e.g. [4, p. 23]). If  $\Omega(t) = \Omega_0$  and  $\Sigma(t) = 0$  ( $0 \le t \le \varepsilon$ ), then the ness of  $C_2$  is immediately caused by Arzela's theorem (cf. e.g. [5, p. 20]). Thus any bounded subset of  $\mathcal{X} \subset C^1[0, 1]$  is transformed into a compact subset of  $C[0, 1]$  by applying the operator  $C'$ . This provides be injective whenever such an  $\varepsilon > 0$  exists. Finally, the stated sufficient condition for the injectivity of  $C'$  comes from Lemma 4 proved below  $\blacksquare$ 

Lemma  $4:$  *Let*  $x \in C^1[0, 1]$  *and*  $y \in C[0, 1]$  *be functions so that* 

for the injectivity of C' comes from Lemma 4 proved below  
\nLemma 4: Let 
$$
x \in C^1[0, 1]
$$
 and  $y \in C[0, 1]$  be functions so that  
\n
$$
\int_a^t x(t - \tau) dy(\tau) = 0 \qquad (0 \le t \le 1).
$$
\nThen  $x(t) = 0$   $(0 \le t \le 1)$  whenever there is  $n \circ \varepsilon > 0$  such that  $y(t) = \text{const } (0 \le t \le \varepsilon)$ .

Proof: As already discussed above we can write

$$
x(t) = 0 \ (0 \le t \le 1) \text{ whenever there is no } \varepsilon > 0 \text{ such that } y(t) = \text{const } (0 \le t \le 0 \text{ of: As already discussed above we can write}
$$
\n
$$
\int_{0}^{t} x'(t-\tau) z(\tau) d\tau + x(0) z(t) = 0, \qquad z(t) = y(t) - y(0) \qquad (0 \le t \le 1)
$$

instead of (10). For  $x(0) \neq 0$ , this would contradict the well-known fact that Volterra integral operators with continuous kernels cannot-have nonzero eigenvalues (cf. e.g. [5, p. 435]). Therefore, (10) may be expressed by the couple of equations

$$
x(0) = 0
$$
 and  $\int_{0}^{t} x'(t - \tau) z(\tau) d\tau = 0$   $(0 \le t \le 1).$ 

When applying Tichmarsh's theorem in the form of  $[1, p. 138]$  we obtain  $x'(t) = 0$  $x(0) = 0$  and  $\int_{0}^{t} x'(t-\tau) z(\tau) d\tau =$ <br>When applying Tichmarsh's theorem in the  $(0 \le t \le 1)$  and thus  $x(t) = 0$   $(0 \le t \le 1)$ 

At the end of this section we still remark that *A', B', C'* are the partial Fréchet derivatives *a*<sub>*AF(x*)  $\alpha$ <sub>*z*</sub> *a*<sub>*z*</sub> *exting a*<sub>*z*</sub> *exting a*<sub>*z*</sub> *exting a*<sub>*z*</sub> *F(ti)*, *a*<sub>*z*</sub>*F(ti)*, *a*<sub>*z*</sub>*F(ti)*, *a*<sub>*z*</sub>*F(ti)*, *e*<sub>*z*</sub>*F(ti)*, *e*<sub>*z*</sub>*F(ti) exting F* with respect to *n*, *v* and</sub>

$$
\partial_p F(\cdot, v, x) = A'(\cdot), \qquad \partial_v F(p, \cdot, x) = B'(\cdot), \qquad \partial_x F(p, v, \cdot) = C'(\cdot)
$$

 $\epsilon$ 

of the operator  $F$  with respect to  $p$ ,  $v$  and  $x$ , respectively.

### 3. A particular maximum principle

Now we return to triples  $(p, v, x) \in \mathcal{P} \times \mathcal{V} \times \mathcal{X}$  satisfying equation (1). As we will show, all values  $p(t)$  ( $0 \le t \le 1$ ) associated with such triples are uniformly bounded below and above by a couple of positive values *Pmin* and *Pniax•* These upper and lower bounds depend on the given values  $p_0$ ,  $v_0$ ,  $v_{\text{min}}$  and  $v_{\text{max}}$ . However, they are completely independent of the function  $x \in \mathcal{X}$ . Such a behaviour reminds us of the maximum principle established for classes of heat equation problem's. Thus, problem' (P1) is closely related to initial-boundary value problems in parabolic partial differential equations. Furthermore, the particular maximum principle stated below helps studying the correctness of inverse problems (P2) and (P3). On the other hand, it is of particular interest for the theory of nonlinear Volterra integral equations.<br>
Theorem 1: Let  $(p, v, x) \in \mathcal{P} \times \mathcal{V} \times \mathcal{X}$  s hand, it is of particular interest for the theory of nonlinear Volterra integral equations. ndependent of t<br>in principle estable osely related to<br>quations. Further dying the correlation of particular<br>is of particular i<br>e m 1 : Let  $(p, v,$ <br> $\Omega_{\text{min}} = \frac{v_{\text{min}}}{v_0}$   $\Omega_0$ ch a behaviour renunds us of the<br>t equation problems. Thus, problem<br>problems in parabolic partial differ-<br>maximum principle stated below<br>ems (P2) and (P3). On the other<br>of nonlinear Volterra integral equa-<br>ing (1). Then w *V0 V0.*  (P1) is closely related to initial-boundary value problems in parab<br>ential equations. Furthermore, the particular maximum principle<br>helps studying the correctness of inverse problems (P2) and (P<br>hand, it is of particular

**Theorem 1:** Let  $(p, v, x) \in \mathcal{P} \times \mathcal{V} \times \mathcal{X}$  satisfying (1). Then we have

is of particular interest for the theory of nonlinear Volterra integral equa-  
\n*em* 1: Let 
$$
(p, v, x) \in \mathcal{P} \times \mathcal{V} \times \mathcal{X}
$$
 satisfying (1). Then we have  
\n
$$
\Omega_{\min} = \frac{v_{\min}}{v_0} \Omega_0 \le \Omega(t) \le \Omega_{\max} = \frac{v_{\max}}{v_0} \Omega_0 \quad (0 \le t \le 1)
$$
\n
$$
0 < p_{\min} = \frac{v_{\min}}{v_{\max}} p_0 \le p(t) \le p_{\max} = \frac{v_{\max}}{v_{\min}} p_0 \quad (0 \le t \le 1).
$$
\nChoose  $\hat{t} \in [0, 1]^t$  so that  $\Omega(t) \le \Omega(\hat{t}) \quad (0 \le t \le 1)$ . Owing to  $x'(t) \le 0$ 

*and*

$$
2^{2} \min_{v_{0}} \frac{1}{2} \sum_{i=1}^{n} \frac{1}{2} \sum_{i=1}^{n}
$$

Proof: Choose  $\hat{t} \in [0,1]^t$  so that  $\Omega(t) \leq \Omega(\hat{t})$   $(0 \leq t \leq 1)$ . Owing to  $x'(t) \leq 0$ <br> $(0 \leq t \leq 1)$  we have

and

\n
$$
0 < p_{\min} = \frac{v_{\min}}{v_{\max}} \, p_0 \leq p(t) \leq p_{\max} = \frac{v_{\max}}{v_{\min}} \, p_0 \quad (0 \leq t \leq 1).
$$
\nProof: Choose  $t \in [0, 1]'$  so that  $\Omega(t) \leq \Omega(t)$   $(0 \leq t \leq 1)$ . Owing.

\n
$$
(0 \leq t \leq 1) \text{ we have}
$$
\n
$$
p(t) = p_0 + x(0) \, \Omega(t) - x(t) \, \Omega_0 + \int_0^t x'(t - \tau) \, \Omega(\tau) \, d\tau
$$
\n
$$
\geq p_0 + x(0) \, \Omega(t) - x(t) \, \Omega_0 + \Omega(t) \int_0^t x'(t - \tau) \, d\tau
$$
\n
$$
= p_0 + x(0) \, \Omega(t) - x(t) \, \Omega_0 + \Omega(t) \, (x(t) - x(0))
$$
\n
$$
= p_0 + x(t) \, (\Omega(t) - \Omega_0) \geq p_0.
$$
\nConsequently, we get the right-hand side inequality of (11)

\n
$$
\Omega(t) \leq \frac{v(t)}{p(t)} \leq \frac{v_{\max}}{p_0} = \frac{v_{\max}}{v_0} \, \Omega_0 \quad (0 \leq t \leq 1).
$$
\nThis implies

Consequently, we get the right-hand side inequality of (11) Consequence Conseq

$$
-p_0 + x(0) \sin(\theta) - x(0) \sin(\theta) - p_0
$$
\n
$$
= p_0 + x(\hat{t}) \left( \Omega(\hat{t}) - \Omega_0 \right) \geq p_0.
$$
\nently, we get the right-hand side inequality of

\n
$$
\Omega(t) \leq \frac{v(\hat{t})}{p(\hat{t})} \leq \frac{v_{\text{max}}}{p_0} = \frac{v_{\text{max}}}{v_0} \Omega_0 \quad (0 \leq t \leq 1).
$$
\nlines.

This implies,

$$
\Omega(t) \leq \frac{v(t)}{p(t)} \leq \frac{v_{\text{max}}}{p_0} = \frac{v_{\text{max}}}{v_0} \Omega_0 \quad (0 \leq
$$
  
plies,  

$$
p(t) = \frac{v(t)}{\Omega(t)} \geq \frac{v_{\text{min}}}{v_{\text{max}}} \rho_0 \quad (0 \leq t \leq 1)
$$

This implies,<br>  $p(t) = \frac{v(t)}{Q(t)} \ge \frac{v_{\text{min}}}{v_{\text{max}}} p_0 \quad (0 \le t \le 1)$ <br>
and thus the left-hand side inequality of (12). By choosing *i* so that  $Q(t) \le Q(t)$ <br>  $(0 \le t \le 1)$  the other couple of inequalities is derived in an analog  $p(t) = \frac{v(t)}{\Omega(t)} \ge \frac{v_{\text{min}}}{v_{\text{max}}} p_0$   $(0 \le t \le 1)$ <br>and thus the left-hand side inequality of (12). By choosing *i* so that  $\Omega(t) \le$   $(0 \le t \le 1)$  the other couple of inequalities is derived in an analogous way

 $\label{eq:2.1} \sum_{\substack{p\in\mathbb{Z}^d\\ p\in\mathbb{Z}^d}}\frac{1}{p}\sum_{\substack{p\in\mathbb{Z}^d\\ p\neq p}}\frac{1}{p}\sum_{\substack{p\in\mathbb{Z}^d\\ p\neq p}}\frac{1}{p}\sum_{\substack{p\in\mathbb{Z}^d\\ p\neq p}}\frac{1}{p}\sum_{\substack{p\in\mathbb{Z}^d\\ p\neq p}}\frac{1}{p}\sum_{\substack{p\in\mathbb{Z}^d\\ p\neq p}}\frac{1}{p}\sum_{\substack{p\in\mathbb{Z}^d\\$ 

Thus we are able to confine our considerations in the sequel to the subset of (cf. (12)) Example to confine our consideration<br>  ${p \in \mathcal{P}: 0 < p_{\min} \leq p(t) \leq p_{\max}}$ <br> *Property* that for fixed  $x \in \mathcal{F}$ 

$$
\tilde{\mathcal{P}} = \{p \in \mathcal{P} : 0 < p_{\min} \leq p(t) \leq p_{\max} \quad (0 \leq t \leq 1)\}.
$$

We can also summarize that, for fixed  $x \in \mathcal{X}$ , the operators  $A'(\hat{p})$ ,  $(A'(\hat{p}))^{-1}$ ,  $B'$ , (cf. (12))<br>  $\tilde{\mathcal{P}} = \{p \in \mathcal{P} : 0 < p_{\min} \leq p(t) \leq p_{\max} \quad (0 \leq t \leq 1)\}.$ <br>
We can also summarize that, for fixed  $x \in \mathcal{X}$ , the operators  $A'(\hat{p})$ ,  $(A'(\hat{p}))^{-1}$ ,  $B'$ ,  $(B')^{-1}$  and  $C'$  are uniformly bounded with resp Thus we are able to confine our considerations in the sequel to the subset of  $\mathcal{P}$ .<br>
(cf. (12))<br>  $\tilde{\mathcal{P}} = \{p \in \mathcal{P} : 0 < p_{\min} \leq p(t) \leq p_{\max} \quad (0 \leq t \leq 1) \}.$ <br>
We can also summarize that, for fixed  $x \in \mathcal{X}$ , the A Particular Volterra Stieltjes Convolution Integral<br>
for thus we are able to confine our considerations in the sequel<br>
(cf. (12))<br>  $\tilde{\mathcal{P}} = \{p \in \mathcal{P} : 0 < p_{\min} \leq p(t) \leq p_{\max} \quad (0 \leq t \leq 1)\}.$ <br>
We can also summarize th A Particular Volterra-Stieltjes Convolution Integral Equatio<br>
(cf. (12))<br>  $\tilde{\mathcal{P}} = \{p \in \mathcal{P} : 0 < p_{\min} \leq p(t) \leq p_{\max} \quad (0 \leq t \leq 1) \}$ <br>
We can also summarize that, for fixed  $x \in \mathcal{X}$ , the operators  $A'(\hat{p})$ ,  $\{A'\}$  $\tilde{\mathcal{P}} = \{p \in \mathcal{P} : 0 < p_{\min} \leq p(t) \leq p_{\max} \quad (0 \leq t \leq 1)\}.$ <br>
We can also summarize that, for fixed  $x \in \mathcal{X}$ , the operators  $A'(p)$ ,  $(A'(p))^{-1}$ ,  $B'$ ,  $(B')^{-1}$  and  $C'$  are uniformly bounded with respect to all  $p \in \tilde$ Moreove<br>  $x(0) \leq x$ <br>
formly b<br> **4.** On the First we<br>
Theo<br>
solvable<br>
Proot<br>
where the form *Let*  $\pi$ , if we consider a neighbourhood  $\mathcal{F} \subset \mathcal{X}$  of an element  $\mathcal{E} \in \mathcal{X}$  such that  $\max$  and  $x(1) \ge x_{\min} > 0$  for all  $x \in \mathcal{S}$ , then these operators are also uni-<br>ounded with respect to all  $x \in \mathcal{S}$ 

First we are going to prove an existence and uniqueness theorem.

Theorem 2: For any given  $p_0 > 0$ ,  $v \in \mathcal{V}$  and  $x \in \mathcal{X}$ , the problem (P1) is uniquely solvable with respect to  $p \in \mathcal{P}$ .

**Proof:** For given  $x \in \mathcal{X}$  and  $v \in \mathcal{V}$  we consider the operator equation

$$
L(p,\vartheta)=0,\qquad p\in C[0,1],\quad \vartheta\in[0,1],\tag{13}
$$

where the continuous nonlinear operator *L*:  $C[0, 1] \times [0, 1] \rightarrow C[0, 1]$  is defined by the formulae

$$
I(p, ∅) = \int_{0}^{t} x(t - π) d\theta \, y e^{(x - π)} \, dx
$$
\nwith respect to  $p ∈ P$ .  
\n
$$
I(p, ∅) = 0, \qquad p ∈ C[0, 1], \quad ∂ ∈ [0, 1],
$$
\nthe continuous nonlinear operator  $L : C[0, 1] \times [0, 1]$   
\nthe continuous nonlinear operator  $L : C[0, 1] \times [0, 1]$   
\n
$$
L(p, ∅) = \int_{0}^{t} x(t - π) d\frac{v_{\theta}(π)}{p_{\epsilon}(τ)} - p_{\epsilon}(t) + p_{0}
$$
\n
$$
(0 ≤ t ≤ 1, 0 ≤ ∂ ≤ 1, 0 < ε < p_{min}),
$$
\n
$$
v_{\theta}(t) = v_{0} + ∂(v(t) - v_{0}), \qquad p_{\epsilon}(t) = max (ε, p(t))
$$
\nand 
$$
v_{\theta}(t) = v_{0} + ∂(v(t) - v_{0}), \qquad p_{\epsilon}(t) = max (ε, p(t))
$$

As we have learned from Theorem 1, for any solution  $(p, \vartheta) \in C[0, 1] \times [0, 1]$  of (13) we have  $p \in \overline{\mathcal{P}}$ . If we introduce a set

$$
\tilde{\mathcal{P}}_{\delta} = \{p \in \mathcal{P} : 0 < \varepsilon < p_{\min} - \delta \leqq p(t) \leqq p_{\max} + \delta \ (0 \leqq t \leqq 1) \}
$$

according to a small positive number  $\delta$ , then in view of Section 2 we can state that *L* and its partial Fréchet derivative  $\partial_p L$  are uniformly continuous with respect to **(p, i)**  $\in \mathcal{F}_\delta \times [0, 1]$ . Moreover,  $(\partial_p L(p, \vartheta))^{-1}$  exists and is uniformly bounded with respect to all  $(p, \vartheta) \in \mathcal{F}_\delta \times [0, 1]$ . Namely, we have  $\|(\partial_p L(p, \vartheta))^{-1}\| \leq x(0)/x(1)$  (cf. (4)). Finally, we a priori know t respect to all  $(p, \vartheta) \in \tilde{\mathcal{P}}_{\delta} \times [0, 1]$ . Namely, we have  $\|(\partial_p L(p, \vartheta))^{-1}\| \leq x(0)/x(1)$  (cf. (4)). Finally, we a priori know that (13) has a unique solution with respect to *p* if  $\vartheta = 0$ . If we set  $v_{\min} = v_0 = v_{\max}$ , then it follows from Theorem 1 that  $p_{\min} = p_0$  $p_{\text{max}}$ . Thus, the constant function  $p(t) = p_0$  ( $0 \le t \le 1$ ) is a solution of (1) for  $v(t) = v_0$  ( $0 \le t \le 1$ ). Then under the conditions derived above a well-known corollary of the implicit-function theorem (see e.g. [8, p. 63]) yields that  $L(p, \vartheta) = 0$ is also uniquely solvable with respect to  $p \in C[0, 1]$  if  $\vartheta = 1$ . Therefore, (P1) is uniquely solvable with respect to  $p \in \mathcal{P}$ . In view of Theorem 1 this uniquely deter-<br>mined solution belongs to the subset  $\tilde{\mathcal{P}}$  of  $\mathcal{P}$ we have  $p \in \mathcal{P}$ . It we incroduce a set<br>  $\tilde{\mathcal{P}}_{\delta} = \{p \in \mathcal{P} : 0 < \varepsilon < p_{\min} - \delta \leq p(t) \leq p_{\max} + \delta \ (0 \leq t \leq$ <br>
according to a small positive number  $\delta$ , then in view of Section 2 we  $L$  and its partial Fréchet

In order to complete the proof of well-posedness for problem (Pt), we still **have**  to show the stability of solutions with respect to small changes in  $v$  and  $x$ . For given  $x \in \mathcal{X}$ , let  $p = Pv$ ,  $P: \mathcal{V} \to \tilde{\mathcal{P}}$ , represent the dependence of solutions p to equation (1) upon the element *v*. On the other hand, let  $p = Qx, Q: \mathcal{X} \to \tilde{\mathcal{F}}$ , denote

## **254 В. Ногманн**

the dependence between p and x in (1) for given  $v \in \mathcal{V}$ . As Theorem 2 indicates, both operators P and Q are uniquely determined.

B. HOFMANN 254<br>
the dependence between p and x in (1) for given  $v \in \mathcal{V}$ . As Theorem 2 indicates<br>
both operators P and Q are uniquely determined.<br>
Under the assumptions stated above the implicit-function theorem (cf. e Under the assumptions stated above the implicit-function theorem (cf. e.g. [8<br> *p*. 51]) applies. Thus, the operators *P* and *Q* are continuous and even Fréchet<br>
differentiable. We obtain<br>  $P'(\hat{v}) = (A''(P\hat{v}))^{-1}B'$  ( $\hat{$ 254 B. HOFMANN<br>the dependence between p and x in (1) for given  $v \in \mathcal{V}$ . As Theorer<br>both operators P and Q are uniquely determined.<br>Under the assumptions stated above the implicit-function theorer<br>p. 51]) applies. Thus 254 B. HOFMANN<br>the dependence between p and x in (1) for given  $v \in \mathcal{V}$ . As Theorem 2 in<br>both operators P and Q are uniquely determined.<br>Under the assumptions stated above the implicit-function theorem (cf.<br>p. 51)) app 254 B. Hormany<br>
the dependence between p and x in (1) for given  $v \in \mathcal{V}$ . As<br>
both operators P and Q are uniquely determined.<br>
Under the assumptions stated above the implicit-function<br>
p. 51]) applies. Thus, the operat Figure 1.1 and *Q* are<br> *accuse P* and *Q* are<br> *accuse assumptions*<br> *i* lies. Thus, the<br> *i* lies. Thus, the<br> *i* lies. Thus, the<br> *i*  $(\hat{v}) = (A'(P\hat{v}))^{-1}$ <br> *mean-value the*<br> *i* in 3: Let, for give<br> *y elements of V*  $Q$  are uniquely<br>
ons stated ab<br>
the operators<br>
in<br>  $(2\theta)^{2}$ <br>  $(2\theta)^{2}$ <br> (1) for given  $v \in \mathcal{V}$ . As Theorem 2 indicates,<br>determined.<br>
Ove the implicit-function theorem (cf. e.g. [8,<br>
P and Q are continuous and even Fréchet-<br>  $\vdots$   $\mathcal{V}$ ) and  $Q'(\hat{x}) = (A'(Q\hat{x}))^{-1} C'$  ( $\hat{x} \in \mathcal{X}$ ).<br>
e.g. Under the assumptions stated above the imp<br> *p.* 51]) applies. Thus, the operators *P* and *Q a*<br>
differentiable. We obtain<br>  $P'(\hat{v}) = (A'(P\hat{v}))^{-1} B'$  ( $\hat{v} \in \hat{v}$ ) and *Q*<br>
Due to the mean-value theorem (cf. e.g. [5,

$$
P'(\hat{v}) = (A'(P\hat{v}))^{-1} B' \quad (\hat{v} \in \mathcal{V}) \quad \text{and} \quad Q'(\hat{x}) = (A'(Q\hat{x}))^{-1} C' \quad (\hat{x} \in \mathcal{X}).
$$

Due to the mean-value theorem (cf. e.g.  $[5, p. 535]$ ) these two formulae imply Lip-

Theorem 3: Let, for given  $p_0 > 0$  and  $x \in X$ ,  $\hat{p} = P\hat{v}$  and  $\hat{p} = P\hat{v}$ , where  $\hat{v}$  and  $\hat{v}$  are arbitrary elements of  $V$ . Then,  $\alpha_i = \frac{1}{2}$ <br>  $\alpha_i$  to the mean-value<br>
itz conditions stated<br>  $\alpha_i$  Theorem 3: Let, for<br>  $\alpha_i$  arbitrary elements of<br>  $\|\hat{p} - \hat{p}\| \leq \frac{2}{\pi}$ <br>
the other hand, let  $\hat{p}$ 

$$
\|\hat{p} - \hat{p}\| \leq \frac{2(x(0))^2}{x(1)} p_{\min}^{-1} \|\hat{v} - \hat{v}\|.
$$
 (14)

*On the other hand, let*  $\tilde{p} = Q\tilde{x}$  *and*  $\tilde{p} = Q\tilde{x}$  *for given*  $p_0 > 0$  *and*  $v \in V$ *, where*  $\tilde{x}$  *and*  $\tilde{x}$ 

*2Vmax* max ((0), (0)) - - **li <sup>p</sup>** -I! - -- (15) Pill  *P0*  ruin (2(1), x(l)) **/ -** 

The results of this section show that  $(P1)$ -is well-posed in the sense of Haclamard. In addition, the formulae (14) and (15) yield measures for the sensitivity of solutions Senitz conditions stated as follows.<br>
Theorem 3: Let, for given  $p_0 > 0$  and  $x \in X$ ,  $\hat{p} = P$ <br>
are arbitrary elements of  $V$ . Then,<br>  $\|\hat{p} - \hat{p}\| \leq \frac{2(x(0))^2}{x(1)} p_{\min}^{-1} \|\hat{v} - \hat{v}\|$ .<br>
On the other hand, let  $\tilde{p} =$ 

## 5. A uniqueness and stability theorem-regarding problem (P2)

-As we know, problem (P2) is of inverse nature. Thus, we suspect that it is ill-posed. Indeed, if  $p \in \mathcal{P}$  but  $p \notin \bar{\mathcal{P}}$ , then as a consequence of Theorem 1 there is, independently of the choice of  $x \in \mathcal{X}$ , no element  $v \in \mathcal{V}$  such that equation (1) may be satisfied. Thus, the existence requirement of Hadamard's well-posedness definition is injured. However, it is known that we have at least one element of  $\mathcal{P}$ , namely the constant function  $p(t) = p_0$  ( $0 \le t \le 1$ ), that possesses a solution  $v \in \mathcal{V}$  to problem (P2) (see **5.** A uniqueness and stability theorem regarding problem (P2)<br>
As we know, problem (P2) is of inverse nature. Thus, we suspect that it is ill-posed.<br>
Indeed, if  $p \in \mathcal{P}$  but  $p \notin \tilde{\mathcal{P}}$ , then as a consequence of the proof of Theorem 2). From the example formulated below we will learn that the obviously closed union set of solutions  $v \in \mathcal{V}$  to (P2) over all elements  $p \in \tilde{\mathcal{P}}$ is not necessarily convex. 5. A uniqueness and stability theorem regarding problem (P2)<br>
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Indeed, if  $p \in \mathcal{P}$  but  $p \notin \tilde{\mathcal{P}}$ , then as a consequence of Theorem 1 there is, As we know, problem (P2) is of inverse nature. Thus, we suspect that it is ill-pos<br>
Indeed, if  $p \in \mathcal{P}$  but  $p \notin \tilde{\mathcal{P}}$ , then as a consequence of Theorem 1 there is, independe<br>
ly of the choice of  $x \in \mathcal{X}$ , no el x, no element  $v \in v$  such that<br>uirement of Hadamard's well-<br>hat we have at least one elem<br>i.  $t \leq 1$ ), that possesses a solut<br>2). From the example formul<br>nion set of solutions  $v \in V$  to<br>x.<br>now consider the extremal<br>ins t *uch that equation (1) may be satisfied.*<br> *d's well-posedness definition is injured.*<br> *one element of*  $\mathcal{P}$ *, namely the constant*<br> *s* a solution  $v \in \mathcal{V}$  to problem (P2) (see<br> *v*  $\in \mathcal{V}$  to (P2) over all elem the proof of Theorem 2). From the example formulated below we will learn that<br>the obviously closed union set of solutions  $v \in \mathcal{V}$  to (P2) over all elements  $p \in \tilde{\mathcal{P}}$ <br>is not necessarily convex.<br>Example: Let us now

Example: Let us now consider the extremal case  $x(t) = c > 0$  ( $0 \le t \le 1$ ). t necessarily convex.<br>
cample: Let us now consider the extrem<br>
equation (1) attains the form<br>  $c(\Omega(t) - \Omega_0) = p(t) - p_0$  ,  $(0 \le t \le$ 

$$
c(Q(t) - \Omega_0) = p(t) - p_0 \quad (0 \le t \le 1).
$$
 (16)

$$
p(t) = \frac{p_0}{2} - \frac{c\Omega_0}{2} + \left\{ \left[ \frac{p_0}{2} - \frac{c\Omega_0}{2} \right]^2 + cv(t) \right\}^{1/2} \quad (0 \le t \le 1)
$$
 (17)

verified from (16) form a continuous function  $p \in \tilde{\mathcal{P}}$ . As formula (17) shows,<sup>1</sup> $p(t)$ verified from (16) form a continuous function  $p \in \mathcal{P}$ . As formula (17) shows,  $p(t)$  does not depend on  $v(\tau)$  ( $0 < \tau < t$ ). On the other hand, for given  $p(t) > 0$ , we derive the uniquely determined value  $v(t) = \frac{p^2(t) + (c$ F(c)  $\frac{16}{2}$   $\frac{2}{2}$   $\frac{2}{2}$   $\frac{12}{2}$ <br>
from (16) form a continuous fur<br>
depend on  $v(t)$  ( $0 < \tau < t$ ). On<br>
e uniquely determined value<br>  $v(t) = \frac{p^2(t) + (c\Omega_0 - p_0) p(t)}{c} =$ 

$$
v(t) = \frac{p^{2}(t) + (c\Omega_{0} - p_{0}) p(t)}{c} = p(t) \left( \Omega_{0} + \frac{p(t) - p_{0}}{c} \right) \quad (0 \leq t \leq 1). \quad (18)
$$

Thus, the function *v* may take on even negative values whenever  $0 < p(t) < p_0$ and  $c > 0$  gets sufficiently small. Now let for fixed  $t \in [0, 1]$  the pairs  $(p_1(t), v_1(t))$ and  $(p_2(t), \nu_2(t))$  both belong to the rectangle  $[p_{\min}, p_{\max}] \times [v_{\min}, v_{\max}]$  and together with a pair  $(p_3(t), v_3(t)) (p_3(t)) = (p_1(t) + p_2(t))/2$  satisfy a condition (18) for the dependence between  $v_i(t)$  and  $p_i(t)$ ,  $i = 1, 2, 3$ . Then, it may occur that  $v_3(t) < v_{\min}$ A Particular Volterra-Stieltjes Convolution Integral Equation 255<br>
Thus, the function *v* may take on even negative values whenever  $0 < p(t) < p_0$ <br>
and  $c > 0$  gets sufficiently small. Now let for fixed  $t \in [0, 1]$  the pairs since  $v(t)$  is a quadratic function of  $p(t)$ . A Particular Volterra-Stieltjes Convolution Integral Equation<br>
Thus, the function v may take on even negative values whenever  $0 <$ <br>
and  $c > 0$  gets sufficiently small. Now let for fixed  $t \in [0, 1]$  the pairs (<br>
and  $(p_2(t),$ 

Now we are going to formulate a uniqueness and stability theorem for problem (P2).

Theorem 4: For any given  $v_0 > 0$ ,  $p \in \mathcal{P}$  and  $x \in \mathcal{X}$ , there is a uniquely determined *function*  $v \in C[0, 1]$ ,  $v(0) = v_0$ , such that the equation (1) holds. Consequently, if the problem  $(P2)$  *is solvable with respect to*  $v \in V$ , then this solution is uniquely determined. *Moreover, for given*  $v_0 > 0$  *and*  $x \in \mathcal{X}$ , we have air  $(p_3(t), v_3(t))$   $(p_3$ <br>
(ce between  $v_i(t)$  and is a quadratic function<br>
is  $v \in C[0, 1], v(0) =$ <br>  $P$ eness and state<br>  $\alpha$  and  $x \in \mathcal{X}$ , the equation (1)<br>
then this solution<br>  $\beta$  -  $\hat{p}$ ||<br>
(P2) corresponent because that  $v_3(t) < v_{\min}$ .<br>
(b) theorem for problem<br>
is a uniquely determined<br>
is a uniquely determined.<br>
is uniquely determined.<br>
(19)<br>
ig to  $p \in \tilde{\mathcal{P}}$  and  $\tilde{p} \in \tilde{\mathcal{P}}$ ,  $\mathcal{P}$  and  $x \in \mathcal{X}$ , there is a uniquely de<br>
he equation (1) holds. Consequenti<br>  $\mathcal{P}$ , then this solution is uniquely det<br>
ave<br>  $\|\hat{p} - \hat{p}\|$ <br>  $\|\hat{p} - \hat{p}\|$ <br>  $\|\hat{p} - \hat{p}\|$ <br>  $\|\hat{p}\| \|\tilde{x} - \tilde{x}\|_1$ <br>  $\|\hat{p}\$ Theorem 4: For any given  $v_0 > 0$ ,  $p \in \mathcal{P}$  and  $x \in \mathcal{X}$ , there is a uniquely determinal function  $v \in C[0, 1]$ ,  $v(0) = v_0$ , such that the equation (1) holds. Consequently, if if problem (P2) is solvable with respect

$$
\|\hat{v} - \hat{v}\| \le \frac{x(0) \Omega_0 + 2p_{\max} - p_0}{x(1)} \|\hat{p} - \hat{p}\|
$$
\n(19)

*whenever*  $\hat{v} \in \mathcal{V}$  and  $\hat{v} \in \mathcal{V}$  are solutions of (P2) corresponding to  $\hat{p} \in \tilde{\mathcal{P}}$  and  $\hat{p} \in \tilde{\mathcal{P}}$ , *respectively. On the other hand, for given*  $v_0 > 0$  *and*  $p \in \tilde{\mathcal{P}}$ *, we obtain-* $\begin{array}{ccc} &\|\hat{\boldsymbol{v}}\|-\tilde{\boldsymbol{v}}\| \ \end{array}$ <br> *iever*  $\hat{\boldsymbol{v}}\in\mathcal{V}$  and  $\begin{array}{ccc} &\|\tilde{\boldsymbol{v}}\|=\tilde{\tilde{\boldsymbol{v}}}\|\ \end{array}$  $x(1)$ <br>  $\tilde{v} \in \mathcal{V}$  are solution.<br>
her hand, for given<br>  $\tilde{x}(0) \Omega_0 + p_{\text{max}} - \frac{\tilde{x}(1) \tilde{x}(1)}{\tilde{x}(1)}$ problem (P2) is solve<br>
Moreover, for given  $v$ <br>  $Moreover, for given v$ <br>  $\|\hat{v} - \hat{v}\| \leq$ <br>  $v$ <br>  $v$  whenever  $\hat{v} \in \mathcal{V}$  and<br>  $\text{respectively.}$  On the c<br>  $\|\tilde{v} - \tilde{\tilde{v}}\| \leq$ <br>  $v$  whenever  $\tilde{v} \in \mathcal{V}$ ,  $\tilde{v} \in \text{tively}$ <br>  $\text{Proof:}$ 

$$
\|\tilde{v} - \tilde{\tilde{v}}\| \le \frac{\tilde{x}(0) \; \Omega_0 + p_{\max} - p_0}{\tilde{x}(1) \; \tilde{\tilde{x}}(1)} \; \|p\| \, \|\tilde{x} - \tilde{x}\|_1 \qquad (20)
$$

*whenever*  $\tilde{v} \in \mathcal{V}$ ,  $\tilde{v} \in \mathcal{V}$  are solutions of (P2) corresponding to  $\tilde{x} \in \mathcal{X}$  and  $\tilde{x} \in \mathcal{X}$ , *respectively.* 

Proof: For given  $v_0 > 0$ ,  $x \in \mathcal{X}$  and  $p \in \mathcal{P}$ , the problem (P2) corresponds to the -  $\tilde{v} \in \mathcal{V}$ ,  $\tilde{v} \in \mathcal{V}$  are solutions of (P2) corresponding to  $\tilde{x} \in \mathcal{V}$ . For given  $v_0 > 0$ ,  $x \in \mathcal{X}$  and  $p \in \mathcal{P}$ , the problem (P2).<br>  $(B'v)(t) = x(t)Q_0 + p(t) - p_0$  ( $0 \le t \le 1$ ).<br>
Lemma 2 the linear oper

$$
(B'v) (t) = x(t) \Omega_0 + p(t) - p_0 \qquad (0 \le t \le 1).
$$

Due to Lemma 2 the linear operator  $B'$  is injective. Therefore,  $(P2)$  is uniquely solvable if the requirement  $v \in V$  is weakened to  $v \in C[0, 1]$ . Now consider formula  $\cdot$ (2) as a linear equation with respect to  $\Omega \in C[0, 1]$ : *f* the requirement  $v \in \mathcal{V}$  is weakened to  $v \in C[0, 1]$ . Now consider formula near equation with respect to  $\Omega \in C[0, 1]$ :<br>  $\int x'(t - \tau) \Omega(\tau) d\tau + x(0) \Omega(t) = x(t) \Omega_0 + p(t) - p_0 \qquad (0 \le t \le 1)$ . (21) First the  $\mathcal{U}_s$ . Then we have  $\mathcal{U}_s = \mathcal{U}_s$  and  $p \in \mathcal{P}$ , the problem (P2) corresponds to the<br>equation<br>(B'v) ( $t$ ) =  $x(t) \Omega_0 + p(t) - p_0$  ( $0 \le t \le 1$ ).<br>Due to Lemma 2 the linear operator B' is injective. Therefore

$$
\int_{0}^{t} x'(t-\tau) \Omega(\tau) d\tau + x(0) \Omega(t) = x(t) \Omega_{0} + p(t) - p_{0} \qquad (0 \leq t \leq 1). \quad (21)
$$

For given  $x \in \mathcal{X}$ , let  $\hat{\Omega}$ ,  $\hat{\Omega}$  denote the solutions of (21) according to  $p = \hat{p}$  and  $p = \hat{p}$ ,

on  
\n
$$
(B'v) (t) = x(t) Q_0 + p(t) - p_0 \qquad (0 \le t \le 1).
$$
\n
$$
D \text{ Lemma 2 the linear operator } B' \text{ is injective. Therefore, (P2) is unique}
$$
\n
$$
P = \text{linear operator } B' \text{ is injective. Therefore, (P2) is unique}
$$
\n
$$
P = \text{linear equation with respect to } Q \in C[0, 1].
$$
\n
$$
\int_{0}^{t} x'(t - \tau) Q(\tau) d\tau + x(0) Q(t) = x(t) Q_0 + p(t) - p_0 \qquad (0 \le t \le 1). \qquad (2 \text{ and } p = \text{interval of } \tau).
$$
\n
$$
\int_{0}^{t} x'(t - \tau) Q(\tau) d\tau + x(0) Q(t) = x(t) Q_0 + p(t) - p_0 \qquad (0 \le t \le 1). \qquad (2 \text{ and } p = \text{interval of } \tau).
$$
\n
$$
||\hat{Q} - \hat{Q}|| \le \frac{||\hat{p} - \hat{p}||}{x(1)} \qquad \text{and} \qquad ||\hat{Q}|| \le \frac{x(0) Q_0 + p_{\text{max}} - p_0}{x(1)},
$$
\n
$$
||\hat{p} - \hat{p}|| \le ||\hat{Q}|| ||\hat{p} - \hat{p}|| + ||\hat{p}|| ||2 - \hat{Q}|| \le \frac{x(0) Q_0 + 2p_{\text{max}} - p_0}{x(1)} ||\hat{p} - \hat{p}||.
$$
\n
$$
\text{or given } p \in \tilde{B}, \text{ let } \tilde{Q}, \tilde{Q} \text{ denote the solutions to (21) according to } \tilde{x} \text{ and}
$$
\n
$$
\text{dively. Then}
$$
\n
$$
||\tilde{Q} - \tilde{Q}|| \le \frac{Q_0 ||\tilde{x} - \tilde{x}||}{\tilde{x}(1)} + \frac{\tilde{x}(0) Q_0 + (p_{\text{max}} - p_0) ||\tilde{x}' - \tilde{x}||}{\tilde{x}(1) \tilde{x}(1)}.
$$

Now, for given  $p \in \tilde{\mathcal{P}}$ , let  $\tilde{\Omega}$ ,  $\tilde{\Omega}$  denote the solutions to (21) according to  $\tilde{x}$  and  $\tilde{x}$ , respectively. Then

$$
\|\tilde{\Omega}-\tilde{\Omega}\|\leq \frac{\Omega_0 \|\tilde{x}-\tilde{x}\|}{\tilde{x}(1)}+\frac{\tilde{x}(0)\,\Omega_0+(p_{\max}-p_0)\,\|\tilde{x}'-\tilde{x}'\|}{\tilde{x}(1)\,\tilde{x}(1)}
$$

and finally<br>  $\|\tilde{v}\|$ 

 

$$
\begin{aligned}\n\|\tilde{\Omega}-\tilde{\Omega}\| &\leq \frac{\Omega_0 \left\|\tilde{x}-\tilde{x}\right\|}{\tilde{x}(1)} + \frac{\tilde{x}(0) \Omega_0 + (p_{\max}-p_0) \left\|\tilde{x}'-\tilde{x}'\right\|}{\tilde{x}(1) \left\|\tilde{x}(1)\right} \\
\text{by} \\
\|\tilde{v}-\tilde{v}\| &\leq \|\tilde{\Omega}-\tilde{\Omega}\| \left\|p\right\| \leq \frac{\tilde{x}(0) \Omega_0 + p_{\max}-p_0}{\tilde{x}(1) \left\|\tilde{x}(1)\right\|} \left\|p\right\| \left\|\tilde{x}-\tilde{x}\right\|_1\n\end{aligned}
$$

## 256 В. Но**гма**нк

If a solution  $v \in V$  of problem (P2) according to  $p \in \mathcal{P}$  and  $x \in \mathcal{X}$  satisfies the inequalities B. HOFMANN<br>
vlution  $v \in \mathcal{V}$  of problem (P2) accorders<br>
ies<br>  $v_{\min} < \min_{0 \le t \le 1} v(t), \qquad v_{\max} > \max_{0 \le t \le 1} v(t),$ <br>
estimations (19) and (20) show that

*/.* 

**'** (22)

$$
v_{\min} < \min_{0 \leq t \leq 1} v(t), \qquad v_{\max} > \max_{0 \leq t \leq 1} v(t),
$$

then the estimations (19) and (20) show that **(P2)** is also solvable with respect to  $v \in \mathcal{V}$  when the elements p and x change by a sufficiently small amount. Thus, at least the uniqueness and stability requirements of Hadamard's well-posedness definition are satisfied for problem (P2): 16. If a solution  $v \in \mathcal{V}$  of problem (P2) according to  $p \in \mathcal{P}$  an inequalities<br>  $v_{\min} < \min_{0 \le i \le 1} v(t),$ <br>  $v_{\max} > \max_{0 \le i \le 1} v(t),$ <br>  $v_{\min} < w(t)$ ,  $v_{\max} > \max_{0 \le i \le 1} v(t),$ <br>
then the estimations (19) and (20) show that *v*<sub>min</sub> < min v(t),  $v_{\text{max}} > \frac{v_{\text{max}}}{0 \le t \le 1} v(t)$ ,<br>
oscient the estimations (19) and (20) show that (P2) is also solvable with respect to  $v \in \mathcal{V}$  when the elements  $p$  and  $x$  change by a sufficiently small amoun  $v_{\text{min}} < \min v(t)$ ,  $v_{\text{max}} > \max_{0 \le t \le 1} v(t)$ ,<br>
estimations (19) and (20) show that (P2) is als<br>
en the elements p and x change by a sufficient<br>
uniqueness and stability requirements of Hadar<br>
e satisfied for problem (P2).<br>

the linear equation

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$$
(C'x)(t) = p(t) - p_0 \qquad (0 \le t \le 1).
$$

Therefoie, the properties of solutions **to** problem, (P3) may be derived from the assertions of Lemma 3. It is evident that the existence requirement as well as the stability requirement of Hadamard's well-posedness definition get injured for probleru (P3) since C' is a compact linear operator of  $C^1[0, 1]$  into  $C[0, 1]$ , but  $\mathcal X$  fails to be a compact subset-of C'[0, **1.** As a consequence of the non-closed range of *C'*  the equation (22) can be inconsistent. Moreover, small perturbations of  $p$  and  $v$ may lead to significant changes in the solution x, because the inverse  $(C')^{-1}$  is unbounded whenever it exists. Thus, problem **(P3)** becomes an ill-posed one., However, we can formulate a necessary and sufficient condition for the unique solvability of  $(P3)$  in the consistent case.  $(C'x)(t) = p(t) - p_0$   $(0 \le$ <br>Therefore, the properties of solutions<br>assertions of Lemma 3. It is evident that<br>stability requirement of Hadamard's vertical theory of Hadamard's<br>lem (P3) since C' is a compact linear<br>to be a comp

*Theorem 5: For given*  $p \in \tilde{\mathcal{P}}$  *and*  $v \in \mathcal{V}$ *, let exist a solution*  $x \in \mathcal{X}$  *of problem (P3). - Then, this solution is uniquely determined if and only if there is no real number*  $\epsilon > 0$ of (P3) in the consistent case.<br>
Theorem 5: For given  $p \in G$ .<br> *Then, this solution is uniquely*<br>
such that  $v(t) = v_0$  ( $0 \le t \le \varepsilon$ ). of (P3) in the consistent case.<br>
Theorem 5: For given  $p \in \tilde{\mathcal{P}}$  and  $v \in V$ , let exist a solution  $x \in \mathcal{X}$  of problem (P3).<br>
Then, this solution is uniquely determined if and only if there is no real number  $\varepsilon > 0$ 

lished in Theorem 1. Thus, Theorem 5 immediately follows from Lemma 3  $\blacksquare$ 

In order to identify the function  $x \in \mathcal{X}$  in a unique manner, it suffices to require a non-steady state of *v* for an arbitrarily small initial interval  $t \in [0, \varepsilon]$ . However, in the computational identification of *x* based on observation data of *p* and *v* sub-**Proof:** If  $(p, v, x) \in \tilde{\mathcal{P}} \times \mathcal{V} \times \mathcal{X}$  satisfy the equation (1), then  $v(t) = v_0$  ( $0 \le t \le \varepsilon$ ) is equivalent to  $\Omega(t) = \Omega_0$  ( $0 \le t \le \varepsilon$ ). This is due to the maximum principle established in Theorem 1. Thus, The stantial difficulties arise from the instability of **(P3)** outlined above. ed in Theorem 1. Thus, Theorem 5 immediate<br>
n order to identify the function  $x \in \mathcal{X}$  in a ur<br>
on-steady state of v for an arbitrarily small<br>
he computational identification of x based or<br>
trial difficulties arise from

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