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On the Reduction Procedure for a Nonlinear Integro-Differential Equation

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Es wird eine vollständige Charakterisierung des globalen Lösungszusammenhanges einer speziellen nichtlinearen Integro-Differentialgleichung angegeben. Dazu wird ein globales Reduktionsverfahren angewendet, welches auf die Lösung eines eindimensionalen Problems führt.

Даётся польное описание глобальной связи решений одного специального интегро-дифференциального уравнения. Для этого применяется глобальный метод приведения приводящий к решению некоторой одномерной задачи.

A complete characterization of the global connexion of the solutions of a special nonlinear integro-differential equation is given. To this end, a global reduction method is applied, which leads to the solution of a one-dimensional problem.

1. Introduction. Let G be a domain in $\mathbf{R}^n$ with boundary ∂G. We consider the following boundary value problem for a nonlinear integro-differential operator:

$$Lu + f(\bar{u}) = g \quad \text{in } G, \qquad u = h \quad \text{on } \partial G, \tag{1}$$

where L is a strongly elliptic differential operator of second order,

$$Lu = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j},$$

f is a real-valued function on $\mathbf{R}$, and $\bar{u} = \int_G u \, dx/|G|$, $|G| = \int_G dx$. Detailed assumptions on the domain G, on the coefficients a_{ij}, and on the right-hand sides g, h are formulated in Section 3.

It is our aim to describe the set Σ of all triples (u, g, h) such that u is a solution of (1) for the right-hand sides g, h. We apply the method of global reduction to a one-dimensional problem. Our result will be that Σ has the structure $\Sigma = \sigma \times S$, where $\sigma = \{(s, t) \in \mathbf{R}^2 \mid \varphi(s) = t\}$ with a suitable chosen function φ, and S is a linear manifold.

The method of one-dimensional reduction was used by Berger and Podolak [2] and by Cafagna and Donati [3] for the solution of boundary value problems for semilinear elliptic differential operators. The differentiability of the considered operators and the kind of interaction of the nonlinearity with the spectrum of the linear part of the operator play an important role for these considerations. We can look at the problem (1) as a simplified semilinear elliptic equation. The effect of this simplification is that

(i) we need not to make any assumptions on the nonlinearity f,
(ii) the behavior of the nonlinearity f with respect to the spectrum of the linear part does not play any role.

In Section 2 we present abstract results, which we will apply to the solution of the problem stated above. The solution of the abstract problem is a special case of the well-known Reduction Lemma (cf. BERGER [1; p. 226]). But the situation is so simple in our case that we will describe the elementar reduction procedure for our problem.

Section 3 contains the application of these results to problem (1). The key assumptions are the followings:

(i) the operator defined by the linear problem

$$Lu = g \quad \text{in } G, \qquad u = h \quad \text{on } \partial G$$

is a linear homeomorphism between two function spaces,

(ii) a maximum principle for the operator L holds.

2. The abstract problem. Let X, Y be real Banach spaces and let A be a mapping from X to Y. We make the following **hypotheses:**

(**H1**) A is a linear homeomorphism of X onto Y.

(**H2**) X is the direct sum $X = X_0 \oplus X_1$ of two subspaces X_0 and X_1, where X_0 is of dimension 1.

We denote by P_0 the projector from X onto X_0. Choosing $x_0 \in X_0$, $x_0 \neq 0$, we can look on P_0 as a mapping of X onto $\mathbf{R}$ with $P_0 x_0 = 1$. If we set $Y_0 = A(X_0)$ and $Y_1 = A(X_1)$, then, by (H1) and (H2), we have $Y = Y_0 \oplus Y_1$. Furthermore, Y_0 is a one-dimensional subspace of Y, the base of which we can choose by $y_0 = Ax_0$. We denote by π_0, π_1 the projectors of Y onto Y_0, Y_1, respectively.

Now, let f be a real-valued function on $\mathbf{R}$. We consider the following operator N on X:

$$N = A + B, \quad \text{where} \quad B(x) = f(qP_0x)\, y_0, \qquad x \in X,$$

with a fixed real number q. We want to describe the set

$$\Sigma = \{(x, y) \in X \times Y \mid N(x) = y\}.$$

To this end, we set

$$\sigma = \{(s, t) \in \mathbf{R}^2 \mid s + f(qs) = t\}.$$

Then we have the following

Proposition: *Let $x \in X$, $x = sx_0 + x_1$ with $s \in \mathbf{R}$ and $x_1 \in X_1$, and let $y \in Y$, $y = ty_0 + y_1$ with $t \in \mathbf{R}$ and $y_1 \in Y_1$. Then, $(x, y) \in \Sigma$ if and only if $(s, t) \in \sigma$ and $y_1 = Ax_1$.*

Proof: The inclusion $(x, y) \in \Sigma$ is equivalent to the equation $A(sx_0 + x_1) + f(qs)\, y_0 = y$, by definition of the operator N. Application of the projectors π_0, π_1 to this equation yields the equivalent system $sy_0 + f(qs)\, y_0 = \pi_0 y$, $Ax_1 = \pi_1 y$. Now, one easily sees that this system is equivalent to $(s, t) \in \sigma$ and $y_1 = Ax_1$ ∎

Remark: The preceding proposition can be interpreted as Σ having the structure $\Sigma = \sigma \times S$, where $S = \{(x_1, y_1) \in X_1 \times Y_1 \mid y_1 = Ax_1\}$ is a linear manifold of codimension two in $X \times Y$.

3. Solution of the integro-differential equation. Let G be a bounded region in $\mathbf{R}^n$ with boundary ∂G. $C^k(\overline{G})$ will denote the space of the functions which are k-times continuously differentiable on G and such that the derivatives can be extended continuously on ∂G. With the usual norm $\|u\|_k = \sup\{|D^r u(x)| : x \in G, 0 \leq r \leq k\}$,

$C^k(\bar{G})$ is a Banach space. $C^{k,\alpha}(G)$ $(0 < \alpha < 1)$ will denote the space of those functions $u \in C^k(\bar{G})$ whose k-th derivatives are Hölder-continuous with the exponent α in G. $C^{k,\alpha}(\bar{G})$ is a Banach space with the norm

$$\|u\|_{k,\alpha} = \|u\|_k + \sup_{x \neq y} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^\alpha}.$$

$C^\alpha(\bar{G})$ will denote the space $C^{0,\alpha}(\bar{G})$. We shall say that G *is of class* $C^{k,\alpha}$ if its boundary has, in a neighbourhood of every point, a regular parametrization of class $C^{k,\alpha}$.

We make the following **assumptions:**

(i) G is a bounded region in $\mathbf{R}^n$ of class $C^{2,\alpha}$ for fixed α, $0 < \alpha < 1$.
(ii) $a_{ij} \in C^\alpha(\bar{G})$ for all i, j.
(iii) L is strongly elliptic, i.e., there is a constant $\mu > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi^i \xi^j \geqq \mu \sum_{i=1}^{n} (\xi^i)^2 \quad \text{for all} \quad \xi \in \mathbf{R}^n, \quad x \in \bar{G}.$$

Consider the operator

$$A: C^{2,\alpha}(\bar{G}) \to C^\alpha(\bar{G}) \times C^{2,\alpha}(\partial G)$$

defined through $Au = (g, h)$ with

$$Lu = g \quad \text{in } G, \qquad u = h \quad \text{on } \partial G. \tag{2}$$

The homogeneous equation (2) has only the trivial solution $u = 0$. This follows from the classical maximum principle, cf. GILBARG and TRUDINGER [4]. Therefore, the operator A is a linear homeomorphism, cf. TRIEBEL [5; Section 4.3.4]. Hence, hypothesis (H 1) of Section 2 is fulfilled if we set

$$X = C^{2,\alpha}(\bar{G}), \qquad Y = C^\alpha(\bar{G}) \times C^{2,\alpha}(\partial G).$$

Now, let $u_0 \in C^{2,\alpha}(\bar{G})$ be the solution of the equation $Au = (1, 0)$. Then $\bar{u}_0 = \int_G u_0\, dx / |G| > 0$ holds. This follows from the classical maximum principle, because $Lu_0 = 1 > 0$ implies $u_0 \geqq 0$ in G, and from $u_0 \neq 0$, because A is a linear homeomorphism. Let $X_0 = \text{Lin}\,(u_0)$ be the one-dimensional subspace of X spanned by u_0. We set $X_1 = \{u \in X : \bar{u} = 0\}$. We have for each $u \in C^{2,\alpha}(\bar{G})$ the decomposition

$$u = \frac{\bar{u}}{\bar{u}_0} u_0 + \left(u - \frac{\bar{u}}{\bar{u}_0} u_0\right); \tag{3}$$

where $w = u - (\bar{u}/\bar{u}_0)\, u_0 \in X_1$, because of $\bar{w} = \bar{u} - (\bar{u}/\bar{u}_0)\, \bar{u}_0 = 0$. This decomposition is unique, because $r = \bar{u}/\bar{u}_0$ is the unique solution of the equation $\bar{u} - r\bar{u}_0 = 0$, by $\bar{u}_0 \neq 0$. Hence, hypothesis (H 2) is fulfilled. Especially, from (3) follows

$$P_0 u = \frac{\bar{u}}{\bar{u}_0}, \qquad \bar{u} = \bar{u}_0 P_0 u, \quad \text{and} \quad P_0 u_0 = 1$$

if we look on P_0 as a mapping of X onto $\mathbf{R}$. The decomposition of X induces a decomposition of Y, by $Y_0 = A(X_0)$ and $Y_1 = A(X_1)$. We have $Y_0 = \text{Lin}\,\big((1, 0)\big)$, $(1, 0) \in C^\alpha(\bar{G}) \times C^{2,\alpha}(\partial G)$.

Using this, equations (1) give a nonlinear operator

$$N = A + B : C^{2,\alpha}(\bar{G}) \to C^\alpha(\bar{G}) \times C^{2,\alpha}(\partial G),$$

where

$$B(u) = \big(f(\overline{u}), 0\big) = \big(f(\overline{u}_0 P_0 u), 0\big), \quad u \in C^{2,\alpha}(\overline{G}).$$

The solution set Σ of problem (1) is given by the proposition of Section 2, i.e. $\Sigma = \sigma \times S$, where

$$\sigma = \{(s, t) \in \mathbf{R}^2 \mid s + f(\overline{u}_0 s) = t\},$$

$$S = \big\{\big(u_1, (g_1, h)\big) \in X_1 \times Y_1 \mid Au_1 = (g_1, h)\big\}.$$

Remark: If we are given $(g, h) \in C^\alpha(\overline{G}) \times C^{2,\alpha}(\partial G)$ for problem (1), the preceding considerations give us the following way to get all solutions of this problem:

(i) We solve the equation $Au = (1, 0)$. Let u_0 be the solution of this problem.

(ii) To determine the components of (g, h) with respect to the decomposition $Y = Y_0 \oplus Y_1$, we solve the problem $Au = (g, h)$. If v is the solution of this equation, we have the unique decomposition $v = tu_0 + v_1$ with $t \in \mathbf{R}$ and $v_1 \in X_1$. By definition of Y_0 and Y_1, it follows that

$$(g, h) = (t, 0) + (g_1, h) \qquad \big((t, 0) \in Y_0, (g_1, h) \in Y_1\big),$$

where $g_1 = g - t$ (we consider t as the function $g \equiv t$).

(iii) Now, all solutions of problem (1) with right-hand side (g, h) are $u_\nu = s_\nu u_0 + v_1$, where $\{s_\nu\}$ is the set of all solutions of the equation $s + f(\overline{u}_0 s) = t$.

REFERENCES

[1] BERGER, M. S.: Nonlinearity and functional analysis. New York—San Francisco—London: Academic Press 1977.

[2] BERGER, M. S., and E. PODOLAK: On the solutions of a nonlinear Dirichlet problem. Indiana Univ. Math. J. 24 (1975), 837—846.

[3] CAFGNA, V., and F. DONATI: On the reduction procedure for a nonlinear boundary value problem. Ricerche Mat. 33 (1984), 159—170.

[4] GILBARG, D., and N. S. TRUDINGER: Elliptic partial differential equations of second order. Berlin—Heidelberg—New York: Springer Verlag 1977.

[5] TRIEBEL, H.: Theory of function spaces. Leipzig: Akad. Verlagsges. Geest & Portig K.-G. 1983.

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