$\begin{array}{c} \text{Zettschrift für Analysis} \\ \text{und ihre Anwendungen} \\ \text{Bd. 7 (4) 1988, S. 289 - 307} \end{array}$ Theorems with o -Rates and Markov Processes General Limit Theorems with σ -Rates and Markov Processes under Pseudo-Moment Conditions

P. L. BUTZER and H. KIRSCHFINK

-

-Mit--Hilfe der Dvoretzkyschen Erweiterung der Trotter- Opera toren-Methode wird ein allgemeiner Konvergenzsatz mit q**.** Ordnung für abhiingige, reellwertie Zufallsvariable, die einer Pseudomomentenbedingung genügen, bewiesen. Dieses Ergebnis wird auf allgemeine Grenzwertsätze, einen zentralen Grenzwertsatz sowie auf ein schwaches Gesetz der großen Zahlen für Markovsche Prozesse mit diskretem Zeitparameter angewandt. Ferner werden Pseudomômentenbedingungen diskutiert und die Materie mit Ergebnissen aus der Theorie der \Vahr- - Mit-Hilfe der Dvoretzkyschen Erweiterung demeiner Konvergenzsatz mit a-Ordnung für ale Pseudomomentenbedingung genügen, bewiesen wertsätze, einen zentralen Grenzwertsatz sowie
für Markovsche Prozesse mit diskretem Zeitp
m Mit-Hilfe der Dvoretzkyschen Erweiterung der Trotter-Operatoren-Methode wird ein allgemeiner Konvergenzsatz mit e-Ordnung für abhängige, reellwertige Zufallsvariable, die einer
Pseudomomentenbedingung genügen, bewiesen. D

C помощью расширения Дворецкого метода операторов Троттера доказывается общая теорема сходимости с *о*-порядком для независимых вещественно-значных случайных величин удовлетворяющих одному условию псевдомоментов. Этот результат применнется к общим теоремам сходимости, к одной центральной предельной теореме и к ослаблённому закону больших чисел для процессов Маркова с дискретным временным сравниваются с теорией вероятностных метрик.

Making use of the Dvoretzky extension of the Trotter-operator method, a general convergence theorem with *o*-rates for dependent, real-valued random variables satisfying a pseudo-moment condition is established. Applications are to general convergence theorems, a central limit theorem as well as a weak law of large numbers for Markov processes with discrete time parameter. Further, pseudo-moment conditions are discussed in detail and the results are compared with the theory of probability-metrics.

I. Introduction

The fundamental limit theorems of probability theory are generally concerned with the convergence of sums of random variables towards a given limit random variable. Apart from the type of convergence, the dependency structure of the random variables in questions as well as the particular limit random variable have to be specified. This paper is devoted to limit theorems with σ -rates of convergence for sums of not necessarily independent nor identically distributed random variables, thus for arbitrarily dependent. random variables. In order that the approach is rather broad, the results will be based upon a general limit theorem with rates in form of conver-
gence in distribution (Theorem 1), thus eter. Further, pseudo-moment conditions are discussed in detail and the re-
teter. Further, pseudo-moment conditions are discussed in detail and the re-
with the theory of probability-metrics.

1. Introduction

The fundam *IE* is devoted to limit theorems with
IF independent nor identically do
IE is will be based upon a general lim
distribution (Theorem 1), thus
 $|E[f(T_n + u)] - E[f(Z + u)]| = o_f$ $\frac{1}{n}$ σ -rat

listribu

order ¹

it theo
 $(\varphi(n))^r$

sums **1.** Introduction

The fundamental limit theorems of probability theory are generally concerned with

the convergence of sums of random variables towards a given limit random variable.

Apart from the type of convergence,

$$
|E[f(T_n + u)] - E[f(Z + u)]| = o_f(\varphi(n)^r V(n)) \qquad (n \to \infty), \qquad (1.1)
$$

arbitrary sequence of real, possibly dependent random variables, $\varphi \colon \mathbb{N} \to \mathbb{R}$ is a normalizing function with $\varphi(n) = c(1), n \to \infty$, Z a φ -decomposable random variable (i.e. for each $n \in \mathbb{N}$ there exist independent random variables $Z_i (= Z_{ni})$, $1 \leq i \leq n$, the results will be based upon a general limit t
gence in distribution (Theorem 1), thus
 $|E[f(T_n + u)] - E[f(Z + u)]| = o_f(\varphi(n\alpha))$
for each fixed $u \in \mathbb{R}$; here the normalized su
arbitrary sequence of real, possibly dependen
normal

290 P. L. BUTZER and H. KIRSCHFINK
such that for the distribution P_Z of *Z* one has $P_Z = P_{\varphi(n)} \sum_{i=1}^{n} z_i$
vergence determining class $C^r(\mathbb{R})$ for any $r \in \mathbb{P} = \{0, 1, 2\}$. such that for the distribution P_Z of Z one has $P_Z = P_{\varphi(n)} \sum_{i=1}^n z_i$, *f* belongs to the con-
vergence determining class $C^r(\mathbb{R})$ for any $r \in \mathbb{P} = \{0, 1, 2, ...\}$ (see (2.1) for the vergence determining class $C^r(\mathbb{R})$ for any $r \in \mathbb{P} = \{0, 1, 2, ...\}$ (see (2.1) for the definition), and $V(n)$ is defined in terms of the absolute pseudo-moments of order r. of the random variables in question (see (2.12ii) for the definition). This general theorem will-be applied in particular to (sums of) random variables X_i , $i \in \mathbb{N}$, which namely to a stochastic process distinguished by the Markov' property (see (4.1)).

form a non-homogeneous Markov process with a discrete time parameter (Theorem 2),
namely to a stochastic process distinguished by the Markov property (see (4.1)).
Until 1975 most of the results known in this direction, at Until 1975 most of the results known in this direction, at least in the particular case of the central limit theorem (for which the limit random variable $Z = X^*$, the normally distributed shed by the mark this direction, at least

andom variable $Z =$ mamely to a stochastic process distinguished by the Markov property (see (4.1)).

Until 1975 most of the results known in this direction, at least in the particular case of the

central limit theorem (for which the limit

necessarily independent random variables that satisfy Doeblin's condition or the so-called φ -mixing condition. Both additional assumptions roughly mean that the random variables are "asymptotically independent"; they are rarely satisfied for Markov processes. In this respect,' for example, LIFSHITS [26] established under a condition upon the maximum coefficient of the correlation (related to the above assumptions [27]) the result sup $|F_n(x) - F_{X^{\bullet}}(x)| = o(1)$, *zEfl?* central limit theorem (for which the limit random variable $Z = X^*$, the normally distributed
random variable with mean 0 and variance 1 and $\varphi(n) = \left(\sum_{i=1}^n \text{Var}(X_i)\right)^{-1/2}$, dealt with not
necessarily independent random convergence of the distribution function of $\left(\sum_{i=0}^{n} f(X_i(\omega)) - \sum_{i=0}^{n} \pi_k \right) / \sqrt{n \pi k}$, where π_k is defin-

ed as the reciprocal of the mean recurrence time of the state *k* and $j = 1_{\{i\}}$, towards $F_{\chi^{\bullet}}$, obtaining the order $\mathcal{O}(n^{-\alpha})$ for each $\alpha < 1/4$ under just a weak third moment condition. In this regard BOLTHAUSEN [5] established the rate $o(n^{-\epsilon})$ for $\epsilon < 1/3 - 1/6(c + 1)$ and some $c \geq 3$ for which $E[|X_i|^c] < \infty$. Both papers actually deal with positive recurrent irreducible Markov chains with countable state space. Fraction interior of $\begin{pmatrix} 2 \ k \end{pmatrix}$

mean recurrence time

ch $\alpha < 1/4$ under just

d the rate $o(n^{-c})$ for

rs actually deal with
 \therefore
 \therefore

mtral limit theorer

e estimate
 $X_i + u \begin{pmatrix} 2 \ k^{-1} \end{pmatrix} - E[f(X \begin{pmatrix} 2 \ k^{-$

Our version of the central limit theorem for Markov 'processes (Theorem 3) gives for each fixed $u \in \mathbb{R}$ the estimate

SEN [5] established the rate
$$
o(n^{-c})
$$
 for $ε < 1/3 - 1/6(c + 1)$ and some $c ≥ 3$ for $ε < ∞$. Both papers actually deal with positive recurrent irreducible Markov
stable state space.
ersion of the central limit theorem for Markov processes (Theorem 3)
fixed $u ∈ ℝ$ the estimate

$$
\left| E\left[f\left(A_n^{-1} \sum_{i=1}^n X_i + u\right)\right] - E[f(X^* + u)] \right| = o_f(A_n^{-r}M(n)) \quad (n \to ∞)
$$

5'. for $f \in C^r$, where $A_n = \left(\sum_{i=1}^n a_i^2\right)^{1/2}$, the a_i being positive reals and $M(n)$ as in (2.12i). In the particular case of Markov processes having stationary, independent increments with mean θ (thus for identically distributed random variables) this estimate with countable state space.

Our version of the central limit theorem for Markov process

for each fixed $u \in \mathbb{R}$ the estimate
 $\left| \mathbf{E} \left[f \left(A_n^{-1} \sum_{i=1}^n X_i + u \right) \right] - \mathbf{E}[f(X^* + u)] \right| = o_f(A_n^{-r}M)$

for $f \in C^r$, where where $A_n = \left(\sum_{i=1}^n a_i^2\right)^{1/2}$, the a_i being positive
ticular case of Markov processes having station
1 0 (thus for identically distributed random v
 $E\left[f\left(n^{-1/2}\sum_{i=1}^n X_i + u\right)\right] - E[f(X^* + u)] = c_f$

$$
\left| \mathbb{E}\left[f\left(n^{-1/2}\sum_{i=1}^n X_i + u\right)\right] - \mathbb{E}[f(X^* + u)] \right| = c_f(n^{(2-r)/2}) \quad (n \to \infty)
$$

for each $u \in \mathbb{R}$ with $A_n = n^{1/2}$.

Concerning the' assumptions needed for our results, moment type conditions will also be required, namely the conditional pseudo-moment condition (3.3) of order *r* as 'vell as a pseudo-Lindeberg condition (2.11) of order *r.* Whereas the dependency structure of the random variable is unrestricted, it will depend indirectly upon moment conditions, in particular upon our pseudo-moment condition. It will in any case cover Markov processes (and martingale difference sequences) for both of which the dependency structure "depends" on the past, i.e., the random variables depend only upon tifeir predecessors; thy will be said to be dependent from below. This aspect will especially be dealt with in Section 3.2. Alternative conditions to the weak pseudo-.

moment condition (3.3) will also be examined $(Section 5)$; it can indeed by replaced by the same non-conditional moment condition already used in BUTZER' and HAHN [8], noting that the functions of *CT* are bounded. In fact, (3.3) can be replaced by the moment condition (3.3) will also be
by the same non-conditional moment
[8], noting that the functions of C^r a
condition
 $\sum_{i=1}^n E[|X_i - Z_i^i|] = c \left(\varphi(n)^r\right)$ General Limit Theorems with o -1
dition (3.3) will also be examined (Section 5); it can indee
non-conditional moment condition already used in Burzz
nat the functions of C^r are bounded. In fact, (3.3) can be re
 $E[|X_i -$

$$
\sum_{i=1}^n \mathbb{E}[|X_i^j - Z_i^j|] = c \left(\varphi(n)^r \sum_{i=1}^n \mathbb{E}[|X_i - Z_i|^r]\right) \qquad (1 \leq j \leq r).
$$

This will enable one to extend assertion (1.1), valid pointwise in $u \in \mathbb{R}$, to the uniform assertion (5.8) of Theorem 6.

It is rather interesting to note that results such as these may also be deduced from the well-known Kantorovitch-Rubinstein-Dudley theorem (see e.g. [17, 37]), or for the analogous $M(n)$ -version by a lemma' and a theorem of ZOLOTAREV [36]. The Kantorovitch-Rubinstein theorem gives for the distance of two random variables the moment condition (3.3) will also be
by the same non-conditional mome
[8], noting that the functions of C^r
condition
 $\sum_{i=1}^{n} \mathbf{E}[|X_i - \hat{Z}_i^i|] = c \left(\varphi(n \right)$
This will enable one to extend asset
form assertion (5.8) o ther interesting to note that results such as these may also be deduced from
known Kantorovitch-Rubinstein-Dudley theorem (see e.g. [17, 37]), or for
gous $M(n)$ -version by a lemma and a theorem of ZOLOTAREV [36]. The
vitc

$$
\sup_{u\in\mathbb{R}}\{|E[f(X+u)]-E[f(Z+u)]|\}\leq E[|X-Z|^s],\qquad 0
$$

where $f \in D_s$, D_s as in (6.5). Applying it to our situation, one has for $X = T_n$ and $Z = Z$ the estimate (compare Theorem 8)
 $\sup_{u \in \mathbb{R}} |\{\mathbb{E}[f(T_n + u) - f(Z + u)]\}| \le \sum_{i=1}^n \varphi(n)^s \mathbb{E}[|X_i - Z_i|^s].$ $Z = Z$ the estimate (compare Theorem 8).

$$
\sup_{u \in \mathbb{R}} \left\{ |E[f(X + u)] - E[f(Z + u)]| \right\} \leq E[|X - Z|^s], \qquad 0 < u \in \mathbb{R}
$$
\n
$$
\in D_s, D_s \text{ as in (6.5). Applying it to our situation, one has
$$
\n
$$
\text{the estimate (compare Theorem 8)}
$$
\n
$$
\sup_{u \in \mathbb{R}} \left\{ |E[f(T_n + u) - f(Z + u)]| \right\} \leq \sum_{i=1}^n \varphi(n)^s \, E[|X_i - Z_i|^s].
$$
\n
$$
\text{with } \varphi(n) = \sum_{i=1}^n \varphi(n)^s \, E[|X_i - Z_i|^s].
$$

However, it is important that this version of (1.2) is only valid for independent random variables X_i and Z_i . Thus our Theorem 6 can be regarded as a certain generalization of' the Kantorovitch-Rubinstein-Dudley theorem to the case of dependent random variables and stochastic processes. This matter is elaborated upon in Section 6. It should be mentioned that BERGSTRoM [4] used pseudo-moment conditions in the case of independent identically distributed random variables already in 1953. The recent papers by PADITZ [30] and SAzoNov and ULYANOV [33] re-emphasize the importance of such conditions.

Not only will limit theorems be studied for the sums $T_n = \sum_{i=1}^n \varphi(n) X_i$ but also

for the processes X_n themselves for $n \to \infty$, as indicated. The latter will depend upon the structure of the increments. Whereas the literature abounds with results concerning the rate of convergence of sums of random' variables which are connected in a Markov process, at least in the case of the central limit theorem, the results for the X_n mostly deal with the behaviour of *n*-step transition probabilities (cf. CHUNG [16, pp. 10] or GIHMAN and SKOROHOD [19, pp. 282]) whereby in the instance of independent and-stationary increments there is a direct connection between the increments of Markov chains and 1-step transition probabilities (cf. C_{HUNG} [16, p. 10]). It is interesting to observe that in the particular case of independent increments the present résults practically coincide with those of BUTZER and $H_{\text{AHN}}[8]$ of 1978 on the convergence of sums of independent random variables towards a φ -decomposable limit random variable (Section 4.3). The latter. were generalized to the case of not necessarily independent random variables which form martingale difference sequences or arrays in BUTZER, HAHN and ROECKERATH [10, 11], and to more general types of dependent random variables in BUTZER and SCHULZ $[13-15]$, as well as to arbitrary sequences for the particular case of identically distributed random variables in BUTZER and K IRSCHFINK [12]. As far as the authors are aware, there are no comparable Markov process, at least in the case of the central limit theored X_n mostly deal with the behaviour of *n*-step transition probal pp. 10] or GHMAN and SKOROHOD [19, pp. 282]) whereby in the ent and stationary increments

/

19*

• The type of convergence considered in this paper is more general than that **in BUTZER, HAHN** and **ROECRERATH [10],** or in **BUTZER** and **SCHULZ** [13] where the dependency structure is of type of martingale difference sequences. In fact; the convergence considered in (1.1) is that for the difference *^f/(x* + *u) dFT,,(x) — ^f/(x ± u) dFz(x) (1.3)* P. L. BUTZER and H. KIRS

P. L. BUTZER and H. KIRS

THE OF CONVERGED CONSUMED AND AND SURVER EXP.

CONSIDERED IN (1.1) is the first of type $\int \mathbb{R} f(x + u) dF_{T_n}(x) - \int \mathbb{R}$

R in $u \in \mathbb{R}$, and not just d above. This is

$$
\int_{\mathbb{R}} f(x+u) dF_{T_n}(x) - \int_{\mathbb{R}} f(x+u) dF_z(x) \tag{1.3}
$$

pointwise in $u \in \mathbb{R}$, and not just for the particular case $u = 0$ treated in the papers mentioned above. This is even generalized to uniform convergence for $u \in \mathbb{R}$ in Section 5. Although both assertions are equivalent in case of convergence, at least in the case of independent, identically distributed random variables, this equivalence need not remain valid for convergence with rates. If the assertion (1.3) is equipped with the rate $\mathcal{O}(n^{-(r-2)/2})$ for $f \in C^r$, then it is equivalent to Although both assertions are equivalent in case of independent, identically distributed random v
remain valid for convergence with rates. If the a
rate $\mathcal{O}(n^{-(r-2)/2})$ for $f \in C^r$, then it is equivalent to
 $\iint_S f(x) dF_{T$ Figure 1 and that

d SCHULZ [13] where t

quences. In fact, the co

quences. In fact, the co
 (1.4)
 $= 0$ treated in the pape

onvergence for $u \in \mathbb{R}$

onvergence, at least

ariables, this equivalen

ssertion (1.3) dependency structure is of type of martingale difference sequences. If
vergence considered in (1.1) is that for the difference
 $\int f(x + u) dF_T(x) - \int f(x + u) dF_2(x)$
 \mathbb{R}
pointwise in $u \in \mathbb{R}$, and not just for the particula f . Although both a

of independent,
 f remain valid for
 f rate $\mathcal{O}(n^{-(r-2)/2})$
 $\int f(x) dF_{T_n}(x)$
 \mathbb{R}
 $\int x^j dF_{T_n}(x) - \int \mathbb{R}$
 $\mathbb{E}[X^j] = \mathbb{E}[Z^j]$ Section 5. Although both assertions are equivalent in case of convergence, at life case of independent, identically distributed random variables, this equivalend not remain valid for convergence with rates. If the asserti

$$
\int_{\mathbb{R}} f(x) dF_{T_n}(x) - \int_{\mathbb{R}} f(x) dF_z(x) = O(n^{-(r-2)/2}) \qquad (n \to \infty)
$$
\nEach of the two

\n
$$
\int_{\mathbb{R}} x^j dF_{T_n}(x) - \int_{\mathbb{R}} x^j dF_z(x) = O(n^{-(r-2)/2}) \qquad (1 \leq j \leq r - \infty)
$$

rate
$$
\mathcal{O}(n^{-(r-2)/2})
$$
 for $f \in C^r$, then it is equivalent to
\n
$$
\int f(x) dF_{T_n}(x) - \int f(x) dF_Z(x) = \mathcal{O}(n^{-(r-2)/2}) \qquad (n \to \infty)
$$
\nR
\nto each of the two
\n
$$
\int x^j dF_{T_n}(x) - \int x^j dF_Z(x) = \mathcal{O}(n^{-(r-2)/2}) \qquad (1 \leq j \leq r-1),
$$
\nR
\n
$$
E[X^j] = E[Z^j] \qquad (1 \leq j \leq r-1)
$$

in the independent identically distributed case (see BUTZER and HAHN [9]). However, nothing seems to be known in this respect if the random variables are not necessarily independent identically distributed or if the $\mathcal{O}(n^{-(r-2)/2})$ -rates are replaced by σ -rates, which is the situation of the present paper. $\int_R f(x) dF_{T_n}(x) - \int f(x) dF_z(x) = \mathcal{O}(n^{-(r-2)/2})$ $(n \to \infty)$

or even to each of the two
 $\int_x x^j dF_{T_n}(x) - \int_x x^j dF_z(x) = \mathcal{O}(n^{-(r-2)/2})$ $(1 \leq j \leq r - 1)$,
 $\mathbb{E}[X^j] = \mathbb{E}[Z^j]$ $(1 \leq j \leq r - 1)$

in the independent identically d $\int f(x) dF_{T_n}(x) - \int f(x) dF_z(x) = \mathcal{O}(n^{-(r-2)/2})$ $(n \to \infty)$

or even to each of the two
 $\int x^j dF_{T_n}(x) - \int x^j dF_z(x) = \mathcal{O}(n^{-(r-2)/2})$ $(1 \leq j \leq r - 1)$
 \mathbb{R}
 $E[X^j] = E[Z^j]$ $(1 \leq j \leq r - 1)$

in the independent identically distr $E[X^j] = E[Z^j]$

a the independent identical

ver, nothing seems to be lecessarily independent iden

y *o*-rates, which is the situal

y *o*-rates, which is the situal

. Notations and preliminari

et $C = C(\mathbb{R})$ denote the $dF_{T_n}(x) - \int x^j dF_z(x) = \mathcal{O}(n^{-(r-2)/2})$ $(1 \leq j \leq r - 1),$
 \mathbb{R}
 $[1] = \mathbb{E}[Z^j]$ $(1 \leq j \leq r - 1)$

ndent identically distributed case (see BUTZER and HARN [9]). How

seems to be known in this respect if the random varia $E[X^j] = E[Z^j]$ $(1 \leq j \leq r - 1)$
dependent identically distributed case (see BUTZER and HAR
hing seems to be known in this respect if the random varialy
independent identically distributed or if the $\mathcal{O}(n^{-(r-2)/2})$ -rates

•

Let $C = C(\mathbb{R})$ denote the class of all real-valued, bounded, uniformly continuous **functions and preliminaries**
functions defined on the reals IR, endowed with norm $||f|| := \sup_{x \in \mathbb{R}} |f(x)|$. For $r \in \mathbb{R} \cup \{0\}$ set functions defined on the reals **R**, endowed with norm $||f|| := \sup_{x \in \mathbb{R}} |f(x)|$. For $r \in \mathbb{P}$
= N u $\{0\}$ set

$$
C^0 = C, \ \ C^r = \{g \in C; \ g^{(j)} \in C, \ 1 \leq j \leq r\},\tag{2.1}
$$

the seminorm on *Cr* being given by $|g|_{C} = ||g^{(r)}||$. Concerning the random variables in question, the class $\mathfrak{Z}(\Omega, \mathfrak{A})$ plays an importent role. If $(\Omega, \mathfrak{A}, P)$ is a probability space, then $\mathfrak{Z}(\Omega, \mathfrak{A}) := \{X; X \text{ is } \mathfrak{A} \text{-}\mathfrak{B}\text{-measurable}\}$, where \mathfrak{B} is the Borel σ -algebra on $C^0 = C$, $C^r = \{g \in C; \dot{g}^{(r)} \in C, 1 \leq j \leq r\}$, (2.1)
the seminorm on C^r being given by $|g|_{C^r} = ||g^{(r)}||$. Concerning the random variables
in question, the class $\mathfrak{Z}(\Omega, \mathfrak{A})$ plays an importent role. If $(\Omega, \math$ real-valued, bounded, uniformly co
dowed with norm $||f|| := \sup_{x \in \mathbb{R}} |f(x)|$. F
 \therefore 1 $\leq j \leq r$,
 \therefore
 \therefore 1 $\leq j \leq r$,
 \therefore
 \therefore 7 = $||g^{(r)}||$. Concerning the random v
importent role. If $(\Omega, \mathfrak{A}, P)$ is a pro-
eas **R.** If $X \in \mathcal{B}(\Omega, \mathcal{U})$, and \mathfrak{E} a sub- σ -algebra of \mathfrak{A} , then the \mathfrak{E} -measurable function $\mathbb{E}[X \mid \mathfrak{E}]$, defined by $\int \mathbb{E}[X \mid \mathfrak{E}] dP = \int X dP$ for every $F \in \mathfrak{E}$, is known as the *conditional expectation* of *X*, given $\mathfrak E$. Its properties will also be needed. Let *X*, *Y* $\in \mathcal{B}(\Omega, \mathfrak{A})$ such that $\mathbb{E}[X] < \infty$, $\mathbb{E}[Y] < +\infty$, and $\mathfrak{E} \subset \mathfrak{A}$. They read: **Notations and preliminaries**
 $\mathcal{X} \subset C = C(\mathbb{R})$ denote the class of all real-valued, bounded, uniformly continuous

netions defined on the reals \mathbb{R} , endowed with norm $||f|| := \sup_{x \in \mathbb{R}} |f(x)|$. For $r \in \mathbb{P}$
 \mathbb = \mathbb{R}^3 o { the semiin question
in question
space, the \mathbb{R} . If X
 $\mathbb{E}[X \mid \mathbb{G}],$
condition
 $\in \mathfrak{Z}(\Omega, \mathfrak{A})$ *X*² set
 *X*² = *X*, *C*^{*r*} = { $g \in C$; $g^{(j)} \in C$, $1 \leq j \leq r$ },
 X a.s; $X = f(X; X)$ is $\mathbb{R}^2 \cup \mathbb{R}^2$ a.s. implies the set \mathbb{R}^2 on, the class \mathbb{R}^2 \mathbb{R}^2 plays an importent role. If \mathbb{R}^2 the seminorm on C^r being given by $|g|_{C'} = ||g^{(r)}||$.

n question, the class $\mathfrak{Z}(\Omega, \mathfrak{A})$ plays an importent

space, then $\mathfrak{Z}(\Omega, \mathfrak{A}) := \{X : X \text{ is } \mathfrak{A} \cdot \mathfrak{B}$ -measurable),
 R. If $X \in \mathfrak{Z}(\Omega, \mathfrak{A})$, 1 norm $||f|| := \sup_{x \in \mathbb{R}} |f(x)|$

1 norm $||f|| := \sup_{x \in \mathbb{R}} |f(x)|$

1 . Concerning the rance role. If $(\Omega, \mathfrak{A}, P)$ is where \mathfrak{B} is the Bore then the \mathfrak{C} -measur revery $F \in \mathfrak{C}$, is k ies will also be need \mathfrak Fig. 2013

Eq. $C^r = \{g \in C; g^{(j)} \in C, 1 \leq j \leq r\}$,

norm on C^r being given by $|g|_{C^r} = ||g^{(r)}||$. Concerning the random van

on, the class $\mathfrak{Z}(\Omega, \mathfrak{A})$ plays an importent role. If $(\Omega, \mathfrak{A}, P)$ is a probs

cn Example 2. The sum of $C = \{g \in C : g^{(j)} \in C, 1 \leq j \leq r\}$, $C^j = C, C^r = \{g \in C : g^{(j)} \in C, 1 \leq j \leq r\}$, $C^0 = C, C^r = \{g \in C : g^{(j)} \in C, 1 \leq j \leq r\}$, (2.1)

norm on C^r being given by $|g|_{C^r} = ||g^{(r)}||$. Concerning the random v norm on C' being given by $|g|_{C'} = ||g^{(v)}||$. Concerning the random, the class $\mathfrak{Z}(\Omega, \mathfrak{A})$ plays an importent role. If $(\Omega, \mathfrak{A}, P)$
en, $\mathfrak{Z}(\Omega, \mathfrak{A}) := \{X; X \text{ is } \mathfrak{A} \cdot \mathfrak{B} \text{ is the Borel}$
 $\in \mathfrak{Z}(\Omega, \mathfrak{A})$

$$
X \geq Y \text{ a.s. implies } \mathbb{E}[X \mid \mathfrak{E}] \geq \mathbb{E}[Y \mid \mathfrak{E}] \text{ a.s.}
$$
\n(2.2)

$$
X = c \text{ a.s. } (c \text{ a constant}) \text{ implies } E[X \mid \mathfrak{E}] = c \text{ a.s.}
$$
\n(2.3)

$$
E[\alpha X + \beta Y | \mathfrak{E}] = \alpha E[X | \mathfrak{E}] + \beta E[Y | \mathfrak{E}] \text{ a.s. } (\alpha, \beta \in \mathbb{R}). \tag{2.4}
$$

(2.6)

$$
\mathrm{E}[\mathrm{E}[X \mid \mathfrak{E}]] = \mathrm{E}[X].
$$

Ceneral Limit Theorems with δ -Rates 293
 \mathfrak{F} -measurable and $\mathbb{E}[XY] < \infty$, then $\mathbb{E}[XY | \mathfrak{G}] = X \mathbb{E}[Y | \mathfrak{G}]$ Let X be E-measurable and $E[XY] < \infty$, then $E[XY | \mathfrak{E}] = XE[Y | \mathfrak{E}]$ General Limit Theorems with δ -Rates 293

Let X be G-measurable and $E[XY] < \infty$, then $E[XY | \mathfrak{E}] = XE[Y | \mathfrak{E}]$

a.s. (2.7)

Let $\mathfrak{A}(X)$ and \mathfrak{E} be independent; then $E[X | \mathfrak{E}] = E[X]$ a.s. (2.8)

Let $\mathfrak{A}(\mathfr$ General Limit Theorems with δ -Rates 293

Let X be G-measurable and $E[XY] < \infty$, then $E[XY | \mathfrak{G}] = XE[Y | \mathfrak{G}]$

a.s. (2.7)

Let $\mathfrak{A}(X)$ and \mathfrak{G} be independent; then $E[X | \mathfrak{G}] = E[X]$ a.s. (2.8)

Let $\mathfrak{A}(\mathfr$

a.s. (2.7)
Let $\mathfrak{A}(X)$ and \mathfrak{G} be independent; then $E[X \mid \mathfrak{G}] = E[X]$ a.s. (2.8)
Let $\mathfrak{A}(\mathfrak{A}(X), \mathfrak{G}_1)$ and \mathfrak{G}_2 be independent for two sub- σ -algebras
 $\mathfrak{G}_1, \mathfrak{G}_2 \subset \mathfrak{A}$. Then $E[X \mid \math$

Let $\mathfrak{A}(\mathfrak{A}(X), \mathfrak{E}_1)$ and \mathfrak{E}_2 be independent for two sub- σ -algebras

(2.9)

For the proofs of these properties see LAnA and **ROHATGI** [23, pp. 358], BAIJER [3, pp. 289] or **GAENSSLER** and **STUTE** [18, pp. 185].

The following generalizations of the well-known Lindeberg condition will play an

Let X be \mathfrak{G} -measurable and $E[XY] < \infty$, then $E[X] \in E[X]$
a.s.
Let $\mathfrak{A}(X)$ and \mathfrak{G} be independent; then $E[X \mid \mathfrak{G}] = E[X]$ a.s.
Let $\mathfrak{A}(\mathfrak{A}(X), \mathfrak{G}_1)$ and \mathfrak{G}_2 be independent for two sub-*σ*-al Definition 1: The sequence $(X_i)_{i\in\mathbb{N}}$ of real random variables having a finite $\mathfrak{E}_1, \mathfrak{E}_2 \subset \mathfrak{A}$. Then $E[X \mid \mathfrak{E}] = E[X \mid \mathfrak{E}_1]$ a.s. with $\mathfrak{E} = \mathfrak{A}(\mathfrak{E}_1, \mathfrak{E}_2)$. (2.9)
For the proofs of these properties see LARA and ROHATGI [23, pp. 358], BAUER [3,
pp. 289] or GAENSSLER an moment of order r, some $0 < r < \infty$
condition of order r if, for every $\delta > 0$,

Let X be G-measurable and
$$
E[XY] < \infty
$$
, then $E[XY | \mathfrak{E}] = XE[Y | \mathfrak{E}]$ a.s. (2.7) Let $\mathfrak{A}(X)$ and \mathfrak{E} be independent; then $E[X | \mathfrak{E}] = E[X]$ a.s. (2.8) Let $\mathfrak{A}((X), \mathfrak{E}_1)$ and \mathfrak{E}_2 be independent for two sub- $-algebras$ \mathfrak{E}_1 , $\mathfrak{E}_2 = \mathfrak{A}$. Then $E[X | \mathfrak{E}] = E[X | \mathfrak{E}_1]$ a.s. with $\mathfrak{E} = \mathfrak{A}(\mathfrak{E}_1, \mathfrak{E}_2)$. (2.9) If the proofs of these properties see LATA and ROHATGI [23, pp. 358], BAUER [3, 289] or GAESSSEER and STTTE [18, pp. 185]. The following generalizations of the well-known Lindeberg condition will play an portant role in the proofs. \nDefinition 1: The sequence $(X_i)_{i \in \mathbb{N}}$ of real random variables having a finite function of order r, some $0 < r < \infty$, is said to satisfy the generalized Lindeberg addition of order r if, for every $\delta > 0$, \n $\sum_{i=1}^n |x|^r dF_X(x) \longrightarrow 0 \quad (n \to \infty)$. (2.10) $\sum_{i=1}^n |E[X_i|]$ $\longrightarrow 0 \quad (n \to \infty)$. (2.11) $\sum_{i=1}^n |E[X_i|]$ $\longrightarrow 0 \quad (n \to \infty)$. (2.12) $\sum_{i=1}^n |E[X_i|]$ $\longrightarrow 0 \quad (n \to \infty)$. (2.13) If the absolute moments of order r or (ii) $E[(X_i - Z_i|] < \infty$, e.s. and to satisfy a generalized pseudo-Lindeberg condition of order r if, for every δ , 0, e.g. said to satisfy a generalized pseudo-Lind

The case $r = 2$ with $\varphi(n) = \left(\sum_{i=1}^{n} \mathbb{E}[X_i^2]\right)^{1/2}$ reduces to the usual Lindeberg condition (cf. BUTZER, HAHN and WESTPHAL [7]).

Definition 2: Two sequences $(X_i)_{i\in\mathbb{N}}$ and $(Z_i)_{i\in\mathbb{N}}$ of real random variables with

ii) $\mathbb{E}[|X_i - Z_i|^r] \leq \infty$,

are said to satisfy a *generalized pseudo-Lindeberg condition of order r* if, for every

tion (cf. BUTZER, HARN and WESTPHAL [7]).
\nDefinition 2: Two sequences
$$
(X_i)_{i \in \mathbb{N}}
$$
 and $(Z_i)_{i \in \mathbb{N}}$ of real random variables with
\ni) finite absolute moments of order *r* or
\nii) $E[|X_i - Z_i|^r] \leq \infty$,
\nare said to satisfy a generalized pseudo-Lindeberg condition of order *r* if, for every
\n $\delta > 0$,
\n
$$
\sum_{i=1}^{n} \int_{|x|^r} |x|^r d(F_{X_i}(x) - F_{Z_i}(x)) = \begin{cases} c_{\delta}(M(n)) \text{ or } & (n \to \infty), \\ c_{\delta}(V(n)) & (n \to \infty), \end{cases}
$$
\n(2.11ii)
\nwhere
\n
$$
M(n) = \sum_{i=1}^{n} (E[|X_i|^r] + E[|Z_i|^r]),
$$
\n(2.12i)
\nThere is the following trivial connection between the generalized Lindeberg and the
\ngeneralized pseudo-Lindeberg condition (2.11i).
\n1. em ma 1: Let $(X_i)_{i \in \mathbb{N}}$ and $(Z_i)_{i \in \mathbb{N}}$ be two sequences in $\mathfrak{Z}(\Omega, \mathfrak{A})$ having finite absolute
\nmoments of order *r*, $0 < r < \infty$. If each of the sequences fulfils a generalized Lindeberg

-:

'

$$
\sum_{i=1}^{n} \int_{|x| \ge \delta/\varphi(n)} |x|^r d(F_{X_i}(x) - F_{Z_i}(x)) =\begin{cases} c_{\delta}(M(n)) \text{ or } & (n \to \infty), \\ c_{\delta}(V(n)) & (2.11ii) \end{cases}
$$
\n
$$
M(n) = \sum_{i=1}^{n} (E[|X_i|^r] + E[|Z_i|^r]),
$$
\n
$$
V(\vec{n}) = \sum_{i=1}^{n} (E[|X_i - Z_i|^r]).
$$
\n
$$
(2.12i)
$$
\nthe following trivial connection between the generalized Independent and the

$$
V(\vec{n}) = \sum_{i=1}^{n} (E[|X_i - Z_i|^r]).
$$
\n(2.12ii)

There is the following trivial connection between the generalized Lindeberg and the

Lemma 1.: Let $(X_i)_{i\in\mathbb{N}}$ and $(Z_i)_{i\in\mathbb{N}}$ be two sequences in $\mathfrak{Z}(\Omega,\mathfrak{A})$ having finite absolute *moments of order r,* $0 < r < \infty$. If each of the sequences fulfils a generalized Lindeberg *condition of order r, then both together fulfil a generalized pseudo-Lindeberg condition* $(2.11i)$ of order r . = $\sum_{i=1}^{n} (E[|X_i|^r] + E[|Z_i|^r]),$

= $\sum_{i=1}^{n} (E[|X_i - Z_i|^r]).$

llowing trivial connection between the generalized Lindeberg

udo-Lindeberg condition (2.11i).

Let $(X_i)_{i \in \mathbb{N}}$ and $(Z_i)_{i \in \mathbb{N}}$ be two sequences in \math

:.

3. General limit theorems with σ -rates

The following main approximation theorem for sums of not necessarily independent random variables will be established by a modification of the Lindeberg-Trotter operator-theoretic approach as tailored to the situation of dependent random variables by means of Dvoretsky's telescoping argument. For this purpose the assumtions are the generalized pseudo-Lindeberg condition of order *r* for the random variables X_i and the decomposition components Z_i as well as a conditional pseudo-moment 294 P. L. BUTZER and H. KIRSCHFINK

294 P. L. BUTZER and H. KIRSCHFINK

2014 Chence **P. 10** Chence **r.**

2016 Chence **r.** The following main approximation theorem for sums of not necessarily

random variables will be esta 3. General
The follow
random va
operator-tll
les by mea
are the general
 X_i and the
condition of
Theore
uted nor i
bat
or
 $\begin{bmatrix} v_r \\ v_r \end{bmatrix}$
for some r *riandom variab-*
riang argument. For this purpose the assumptions
 remain to order r for the random variables
 riandom variable with $E[Z] = 0$, *such riangle andom variable with* $E[Z] = 0$, *such*
 ri = $E[|Z_i|^r < \in$ 3. General limit theorems with o -rates

The following main approximation theorem for su

random variables will be established by a modi

operator-theoretic approach as tailored to the situal

les by means of Dvoretsky's 3. Gene The foll random
operator les by m
are the X_i and condition
 $\frac{1}{2}$ and condition
 $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ are $\frac{1}{2}$ and $\frac{1}{2}$ are $\frac{1}{2}$ and $\$ The following main approxandom variables will be
operator-theoretic approaches will be
operator-theoretic approaches
les by means of Dvoretsky
are the generalized pseud
 X_i and the decomposition
condition of order r.
The mit theorems with *c*-rates

g main approximation theorem for sums of not necessarily independent

ables will be established by a modification of the Lindeberg-Trotter

oretic approach as tailored to the situation of depe *composition components (Z_i)_{iex} together satisfy</sub> interperator-theoretic approach as tailored to the situation of the Lindeberg-Trott
operator-theoretic approach as tailored to the situation of dependent random varia
 theoretic approach as tallored to the situation of dependent random variab-

ans of Dvoretsky's telescoping argument. For this purpose the assumptions

eneralized pseudo-Lindeberg condition of order r for the random vari order is by means of Dvoretsky's telescoping argument. For this pare the generalized pseudo-Lindeberg condition of order* r *for a* X_i *and the decomposition components* Z_i *as well as a condition of order* r *.
Theorein* are the generalized pseudo-Lindeberg condition of order r for the rand X_i and the decomposition components Z_i as well as a conditional psecondition of order r.

Theorem 1: Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of \mathfrak{Z}

Theorem 1: Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of $\mathfrak{Z}(\Omega, \mathfrak{A})$ (not necessarily identically distrib*uted nor independent*) and Z a φ -decomposable random variable with $E[Z] = 0$, such *that* $\begin{aligned} \textbf{1} &\text{1:} \text{ Let } (X_i)_{i \in \mathbb{N}} \text{ be a sequence} \\ \text{dependent) and } Z \text{ a } \varphi \text{-decod} \\ & = \text{E}[|X_i|^r] < \infty \text{ and } \xi \end{aligned}$ condition of order *r*.

Theorem 1: Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of $\mathfrak{Z}(\Omega, \mathfrak{A})$ (not necessarily identic
 uted nor independent) and Z a φ -decomposable random variable with E[Z

that
 $\xi_{ri} = \mathrm{E}[|X_i|^r] <$ r riy identically e with $\mathbb{E}[Z] =$
 \ldots

$$
\zeta_{ri} = \mathbb{E}[|X_i|^r] < \infty \quad and \quad \xi_{ri} = \mathbb{E}[|Z_i|^r < \infty), \tag{3.1}
$$

$$
v_{ri} = \mathbf{E}[|X_i - Z_i|^r] < \infty
$$

for some r \geq 2. *Assume further that the sequences of random variables* $(X_i)_{i \in \mathbb{N}}$ and detain components $(Z_i)_{i \in \mathbb{N}}$ *softer satisfy*

the generalized pseudo-Lindeberg condition
$$
(2.11\,\mathrm{ii})
$$
 $(3.2\,\mathrm{ii})$

S
S

 $\mathcal{S} = \begin{cases} \mathcal{S} & \text{if } \mathcal{S} \leq \mathcal{S} \\ \mathcal{S} & \text{if } \mathcal{S} \leq \mathcal{S} \end{cases}$

for some
$$
r \geq 2
$$
. Assume further that the sequences of random variables $(X_i)_{i \in \mathbb{N}}$ and de-
composition components $(Z_i)_{i \in \mathbb{N}}$ together satisfy
the generalized pseudo-Lindeberg condition (2.11i)
or
the generalized pseudo-Lindeberg condition (2.11ii)
of orders r, as well as

$$
\sum_{i=1}^{n} E[(X_i^j - Z_i^j) | \mathfrak{A}_{n_i}] (\omega) = \begin{cases} c(\varphi(n)^r M(n)), & \text{a.s.} \\ c(\varphi(n)^r V(n)), & \text{a.s.} \end{cases}
$$
(3.3i)
where $\mathfrak{A}_{n_i} = \mathfrak{A}(X_1, ..., X_{i-1}, Z_{i+1}, ..., Z_n)$.
Then for any $f \in C^r$ there holds for each fixed $u \in \mathbb{R}$ the estimate

E $\mathfrak{A}_{n} = \mathfrak{A}(X_1, \ldots, X_{i-1}, Z_{i+1}, \ldots, Z_n)$.
Then for any $f \in C^r$ there holds for each fixed $\mu \in \mathbb{R}$ the estimate

$$
v_{ri} = E[|X_i - Z_i|^r] < \infty
$$
\n(3.1 ii)
\n
$$
v_{ri} = E[|X_i - Z_i|^r] < \infty
$$
\n(3.1 ii)
\n
$$
v_{\text{m}} \geq 2. \text{ Assume further that the sequences of random variables } (X_i)_{i \in \mathbb{N}} \text{ and } de_{\tau}
$$
\n
$$
mposition \text{ components } (Z_i)_{i \in \mathbb{N}} \text{ together satisfy}
$$
\nthe generalized pseudo-Lindeberg condition (2.11 ii)
\n
$$
v_{\text{m}} \geq 2. \text{ Assume further that the sequences of random variables } (X_i)_{i \in \mathbb{N}} \text{ and } de_{\tau}
$$
\n(3.2 i)
\nthe generalized pseudo-Lindeberg condition (2.11 ii)
\n
$$
v_{\text{m}} \geq 2. \text{ Assume further that the sequences of random variables } (X_i)_{i \in \mathbb{N}} \text{ and } de_{\tau}
$$
\n(3.2 i)
\n
$$
v_{\text{m}} \geq 2. \text{ Assume further that the sequences of random variables } (X_i)_{i \in \mathbb{N}} \text{ and } de_{\tau}
$$
\n(3.2 i)
\n
$$
v_{\text{m}} \geq 2. \text{ Assume further that the sequences of random variables } (X_i)_{i \in \mathbb{N}} \text{ and } de_{\tau}
$$
\n(3.2 i)
\n
$$
\text{or} \quad \text{the generalized pseudo-Lindeberg condition (2.11 ii)\n
$$
\text{or} \quad (2.11 ii)\n
$$
\text{or} \quad (3.2ii)\n
$$
v_{\text{m}} \geq 2. \text{ Assume further that the sequences of random variables } (X_i)_{i \in \mathbb{N}} \text{ and } de_{\tau}
$$
\n(3.2 i)
\n
$$
\text{or} \quad (3.2ii)\n
$$
\text{or} \quad \text{the generalized pseudo-Lindeberg condition (2.11 ii)\n
$$
\text{or} \quad (3.2ii)\n
$$
\text{or} \quad (3.3iii)\n
$$
\text{or} \quad \text{the generalized pseudo-Lindeberg condition (2.11 ii)\n
$$
\text{or} \quad (3
$$
$$
$$
$$
$$
$$
$$
$$
$$

 \ddotsc

ke estimate

(n)

(n $\rightarrow \infty$)

(n $\rightarrow \infty$)

(n $\sum_{k=1}^{n} X_k + \sum_{k=i+1}^{n} X_k$

(ation of Taylo *f* $\sum_{i=1}^{n} E[(X_i^j - Z_i^j) | \mathfrak{A}_{n_i}] (\omega) = \begin{cases} \circ(\varphi(n)^r M(n)), & \text{or} \\ \circ(\varphi(n)^r V(n)), & \text{where } \mathfrak{A}_{n_i} = \mathfrak{A}(X_1, \ldots, X_{i-1}, Z_{i+1}, \ldots, Z_n). \end{cases}$
 Then for any $f \in C^r$ *there holds for each fixed* $u \in \mathbb{R}$ *the estimat*
 $|E[f(T_n$

$$
\sum_{i=1}^{n} E[(X_i^j - Z_i^j) | \mathfrak{A}_{n_i}] (\omega) = \begin{cases} \partial(\varphi(n) \cap \mathcal{U}(n)) & a.s. & (3.31) \\ \partial(\varphi(n) \cap V(n)) & a.s. & (3.31) \end{cases}
$$
\nwhere $\mathfrak{A}_{n_i} = \mathfrak{A}(X_1, ..., X_{i-1}, Z_{i+1}, ..., Z_n)$.
\nThen for any $f \in C^r$ there holds for each fixed $u \in \mathbb{R}$ the estimate\n
$$
|E[f(T_n + u)] - E[f(Z + u)]| = \begin{cases} \partial(\varphi(n) \cap M(n)) & (n \to \infty) \\ \partial(\varphi(n) \cap V(n)) & (n \to \infty) \end{cases}
$$
\n(3.4i)
\nProof: Regarding the first case, setting $R_{n_i} = \sum_{i=1}^{i-1} X_k + \sum_{i=1}^{n} Z_k$, $1 \leq i \leq n$,
\n $n \in \mathbb{N}$, the telescoping argument and a double application of Taylor's formula for\n $f \in C^r$ yields for each $u \in \mathbb{R}$ the identity and estimate\n
$$
|E[f(T_n + u)] - E[f(Z + u)]|
$$
\n
$$
= \begin{vmatrix} \sum_{i=1}^{n} \rho(n) X_i + u \end{vmatrix} - f \left(\sum_{i=1}^{n} \varphi(n) Z_i + u \right) \end{vmatrix}
$$
\n
$$
= \begin{vmatrix} \sum_{i=1}^{n} \sum_{j=1}^{r} \frac{\varphi(n)^j}{j!} E[f(\varphi(n) R_{n_i} + u) (X_i^j - Z_i^j)] \end{vmatrix}
$$
\n
$$
\leq \begin{vmatrix} \sum_{i=1}^{n} \sum_{j=1}^{r} \frac{\varphi(n)^j}{j!} E[f(\varphi(n) R_{n_i} + u) (X_i^j - Z_i^j)] \end{vmatrix}
$$
\n
$$
+ \sum_{i=1}^{n} \frac{1}{(r-1)!} \begin{vmatrix} 1 & (1-t)^{r-1} E[f(\varphi(n) R_{n_i} + u + t\varphi(n) X_i) (\varphi(n) X_i)^r - f(\varphi(n) R_{n_i} + u) (\varphi(n) X_i
$$

 $\mathcal{S}(\mathbf{x})$

General Limit Theorems with *o* Rates 295
\n
$$
+ \sum_{i=1}^{n} \frac{1}{(r-1)!} \left| \int_{0}^{1} (1-t)^{r-1} E[f^{(r)}(\varphi(n) R_{n_i} + u + t\varphi(n) Z_i) \right|
$$
\n
$$
\times (\varphi(n) Z_i)^r - f^{(r)}(\varphi(n) R_{n_i} + u) (\varphi(n) Z_i)^r \right] dt.
$$
\n(3.5)
\nTo estimate the first term, namely
\n
$$
\sum_{i=1}^{n} \sum_{j=1}^{r} \frac{\varphi(n)^j}{j!} E[f^{(j)}(\varphi(n) R_{n_i} + u) (X_i^j - Z_i^j)],
$$
\n(3.6)
\none has on account of (2.4)–(2.6), noting the \mathfrak{A}_{n_i} -measurableity of the R_{n_i} , and that
\n $f^{(j)}(x)| \leq N_f^{(j)}, x \in \mathbb{R}$, for $1 \leq j \leq r$ since $f \in C^r$, together with condition (3.3), for
\neach $u \in \mathbb{R}$;
\n
$$
\sum_{i=1}^{n} E[f^{(j)}(\varphi(n) R_{n_i} + u) (X_i^j - Z_i^j)]
$$
\n
$$
= \sum_{i=1}^{n} E[f^{(j)}(\varphi(n) R_{n_i} + u) E[(X_i^j - Z_i^j) | \mathfrak{A}_{n_i}]]
$$

$$
\times (\varphi(n) Z_i)^r - f^{(r)}(\varphi(n) R_{ni} + u) (\varphi(n) Z_i)^r] dt.
$$
\n(3.5)
\nate the first term, namely\n
$$
\sum_{i=1}^{n} \sum_{j=1}^{r} \frac{\varphi(n)^j}{j!} E[f^{(i)}(\varphi(n) R_{ni} + u) (X_i^j - Z_i^j)],
$$
\n(3.6)
\nthe second of (3.4), (3.6) noting the W , $[\text{measurable}]$ the R , and that

one has on account of $(2.4) - (2.6)$, noting the \mathfrak{A}_n -measurability of the R_{ni} , and that $|f^{(i)}(x)| \leq N_f^{(i)}$, $x \in \mathbb{R}$, for $1 \leq j \leq r$ since $f \in C^r$, together with condition (3.3), for

To estimate the first term, namely
\n
$$
\sum_{i=1}^{n} \sum_{j=1}^{r} \frac{\varphi(n)^j}{j!} \mathbb{E}[f^{(i)}(\varphi(n) R_{ni} + u) (X_i^i - Z_i^j)],
$$
\none has on account of $(2.4) - (2.6)$, noting the \mathfrak{A}_{n_i} -measurable to the R_{n_i} , an $|f^{(i)}(x)| \leq N_f^{(i)}$, $x \in \mathbb{R}$, for $1 \leq j \leq r$ since $f \in C^r$, together with condition (3 each $u \in \mathbb{R}$),
\n
$$
\sum_{i=1}^{n} \mathbb{E}[f^{(i)}(\varphi(n) R_{ni} + u) (X_i^j - Z_i^j)]
$$
\n
$$
= \sum_{i=1}^{n} \mathbb{E}[f^{(i)}(\varphi(n) R_{ni} + u) \mathbb{E}[(X_i^j - Z_i^j) | \mathfrak{A}_{ni}]]
$$
\n
$$
\leq N_f^{(i)} \sum_{i=1}^{n} |\mathbb{E}[\mathbb{E}[(X_i^i - Z_i^j) | \mathfrak{A}_{ni}] (\omega)]|
$$
\n
$$
= N_f^{(i)} \circ (\varphi(n)^r M(n)) \text{ a.s.}
$$
\nMultiplying this estimate by $\varphi(n)^i/j!$ and summing over $1 \leq j \leq r$ yields the

 r yields that (3.6) is of the order $c(N_f(r, \varphi) \varphi^r(n) M(n))$ with $N_f(r, \varphi) := \sum_{i=1}^r \varphi(n)^i N_f^{(i)}/j!$. Concerning the second and third terms of (3.5), let us show that they are of order $o(1)$ after being multiplied by $\varphi(n)^{-r} M(n)^{-1}$. Indeed, since $f \in C^r$, $|f^{(r)}(\varphi(n) R_{n_i} + u + t\varphi(n) X_i - f^{(r)}(\varphi(n) R_{n_i} + u)| < \varepsilon$ for $|X_i| < \delta/\varphi(n)$ s multiplied by $\varphi(n)^{-r} M(n)^{-1}$. Indeed, since $f \in C^r$, $\left|f^{(r)}(\varphi(n) R_{ni} + u + t\varphi(n) X_i\right|$ $\leq N_f^{(i)} \sum_{i=1}^{n}$
 $= N_f^{(i)} \circ \left(\sum_{i=1}^{n} N_i^{(i)} \right)$
 Multiplying this e
 is of the order of
 indiplied by $\varphi(\sum_{i=1}^{n} f^{(i)}(\varphi(n) R_{ni} + \sum_{i=1}^{n} D E(t))$ $||u|| < \varepsilon$ for $|X_i| < \delta/\varphi(n)$ since $0 < t < 1$. Hence d and thin
d by $\varphi(n)$
e) $R_{ni} + u$
DE(t) := $\begin{aligned} &\mathbb{P}_f(r,\,\varphi)\;\varphi^r(n)\;M(r)\ &\text{d terms of (3.5)},\ &\mathbb{P}^r\;M(n)^{-1}.\quad\text{Ind}\ &\mathbb{P}_f\left[\left\{f^{(r)}\big(\varphi(n)\;R_{n}\big)\right\}\right] &\leq \varepsilon\;\text{for}\;|X_i|\ &\leq \mathbb{E}\big[\left|\left\{f^{(r)}\big(\varphi(n)\;R_{n}\big)\right\}\right] &\times\;\{\mathbb{I}_{|x|<\delta/\varphi(n)}\,+\,1\big\} \end{aligned}$ $\begin{array}{c} .6) \ \hline \text{ng} \ \text{ng} \ \text{g} \ \text{g} \ \text{g} \ \text{g} \ \text{g} \end{array}$ $c(N_f(r, \varphi) \varphi^r(n) M(n))$ with $N_f(r, \varphi) := \sum_{j=1}^r \varphi(n)^j$

third terms of (3.5), let us show that they are of $\varphi(n)^{-r} M(n)^{-1}$. Indeed, since $f \in C^r$, $|f^{(r)}(\varphi(n) + u)| < \varepsilon$ for $|X_i| < \delta/\varphi(n)$ since $0 < t < 1$. Her
 $:= \mathbb{E}[|\{f$ $\begin{aligned}\n&= N_f^{(1)} \circ (\varphi(n) \cdot M(n)) \quad \text{a.s.} \\
&\text{arg this estimate by } \varphi(n) / j! \text{ and summing over } 1 \leq j \leq r \text{ yield} \\
&\text{order } \circ (N_f(r, \varphi) \cdot \varphi^r(n) M(n)) \text{ with } N_f(r, \varphi) := \sum_{j=1}^r \varphi(n)^j N_f^{(j)} / j! \\
&\text{d and third terms of (3.5), let us show that they are of order } \circ (1) \text{ by } \varphi(n)^{-r} M(n)^{-1}. \text{ Indeed, since } f \in C^r, |f^{(r)}(\varphi(n) R_{n$

$$
DE(t) := E\left[\left\{f^{(r)}(\varphi(n) R_{n_i} + u + t\varphi(n) X_i) - f^{(r)}(\varphi(n) R_{n_i} + u)\right\} X_i\right]
$$

$$
\times \left\{1_{|x| < \delta/\varphi(n)} + 1_{|x| \ge \delta/\varphi(n)}\right\} \le \varepsilon \zeta_{ri} + 2 \left|f\right|_{C^r} \int_{x \ge \delta/\varphi(n)} |x|^r dF_{X_i}(x).
$$

This gives

the second and third terms of (3.5), let us show that the zero of order
$$
o(1)
$$
 after being multiplied by $\varphi(n)^{-r} M(n)^{-1}$. Indeed, since $f \in C^r$, $[f^{(r)}(\varphi(n) R_{n_1} + u + t\varphi(n) X_1) - f^{(r)}(\varphi(n) R_{n_1} + u + t\varphi(n) X_1) - f^{(r)}(\varphi(n) R_{n_1} + u + t\varphi(n) X_1)$ $= F\left[\left|\left\{f^{(r)}(\varphi(n) R_{n_1} + u + t\varphi(n) X_1) - f^{(r)}(\varphi(n) R_{n_1} + u)\right\} X_1\right\}\right]$ $\times \{1\}_{|x| < \delta/\varphi(n)} + 1_{|x| \ge \delta/\varphi(n)}\right] \le \varepsilon_{r_1}^r + 2|f|_{C^r} \int |x|^r dF_{X_1}(x)$
\nThis gives\n
$$
\frac{1}{M(n)} \sum_{i=1}^n \frac{1}{(r-1)!} \int_0^1 (1-t)^{r-1} DE(t) dt
$$
\n
$$
\leq \frac{1}{M(n)} \sum_{i=1}^n \frac{1}{(r-1)!} \left\{ \int_0^1 (1-t)^{r-1} \left\{ \varepsilon_{r_1}^r + 2|f|_{C^r} \int_{|x|^r} |x|^r dF_{X_1}(x) \right\} dt \right\}
$$
\n
$$
= \frac{\varepsilon}{r!} + 2|f|_{C^r} \frac{1}{M(n)} \sum_{i=1}^n \left\{ \int_{|x| \ge \delta/\varphi(n)} |x|^r dF_{X_i}(x) \right\}.
$$
\nOn account of the estimate of (3.6), as well as of (3.7) and its counterpart for Z_i , this yields a constant Z_i in Z_i .

On account of the estimate of (3.6), as well as of (3.7) and its counterpart for Z_i , this yields for each $u \in \mathbb{R}$ in view of the generalized pseudo-Lindeberg condition (2.11i) for the random variables X_i and Z_i ,

296 P. L. BUTZER and H. KIRSCHFINR
\nfor the random variables
$$
X_i
$$
 and Z_i ,
\n
$$
\frac{1}{\varphi(n)^r M(n)} \{E[(T_n + u)] - E[(Z + u)]\} \leq c_f(1) + \frac{\varepsilon}{r!} + \frac{2||\rho_r}{r!}
$$
\n
$$
\times \frac{1}{M(n)} \int_{|z| \geq \delta/\varphi(n)} |x|^r d(F_X(x) - F_{Z_i}(x)) = c_f(1).
$$
\nThe proof for the second case follows similarly as above by replacing conditions
\n(3.1 i), (3.2 i), (3.3 i) by (3.1 ii), (3.2 ii), (3.3 ii), and $M(n)$ by $V(n)$, respectively
\nRemark 1: An analogous proof for the same result could also be carried out by using the
\nrandom variables
\n
$$
R_{ni}^* = \sum_{k=1}^{n} Z_i + \sum_{k=i+1}^{n} X_i, \quad \mathfrak{N}_{ni}^* = \mathfrak{A}(Z_1, ..., Z_{i-1}, X_{i-1}, ..., X_n)
$$
\ninstead of the R_{ni} , \mathfrak{N}_{ni} and, in place of condition (3.3 i),
\n
$$
\sum_{i=1}^{n} E[X_i - Z_i] \mathfrak{N}_{ni}^* = o(\varphi(n)^r M(n))
$$
 a.s. $(n \to \infty)$.
\n(3.3 i*)
\nA $V(n)$ -version follows analogously. Both results are due to the \mathfrak{N}_{ni}^* measuring infinity of the R_{ni}^* .
\nCorollary 1: If the random variables, X_i as well as the decomposition components

The proof for the second case follows similarly as above by replacing conditions (3.1 i),(3.2i), (3.3i) by (3.1 ii), (3.2ii), (3.3ii), and *M(n)* by V(n), respectively **^U**

Remark 1: An analogous proof for the same result could also be carried out by using the random variables

.21), (3.31) by (3.1ii), (3.2ii), (3.3ii), and
$$
M(n)
$$
 by $V(n)$, respectively, the result is given by R_{ni}^* and $R_{ni}^* = \sum_{k=1}^{i-1} Z_i + \sum_{k=i+1}^{n} X_i$, $\mathfrak{A}_{ni}^* = \mathfrak{A}(Z_1, \ldots, Z_{i-1}, X_{i-1}, \ldots, X_n)$ is the R_{ni} . The result is R_{ni}^* and R_{ni} is the R_{ni} .

$$
\sum_{i=1}^{n} \mathbb{E}[X_i^j - Z_i^j \mid \mathfrak{A}_{ni}^*] = o(\varphi(n)^r \, M(n)) \text{ a.s.} \qquad (n \to \infty).
$$
 (3.31^{*})

A $V(n)$ version follows analogously. Both results are due to the \mathfrak{A}_{ni}^* measurability of the R_{ni}^* .

Corolla'ry 1: If the random variables, X_i as well as the decomposition components Z_i , $i \in \mathbb{N}$, are in addition identically distributed and φ is such that $\varphi(n) = o(1)$, $n \to \infty$, *then under assumption* $(3.3i)$ *one has for* $f \in C^r$ *and each* $u \in \mathbb{R}$ $\sum_{i=1}^{\infty} E[X_i - Z_i] \mathbb{I}_{n_i}^* = o(\varphi(n)^r M(n))$ a.s. $(n \to \infty)$.
 $\Delta V(n)$ -version follows analogously. Both results are due to the $\mathbb{I}_{n_i}^*$
 $\sum_{i}^{\infty} V_i \in \mathbb{N}$, are in addition identically distributed and φ is suc \cdot measurabili $decompositi \ \textit{that} \ \varphi(n) = \mathbb{R} \ \left(\frac{\varphi(n)^r}{n}\right)$

$$
|E[f(T_n+u)]-E[f(Z+u)]|=c_f(\varphi(n)^r n(\zeta_{r1}+\xi_{r1}))=c_f\left(\frac{\varphi(n)^r}{n}\right)\quad (n\to\infty).
$$

The result follows from Theorem-1 if the pseudo-Lindeberg condition for the X_i and Z_i can be shown to follow for $\varphi(n) = \varphi(1)$. But for identically distributed random variables this condition reduces to $\int |x|^r dF_{X_i}(x) \to 0$ for each $\delta > 0$, which is

satisfied automatically since $\delta/\varphi(n) \to \infty$, $n \to \infty$

Remark 2: 1. The term $|E|/(T_n + u) - E[(Z) + u]|$ in (3.4) tends to zero for $n \to \infty$ if $\varphi(n)^{r} M(n)$ (or $\varphi(n)^{r} V(n)$) is bounded. In the identically distributed case this is fulfilled for $\varphi(n) = n^{-1/r}$. 2. According to our knowledge, no results directly comparable to those of Theo**rem I seem to** be **contained in the literature. However,** results are known for more particular **• sequences** 'bf random variables for which the dependency structure is fixed (which are also subsumed under Theorem 1). In the case of martingale difference sequences let us refer to the papers of **BASU [2]. KATO [21], PRAKASA RAO [31], RYCHLIK [32], SCOTT** [34] and **STROBEL [35]. 3.** The constant $2 \frac{|f|}{c}r!$ in the estimate (3.7) has in the case of independent random variables From 1 seem to be contained in the literature. However
sequences of random variables for which the depends
ubsumed under Theorem 1). In the case of martingale
papers of Basu [2], KaTo [21], PRAKASA RAO [31], RYC
3. The From 1 seem to be contained in the literature. However, results are known for ones on 1 reo-
rem 1 seem to be contained in the literature. However, results are known for once particular
sequences of random variables for w

in regard to O-estimates been improved by the factor $\sum c_j ||f^{(i)}||_C/j! + 2L_f/r!$, where L_f is the $\tilde{j}=0$

Since most of the applicable structures are actually dependencies upon the past, the following differentiation is meaningful, especially for Markov processes. Part c) was formulated in the case of Banach-valued random variables in $[12]$ in connection with Donsker's weak invariance principle.

Definition 3: Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of real random variables on some probability space $(\Omega, \mathfrak{A}, P)$. It is said to be

b) dependent from above if, for each $1 \leq i \leq n, n \in \mathbb{N}$,

 $P(X_i \in B \mid X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) = P(X_i \in B \mid X_{i+1}, \ldots, X_n)$ a.s. $(B \in \mathfrak{B})$. (3.9)

General Limit Theorems with *o*-Rates
\nc) *expectationally dependent* from below or from above if, for each
$$
1 \le i \le n, n \in \mathbb{N}
$$
,
\n
$$
E[X_i | X_1, ..., X_{i-1}, X_{i+1}, ..., X_n] = \begin{cases} E[X_i | X_1, ..., X_{i-1}] & a.s. \\ E[X_i | X_{i+1}, ..., X_n] & a.s. \end{cases}
$$
\n(3.10)
\nExamples for random variables that are expectationally dependent from below
\ne martingale difference sequences (see [12]) and Markov processes (see below);

Examples for random variables that are expectationally dependent from below, are martingale difference sequences (see [12]) and Markov processes (see below); dependent from above are inverse martingale difference sequences and inverse Markov chains. gale difference sequences (see [12]) and Marko

rom above are inverse martingale difference seq

2: *Examination of condition* (3.3) in the light of
 $\lfloor \frac{2}{3} \rfloor$ *if* $\lfloor \frac{2}{3} \rfloor$ *for all* $B \in \mathfrak{B}$ *implies*

Lemma 2: *Examination of condition (3.3) in the light of Definition* 3 *leads to the following statements:*

a) If X is any random variable, \mathfrak{E} , \mathfrak{F} are two sub- σ -algebras of \mathfrak{A} , then $P(X \in B \mid \mathfrak{E})$ $P(X \in B \mid \mathfrak{F})$ for all $B \in \mathfrak{B}$ implies $E[X \mid \mathfrak{F}]=E[X \mid \mathfrak{F}]$ a.s.

b) If $(X_i)_{i \in \mathbb{N}}$ is a sequence of random variables that is dependent from below, then it is *expect ationally dependent from below.*

c) If $(X_i)_{i \in \mathbb{N}}$ *is dependent from above, then it is expectationally so.*

 $\label{eq:2.1} \frac{1}{\sqrt{2\pi}}\int_{0}^{\pi} \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{2\pi}}\right)^{2}d\mu$

Proof: The hypothesis gives in particular for $B = (-\infty, t]$ that $P(X \leq t | \mathfrak{E})$ *zinclinally dependent from below.*

c) If $(X_i)_{i \in \mathbb{N}}$ is dependent from above, then it is expectationally so.

Proof: The hypothesis gives in particular for $B = (-\infty, t]$ that $P(X \le t |$
 $= P(X \le t | \mathfrak{F})$, so that $F_X(t | \$ **Proof:** The hypothesis gives in particular for $B = (-\infty, t]$ that $P(X \le t | \mathfrak{F})$
 $= P(X \le t | \mathfrak{F})$, so that $F_X(t | \mathfrak{F}) = F_X(t | \mathfrak{F})$. This yields $E[X | \mathfrak{F}] = \int x dF_X(x | \mathfrak{F})$
 $= E[X | \mathfrak{F}]$, which completes the proof of Proof: The hypothesis gives in particular for $B = (-\infty)$
 $= P(X \le t | \mathfrak{F})$, so that $F_X(t | \mathfrak{G}) = F_X(t | \mathfrak{F})$. This yields E(
 $= \mathbb{E}[X | \mathfrak{F}]$, which completes the proof of part a). Parts b

from a)

Lemma 3: If the se

 $=$ E[X | \mathfrak{F}], which completes the proof of part a). Parts b) and c) follow directly from a) \blacksquare $f(x) = F[X | \mathfrak{F}],$ which contribute the set of $f(x) = f(x)$ is the set of $f(x) \leq f \leq r, n \to \infty$, $= E$
from
 L_0
for 1

Lemma 3: *If the sequence* $(X_i)_{i \in \mathbb{N}}$ *is expectationally dependent from below such that,* for $1 \leq j \leq r, n \to \infty$,

\n- statements:
\n- is any random variable,
$$
\mathfrak{F}
$$
, \mathfrak{F} are two sub- σ -algebras of \mathfrak{A} , then $P(X \in B \mid \mathfrak{F})$ for all $B \in \mathfrak{B}$ implies $E[X \mid \mathfrak{F}] = E[X \mid \mathfrak{F}]$ a.s.
\n- K_i is a sequence of random variables that is dependent from below, then it is nullly dependent from below.
\n- K_i is a dependent from above, then it is expectationally so.
\n- The hypothesis gives in particular for $B = (-\infty, t]$ that $P(X \leq t \mid \mathfrak{F})$ for all $F(X \leq t \mid \mathfrak{F})$.
\n- The hypothesis gives in particular for $B = (-\infty, t]$ that $P(X \leq t \mid \mathfrak{F})$.
\n- The hypothesis gives in particular for $B = (-\infty, t]$ that $P(X \leq t \mid \mathfrak{F})$.
\n- The hypothesis gives an particular for $B = (-\infty, t]$ that $P(X \leq t \mid \mathfrak{F})$.
\n- The hypothesis gives the proof of part a). Parts b) and c) follow directly
\n- S .
\n- The function S is a set of S and S is a set of S .
\n- S .
\n- F : $E[(X_i - Z_i^i) \mid \mathfrak{F}_{i-1}] = \begin{cases} \circ(\varphi(n)^r M(n)), & \text{if } a.s. \\ \circ(\varphi(n)^r V(n)), & \text{if } a.s. \end{cases}$.
\n- Theorem 1, S is a set of S and S is a set of S and S is a set of S and S is a set of $$

then condition. (3.3 i) or (3.3ii) *is satisfied.*

Proof: By Lemma 2b, and the independence of the Z_i from \mathfrak{A}_{ni} , one has with (2.4), (2.8) and (2.9),

$$
\sum_{i=1}^{n} \mathbf{E}[(X_i^j - Z_i^j) | \mathfrak{E}_{i-1}] =\begin{cases} \sigma(\varphi(n) \text{ for } n(n)) & \text{if } n \leq n \leq n \end{cases}
$$
\n
$$
a = \mathfrak{A}(X_1, \ldots, X_{i-1}), \text{ and } Z_i \text{ are the decomposition component}
$$
\n
$$
a = \mathfrak{A}(X_1, \ldots, X_{i-1}), \text{ and } Z_i \text{ are the decomposition component}
$$
\n
$$
a = \mathfrak{A}(X_i, \ldots, X_{i-1}), \text{ and } Z_i \text{ are the decomposition component}
$$
\n
$$
a = \mathfrak{A}(X_i)
$$
\n
$$
a = \math
$$

Remark 3: 1. On account of Remark 1 one-can also formulate both versions of Lemma 3 for sequences of random variables that are expectationally dependent from above, employing condition $(3.3)^*$ in place of (3.3) . 2. The conditions "dependent from below" or "dependent from above" are rather severe restrictions upon dependence. For a sequence $(X_i)_{i\in\mathbb{N}}$ that is dependent from below it means that for each X_n the past is not influenced by the future. In this sense dependence from below may be more restrictive than (general) Markov dependence. In the next section, where only a special Markov-property is allowed, Markov-dependence is an example of dependence from below.

'4.' General limit theorems for Markov processes

A Markov process with discrete time parameter is a sequence of random variables $(X_i)_{i \in \mathbb{N}}$ on some probability space $(\Omega, \mathfrak{A}, P)$, where each random variable X_i is only) restricted by the Markov property *P(X_i* E) *P(X_*

$$
P(X_i \in B \mid X_1, \ldots, X_{i-1}) = P(X_i \in B \mid X_{i-1}) \qquad (B \in \mathfrak{B}; i \geq 2).
$$
 (4.

It is obvious that such a Markov process is dependent from below according to Definition 1. A Markov process with (possibly) dependent increments is a process $(X_i)_{i \in \mathbb{N}}$ for which the sequence of increments *Y1* with 298 P. L. BUTZER and

It is obvious that such a

nition 1. A Markov proce

for which the sequence c
 $Y_i := X_i - X_i$

is (possibly) dependent.

sarily dependent, they are

Lemma 4: If $(X_i)_{i \in \mathbb{N}}$

is expectationally depend **P. L. BUTZER and H. KIRSCHFINK**

vious that such a Markov process is dependent from below according to Defi-

A Markov process with (possibly) dependent increments is a process $(X_i)_{i \in \mathbb{N}}$

is the sequence of increme

$$
Y_i := X_i - X_{i-1}, \qquad X_0 = 0 \qquad \text{a.s.} \tag{4.2}
$$

is (possibly) dependent. Whereas for a Markov process the increments are not necessarily dependent, they are nevertheless expectationally so as is seen by the following

Lemma⁴: If $(X_i)_{i \in \mathbb{N}}$ *is a Markov process, then/the sequence of increments* $(Y_i)_{i \in \mathbb{N}}$ *is expectationally dependent from below.*

In fact, (X_i) being a Markov process,.

 $E[Y_i | Y_1, ..., Y_{i-1}, Y_{i+1}, \ldots, Y_n] = E[X_i | Y_1, ..., Y_{i-1}, Y_{i+1}, ..., Y_n]$ **•** $\mathbf{F}[X_{i-1} | Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n] = \mathbf{E}[X_i | Y_{i-1}] - \mathbf{E}[X_{i-1} | Y_{i-2}, Y_{i-1}]$ $=$ $E[(X_i - X_{i-1}) | Y_{i-2}, Y_{i-1}] = E[Y_i | Y_{i-2}, Y_{i-1}].$

In particular, for a Markov process with independent increments the Y_i are independent (see below).

The following limit theorems with rates will be formulated for three instances, namely for sums of Markovian dependent random variables, for Markov processes with dependent as well as with independent increments.

4.1 General limit theorem and central limit theorem

At first to the general result.

Theorem 2: Let $(X_i)_{i \in \mathbb{N}}$ be Markov-dependent and Z be a φ -decomposable random *variable with* $E[Z] = 0$ *such that* (3.1i) *or* (3.1ii) *hold for* $i \in \mathbb{N}$ *and* $r \geq 2$. If the se*quences* $(X_i)_{i \in \mathbb{N}}$ and $(Z_i)_{i \in \mathbb{N}}$ together satisfy the generalized pseudo-Lindeberg condition $(2.11 i)$. *or* $(2.11 ii)$ *of order r, as well as condition* $(3.12 i)$ *or* $(3.12 ii)$, *then any* $f \in C^{\tau}$ *again implies the estimate* (3.4i) *or* (3.4ii). Theorem 2: Let $(X_i)_{i \in \mathbb{N}}$ be Markov-dependent and Z be a variable with $E[Z] = 0$ such that $(3.1i)$ or $(3.1ii)$ hold for $i \in$ quences $(X_i)_{i \in \mathbb{N}}$ and $(Z_i)_{i \in \mathbb{N}}$ together satisfy the generalized ps $(2.11i)$ $(X_i)_{i \in \mathbb{N}}$ be *Markov-dependent and*
 $(X_i)_{i \in \mathbb{N}}$ be *Markov-dependent and*
 $(Z_i)_{i \in \mathbb{N}}$ together satisfy the general order r, as well as condition (3.
 imate (3.4i) or (3.4ii).
 a immediately from Theore From 2: Let $(X_i)_{i \in \mathbb{N}}$ be Markov-dependent and Z be a φ -decomposable random
with $E[Z] = 0$ such that $(3.1i)$ or $(3.1ii)$ hold for $i \in \mathbb{N}$ and $r \geq 2$. If the se-
 $(X_i)_{i \in \mathbb{N}}$ and $(Z_i)_{i \in \mathbb{N}}$ together

The proof follows immediately from Theorem 1 and Lemma 3 I

Now to a handy, version of the central limit theorem for Markov processes. Here we will apply Theorem 2 to a concrete limiting random variable *Z,* namely to *X*.*

Theorem 3: Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of random variables which form a Markov *process. such that* $\mathbb{E}[|X_i|^r] < \infty$, $i \in \mathbb{N}$, for any $r \geq 2$ as well as satisfy a generalized *Lindeberg condition* (2.10) *of order r. Assume further that* $(a_i)_{i \in \mathbb{N}}$ *is any sequence of positive reals wich satisfies a Feller-type condition* boot follows immediately from Theorem 1

a handy, version of the central limit the

ply Theorem 2 to a concrete limiting rand
 $\text{em } 3$: Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of rand

ch that $\text{E}[|X_i|^r] < \infty$, $i \in \mathbb{N}$, *fixed u in mpines the estimate* (3.41) or (3.411).

The proof follows immediately from Theorem 1 and Lemma 3 **I**

Now to a handy, version of the central limit theorem for Markov processes. I

we will apply Theorem 2 to a

$$
\lim_{n \to \infty} \max_{1 \le i \le n} \frac{a_i}{A_n} = 0 \quad with \quad A_n = \left(\sum_{i=1}^n a_i^2\right)^{1/2}.
$$
 (4.3)

Inndeberg condition (2.10) *of order r. Assume further that* $(a_i)_{i \in \mathbb{N}}$ *is any sequence of* positive reals wich satisfies a Feller-type condition
 $\lim_{n \to \infty} \max_{1 \le i \le n} \frac{a_i}{A_n} = 0$ with $A_n = \left(\sum_{i=1}^n a_i^2\right)^{1$

$$
\lim_{n \to \infty} \max_{1 \le i \le n} \frac{a_i}{A_n} = 0 \quad \text{with} \quad A_n = \left(\sum_{i=1}^n a_i^2\right)^{1/2}.
$$
 (4.3)
If $E[|Z_i|^r] < \infty$ as well as (3.12i) holds with $P_{Z_i} = P_{a_i}x$, then $f \in C^r$ yields for each fixed $u \in \mathbb{R}$

$$
\int_{\mathbb{B}} \left| E \left[f \left(A_n^{-1} \sum_{i=1}^n X_i + u \right) \right] - E[f(X^* + u)] \right| = o_f(A_n^{-r}M(n)) \quad (n \to \infty) \text{ (4.4)}
$$
Corcerning the proof, X^* is φ -decomposable for each $n \in \mathbb{N}$ into n independent normally distributed random variables Z_i with $P_{Z_i} = P_{a_i}x$, since

Concerning the proof, X^* is φ -decomposable for each $n \in \mathbb{N}$ into *n* independent.

 $P_{X^*} = P_{A_n^{-1} \sum_{i=1}^n Z_i}$ with $\varphi(n) = A_n^{-1}$

(see [14]). Further, (4.3) in connection with Lemma 1 yields the generalized pseudo-Lindeberg condition (2.11i). So Theorem 2 may be applied **I**

Remark 4: There exists a further version of Theorem 3 in case the finiteness of the r-th absolute moments of the X_i and Z_i is replaced by the finiteness of the r-th pseudo-moments of absolute moments of the X_i and Z_i is replaced by the finiteness of the r-th pseudo-moments of $(X_i - Z_i)$, and the generalized Lindeberg condition for X_i together with the Feller condition For a is a furthermorphism of the X_i and Z_i
 $(X_i - Z_i)$, and the generalized Linfor a_i is replaced by
 $\frac{1}{V'(n)} \sum_{i=1}^n \int |x|^r d(F)$ ere exist

of the 2

c genera

y

y

<u>n</u>
 $\sum_{r=1}^{n}$ **k** 4: There ex

and the gene

placed by
 $\frac{1}{V'(n)} \sum_{i=1}^{n}$
 $\frac{1}{|x| \ge \delta}$ **IVENTIFY CONTROVER EXECUTE:**

It is a further version of Theorem 3 in X_i and Z_i is replaced by the finiteness

ralized Lindeberg condition for X_i toge
 $\int_{\varphi(n)} |x|^r d(F_X(x) - F_{a_iX^*}(x)) = o_{\delta}(1)$
 $\varphi(n)$
 $\qquad''(n) := \sum_{i$ Remark 4: There exists a further version of Theorem 3 in case the finiteness of the r-th
absolute moments of the X_i and Z_i is replaced by the finiteness of the r-th pseudo-moments of
 $(X_i - Z_i)$, and the generalized Lind

\n
$$
\frac{1}{V'(n)} \sum_{i=1}^{n} \int_{|x|^2} |x|^r \, d(F_{X_i}(x) - F_{a_i X^*}(x)) = o_{\delta}(1)
$$
\n
$$
= o_{\delta}(1)
$$
\n\n (4.3*)

\n
$$
= o_{\delta}(1)
$$
\n\n (4.3*)

\n
$$
= o_{\delta}(1)
$$
\n
$$
= o_{\delta}(1)
$$
\n\n (4.3*)

\n
$$
= o_{\delta}(1)
$$

 $(3.12i)$ with $V(n)$ replaced by $V'(n)$, then assertion (4.4) reads. for each $\delta > 0$, where $V'(n) := \sum_{i=1}^{n} \mathbb{E}[|X_i - a_i X^*|^r]$. If moreover (3.12ii) takes th

(3.12i) with $V(n)$ replaced by $V'(n)$, then assertion (4.4) reads.
 $\mathbb{E}\left[f\left(A_n^{-1}\sum_{i=1}^n X_i + u\right)\right] - \mathbb{E}[f(X^* + u)]\right| = {}_s^o f(A_n^{-1$

$$
\left| \mathbb{E}\left[f\left(A_{n^{-1}}\sum_{i=1}^{n}X_{i}+u\right)\right]-\mathbb{E}[f(X^{*}+u)]\right| =\underset{\iota}{\circ}f(A_{n^{-1}}V'(n)) \qquad (n\to\infty).
$$

Corollary 2 : a) *If, in addition to the assumptions of Theorems 3, the* X_i *are iden-*

$$
\begin{aligned}\n\left| \mathbf{E} \left[f \left(A_n^{-1} \sum_{i=1}^n X_i + u \right) \right] - \mathbf{E} [f(X^* + u)] \right| &= \varepsilon_f (A_n^{-\tau} V'(n)) \qquad (n \to \infty). \\
\text{Corollary 2: a) } & If, \text{ in addition to the assumptions of Theorems 3, the } X_i \text{ and } \\ & \text{ally distributed, then for } f \in C^r \text{ and each } u \in \mathbb{R} \\
& \qquad \left| \mathbf{E} \left[f \left(A_n^{-1} \sum_{i=1}^n X_i + u \right) \right] - \mathbf{E} [f(X^* + u)] \right| &= \varepsilon_f (n A_n^{-\tau}) \qquad (n \to \infty). \\
\text{b) } & If \text{ further } a_i := \text{Var } X_i, \ 1 \leq i \leq n, \text{ then for } f \in C^r \text{ and } u \in \mathbb{R} \\
& \qquad \qquad \left| \mathbf{E} \left[f \left(\left(\sum_{i=1}^n \text{Var } X_i \right)^{-1/2} \sum_{i=1}^n X_i + u \right) \right] - \mathbf{E} [f(X^* + u)] \right| &= \varepsilon_f (n^{(2-\tau)/2}).\n\end{aligned}
$$

$$
\left| \mathbf{E} \left[f \left(\left(\sum_{i=1}^{n} \text{Var } X_i \right)^{-1/2} \sum_{i=1}^{n} X_i + u \right) \right] - \mathbf{E} [f(X^* + u)] \right| = c_f(n^{(2-r)/2}).
$$

\n*n* particular Var $X_i = 1, 1 \leq i < n$, then for $f \in C^2$ and $u \in \mathbb{R}$.
\n
$$
\left| \mathbf{E} \left[f \left(n^{-1/2} \sum_{i=1}^{n} X_i + u \right) \right] - \left| \mathbf{E} [f(X^* + u)] \right| = c_f(1) \qquad (n \to \infty).
$$

c) *If in particular* $Var X_i = 1, 1 \leq i < n$, then for $f \in C^2$ and $u \in \mathbb{R}$

$$
\mathbf{E}\left[f\left(n^{-1/2}\sum_{i=1}^n X_i + u\right)\right] - \mathbf{E}[f(X^* + u)]\right| = c_f(1) \qquad (n \to \infty).
$$

Apart from the papers [5, 24, 26] mentioned in the introduction, there exist many further ones dealing with the central limit theorem for Markovian dependent random variables. Generally homogeneous Markov chains are studied. Thus NAOAEV [28, 29] considered convergence in regard to the central limit theorem for chains, comparable to Corollary 2c) with 0-rates. Additional papers in this respect are BOLrHAtJSEN [6], GUDYNAS [20) and LirsiriTs [25].

4.2 Processes with dependent increments

This subsection is devoted to the behaviour of the process $\varphi(n)$ $X_n = \sum_{n=1}^{\infty} \varphi(n)$ Y for $n \to \infty$, the increments Y_i being assumed to be dependent.

Theorem 4: Let $(X_i)_{i \in \mathbb{N}}$ be a Markov process with dependent increments $(Y_i)_{i \in \mathbb{N}}$ as *in* (4.2) with $X_0 := 0$ *a.s.* Let Z be a φ -decomposable random variable (with respect to $(X_i)_{i \in \mathbb{N}}$ *such that* $E[Z] = 0$ *and* $v_i^* := E[|Y_i - Z_i|^r] < \infty$, $r \geq 2$. If the sequences $(Y_i)_{i \in \mathbb{N}}$ and $(Z_i)_{i \in \mathbb{N}}$ together satisfy a pseudo-Lindeberg condition (2.11 ii) of order r as *wellet all y londgeneous mallow chains at gence in regard to the central limit thee*
 Wellet as Additional papers in this respect a

4.2 Processes with dependent increme

This subsection is devoted to the l

for $n \to \in$ *l* $(X_i)_{i \in \mathbb{N}}$ *be a Markov process with dependent increments* $(Y_i)_{i \in \mathbb{N}}$ *as*
 $= 0$ *a.s.* Let Z *be a q-decomposable random variable (with respect to*
 $E[Z] = 0$ *and* $v_i^* := E[|Y_i - Z_i|^r] < \infty$, $r \geq 2$. If the *with* $\sum_{i=1}^{n} W_i(x_i) = 0$ *(Y_i*) *i*, *such that* $E[Z] = 0$ *and* $v_{ri}^* := E[|Y_i - Z_i|^r] < \infty$, $r \ge 2$. *If the sequel (Y_i)*_{ie} *x and* $(Z_i)_{i \in \mathbb{N}}$ *together satisfy a pseudo-Lindeberg condition* (2.11ii) *of ord* section is devoted to the behaviour of the process $\varphi(n) X_n = \sum_{i=1}^n$

b, the increments Y_i being assumed to be dependent.
 $\lim_{i\to\infty} 4: Let (X_i)_{i\in\mathbb{N}}$ be a Markov process with dependent increments (1)
 $\lim_{i\to\infty}$

$$
\sum_{i=1}^{n} \mathrm{E}[(Y_i^j - Z_i^j) \mid \mathfrak{E}_{i-1}^*] = o(\varphi(n)^r V^*(n)) \qquad (1 \leq j \leq r; n \to \infty) \ a.s. \qquad (4.5)
$$

$$
\big|E[f(\varphi(n) X_n + u)] - E[f(Z + u)]\big| = c_f(\varphi(n)^\intercal V^*(n)) \qquad (n \to \infty).
$$

The proof follows directly from Theorem 1 and Lemmata 2 and 3 **1**

4.3 Processes with independent increments

If in the results of Section 4.1 the increments $Y_i = X_i - X_{i-1}$ are assumed to be independent, which is often the, situation in applications (e.g. queueing theory and simulation), then our problem reduces to a study of the rate of convergence for sums of independent random variables. In this sense Markovian dependency is a generaliof independent random variables. In this sense markovian dependency is a general-
zation of independency, as $CHUNG$ [16, p. 10] remarks. Now it is quite surprising that the results obtained by BUTZER and HAHN [8] for sums of independent random variables (by means of Trotter operator-theoretic arguments) resemble those of example in the same of the rate of convergence for sums of independent random variables. In this sense Markovian dependency is a generalization of independency, as CHUNG [16, p. 10] remarks. Now it is quite surprising that Theorem 4 both in regard to the hypotheses and conclusion; there one managed to get along with a pseudo-moment condition instead of a conditional condition of this type and the convergence was uniform in $u \in \mathbb{R}$. In this situation one has the following result; it is comparable to [8, Theorem 12]. marks. Now it is quarks.

(8) for sums of independent of \mathcal{E}

(8) for sums of independent)

(ii) - on this situation

(*i*) the conditional post of the conditional post of i

($i \rightarrow \infty$),

Theorem 5: Let $(X_i)_{i \in \mathbb{N}}$ **be a Markov process with independent increments** Y_i **. Then** *the conclusions* **of** *Theorem 4 even remain valid if* the *conditional pseudomoment condition (4.5) is replaced by* ; it is comparable to [8, Theorem 12].

em 5: Let $(X_i)_{i \in \mathbb{N}}$ be a Markov process

sions of Theorem 4 even remain valid i

is replaced by
 $\sum_{i=1}^{n} \mathbf{E}[(Y_i - Z_i^i)] = c(\varphi(n)^r V^*(n))$

$$
\sum_{i=1}^{n} \mathrm{E}[(Y_{i}^{j}-Z_{i}^{j})]=c(\varphi(n)^{r} V^{*}(n)) \qquad (n \to \infty), \qquad (4.5^{*})
$$

the others remaining unchanged.

 $\sum_{\substack{\alpha\in\mathbb{Z}^d\\ \alpha\neq\beta\neq\beta}}\frac{1}{\sqrt{2\pi}}$

In fact, (4.5) reduces to (4.5*) in case the Y_i are independent in view of (2.8)

I. 5. Pseudo-moments and generalizations

If one looks at the proof of Theorem 1 more closely one sees that the main problem is the estimation of the first term (3.6) , thus to show that the double sum of (3.6) is in some way or other of the maximal order $c_j(\varphi(n)^r M(n))$ or $c_j(\varphi(n)^r V(n))$. This was achieved there by employing condition (3.3i) or (3.3'ii) together with properties of conditional expectations in connection with the admissible dependency and measurability properties. In order to simplify this proof, thus to estimate (3.6) with the

desired order, it obviously suffices to assume the difficult looking condition *(uEIR;1 j;5r;i-.-).oc), E[/(i)((n) R i + u)(X - Z1)JI* = *j((n)r M(n))*

the $V(n)$ -version being analogous. This is an implicit pseudo-moment condition in the sense that there is a "weighted" difference of the random variables X_i^j and Z_{ni}^j . Now many estimates of pseudo-moments are known in the literature. Let us first define some types of pseudo-moments and consider their properties (compare ZOLOTAREV **Example 10**
 Example 10 the $V(n)$ -version being analogous. This is an implicit pseudo-moment condition in the sense that there is a "weighted" difference of the random variables X_i and Z_n . No many estimates of pseudo-moments are known in the

Definition 4: Let *X*, *Z* be two random variables. The *pseudo-moment* $v(X, Z)$ is. *Definition 4: Let X, Z be two random variables. The pseudo-moment* $v(X, Z)$ *is* defined by $v(X, Z) = |E[X - Z]|$, and the *conditional pseudo-moment* $\tau(X, Z; \mathcal{G})$ by

Lemma $5: For $c \in \mathbb{R}$ there hold$ $\langle X, Z; \mathfrak{G} \rangle = |\mathfrak{E}[(X - Z)| \mathfrak{G}]|$ where \mathfrak{G} .

Lemma 5: For $c \in \mathbb{R}$ there hold
 $\langle Y \rangle$
 $\langle cX, c \cdot Z \rangle \leq |c| \nu(X, Z),$

-
- ii) $v(X, Z) = \mathbb{E}[\tau(X, Z; \mathbb{G})],$

 \mathfrak{m}
iii) $\tau(c \cdot X, c \cdot Z; \mathfrak{B}) \leq |c| \tau(A)$ iii) $\tau(c \cdot X, c \cdot Z; \mathcal{G}) \leq |c| \tau(X, Z; \mathcal{G}),$

General Li

iii) $\tau(c \cdot X, c \cdot Z; \mathfrak{G}) \leq |c| \tau(X, Z; \mathfrak{G}),$

iv) $\tau(c_1X, c_1Z; \mathfrak{G}) \leq \tau(c_2X, c_2Z; \mathfrak{G}),$
 Proof: We have i) [Fig. X, e.g. $Z| < |c|$]

General Limit Theorems with o Rates

iii) $\tau(c \cdot X, c \cdot Z; \mathcal{G}) \leq |c| \tau(X, Z; \mathcal{G}),$

iv) $\tau(c_1X, c_1Z; \mathcal{G}) \leq \tau(c_2X, c_2Z; \mathcal{G}),$
 $c_1 \leq c_2.$

Proof: We have i) $|E[c \cdot X - c \cdot Z]| \leq |c| E[|X - Z|]$ and ii) $|E[X - E|E[(X - Z) | \mathcal{G}]|$ $\begin{aligned} \text{Tr}[V] \ \tau(c_1 X, c_1 Z; \mathfrak{G}) &\leq \tau(c_2 X, c_2 Z; \mathfrak{G}), \qquad c_1 \leq c_2. \end{aligned}$
 $\begin{aligned} \text{Proof:} \ \text{We have i} \ \vert \ E[c \cdot X - c \cdot Z] \vert \leq \vert c \vert \ E[\vert X - Z \vert] \ \text{and ii} \ \vert \ E[X - Z] \vert \\ &= \vert E[E[(X - Z) \mid \mathfrak{G}]\vert] = E[\tau(X, Z; \mathfrak{G})]; \ \text{iii} \ \text{follows as in i}; \ \text{iv} \ \text{follows$ $\left| \mathrm{E}[(c_1 X - c_1 Z) \mid \mathfrak{G}] \right| = |c_1 \mathrm{E}[(X - Z) \mid \mathfrak{G}]| \leq |c_2 \mathrm{E}[(X - Z) \mid \mathfrak{G}]| = \tau(c_2 X, c_2 Z; \mathfrak{G})$ iii) $\tau(c \cdot X, c \cdot Z; \mathcal{G}) \leq |c| \tau(X, Z; \mathcal{G}),$

iv) $\tau(c_1 X, c_1 Z; \mathcal{G}) \leq \tau(c_2 X, c_2 Z; \mathcal{G}),$

iv) $\tau(c_1 X, c_1 Z; \mathcal{G}) \leq \tau(c_2 X, c_2 Z; \mathcal{G}),$

Proof: We have i) $|E[c \cdot X - c \cdot Z]| \leq |c| E[|X -$
 $= |E[E[(X - Z) | \mathcal{G}]]| = E[\tau(X, Z; \mathcal{G})];$ ii $E[\tau(X, Z; \mathcal{G})];$ iii) $f = |c_1 E[(X - Z) | \mathcal{G}]| \leq |c_2$
condition (5.1) reads (each
 $R_{ni} + u) X_i^j f^{(i)}(\varphi(n) R_{ni})$ iv) $i(t_1A, t_1B, \omega) \le t(t_2A, t_2B, \omega)$; $t_1 \le t_2$.

Proof: We have i) $|E[c \cdot X - c \cdot Z]| \le |c| E[|X - Z|]$ and ii)
 $|E[E[(X - Z) | \mathcal{G}]|| = E[\tau(X, Z; \mathcal{G}])$; iii) follows as in i); iv)
 $[(c_1X - c_1Z) | \mathcal{G}]| = |c_1E[(X - Z)| \mathcal{G}]| \le |c_2E[(X - Z)| \math$

$$
|E[(c_1X - c_1Z) \mid \emptyset]| = |c_1E[(X - Z) \mid \emptyset]| \leq |c_2E[(X - Z) \mid \emptyset]| = \tau(c_2X, c_2Z; \emptyset)
$$
\nIn this terminology condition (5.1) reads (each $u \in \mathbb{R}$)\n
$$
\sum_{i=1}^{n} \nu(f^{(i)}(\varphi(n) R_{ni} + u) X_i^i, f^{(i)}(\varphi(n) R_{ni} + n) Z_n^i) = c_f(\varphi(n)^r M(n))
$$
\nLet us now give some conditions which are sufficient for condition (5.2) to hold, and so suffice for the proof of Theorem 1.

so suffice for the proof of Theorem 1. Let us now give some conditions which are sufficient for condition (5.2) to hold, and In this termin
 $\sum_{i=1}^{n} v(f^{(i)})$

Let us now give

so suffice for the

Lemma 6: Le

i) If for the pse
 $v(X_i^i, Z_i^i)$

then
 $\sum_{i=1}^{n} v(X_i^i)$

Lemma 6: Let
$$
f \in C^r
$$
, $M(n) := M(n, r) = \sum_{i=1}^n {\{E[|X_i|^r] + E[|Z_{ni}|^r]\}}$. There hold.
\ni) If for the pseudo-moment
\n $v(X_i^i, Z_{ni}^i) = c(n^{-1}\varphi(n)^r M(n))$ $(n \to \infty)$

$$
\nu(X_i^j, Z_{ni}^j) = c\big(n^{-1}\varphi(n)^r M(n)\big) \qquad (n \to \infty) \tag{5.3}
$$

•

 $\frac{1}{2}$

•

$$
r \text{ the pseudo-moment}
$$
\n
$$
v(X_i^j, Z_{ni}^j) = c(n^{-1}\varphi(n)^r M(n)) \qquad (n \to \infty)
$$
\n
$$
\sum_{i=1}^n v(X_i^j, Z_{ni}^j) = c(\varphi(n)^r M(n)). \qquad (5.3)
$$
\n
$$
(5.4)
$$

$$
\sum_{i=1}^{n} v(f^{(i)}(\varphi(n) R_{ni} + u) X_i^j, f^{(i)}(\varphi(n) R_{ni} + n) Z_n^j) = c_f(\varphi(n)^r M(n))
$$
\nLet us now give some conditions which are sufficient for condition (5.2) to hold,
\nso suffice for the proof of Theorem 1.
\nLemma 6: Let $f \in C^r$, $M(n) := M(n, r) = \sum_{i=1}^{n} \{E[|X_i|^r] + E[|Z_{ni}|^r]\}$. There hold:
\n,i) If for the pseudo-moment
\n $v(X_i^j, Z_n^j) = c(n^{-1}\varphi(n)^r M(n))$ $(n \to \infty)$
\nthen
\n
$$
\sum_{i=1}^{n} v(X_i^j, Z_n^i) = c(\varphi(n)^r M(n))
$$
\nii) There exists a constant $N_f^{(i)}$, $1 \leq j \leq r$, such that
\n
$$
\sum_{i=1}^{n} v(f^{(i)}(\varphi(n) R_{ni} + u) X_i^j, f^{(i)}(\varphi(n) R_{ni} + u) Z_n^j) \leq |N_f^{(i)}| \sum_{i=1}^{n} v(X_i^j, Z_n^j).
$$
\niii) $v(f^{(i)}(\varphi(n) R_{ni} + u) X_i^j, f^{(i)}(\varphi(n) R_{ni} + u) Z_n^j)$
\n
$$
= E[f^{(i)}(\varphi(n) R_{ni} + u) \tau(X_i^j, Z_n^j; \mathfrak{A}_{ni})].
$$

\niv) There exists a constant $N_f^{(i)}$, $1 \leq j \leq r$, with

$$
v(f^{(j)}(\varphi(n) R_{ni} + u) X'_{i}, f^{(j)}(\varphi(n) R_{ni} + u) Z'_{ni})
$$
\n
$$
= E[f^{(j)}(\varphi(n) R_{ni} + u) \tau(X'_{i}, Z'_{ni}; \mathfrak{A}_{ni})].
$$
\n
$$
iv) There exists a constant $N_{j}^{(j)}$, $1 \leq j \leq r$, with\n
$$
\sum_{i=1}^{n} E[\tau(f^{(j)}(\varphi(n) R_{ni} + u) X'_{i}, f^{(j)}(\varphi(n) R_{ni} + u) Z'_{ni}; \mathfrak{A}_{ni})]
$$
\n
$$
\leq \sum_{r=1}^{n} |N_{j}^{(j)}| \nu(X'_{i}, Z_{ni}^{(j)}).
$$
\n
$$
v) If for the conditional pseudo-moment
$$
\n
$$
\tau(X'_{i}, Z'_{ni}; \mathfrak{A}_{ni}) = o(n^{-1}\varphi(n)^{r} M(n)) \quad a.s. \quad (n \to \infty)
$$
\nthen (recall condition (3.3i) of Theorem 1)\n
$$
\sum_{i=1}^{n} \tau(X'_{i}, Z'_{ni}; \mathfrak{A}_{ni}) = o(\varphi(n)^{r} M(n)) \quad a.s.
$$
\n
$$
vi) If (5.5) holds, so does (5.3), and if (5.6) holds, so does (5.4).
$$
\nProof: i) We have\n
$$
\sum_{i=1}^{n} \nu(X'_{i}, Z'_{ni}) = \sum_{i=1}^{n} c(n^{-1}\varphi(n)^{r} M(n)) = o(\varphi(n)^{r} M(n)).
$$
\n
$$
\therefore b \qquad i = 1
$$
$$

$$
\tau(X_i^j, Z_{ni}^j; \mathfrak{A}_{ni}) = c\big(n^{-1}\varphi(n)^r M(n)\big) \quad a.s. \qquad (n \to \infty)
$$
\n
$$
(5.5)
$$

then (recall condition (3.31) *of* Theorem 1)

$$
\sum_{i=1}^{n} \tau(X_i^j, Z_{ni}^j; \mathfrak{A}_{ni}) = c(\varphi(n)^r M(n)) \quad a.s.
$$
 (5.6)

vi) If (5.5) holds, so does (5.3), and if (5.6) holds, so does (5.4)
\nProof: i) We have
\n
$$
\sum_{i=1}^{n} v(X_i^j, Z_{ni}^j) = \sum_{i=1}^{n} c(n^{-1}\varphi(n)^r M(n)) = o(\varphi(n)^r M(n)).
$$

ii) Since $f \in C^r$, $f^{(j)}(x)$ is bounded by a constant $N_f^{(j)}$, and there holds

1) Since
$$
f \in C^r
$$
, $f^{(j)}(x)$ is bounded by a constant $N_j^{(j)}$, and $\sum_{i=1}^n \nu(f^{(j)}(\varphi(n) \cdot R_{ni} + u) \cdot X_i^j, f^{(j)}(\varphi(n) \cdot R_{ni} + u) \cdot Z_{ni}^j)$ $\leq \sum_{i=1}^n \nu(N_f^{(j)} X_i^j, N_f^{(j)} Z_{ni}^j) \leq |N_f^{(j)}| \sum_{i=1}^n \nu(X_i^j, Z_{ni}^j).$
\niii) There holds with Lemma 5ii) and (2.7) that $\nu(f^{(j)}\varphi(\varphi(n) \cdot R_{ni} + u) \cdot X_i^j, f^{(j)}(\varphi(n) \cdot R_{ni} + u) \cdot Z_{ni}^j)$ $= \mathbb{E}[\tau(f^{(j)}(\varphi(n) \cdot R_{ni} + u) \cdot X_i^j, f^{(j)}(\varphi(n) \cdot R_{ni} + u) \cdot Z_{ni}^j)]$

iii) There holds with Lemma 5ii) and (2.7) that

$$
\leq \sum_{i=1}^{n} v(N_f^{(i)} X_i^j, N_f^{(i)} Z_{ni}^j) \leq |N_f^{(i)}| \sum_{i=1}^{n} v(X_i^j, Z_{ni}^j).
$$

\ne holds with Lemma 5ii) and (2.7) that
\n
$$
v(f^{(i)} \varphi(\varphi(n) R_{ni} + u) X_i^j, f^{(i)}(\varphi(n) R_{ni} + u) Z_{ni}^j)
$$
\n
$$
= \mathbb{E}[\tau(f^{(i)}(\varphi(n) R_{ni} + u) X_i^j, f^{(i)}(\varphi(n) R_{ni} + u) Z_{ni}^i; \mathfrak{A}_{ni})]
$$
\n
$$
= \mathbb{E}[f^{(i)}(\varphi(n) R_{ni} + u) \tau(X_i^j, Z_{ni}^j; \mathfrak{A}_{ni})].
$$

\n*ks* as does ii); v) follows as i); vi) follows by Lemma 5ii

iv) follows' as does ii); v) follows as i); vi) follows by Lemma 5ii) and (2.3) **^I**

Lemma 6 could obviously be formulated also for the $V(n)$ **-case; whereas Lemma 6** refers to Theorem 1, versions attached to Theorems 3 and 4 are also possible. So one can see that conditions (5.2) , (5.4) , (5.5) and (5.6) suffice for (5.2) or (5.1) , and so one can formulate a weaker version of Theorem 1. The weakest alternative condition to (3.3) (or (5.1)) is condition (5.4) , and will now be employed. *(xi, fii*)($\varphi(n) R_{ni} + u$) Z_{ni}
 $(+ u) X_i^i$, $f^{(i)}(\varphi(n) R_{ni} + u) Z_{ni}^i$; \mathfrak{A}_{ni})]
 $(u) \tau(X_i^i, Z_{ni}^i; \mathfrak{A}_{ni})$].

lows as i); vi) follows by Lemma 5ii) and (2.3) **I**

be formulated also for the $V(n)$ -case; whereas Le $E[\tau(f^{(j)}(\varphi(n) R_{ni} + u) X)]$
 $E[f^{(j)}(\varphi(n) R_{ni} + u) \tau(X)]$

as does ii); v) follows as

is could obviously be form

heorem 1, versions attac

that conditions (5.2), (5.

mulate a weaker version

(5.1)) is condition (5.4),

m 6: $\left[\frac{1}{n}, f^{(j)}(p(n) R_n) \right],$
 $\left[\frac{1}{n}, Z_{ni}^j; \mathfrak{A}_{ni}\right].$
 $\left[\frac{1}{n}, Z_{ni}^j; \mathfrak{A}_{ni}\right].$
 $\left[\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right].$
 $\left[\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right],$
 $\left[\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right],$
 $\left[\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right]$ $f(\mathbf{X}_i^j, \mathbf{Z}_{ni}^j; \mathfrak{A}_{ni}; \mathfrak{A}_{ni})$
 (r) (x) (x) (i) (i the $V(n)$ -case; whereas Lemma 6

is 3 and 4 are also possible. So

i) suffice for (5.2) or (5.1), and so

the weakest alternative condition

employed.

there holds, in case condition (3.3)

(5.7i)

(n refers to Theorem 1, versions attached to 1
one can see that conditions (5.2), (5.4), (5.5) a
one can formulate a weaker version of Theore
to (3.3) (or (5.1)) is condition (5.4), and will i
Theorem 6: *Under the assumptio*

Theo rem 6: Under the assumptions of Theorem 1 there holds, in case condition (3.3) is replaced by

Theorem 6: Under the assumptions of Theorem 1 there holds, in case condition (3.3)
\nis replaced by
\n
$$
\sum_{i=1}^{n} v(X_i^j, Z_{ni}^j) = \begin{cases}\n\circ \left(\frac{\varphi(n)^r}{(r-1)!} M(n) \right) & \text{or} \\
\circ \left(\frac{\varphi(n)^r}{(r-1)!} V_{j}(n) \right), & \text{.} \\
\text{for each } j \in C^r \text{ the estimate }\n\end{cases}
$$
\n(5.7ii)
\n
$$
\text{for each } j \in C^r \text{ the estimate }
$$
\n
$$
\sup_{u \in \mathbb{R}} |E[f(T_n + u)] - E[f(Z + u)]| = \begin{cases}\n\circ_f \left(\frac{\varphi(n)^r}{(r-1)!} M(n) \right) & \text{or} \\
\circ_f \left(\frac{\varphi(n)^r}{(r-1)!} V(n) \right) & \text{.} \\
\circ_f \left(\frac{\varphi(n)^r}{(r-1)!} V(n) \right) & \text{.} \\
\text{.} & \text{.} \\
\text{.} & \text{.} \\
\end{cases}
$$
\n(5.8ii)
\nProof: Checking the proof of Theorem 1, one just needs to re-examine the esti-

$$
\left(\circ \left(\frac{\varphi(n)}{(r-1)!} \right) V_j(n) \right), \tag{5.7ii}
$$

for each $f \in C^r$ *the estimate*

5.5.7.7) is conditional (5.4), and with flow be employed.
\n6. Under the assumptions of Theorem 1 there holds, in case condition (3.3)
\n
$$
P(X_i^j, Z_{ni}^j) = \begin{cases}\n\sigma\left(\frac{\varphi(n)^r}{(r-1)!} M(n)\right) & or & (5.7i) \\
\sigma\left(\frac{\varphi(n)^r}{(r-1)!} V_n(n)\right), & (5.7ii)\n\end{cases}
$$
\n5.7.7.7.7.7.8.7.8.7.9. (5.8.7.9.1)

Proof: Checking the proof of Theorem '1, one just needs to re-examine the estimate of the first term of (3.5), namely (3.6). But it follows by Lemma 6ii) and condition (5.7) that (3.6) is of order $c(\varphi(n)^r M(n))$ $(c(\varphi(n)^r V(n)))$ uniformly for all $u \in \mathbb{R}$, as desired.-So the proof is complete I

Remark 5: 1. In the particular case of independent random variables X_i Theorem. 6 coincides, with Theorem 12 in [8] so that the former is a true generalization of our earlier results for the independent case. Pseudo-moment conditions are again of decisive importance. 2. As the proof of Theorem 6 reveals, all of the estimates derived in Sections 3 and 4 are valid not only for each individual $u \in \mathbb{R}$ but uniformly in $u \in \mathbb{R}$ provided condition (5.4) of Lemma 6 would be employed throughout. 3. SAZONOV and ULYANOV [33] established an O -estimate for the rate in the multidimensional central limit theorem for independent identically distributed random variables also in terms of pseudo-moments by using a Taylor expansion for functions $f \in C^3(\mathbb{R}^k)$. Exmark 3: 1. In the particular case of independent random variables X_i Theorem.

ided is with Theorem 12 in [8] so that the former is a true generalization of our earlier rest.

the independent case. Pseudo-moment condi

6. Probability metrics; comparisons with known results

The exploitation of pseudo-moments in the case of limit theorems in probability theory is part of the 'theory of probability metrics (see e.g. [17, 38]). In this section we will give a short survey in connection with different probability metrics. 45

• Definition 5: The distance μ of two random variables X, Y, with μ : $\mathfrak{Z}(\Omega, \mathfrak{A})$ \times 3(Ω , \mathfrak{A}) \rightarrow [0, ∞], is called a *probability metric* if *Definition 5:* The distance $\mathfrak{Z}(\Omega, \mathfrak{A}) \to [0, \infty]$, is called *i*
i) $P(X = Y) = 1$ implies μ
ii) $\mu(X, Y) = \mu(Y, X)$,
iii) $\mu(X, Y) \le \mu(X, Z) + \mu(X)$ General Limit Theorems with o -Rates 30;

Definition 5: The distance μ of two random variables X, Y, with $\mu: \mathfrak{Z}(\Omega, \mathfrak{A})$
 $\mathfrak{Z}(\Omega, \mathfrak{A}) \to [0, \infty]$, is called a *probability metric* if

i) $P(X = Y) = 1$ implie

- i) $P(X = Y) = 1$ implies $\mu(X, Y) = 0$,
-

iii) $\mu(X, Y) \leq \mu(X, Z) + \mu(Y, Z)$ for a random variable $Z \in \mathcal{B}(\Omega, \mathcal{X})$.

Lemma *7:.ln regard to the above definition one has:*

b) The conditional pseudo-moment τ is a pseudo-probability metric, i.e., conditions ii) *and* iii) *of Definition* 5 *are fulfilled, but condition* 1) *holds only a.s.*

Proof: Concerning part a), the proof is evident. Concerning b), we have: iii) $\mu(X, Y) \leq \mu(X, Z) + \mu(Y, Z)$ for a random variable
Lemma 7: *In regard to the above definition one has:*
a) The pseudo-moment v is a probability metric.
b) The conditional pseudo-moment r is a pseudo-probabili
) and iii) $(x)| = 0$ a.s. since $F_{X|\mathfrak{G}}$ $\Pr_{\text{i)} \text{i}} \ = \frac{F}{F}$ $=$ $F_{Y|0}$ a.s.; *Proof:* Concerning part a), the proof is cvident. Concerning b), we have:
 i) $P(X = Y) = 1$ $\Rightarrow \tau(X, Y; \mathcal{G}) = | \int x d(F_{X|\mathcal{G}} - F_{Y|\mathcal{G}}) (x) | = 0$ a.s. since *F*
 *F*_{Y|}(\mathcal{G} a.s.;
 ii) evident, since $| \int x d(F_{X|\mathcal{G}} -$ *• L(X)* = *(YE 3(Q,9[); P* = *P}.* •

ii) evident, since $| \int x d(F_{X|G} - F_{Y|G}) (x) | = | \int x d(F_{Y|G} - F_{X|G}) (x) |;$
iii) $\tau(X, Y; G) = | \int x d(F_{X|G} - F_{Z|G} + F_{Z|G} - F_{Y|G}) (x) | \le \tau(X, Y; G)$ $+ \pi(Y, Z; \mathcal{B})$ **I**

In the following, we will establish two results **to** be compared with Theorem 6; they will be deduced by well-known results of other authors. following, we will establish two results to
following, we will establish two results to
educed by well-known results of other a
ition 6: Let $X, Y \in \mathfrak{Z}(\Omega, \mathfrak{A})$. The law
 $L(X) = \{Y \in \mathfrak{Z}(\Omega, \mathfrak{A})\colon P_X = P_Y\}.$
classica

Definition 6: Let X, $Y \in \mathfrak{Z}(\Omega, \mathfrak{A})$. The *law of X, L(X)*, is defined by

$$
L(X) = \{Y \in \mathfrak{Z}(\mathfrak{Q}, \mathfrak{A}); P_X = P_Y\}.
$$

If d is a classical metric, then $y(X, Y; d) := \mathbb{E}[d(X, Y)]$ defines a probability metric; the so-called *minimal metric* with respect to γ is defined by $L(X) = \{Y \in \mathfrak{Z}(\Omega, \mathfrak{A})\, ;\, P_X = P_Y\}.$

If d is a classical metric, then $\gamma(X, Y, d) := \mathrm{E}[d(X, Y)]$ def

the so-called minimal metric with respect to γ is defined by
 $\hat{\gamma}(X, Y, d) = \inf \{ \gamma(X, Y, d) \, ;\, L(X, Y) \in \mathfrak{Z}(\Omega, \mathfrak{A})$

 $\hat{\gamma}(X, Y; d) = \inf \{\gamma(X, Y; d); L(X, Y) \in \mathfrak{Z}(\Omega, \mathfrak{A}) \times \mathfrak{Z}(\Omega, \mathfrak{A}),\}$

$$
L(X) = P_X \text{ and } L(Y) = P_Y
$$

In the particular case $d(x, y) = |x - y|^s$, $s > 0$, γ is called a *Wasserstein metric*, and If *d* is a classical metric, then $y(X, Y; d) := \mathbb{E}[d(X, Y)]$ def
the so-called *minimal metric* with respect to *y* is defined by
 $\hat{y}(X, Y; d) = \inf \{y(X, Y; d) : E(X, Y) \in \mathcal{B}(\Omega, \mathcal{Y})\}$
 $L(X) = P_X$ and $L(Y) = P_Y\}$.
In the particular c $f(X) = P_X$ and $L(Y) = P_Y$.

ticular case $d(x, y) = |x - y|^s$, $s > 0$, γ is called a *Wasserstein metric*, an

by $W_s(P_X, P_Y)$. The *metric* ξ is defined by
 $(X, Y, F) = \sup_{f \in F} \{ |f(x) d(P_X - P_Y)(x)| \},$ Let *Final* **IE 12 IE 12**

$$
\xi(X, Y; F) = \sup_{f \in F} \{ |f(x) d(P_X - P_Y)(x)| \},
$$

where F is any function-class. A particular version of ξ is given by

$$
\xi_s(X, Y) = \xi(X, Y; D_s),
$$

In the particular case
$$
d(x, y) = |x - y|^s
$$
, $s > 0$, $\hat{\gamma}$ is called a *Wasserstein metric*, a is denoted by $W_s(P_X, P_Y)$. The metric ξ is defined by\n
$$
\xi(X, Y; F) = \sup_{f \in F} \{ |f(x) d(P_X - P_Y)(x)| \},
$$
\nwhere F is any function-class. A particular version of ξ is given by\n
$$
\xi_s(X, Y) = \xi(X, Y; D_s),
$$
\nwhere\n
$$
D_s = \left\{ f \in C, \int_{f^{(r)} \in C} \cap \text{Lip } s \}, \qquad 0 < s \leq 1,
$$
\nand (see e.g. [38]) Lip $\alpha := \{ f \in C; |f(x) - f(y)| \leq |x - y|^s \}.$

Under these notations, the Kantorovitch-Rubinstein-Dudley Theorem reads (see e.g. $[37]$)

is denoted by $W_s(P_X, P_Y)$. The metric ξ is different $\xi(X, Y; F) = \sup_{f \in F} \{ |f(x)| d(P_X - P) \}$

where F is any function-class. A particular $\xi_s(X, Y) = \xi(X, Y; D_s)$,

where
 $D_s = \left\{ f \in C, \int_{[T)} \xi C \cap \text{Lip } s \right\},$

and (see e.g. [38] Theorem 7: For any measurable metric d and any $X, Y \in \mathcal{B}(\Omega, \mathfrak{A})$; there holds Under these notations, the Kantorovitch-Rubinstein-I
g. [37])
Theorem 7: For any measurable metric d and any λ
 $(X, Y; d) = \xi(X, Y; G(d)),$ where $G(d) = \{f \in C; |f(x) -$
For the application of Theorem 7 to our results anot $f(y) \leq d(x, y)$. Theorem 7: For any measurable metric d an $\mathcal{P}(X, Y, d) = \xi(X, Y; G(d))$, where $G(d) = \{f \in C$
For the application of Theorem 7 to our rest important.

For the application of Theorem 7 to our results, another property of metrics is

Definition 7: A metric μ is called *ideal* of order $s > 0$ if for any random variables *X*, *Y*, *Z* \in 3(Ω , \mathfrak{A}) and any constant $c \in \mathbb{R}$, $c \neq 0$. $\begin{array}{c|cc}\n & 304 \\
 & \times & 1 \\
 & \times & \text{i} \\
 & \text{i} & \text{j}\n\end{array}$ **Definition 7:** A metric μ is call
 (Y, Z \in 3(Ω , \mathfrak{A}) and any constate
 i) $\mu(X + Z, Y + Z) \leq \mu(X, Y)$
 ii) $\mu(c \cdot X, c \cdot Y) \leq |c|^s \mu(X, Y)$

) $\mu(X + Z, Y + Z) \leq \mu(X, Y),$

where Z is independent of \hat{X} and Y .

In this respect there holds (see [36, Lemma 3]).

Lemma 8: For any $s > 0$, ξ , is an ideal metric of order s in the maximal subset of $\mathfrak{Z}(\Omega, \mathfrak{A})$ within which the values of ξ_s are finite.

Since for the Kantorovitch-Rubinstein theorem the function class $G(d)$ is needed, and since the metric ξ is ideal only for the particular function class D_s , $s > 0$, for our application our function class has to fulfil both conditions. This means that $D_{\epsilon} \cap G(d)$ $=$ D_s , $0 < s \leq 1$. So our application of the Kantorovitch-Rubinstein theorem is only valid for the metric ξ with $0 < s \leq 1$. is independent of \hat{X} and Y .

respect there holds (see [36, Lemma 3])

ia 8: For any $s > 0$, ξ_s is an ideal metric of order s in the maximal subset of

yithin which the values of ξ_s are finite.

or the Kanto

Theorem 8: Let
$$
X, Y \in \mathcal{G}(\Omega, \mathcal{X})
$$
 and $f \in D_s, 0 < s \leq 1$. Then

\n
$$
\sup_{u \in \mathbb{R}} \{|E[f(X + u) - f(Y + u)]\}| \leq E[(X - Y)^s].\tag{6.1}
$$

Proof: One has by Lemma 8 and Theorem 7 that the left-hand side of (6.1) is estimated by

$$
\sup_{u\in\mathbb{R}} \left\{ |E[f(X+u)-f(Y+u)]| \right\} \leq E[(X-\dot{Y})^2].
$$
\n
$$
\text{where } \lim_{u\in\mathbb{R}} \left\{ |E[f(X+u)-f(Y+u)]| \right\} \leq \lim_{x\to\infty} \left\{ |E[f(X+u)] - E[f(Y+u)]| \right\} \leq \sup_{f\in D_s} \left\{ |E[f(X)] - E[f(Y)]| \right\}
$$
\n
$$
\leq \sup_{f\in D_s} \left\{ |E[f(X+u)] - E[f(Y+u)]| \right\} \leq \sup_{f\in D_s} \left\{ |E[f(X)] - E[f(Y)]| \right\}
$$
\n
$$
= \xi_s(X, Y) = W_s(P_X, P_Y) \leq E[|X - Y|^s] \blacksquare
$$

Observe that Theorem 8 is somewhat comparable to the $V(n)$ -version of Theorem 6 with $X = T_n$, $Y = Z$. However, the function class $D_s = C \cap \text{Lip } s$ is larger than that in Theorem 6, namely C_r , P_Y) \leq E[|X - Y|^s] **I**

Observe that Theorem 8 is somewhat comparable to the $V(n)$ -version of Theorem 6 with $X = T_n$, $Y = Z$. However, the function class $D_s = C \cap \text{Lip } s$ is larger than that respect see also a further concretization in Theorem 9. From another point of view there states a lemma of ZOLOTAREV (see [36, Lemma 2]) that ER (6.1) is

by

(one has by Lemma 8 and Theorem 7 that the left-hand side of (6.1) is

by

(up sup $\{|E[f(X + u)] - E[f(Y + u)]\}| \le \sup_{f \in D_s} \{|E[f(X)] - E[f(Y)]\}|$
 $= \xi_s(X, Y) = W_s(P_X, P_Y) \le \mathbb{E}[|X - Y|^s]$

b), $\exists \xi_s(X, Y) = W_s(P_X, P_Y) \le \mathbb{E}[|X - Y|^s]$
 sup sup $\{|\mathbf{E}[f(X + u)] - \mathbf{E}[f(Y + u)]\}| \le \sup_{f \in D_s} \{|\mathbf{E}[f(X)] - \mathbf{E}[f(Y)]|\}$
 $= \xi_s(X, Y) = W_s(P_X, P_Y) \le \mathbf{E}[|X - Y|^s] \blacksquare$

Observe that Theorem 8 is somewhat comparable to the $V(n)$ -version of The

with $X = T_n, Y = Z$. However, the fun $f \in \mathcal{D}_s u \in \mathbb{R}$
 $= \xi_s(X, Y) = W_s(P_X, P_Y) \leq \mathbb{E}[[X - Y]^s]$

Observe that Theorem 8 is somewhat comparable twith $X = T_n$, $Y = Z$. However, the function class *D*

in Theorem 6, namely *C*^r, $r \geq 2$; the large *O* order

r 8 is somewhat comparable to the $V(n)$ -version of Theorem 6

wever, the function class $D_s = C \cap \text{Lip } s$ is larger than that
 $\forall r, r \geq 2$; the large- C order is replaced by little- o . In this

concretization in Theorem 9.

$$
\xi_s(X, Y) \leq \frac{\Gamma(1+\alpha)}{\Gamma(1+s)} B_s(X, Y) \qquad (s > 0), \qquad (6.2)
$$

$$
B_s(X, Y) := \{E[|X|^s] + E[|Y|^s], s > 0, s = r + \alpha, r \in \mathbb{N}, \alpha \in (0, 1],
$$

- *i*) $B_s(X, Y) < \infty$,
- where
 $B_s(X, Y) := \{E[|X|^s] + E[|Y|\]$

(6.2) being valid provided \rightarrow

i) $B_s(X, Y) < \infty$,

ii) $E[X^k] = E[Y^k] + (1 \le k \le r).$

Theorem 9: Let $s > 0$, $s = r + \alpha$, $\alpha \in (0, 1]$, $r \in \mathbb{N}$. For two sequences of inde*pendent random variables with* $T_n = \sum_{i=1}^{n} \varphi(n) X_i$ and $Z = \sum_{i=1}^{n} \varphi(n) Z_n$, for which the *x*, Y $\leq \infty$,
 \mathbb{R}^k] = $E[Y^k]$ \qquad $(1 \leq k \leq r)$.
 $e \mod n$ ≥ 0 , $s = r + \alpha$, $\alpha \in (0, 1]$, $r \in \mathbb{N}$. For two sequences of inde-
 random variables with $T_n = \sum_{i=1}^n \varphi(n) X_i$ and $Z = \sum_{i=1}^n \varphi(n) Z_{ni}$, fo

$$
X_i, Z_{ni} satisfy (6.3), one has for $f \in D_s$ (recall Definition 6),
\n
$$
\sup_{u \in \mathbb{R}} \{|E[f(T_n + u)] - E[f(Z + u)]\}| \leq \frac{\Gamma(1 + \alpha)}{\Gamma(1 + s)} |\varphi(n)|^s B_s(X_i, Z_{ni}).
$$
 (6.4)
$$

Proof: It follows by Lemma 8, (6.2) and a further Theorem of ZOLOTAREV [36 Theorem 2], namely $\xi_s(T_n, Z) \leq |\varphi(n)|^s \sum \xi_s(X_i, Z_{ni}),$ that If follows by Lemma 8, (6.2) and a f

2], namely $\xi_s(T_n, Z) \leq |\varphi(n)|^s \sum_{i=1}^n \xi_s(X_i,$
 $\sup_{u \in \mathbb{R}} {\left(|E[f(T_n + u)] - E[f(Z + u)]|\right)} \leq \xi$ General Limit Incorems with λ

3.2) and a further Theorem of 2
 $|\mathbf{S}^n| \sum_{i=1}^n \xi_i(X_i, Z_{ni})$, that
 $+ u)$]] $\leq \xi_3(T_n, Z)$
 $(n)|^s \sum_{i=1}^n \frac{\Gamma(1 + \alpha)}{\Gamma(1 + s)} B_s(X_i, Z_{ni})$ General Limit Theorems with o^t Rates

Proof: It follows by Lemma 8, (6.2) and a further Theorem of ZoloraREV

eorem 2], namely $\xi_s(T_n, Z) \leq |\varphi(n)|^s \sum_{i=1}^n \xi_s(X_i, Z_{ni})$, that
 $\sup_{u \in \mathbb{R}} \{ |E[f(T_n + u)] - E[f(Z + u)] |] \leq \xi_s(T_n, Z)$ $\begin{array}{rcl}\n\text{deorem 2} \\
\text{se} \\
\text{se} \\
\text{se} \\
\text{Hermark} \\
\text{2} \\
\text{se} \\
\text{e} \\
\text{2} \\
\text{e} \\
\text{2} \\
\text{d} \\
\text{he} \\
\text{2} \\
\text{e} \\
\text{e} \\
\text{e} \\
\text{e} \\
\text$ al Limit Theorems with o^{\prime} Rates

(1) $\left\{ \begin{aligned}\n &\text{for all } i \in \mathbb{Z}_n, \text{ and } i$ Proof: It follows by Lemma 8, (6.2) and a further Theorem of ZotoTakev |

Theorem 2], namely $\xi_s(T_n, Z) \leq |\varphi(n)|^s \sum_{i=1}^n \xi_i(X_i, Z_{ni})$, that
 $\sup_{u \in \mathbb{R}} {\vert \vert \mathbb{E}[f(T_n + u)] - \mathbb{E}[f(Z + u)] \vert \vert \geq \xi_s(T_n, Z)}$
 $\leq |\varphi(n)|^s \sum_{i=1}^n \xi$

General Limit Theorems
\n
$$
f: \text{ It follows by Lemma 8, (6.2) and a further Theorem}
$$
\n
$$
a \text{ 2}, \text{ namely } \xi_s(T_n, Z) \leq |\varphi(n)|^s \sum_{i=1}^n \xi_s(X_i, Z_{n_i}), \text{ that}
$$
\n
$$
\sup_{u \in \mathbb{R}} \{ |\mathbb{E}[f(T_n + u)] - \mathbb{E}[f(Z + u)]| \} \leq \xi_s(T_n, Z)
$$
\n
$$
\leq |\varphi(n)|^s \sum_{i=1}^n \xi_s(X_i, Z_{n_i}) \leq |\varphi(n)|^s \sum_{i=1}^n \frac{\Gamma(1 + x)}{\Gamma(1 + s)} B_s(X_i, Z_n)
$$
\n
$$
\text{rk } 6: \text{ If one would replace the pseudo-moment condition (5.2) is}
$$

$$
\leq |\varphi(n)|^s \sum_{i=1}^n \xi_s(X_i,Z_{ni}) \leq |\varphi(n)|^s \sum_{i=1}^n \frac{\Gamma(1+\alpha)}{\Gamma(1+s)} B_s(X_i,Z_{ni}) \quad \blacksquare
$$

$$
\leq |\varphi(n)|^s \sum_{i=1} \xi_s(X_i, Z_{ni}) \leq |\varphi(n)|^s \sum_{i=1} \frac{1}{\Gamma(1+s)} B_s(X_i, Z_{ni})
$$
\n
\nRemark 6: If one would replace the pseudo-moment condition (5.2) by\n
$$
\sum_{i=1}^n E[X_i^j - Z_{ni}^j] = O\left(\frac{\varphi(n)^r}{(r-1)!} M(n)\right), \qquad (6.5)
$$
\n
\n'it could be shown (compare [12]) that for functions f from Lip^(r-1) $\alpha := \{f \in C; f^{(r-1)} \in \text{Lip } \alpha\}$,\n
$$
\alpha \in (0, 1], \text{ and for possibly dependent random variables } X, \text{ there holds}
$$

$$
\leq |\varphi(n)|^s \sum_{i=1}^n \xi_s(X_i, Z_{ni}) \leq |\varphi(n)|^s \sum_{i=1}^n \frac{\Gamma(1+\alpha)}{\Gamma(1+s)} B_s(X_i, Z_{ni}) \blacksquare
$$
\n
\nrk 6: If one would replace the pseudo-moment condition (5.2) by\n
$$
\sum_{i=1}^n E[X_i^j - Z_{ni}^j] = \mathcal{O}\left(\frac{\varphi(n)^r}{(r-1)!} M(n)\right), \qquad (6.5)
$$
\nbe shown (compare [12]) that for functions f from Lip^(r-1) $\alpha := \{f \in C; f^{(r-1)} \in \text{Lip } \alpha\},$,\n], and for possibly dependent random variables X_i there holds\n
$$
\sup_{u \in \mathbb{R}} \{ |E[f(T_n + u)] - E[f(Z + u)] | \} = \mathcal{O}\left(\left\{\frac{\varphi(n)^r}{2(r-1)!} M(n)\right\}^{-1-r} \right). \qquad (6.6)
$$
\n
$$
\text{and now (6.6) with the estimate in Theorem 9, the right-hand side of (6.4) in the parti-
$$

Comparing now (6.6) with the estimate in Theorem 9, the right-hand side of (6.4) in the particular case $s = (r - 1) + 1$ would be equal to $\sum_{i=1}^{r} B_{r-1}(X_i, Z_{ni}) = M(n, r - 1)$, and the function classes in both results, namely D_r and $Lip^{(r-1)}\alpha$ with $\alpha = 1$, would also be equal; like wise are the orders, namely $\varphi(n)^r M(n, r)^1$ (noting $1 - (1 - \alpha)/r = 1$ there). However, the estimate in (6.6) is more gener wise are the orders, namely $\varphi(n)^r M(n, r)^1$ (noting $1 - (1 - \alpha)/r = 1$ there). However, the estimate in (6.6) is more general in the sense that it does not only hold for independent random variables X_i as in the case of Theorem 9 but also for possibly dependent ones. Further, the pseudo-moment condition (6.5) is much weaker than (6.3), namely $\text{E}[X_i^j] = \text{E}[\tilde{Z}_{ni}^j], 1 \leq j \leq r$, $i \in \mathbb{N}$. function classes in both results, namely D_r and $\text{Lip}^{(r-1)}$ as with $\alpha = 1$, would also b
wise are the orders, namely $\varphi(n)^r M(n, r)^1$ (noting $1 - (1 - \alpha)/r = 1$ there). If
variables X_i as in the case of Theorem 9 but al

Concludingly one can say that our results of Sections $3-5$ generalize known results on the distance of random variables to the ease of general limit theorems for dependent random variables.

The authors would like to thank Dr. Dietmar Pfeifer, Heisenberg Professor, Aachen, for his critical reading of the manuscript and valuable suggestions.

- [1] ANASTASSIOU, G. A.: An improved general stochastic inequality. Bull. Soc. Math. Grèce for his critical reading of

• REFERENCES

[1] ANASTASSIOU, G. A.: Ar

(N. S.) 24 (1983), 1 – 11.

[2] BASU, A. K.: On the rat

dom variables and rand
- [2] BASU, A. K.: On the rate of convergence in the central limit theorem for dependent random variables and random vectors. J. Multivariate Anal. 10 (1980), 565-578. (2) BASU, A. K.: On the rate of convergence in the central limit theorem for dependent random variables and random vectors. J. Multivariate Anal. 10 (1980), 565-578.

(3) BAUER, H.: Wahrscheinlichkeitstheorie und Grundzüg
	- [3] BAUER, **H.:** Wahrschcinlichkeitstheorie und Grundzüge der MaBtheorie (2. AufI.). Berlin: De Gruyter 1974.
	- [4] BEROSTROEM, **H.:** On distribution functions with a limiting stable distribution function. Arkiv Mat. 2 (1952), 463-474. • De Gruyter 1974.

	• De Gruyter 1974.

	• De Gruyter 1974.

	• (4) ВЕRGSTROEM, H.: On distribution functions with a limi

	• Arkiv Mat. 2 (1952), 463-474.

	• [5] ВОLТНАUSEN, E.: On rates of convergence in a random

	• centra
		- [5] BOLTHAUSEN, E.: On rates of convergence in a random central limit theorem and in the central limit theorem for Markoff chains. **Z.** Wahrsch. Verw. Gebiete 38(1977), 279-286.
		- Wàhrsch. Verw. Gehiete **54** (1980) ⁹ ⁵⁹ -73.
		- [7] BUTZEH, **P. L.,** HAHN, **L.,** and U. WESTFHAL: On the rate of approximation in the central
		- [8] BUTZER. P. L., and L. HAHN: General theorems on rates of convergence in distributions of random variables. 1: General limit theorems. **J.** Multivariate Anal. 8(1978), **181-201.**

 $1/20$ -Analysis Bd. 7, Heft 4 (1988)

P. L. BUTZER and H. KIRSCHFINK

- [9] BUTZER, P. L., and L. HAHN: On Connections Between the Rates of Norm and Weak Convergence in the Central Limit Theorem. Math. Nachr. 91 (1979), 245–251.
[10] BUTZER, P. L., HAHN, L., and M. ROECKERATH: General theorems on "little-o" rates of
- F. L. BUTZER and H. KIRSCHFINK

(9) BUTZER, P. L., and L. HAHN: On Connections Between the Rates of Norm

Convergence in the Central Limit Theorem. Math. Nachr. 91 (1979), 245-251.

[10] BUTZER, P. L., HAHN, L., and M. ROE closeness of *two* weighted sums of independent Hilbert space valued random variables with applications. J. Multivariate-Anal. *9* (1979), 487-510.
	- [11] BUTZER, P. L., HAHN, L., and M. ROECKERATH: Central limit theorem and weak law of large numbers with rates for martingales in Banach spaces. J. Multivariate Anal. 13 (1983), $287 - 301.$ Deseness of two weighted sums of independent Hilbert space valued random variable
closeness of two weighted sums of independent Hilbert space valued random variable
with applications J. Multivariate Anal. 9 (1979), 487-510
	- [12] BUTZER, P. L., and H. KIRSCHFINK: Donsker's weak invariance principle with rates for *C*[0, 1]-valued, dependent random-functions. Approx. Theory Appl. 2 (1986), 55-77.
	- [13] BUTZER, P. L., and D. SCHULZ: General Random Sum Limit Theorems for Martingales with Large-O Rates. Z. Anal. Anw. 2 (1983), $97-109$.
	- [14] BUTZER, P. L., and D. SCHULZ: The random martingale central limit theorem and weak

	⁹ law of large numbers with *o*-rates. Acta Sci. Math. (Szeged) 45 (1983), $81-94$.
	- [15] BUTZER, P. L., and D. SCHULZ: Approximation theorems for martingale difference arrays with applications to randomly stopped sums. In: Math, structures $-$ computational math. math, modelling 2: Papers dedicated to L. Iliev's 70th anniversary (Ed.: BI. .Sendov). Sofia: Bulg. Akad. Sci. 1984, pp. 121-130. BUTZER, P. L., and D. SCHULZ: Approximation theo
with applications to randomly stopped sums. In: Ma
— math. modelling 2: Papers dedicated to L. Iliev's
Sofia: Bulg. Akad. Sci. 1984, pp. 121 – 130.
CHUNG, K. L.: Lectures fr
	- [16] CHUNG, K. L.: Lectures from Markov Processes to Brownian Motion. New York: Springer-Verlag 1982.
	- [17] DUDLEY, R. M.: Probabilities and Metrics: convergence of laws on nietric spaces, with a view to statistical testing (Lect. Notes Series: No. 45). Aarhus: Aarhus Univ. 1976.
	- [18] GAENSSLER, P., and W. STUTE: Wahrscheinlichkeitstheorie. Berlin: Springer-Verlag 1977.
	- [19] GIHMAN, I. I., and A. V. SKOROHOD: The Theory of Stochastic Processes II. Berlin: Springer-Verlag 1975. Симла, I.
Špringer-Ve
Gudynas, I
Lith. Math.
Lith. Math. Stat.
Math. Stat.
1976.
Lана, R. G
1979.
	- [20] GUDYNAS, P.:' Refinements of the entral limit theorem for a homogeneous Markov chain.
	- [21] KATO, J.: Convergence rates in the central limit theorem for martingale differences. Bull. Math. Stat. $18(1979)$, $1-9$.
	- [22]'KEMENY, J. G., and J. L. SNELL: Finite Markov Chains. New York: Springer-Verlag
	- [23] LAHA, R. G., and V. K. ROHATGI: Probability Theory. New York: John Wiley & Sons 1979.
	- [24] LANDERS, D., and L. ROGGE: On the rate of convergence in the central limit theorem for Markov chains. *Z.* Wahrsch.' Verw. Gebiete 35 (1976), 57-63.
	- [25] LTFsmTs, B. A.: On a central limit theorem for sums' of random variables connected' in a Markov chain. Dokl. Akad. Nauk SSSR 219 (1974), 797-799.
	- [26] LIFSIIITS, B. A.: On the central limit theorem for Markov chains. Theory Probab. AppI. 23 (1978), 279-296.
	- [27] LJFSHITS, B. A.: Invariance principle for weakly dependent 'variables. Theory Probab. Appl. 29 (1985) , $33-40$.
	- [28] NAGAEV, S. V.: Some limit for stationary Markov chains. Theory Probab. AppI. 2 (1957), $378 - 406.$
	- [29] NAGAEV. S. V.: More exact statements of limit theorems for homogeneous Markov chains. Theory Probab. Appl. 6 (1961), $62-81$.
	- [30] PADITZ, L.: Über die Annäherung von Summenverteilungsfunktionen gegen unbegrenzt 4eilbare Verteilungsfunktionen in der Terminologie der Pseudomomente. Wiss. *Z.* Techn. • [30] PADITZ, L.: Uber die Annäherung von Summenverteilungsfunktionen gegen unbegrenz

	• veilbare Verteilungsfunktionen in der Terminologie der Pseudomomente. Wiss. Z. Techn

	• Univ. Dresden 27 (1978), 1129–1133.

	• [31]
		- Univ. Dresden 27 (1978), 1129–1133.
[31] Prakasa Rao, B. L. S.: On central limit theorems, invariance principle and rates of convergence for backward martingale arrays. Litovsk. Mat. Sb. 19 (1979) 4, 153 - 165, 212.
		- [32] RYCHLTK, *Z.:* The order of approximation in the random central limit theorem. Lect. Notes Math. 656 (1978), 225-236.
		- [33] SAZONOV, V. V., and V. *V.* ULYANOV: On the accuracy of normal approximation. J.,Multivariate Anal. 12 (1982), 371-384.
		- [34] SCOTT, $D. J.:$ Central limit theorems for martingales and for processes with stationary increments using a Skohorod representation approach. Adv. AppI. Probab. 5 (1973), 119-137.

[35] STROBEL, J.: Könvergenzraten in zentralen Grenzwertsätzen für Martingale. Dissertation. Bochum: Univ. Bochum 1978. General Limit Theorems with o -Rates 307.

(35) STROBEL, J.: Könvergenzraten in zentralen Grenzwertsätzen für Martingale. Dissertation.

Bochum: Univ. Bochum 1978.

(36) ZOLOTAREV, V. M.: Approximation of distributions of

[36] ZOLOTAREV, V. M.: Approximation of distributions of sums of independent random variab les with values in infinite-dimensional spaces. Theor. Probab. Appl. 21 (1976), 721–737. [37] ZOLOTAREV, V. M.: On pseudomoments. Theor. Probab. Appl. 23 (1978), 269–278.

[38] ZOLOTAREV, V. M.: Ideal metrics in the problems of probability theory and mathematical statistics. Austral. J. Statist. 21 (1979), 193-208. • Bochum : Univ. Bochum 1978.

[36] ZOLOTAREV, V. M.: Approximation of distributions

les with values in infinite-dimensional spaces. The

[37] ZOLOTAREV, V. M.: On pseudomoments. Theor. P

[38] ZOLOTAREV, V. M.: Ideal met - EDGIMIN: UNIV. Boordin 1978.

26 ELOLOTAREV, V. M.: Approximation of distributions of sums of independent random variab.

18 El Soutra Rev. V. M.: On pseudomoments. Theor. Probab. Appl. 23 (1978), 721 – 737.

37] ZOLOTA OLOTAREV, V. M.: On pseudomoments. Theor. Prob.

(OLOTAREV, V. M.: Ideal metrics in the problems of

itatistics. Austral. J. Statist. 21 (1979), 193–208.

Manuskripteingang: 28. 09. 1987

VERFASSER:

Prof. Dr. h. c. PAUL L

- Manuskripteingang: 28. 09. 1987

BEL, J.: Könvergenzraten

1m: Univ. Bochum 1978.

IAREV, V. M.: Approximat

th values in infinite-dimentaries.

TAREV, V. M.: Ideal metric

TAREV, V. M.: Ideal metric

dies. Austral. J. Statist. 21

Manuskripteingang: 28. • SOLUTION, V. M. Hearthcurcs in the problems of probability theory and m

statistics. Austral. J. Statist. 21 (1979), 193–208.

• Manuskripteingang: 28. 09. 1987

• VERFASSER:

• Prof. Dr. h. c. Paul L. Butzer und Dipl.-M AREV, V. M.: Approximation of distributions of sums of independent and values in infinite-dimensional spaces. Theor. Probab. Appl. 21

TAREV, V. M.: Ideal metrics in the problems of probability theor

FAREV, V. M.: Ideal m Very Extrinsity, V. M.: Ideal Heenes in the

statistics. Austral. J. Statist. 21 (1979),

Manuskripteingang: 28. 09. 1987

VERFASSER:

Prof. Dr. h. c. PAUL L. BUTZER un

Lehrstuhl A für Mathematik

der Rheinisch-Westfälisc Prof. Dr. h. c. PAUL L. BUTZER und Dipl. Math. HERIBERT KIRSCHFINK
Lehrstuhl A für Mathematik
der Rheinisch-Westfälischen Technischen Hochschule

•

 $\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}$, where $\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}$ is the set of $\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}$, where $\mathcal{L}^{\mathcal{L}}_{\mathcal{L}}$