Zeitschrift für Analysis
und ihre Anwendungen
Bd. 7 (4) 1988, S. 337 – 346

Solvability of Boundary Value Problems for the Inclusion $u_{tt} - u_{xx} \in g(t, x, u)$ via the Theory of Multi-Valued A-Proper Maps

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Es wird die Existenz eines Koinzidenzpunktes x für die Inklusionsbeziehung

$$Lx \in \Gamma(x) \tag{$*$}$$

untersucht, wobei $L: D(L) \subset E \to F$ ein linearer hyperbolischer Operator und $\Gamma: E \to 2^F$ eine konvex-mengenwertige Abbildung ist. Es wird gezeigt, daß jede solche monotone halbstetige Abbildung Γ schwach A-eigentlich ist. Es werden verschiedene Existenzsätze für ($*$) bewiesen und die Resultate auf eine Randwertaufgabe für die Inklusionsbeziehung $u_{tt} - u_{xx} \in g(t, x, u)$ angewandt.

Изучается существование точки коинцидентности x для включения

$$Lx \in \Gamma(x), \tag{$*$}$$

где $L: D(L) \subset E \to F$ — гиперболический линейный оператор и $\Gamma: E \to 2^F$ — выпукло-множествозначное отображение. Показывается, что каждое такое монотонное полу-непрерывное отображение Γ является слабо A-собственным. Устанавливаются несколько теорем существования для ($*$) и эти результаты применяются к граничной задачи для включения $u_{tt} - u_{xx} \in g(t, x, u)$.

The existence of a coincidence point x for the inclusion

$$Lx \in \Gamma(x) \tag{$*$}$$

is studied where $L: D(L) \subset E \to F$ is a linear hyperbolic operator and $\Gamma: E \to 2^F$ is a convex-valued map. It is shown that any such monotone demi-continuous map Γ is weakly A-proper. Some existence theorems for ($*$) are established and the results are applicated to a boundary value problem for the inclusion $u_{tt} - u_{xx} \in g(t, x, u)$.

0. Let E, F be Banach spaces, $L: D(L) \subset E \to F$ a linear operator, and $\Gamma: E \to 2^F$ a convex-valued map. The aim of this paper is to establish some existence theorems for the coincidence inclusion

$$Lx \in \Gamma(x), \tag{0.1}$$

and to present an application of those results to a boundary value problem for the inclusion

$$u_{tt} - u_{xx} \in g(t, x, u). \tag{0.2}$$

For a Fredholm operator L, the problem (0.1) has been studied by many authors, see [4, 9, 17, 19]. That case can be applied, for example, to differential inclusions of the type $Lu \in g(x, u, Du)$ where L is elliptic or, in particular, $Lu = u''$ (see [10, 18])

1) Supported by the NSERCC postdoctoral fellowship.

but not to hyperbolic operators, as it is the case with (0.2). Of our interest is to extend to the multi-valued case some of the results presented in [11] where L was a closed operator with $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = \infty$, and where $\Gamma = f: E \to F$ was a single-valued map. We refer the reader to [11] for the extensive references to that topic.

The main result of Section 1, Theorem 1.3, is an analogue of the Leray-Schauder Theorem on a priori bounds. The proof is based on

a) the method of Topological Transversality introduced and developed by Granas [6, 8],

b) the coincidence theory of Fredholm operators of index zero with multi-valued maps presented in [9],

c) adopting, to multi-valued maps, the technique of A-proper maps originally due to Petryshyn ([15]; see also [16]) who was later joined by Browder [2] in the research of that class of maps.

In Section 2, we show that any monotone demi-continuous convex-valued map is weakly A-proper. For a single-valued map, results of this type appear in most of papers using, explicitly or not, A-proper mapping techniques, e.g. in [11—14, 20]. Our direct approach, however, seems to be particularly simple.

In Section 3, we study generalized solutions of a periodic-Dirichlet problem for the inclusion (0.2). Its formulation is modelled on a result of Mawhin [12].

In Section 4, we derive the existence of a generalized optimal solution of the periodic-Dirichlet problem for the equation $u_{tt} - u_{xx} = f(t, x, u)$, where f is discontinuous in all variables (normally, f is assumed measurable in (t, x) and continuous in u). The growth condition on f in Corollary 4.2 comes from [12]. Differential inclusions of first order, as a tool for investigating equations with discontinuous right-hand side had been first considered by Filippov [7] and they became a frequent tool in the optimal control theory (see e.g., [1: Chapter II]). For the elliptic boundary value problems, the concept of what we call optimal solution (also called solution in the sense of Filippov) has been used by Chang [5]. To the authors' best knowledge, optimal solutions of hyperbolic equations with discontinuous right-hand side have not been previously studied.

1. In what follows, E, F are Banach spaces and $L: \mathrm{D}(L) \subset E \to F$ is a densely defined linear (not necessarily bounded) operator. We assume the following conditions on L:

(L1) L is closed (i.e. the graph of L is closed in $E \times F$);

(L2) $\operatorname{Ker} L$ and $\mathrm{R}(L)$ are closed and topologically complemented, i.e. $E = \operatorname{Ker} L \oplus E_0$, $F = F_0 \oplus \mathrm{R}(L)$, E_0 and F_0 closed;

(L3) $\dim \operatorname{Ker} L = \operatorname{codim} \mathrm{R}(L) = \infty$.

With the decomposition given in (L2) we associate the linear projections $P: E \to E_0$, $Q: F \to \mathrm{R}(L)$. By $K: \mathrm{R}(L) \to E_0$ we denote the right inverse of L, i.e. the inverse of the operator $L_{|E_0 \cap \mathrm{D}(L)}: E_0 \cap \mathrm{D}(L) \to \mathrm{R}(L)$. Since L is closed, K is a linear bounded operator.

We associate with L a *Fredholm Factorization* $\Pi = \{E_n, P_n, F_n, Q_n\}_{n \in \mathbb{N}}$ defined in [11]. We recall what this means. First, by $\{E_n'\}$ and $\{F_n'\}$ we denote the dense finite-dimensional filtrations of $\operatorname{Ker} L$ and F_0 respectively, i.e. $E_n \subset E_{n+1}$ and $F_n \subset F_{n+1}$ for $n \in \mathbb{N}$, $\overline{\bigcup E_n} = \operatorname{Ker} L$, $\overline{\bigcup F_n} = F_0$ and $\dim E_n = \dim F_n < \infty$, $n \in \mathbb{N}$. We suppose that there exist linear projections $P_n': \operatorname{Ker} L \to F_n'$, $Q_n': F_0 \to F_n'$ such that

$$\|P_n'x - x\| = d(x, E_n'),\ x \in \operatorname{Ker} L, \quad \text{and} \quad \|Q_n'y - y\| = d(y, F_n'),\ y \in F_0, \tag{1.1}$$

where $d(\cdot, \cdot)$ denotes the distance from a point to a set. We finally define $E_n := E_n' \oplus E_0$, $F_n := F_n' \oplus \mathrm{R}(L)$, $P_n := P + P_n'(I - P): E \to E_n$, $Q_n := Q + Q_n'(I - Q): F \to F_n$, $n \in \mathbb{N}$. Thus $\Pi = \{E_n, P_n, F_n, Q_n\}_{n\in\mathbb{N}}$ is defined. From (1.1) it follows that

$$\begin{aligned} &P_n'x \to x \quad \text{for} \quad x \in \operatorname{Ker} L; &&Q_n'y \to y \quad \text{for} \quad y \in F_0; \\ &P_nx \to x \quad \text{for} \quad x \in E; &&Q_ny \to y \quad \text{for} \quad y \in F. \end{aligned} \tag{1.2}$$

We also note that, for any n, the operator

$$L_n := L_{|E_n \cap \mathrm{D}(L)}: E_n \cap \mathrm{D}(L) \to F_n$$

is a Fredholm operator of index zero. If $J_n: E_n' \to F_n'$ is any isomorphism, then $T_n = J_n(I - P): E_n \to F_n$ is a Fredholm resolvent of finite rank of L_n, i.e. an operator of finite rank such that $L_n + T_n$ is bijective. We note that $(L_n + T_n)^{-1} = KQ + J_n^{-1}(I - Q): F_n \to E_n$ is a bounded operator which is compact whenever K is compact.

Let us recall that a multi-valued map $\Gamma: X \to 2^Y$ is called *upper semicontinuous* if $\{x \in X \mid \Gamma(x) \subset U\}$ is open for any open U in Y. We are concerned with maps $\Gamma: X \to 2^F$, where $X \subset E$ or $X \subset E \times [0, 1]$ has a non-empty intersection with $E_n \cap \mathrm{D}(L)$ (respectively with $(E_n \cap \mathrm{D}(L)) \times [0, 1]$) for all but finitely many n. In general, Γ is not assumed upper semicontinuous but we always assume that the values of Γ are non-empty closed and convex. We use the notation

$$X_n = X \cap E_n \quad (\text{resp.}, X_n = X \cap (E_n \times [0, 1])), \qquad \Gamma_n = Q_n \circ \Gamma: X_n \to 2^{F_n}.$$

A map $\Gamma: X \to 2^F$ is called *L-compact* if the map $\Gamma_n: X_n \to 2^{F_n}$ is L_n-compact in the sense of [9], for a.e. n. This means that $(L_n + T_n)^{-1} \circ \Gamma_n: X_n \to 2^{E_n}$ is upper semicontinuous and it sends bounded sets to relatively compact sets, where T_n is defined above. Let now $A \subset X$ be a pair of closed bounded subsets of E and let $\Gamma: X \to 2^F$ be an L-compact map. We say that $\Gamma \in \mathcal{K}_L(X, A)$ if, for a.e. n, Γ_n has no coincidence point with L_n in $A_n \cap \mathrm{D}(L)$, i.e. $Lx \notin \Gamma_n(x)$ for all $x \in A_n \cap \mathrm{D}(L)$. (We assume about A that $A_n \cap \mathrm{D}(L) = A \cap E_n \cap \mathrm{D}(L) \neq \emptyset$ for a.e. n.) Such Γ is called *L-essential* if Γ_n is L_n-essential in the sense of [9], for a.e. n. This means that every map $\Phi_n: X_n \to 2^{F_n}$ with $\Phi_{n|A_n} \equiv \Gamma_{n|A_n}$ has a coincidence point with L_n in $X_n \cap \mathrm{D}(L)$. We say that $\mathcal{H}: X \times [0, 1] \to 2^F$ is a *homotopy* between maps $\Gamma, \Phi \in \mathcal{K}_L(X, A)$ if

(i) $\mathcal{H}(\cdot, 0) \equiv \Gamma$ and $\mathcal{H}(\cdot, 1) \equiv \Phi$;

(ii) $\mathcal{H}_n$ is L_n-compact, for a.e. n;

(iii) $Lx \notin \mathcal{H}_n(x, t)$ for all $x \in A_n \cap \mathrm{D}(L)$, $t \in [0, 1]$.

We write $\Gamma \sim \Phi$. It is easy to show that $\sim$ is an equivalence relation.

Proposition 1.1 (Topological Transversality Theorem): *Suppose that $\Gamma \sim \Phi$ in $\mathcal{K}_L(X, A)$. Then Γ is L-essential if and only if Φ is L-essential.*

The proof is an immediate consequence of [9: II. 4.5] and the above definitions ∎

Unlike in [9], an L-essential map may have no coincidence point with L on $X \cap \mathrm{D}(L)$. We must therefore restrict our study to maps defined below. A map $\Gamma: X \to 2^F$ is called *A-proper* (respectively *weakly A-proper*) with respect to the Fredholm factorization Π of L if for any bounded sequence $\{x_k \in X_{n_k} \cap \mathrm{D}(L)\}$ with $d(Lx_k, \Gamma_{n_k}(x_k)) \to 0$ as $n_k \to \infty$, there exists a subsequence $\{x_{k_l}\}$ of $\{x_k\}$ converging (resp. weakly converging) to an element $x \in X \cap \mathrm{D}(L)$ such that $Lx \in \Gamma(x)$. Although this definition depends on Π, we will normally omit saying "with respect to Π" having a given Fredholm factorization of L in mind.

Proposition 1.2: *Let U be an open bounded subset of E and let $\Gamma: E \to 2^F$ be a weakly A-proper/L-compact map. If $\Gamma_{|\bar{U}} \in \mathcal{K}_L(\bar{U}, \partial U)$, we suppose that it is L-essential. Then there exists an $x \in \overline{\mathrm{co}U} \cap \mathrm{D}(L)$ such that $Lx \in \Gamma(x)$. If $\Gamma_{|\bar{U}}$ is not in $\mathcal{K}_L(\bar{U}, \partial U)$, then the same conclusion follows.*

Proof: If $\Gamma_{|\bar{U}}$ is L-essential in $\mathcal{K}_L(\bar{U}, \partial U)$, then, for a.e. n, there exists $x_n \in U_n \cap \mathrm{D}(L)$ with $Lx_n \in \Gamma_n(x_n)$. Since Γ is weakly A-proper, there is a subsequence $x_{n_k} \rightharpoonup x \in E \cap \mathrm{D}(L)$ such that $Lx \in \Gamma(x)$. To reach the conclusion, we just note that a weak limit of a sequence of points in $\bar{U}$ belongs to the closed convex hull $\overline{\mathrm{co}U}$ of U.

The condition $\Gamma_{|\bar{U}} \notin \mathcal{K}_L(\bar{U}, \partial U)$ means that there exists a sequence $n_k \to \infty$, $x_k \in (\partial U)_{n_k} \cap V(L)$, with $Lx_k \in \Gamma_{n_k}(x_k)$. The conclusion again follows from the definition of weakly A-proper map ∎

Theorem 1.3: *Let $S: E \to F$ be an A-proper L-compact linear operator such that $L - S$ is injective and let $\Gamma: E \to 2^F$ be a weakly A-proper L-compact map satisfying the following condition: There exists a constant $M > 0$ (called a priori bound) and $n_0 \in \mathbb{N}$ such that every solution $x \in E_n \cap \mathrm{D}(L)$ of*

$$Lx \in (1 - \lambda) S_n x + \lambda \Gamma_n(x) \tag{1.3}$$

with $n > n_0$ and $\lambda \in (0, 1)$ must have norm less than M. Then there exists an $x \in E \cap \mathrm{D}(L)$ with $\|x\| \leqq M$ such that $Lx \in \Gamma(x)$.

Proof: Let $U = \{x \in E \mid \|x\| < M\}$. We may assume that $\Gamma_{|\bar{U}} \in \mathcal{K}_L(\bar{U}, \partial U)$, since otherwise the conclusion follows directly from Proposition 1.2. Next, we note that $S_{|\bar{U}} \in \mathcal{K}_L(\bar{U}, \partial U)$. Indeed, the contrary would imply the existence of $n_k \to \infty$, $x_k \in E_{n_k} \cap \mathrm{D}(L)$, $\|x_k\| = M$, such that $Lx_k = S_{n_k} x_k$. Since S is A-proper, a subsequence of x_k tends to an x with $Lx = Sx$, $\|x\| = M$. This contradicts injectivity of $L - S$. It now follows from [9: II. 4.9] that S is L-essential in $\mathcal{K}_L(\bar{U}, \partial U)$. It is easy to verify that the formula

$$\mathcal{H}(x, t) := (1 + t) Sx + t\Gamma(x), \qquad x \in \bar{U}, t \in [0, 1],$$

defines a homotopy from S to Γ in $\mathcal{K}_L(\bar{U}, \partial U)$. In fact, since we already know that S and Γ are in $\mathcal{K}_L(\bar{U}, \partial U)$, the property (iii) of homotopy must be only verified for $t \in (0, 1)$ and that is exactly guaranteed by (1.3). The conclusion now follows from Propositions 1.1 and 1.2 ∎

Remark 1.4: Analogous definitions and results can be given for L-condensing multi-valued maps, i.e. such Γ that, for a.e. n, Γ_n is L_n-condensing with respect to a given measure of noncompactness on E_n (see [9, 11]).

2. In this section $E = F = H$ is a separable Hilbert space with a scalar product $(\cdot, \cdot)$, and $L: \mathrm{D}(L) \subset H \to H$ is a self-adjoint operator satisfying (L1)–(L3) of the previous section. This implies that $\mathrm{R}(L) = \mathrm{Ker}\, L^\perp$. Let P be the orthogonal projection on H onto $\mathrm{R}(L)$, and let $\{v_n\}_{n \in \mathbb{N}}$ be an orthonormal set spanning a dense subspace of Ker L. We define

$$H_n' = \mathrm{Lin}\,\{v_1, \dots, v_n\},$$

$$P_n': \mathrm{Ker}\, L \to H_n', \; P_n'x = \sum_{k=1}^{n} (x, v_k)\, v_k,$$

$$H_n = H_n' \oplus \mathrm{R}(L), \qquad P_n = P + P_n'(\mathrm{I} - P): H \to H_n.$$

If follows that $\Pi = \{H_n, P_n, H_n, P_n\}$ is a Fredholm factorization associated with L. A map $\Gamma: H \to 2^H$ is called *monotone* if for any $x_1, x_2, z_1, z_2 \in H$ with $z_1 \in \Gamma(x_1)$ and $z_2 \in \Gamma(x_2)$, we have $(z_1 - z_2, x_1 - x_2) \geqq 0$. It is known that a compact-valued map $\Gamma: H \to 2^H$ is upper semicontinuous if and only if Γ sends relatively compact sets to relatively compact sets and the graph of Γ is closed in $H \times H$. Above, we mean the strong (norm) topology on $H \times H$. By considering either ($H \times H$, weak topology) or (H, strong topology) $\times$ (H, weak topology), we arrive with the following definitions: Let $\Gamma: H \to 2^H$ be a map with non-empty closed convex values which sends bounded sets to bounded sets. Such Γ is called

weakly continuous if for any sequences $x_k \rightharpoonup x$, $z_k \rightharpoonup z$ with $z_k \in \Gamma(x_k)$, $k \in \mathbb{N}$, we have $z \in \Gamma(x)$;

demi-continuous if for any sequences $x_k \to x$, $z_k \rightharpoonup z$ with $z_k \in \Gamma(x_k)$, $k \in \mathbb{N}$, we have $z \in \Gamma(x)$.

Theorem 2.1: *Any weakly continuous map $\Gamma: H \to 2^H$ is weakly A-proper.*

Proof: Let $\{x_k \in H_{n_k} \cap D(L)\}$ be a bounded sequence with $d(Lx_k, \Gamma_{n_k}(x_k)) \to 0$ as $n_k \to \infty$ and let $\{z_k \in \Gamma(x_k)\}$ be a sequence with $Lx_k - P_{n_k} z_k \to 0$. Since $\{(x_k, z_k)\}$ is bounded and $H \times H$ reflexive, we may assume by passing to a subsequence that $(x_k, z_k) \rightharpoonup (x, z)$. Therefore $z \in \Gamma(x)$. We need to show that $Lx = z$. First note that

$$Lx_k = PLx_k = P(Lx_k - P_{n_k} z_k) + Pz_k \rightharpoonup Pz$$

as $k \to \infty$. Since L is closed, it is weakly closed hence $Lx = Pz$. It remains to show that

$$(I - P)\, z = 0. \tag{2.1}$$

Indeed, since $z_k \rightharpoonup z$ and $P'_{n_k}(I - P)\, w \to (I - P)\, w$ for all $w \in H$, we have

$$\left(P'_{n_k}(I - P)\, z_k - P'_{n_k}(I - P)\, z, w\right) = \left(z - z_k, P'_{n_k}(I - P)\, w\right) \to 0 \tag{2.2}$$

for all w. Next, it is verified that $P'_{n_k}(I - P)\, z_k = (P - I)\,(Lx_k - P_{n_k} z_k) \to 0$, so by (2.2), $P'_{n_k}(I - P)\, z \rightharpoonup 0$. On the other hand, $P'_{n_k}(I - P)\, z \to (I - P)\, z$, and (2.1) follows ∎

Theorem 2.2: *Let $\Gamma: H \to 2^H$ be a demi-continuous map such that either Γ or $-\Gamma$ is monotone. Then Γ is weakly continuous. Consequently, Γ is weakly A-proper.*

Proof: Since the weak continuity of $-\Gamma$ is equivalent to that of Γ, it is enough to give the proof for Γ monotone. Let $x_k \rightharpoonup x$, $z_k \rightharpoonup z$, $z_k \in \Gamma(x_k)$, $k \in \mathbb{N}$. We have to show that $z \in \Gamma(x)$. Suppose for contradiction that $z \notin \Gamma(x)$. $\Gamma(x)$ is closed and convex, so the Hahn-Banach separation theorem implies the existence of $y \in H$ and $\alpha \in \mathbb{R}$ such that

$$(y, z) < \alpha < (y, w) \text{ for all } w \in \Gamma(x). \tag{2.3}$$

We put $y_m = x - t_m y$, where $t_m > 0$, $t_m \to 0$, and we choose $\bar{z}_m \in \Gamma(y_m)$. Without loss of generality, $\bar{z}_m \rightharpoonup z_0 \in H$. Since $y_m \to x$ and Γ is demi-continuous, we have $z_0 \in \Gamma(x)$. Γ is monotone, therefore $(z_k - \bar{z}_m, x_k - x + t_m y) \geqq 0$ for all $k, m \in \mathbb{N}$ and, consequently,

$$0 \leqq \varliminf_{k\to\infty} (z_k - \bar{z}_m, x_k - x + t_m y) = \varliminf_{k\to\infty} (z - \bar{z}_m, t_m y),$$

for all $m \in \mathbb{N}$. Thus $0 \leqq \varliminf_{k\to\infty} (z_k - \bar{z}_m, y)$ for all $m \in \mathbb{N}$. Since $\bar{z}_m \rightharpoonup z_0$, passing to the limit as $m \to \infty$ gives

$$0 \leqq \varliminf_{k\to\infty} (z_k - z_0, y). \tag{2.4}$$

By comparing (2.3) with (2.4) and using the fact that $z_k \rightharpoonup z$, we get

$$0 \leqq \varliminf_{k\to\infty} (z_k, y) - (z_0, y) < \varliminf_{k\to\infty} (z_k, y) - \alpha < \varliminf_{k\to\infty} (z_k, y) - (z, y)$$
$$= \varliminf_{k\to\infty} (z_k - z, y) = \lim_{k\to\infty} (z_k - z, y) = 0$$

and the contradiction is reached. The second conclusion now follows from Theorem 2.1 ∎

We complete this section by showing a relation between demi-continuous and upper semicontinuous maps.

Lemma 2.3: *If $A: H \to H$ is a compact linear operator and $\Gamma: H \to 2^H$ a demi-continuous map, then $A \circ \Gamma$ is upper semicontinuous and compact. Consequently, if the right inverse K of L is compact and Γ demi-continuous, then Γ is L-compact.*

Proof: Since $A \circ \Gamma$ sends bounded sets to relatively compact sets, we must only show that the graph of this map is closed in $H \times H$. For, let $x_k \to x$ and $Az_k \to y$, where $z_k \in \Gamma(x_k)$, $k \in \mathbb{N}$. We have to show that $y \in A(\Gamma(x))$. Since $\{z_k\}$ is bounded, there is a subsequence $z_{k_l} \rightharpoonup z$. Γ is demi-continuous, so $z \in \Gamma(x)$. Any continuous linear operator is weakly continuous, hence $Az_{k_l} \rightharpoonup Az$. Consequently $y = Az \in A(\Gamma(x))$. The second conclusion follows from the first one by the comment on the Fredholm resolvent T_n in Section 1 ∎

Remark 2.4: For simplicity of the presentation we have restricted the study to $E = F = H$ a separable Hilbert space. However, the above definitions and results can be easily extended to self-ajoint operators $L: \mathrm{D}(L): E \to E^*$, where E is a separable reflexive Banach space of π_α-type, E^* its dual and the scalar product is replaced by the duality product, see [11].

3. Let us recall that a multifunction $g: D \subset \mathbb{R}^m \to 2^{\mathbb{R}^q}$ is called *measurable* if $\{x \in D \mid g(x) \subset U\}$ is Lebesgue measurable for any open U in $\mathbb{R}^q$. By a *single-valued selection* of g we mean a function $s: D \to \mathbb{R}^q$ such that $s(x) \in g(x)$ for all x. It is known that any measurable multifunction has a single-valued selection, c.f. [3]. A multifunction $g: \mathbb{R}^q \to 2^{\mathbb{R}^q}$ will be called *monotone* if it is monotone in the sense of Section 2 with respect to the Euclidean scalar product $\langle\cdot,\cdot\rangle$. The Euclidean norm is denoted by $|\cdot|$. We use the notation $|g(x)| := \sup\{|z| \mid z \in g(x)\}$. Let $J = (0, 2\pi) \times (0, \pi)$ and let $g: J \times \mathbb{R}^q \to 2^{\mathbb{R}^q}$ be a multifunction with non-empty closed convex values. Such g is called a *Carathéodory multifunction* if

(a) $(t, x) \to g(t, x, u)$ is measurable for all $u \in \mathbb{R}^q$;

(b) $u \to g(t, x, u)$ is upper semicontinuous for all $(t, x) \in J$.

We let $H = L^2(J, \mathbb{R}^q)$. A *Nemitskii (multivalued) operator* for a Carathéodory multi-function g, $\Gamma_g: H \to 2^H$, is defined by

$$\Gamma_g(u) = \{v \in H \mid v(t, x) \in g(t, x, u(t, x)) \text{ for a.e. } (t, x) \in J\}.$$

Lemma 3.1: *Let g be a Carathéodory multifunction (with non-empty closed convex values) and suppose that there exist an $h \in L^2(J, \mathbb{R})$ and a constant $c > 0$ such that* *for all* $u \in \mathbb{R}^q$ *and a.e.* $(x, t) \in J$,

$$|g(t, x, u)| \leqq h(t, x) + c|u|. \tag{3.1}$$

Then $\Gamma_g: H \to 2^H$ *is a demi-continuous map.*

Proof: We refer the reader to Lemma 4.2 in [7]. This result was proved there for functions with values in $\mathbb{R}$ but it immediately follows in this formulation since $H = L^2(J, \mathbb{R}) \times \cdots \times L^2(J, \mathbb{R})$ (q copies) ∎

We are concerned with the existence of solutions u of the inclusion

$$u_{tt} - u_{xx} \in g(t, x, u). \tag{3.2}$$

We say that $u \in H$ is a generalized solution of the periodic-Dirichlet problem for (3.2) if there exists a selection $s \in H$ of the multifunction $(t, x) \to g(t, x, u(t, x))$ such that

$$\int_J \langle u, v_{tt} - v_{xx} \rangle = \int_J \langle s, v \rangle \tag{3.3}$$

for all $v \in C^2(J, \mathbb{R}^q)$ satisfying the boundary conditions

$$\left.\begin{array}{l} v(0, x) = v(2\pi, x), v_t(0, x) = v_t(2\pi, x), x \in [0, \pi] \\ v(t, 0) = v(t, \pi) = 0, \qquad t \in [0, 2\pi] \end{array}\right\}. \tag{3.4}$$

It is verified that the following set of functions is orthonormal in H:

$$v_{m,n,k}(t, x) = \frac{1}{\pi} e^{imt} \sin(nx) e_k, m \in \mathbb{Z}, n \in \mathbb{N}, k = 1, 2, \ldots, q,$$

where $\{e_k\}$ is the standard basis of $\mathbb{R}^q$. We define $L: \mathrm{D}(L) \subset H \to H$ by

$$\mathrm{D}(L) = \left\{u \in H \,\middle|\, \sum_{k=1}^{q} \sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}} \left| (n^2 - m^2)(u, v_{m,n,k})\right|^2 < \infty \right\},$$

$$Lu = \sum_{k=1}^{q} \sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}} (n^2 - m^2)(u, v_{m,n,m}) v_{m,n,k}.$$

By standard arguments (see [15]), it follows that: $\mathrm{D}(L)$ is dense in H, L verifies assumptions which were made in Section 2, the spectrum of L is $\sigma(L) = \{n^2 - m^2 \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$ and the right inverse $K: \mathrm{R}(L) \to H$ of L is compact. Moreover, $u \in H$ is a generalized solution of the periodic-Dirichlet problem for $u_{tt} - u_{xx} = h(t, x)$, $h \in H$, if and only if $u \in \mathrm{D}(L)$ and $Lu \in \Gamma_g(u)$.

Theorem 3.2: *Let* $g: J \times \mathbb{R}^q \to 2^{\mathbb{R}^q}$ *be a Carathéodory multifunction satisfying the following conditions:*

(i) *Either* $u \to g(t, x, u)$ *or* $u \to -g(t, x, u)$ *is monotone for a.e.* $(t, x) \in J$.

(ii) *There exist* $r \in \mathbb{R} \setminus \sigma(L)$, $0 \leqq \delta < d(r, \sigma(L))$, *and* $h \in L^2(J, \mathbb{R})$ *such that* $|g(t, x, u) - ru| \leqq \delta |u| + h(t, x)$, *for all* $u \in \mathbb{R}^q$ *and a.e.* $(t, x) \in J$.

Then there exists $u \in H \cap \mathrm{D}(L)$ *with* $Lu \in \Gamma_g(u)$.

Proof: We shall use Theorem 1.3 for the map Γ_g and the operator $S = rI$. From (i) it follows by integration that Γ_g or $-\Gamma_g$ is monotone and, from (ii), (3.1) follows. Hence, Lemma 3.1 and Theorem 2.2 imply that Γ is weakly A-proper. Since K is compact, both Γ and S are L-compact, by Lemma 2.3. By the choice of r, $L - S$ is bijective with the bounded inverse of norm $\|(L - rI)^{-1}\| = (d(r, \sigma(L)))^{-1} =: \alpha$. In particular, it easily follows that S is A-proper. It remains to determine an a priori bound M on solutions $u \in H_n \cap \mathrm{D}(L)$ of (1.3). For, let $n \in \mathbb{N}$, $\lambda \in (0, 1)$ and $u \in H_n \cap \mathrm{D}(L)$ satisfy $Lu \in (1 - \lambda) rP_n u + \lambda P_n(\Gamma_g(u))$. Since $P_n u = u$, we obtain $u \in \lambda \times (L - rI)^{-1} P_n(\Gamma_g(u) - ru)$. Hence, by using the condition (ii), we get $\|u\| \leqq \|(L - rI)^{-1}\| (\delta \|u\| + \|h\|) = \alpha\delta \|u\| + \alpha \|h\|$, therefore $\|u\| \leqq \alpha \|h\|/(1 - \alpha\delta)$. It remains to choose for M any number greater than the right-hand side of the last inequality ∎

Remark 3.3: In the definition of Carathéodory multifunction, the condition (a) can be replaced by the following weaker condition:

(a') $(t, x) \to g(t, x, u)$ is measurable for all u from a dense subset of $\mathbb{R}^q$.

The conclusion of Lemma 3.1 will remain true, see [10].

4. In what follows, $f: J \times \mathbb{R} \to \mathbb{R}$ is a function which does not satisfy, a priori, any continuity condition. In this case, there is no hope of solving any boundary value problem for

$$u_{tt} - u_{xx} = f(t, x, u) \tag{4.1}$$

in the usual sense but we may look for optimal solutions in the following sense: Let $\underline{f}, \bar{f}: J \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\underline{f}(t, x, u) = \varliminf_{v \to u} f(t, x, v), \qquad \bar{f}(t, x, u) = \varlimsup_{v \to u} f(t, x, v).$$

An optimal solution of (4.1) is a function u verifying

$$\underline{f}(t, x, u) \leqq u_{tt} - u_{xx} \leqq \bar{f}(t, x, u) \tag{4.2}$$

for a.e. $(t, x) \in J$. A generalized solution of the periodic-Dirichlet problem for (4.2) is such $u \in L^2(J, \mathbb{R})$ that

$$\int_J \underline{f}(t, x, u) \cdot v \leqq \int_J u \cdot (v_{tt} - v_{xx}) \leqq \int_J \bar{f}(t, x, y) \cdot v \tag{4.3}$$

for all $v \in C^2(\bar{J}, \mathbb{R})$ satisfying the boundary conditions (3.4).

Theorem 4.1: *Suppose $f: J \times \mathbb{R} \to \mathbb{R}$ verifies the following conditions:*

(i) *The set of those $(t, x) \in J$ that $a < \underline{f}(t, x, u) \leqq \bar{f}(t, x, y) < b$ is measurable for all $a, b, u \in \mathbb{R}$.*

(ii) *$u \to f(t, x, u)$ is either non-decreasing or non-increasing for a.e. $(t, x) \in J$.*

(iii) *There exist $r \in \mathbb{R} \setminus \sigma(L)$, $0 \leqq \delta < d(r, \sigma(L))$ and $h \in L^2(J, \mathbb{R})$ such that, for all $u \in \mathbb{R}$ and a.e. $(t, x) \in J$, $|f(t, x, u) - ru| \leqq \delta|u| + h(t, x)$.*

Then there exists a generalized solution $u \in L^2(J, \mathbb{R})$ of the periodic-Dirichlet problem for (4.2).

Proof: Let $g(t, x, u) = [\underline{f}(t, x, u), \bar{f}(t, x, u)]$. The problem (4.2) is equivalent to (3.2), and (4.3) to (3.3) with $q = 1$. It follows from Proposition 4.4 in [10], from (i) and (iii) that g is Carathéodory multifunction. It instantly follows that g satisfies the hypotheses of Theorem 3.2, hence the conclusion ∎

Corollary 4.2: *Let f be as in Theorem 4.1 with the condition (iii) replaced by the following two:*

a) *For any $M > 0$ there exists $h \in L^2(J, \mathbb{R})$ such that*

$$|f(t, x, u)| \leqq h(t, x) \text{ for a.e. } (t, x) \in J \text{ and all } u \in \mathbb{R} \text{ with } |u| < M.$$

b) *There exists an $a < b$ with $(a, b) \cap \sigma(L) = \emptyset$ such that*

$$a < \underline{\lim}_{|u| \to \infty} \frac{f(t, x, w)}{u} \leqq \overline{\lim}_{|u| \to \infty} \frac{f(t, x, u)}{u} < b \quad \text{for a.e. } (t, x) \in J.$$

Then the conclusion of Theorem 4.1 *remains true.*

Proof: For verification that a) and b) imply the condition (iii) of Theorem 4.1, we refer the reader to [11] ∎

REFERENCES

[1] AUBIN, J. P., and A. CELLINA: Differential Inclusions: Set-Valued Maps and Viability Theory. Berlin—Heidelberg—New York: Springer-Verlag 1984.

[2] BROWDER, F. E.: Approximation solvability of nonlinear functional equations in normed linear spaces. Arch. Rat. Mech. Anal. **26** (1967), 33—41.

[3] CASTAING, C., and M. VALADIER: Convex analysis and measurable multifunctions. Lect. Notes Math. **580** (1977), 1—278.

[4] CELLINA, A., and A. LASOTA: A new approach to the definition of topological degree for multivalued mappings. Rend. Acad. Naz. dei Lincei **47** (1969), 434—440.

[5] CHANG, K. C.: The obstacle problem and partial differential equations with discontinuous nonlinearities. Commun. Pure Appl. Math. **33** (1980), 117—146.

[6] DUGUNDJI, J., and A. GRANAS: Fixed point theory, Vol. I (Monografie Mat. No. 61). Warsaw: Polish Sci. Publ. 1982.

[7] FILIPPOV, A. F.: Differential equations with discontinuous right hand side. Math. Sbornik **51** (1960), 99—128. Engl. Transl.: Transl. Ann. Math. Soc. **42** (1964), 199—232.

[8] GRANAS, A.: Homotopy extension theorem in Banach spaces and some of its applications to the theory of non-linear equations. Bull. Acad. Sci. **7** (1959), 387—394.

[9] KACZYNSKI, T.: Topological transversality and nonlinear equations in locally convex spaces. To appear.

[10] KACZYNSKI, T.: On some applications of fixed point principles for single- and multi-valued maps to differential equations. To appear.

[11] KRAWCEWICZ, W.: Résolution des équations semilinéaires avec la partie linéaire à noyau de dimension infinie via des applications A-propres. To appear.

[12] MAWHIN, J.: Compacité, monotonie et convexité dans l'étude de problèmes aux limites semi-linéaires. Séminaire d' Analyse Moderne, 19. Univ. de Sherbrooke (Quebec, Canada) 1981, 1—144.

[13] MAWHIN, J., and M. WILLEM: Operators of monotone type and alternative problems with infinite dimensional kernel. In: Recent Advances in Diff. Equ., Trieste 1978 (ed.: R. Conti). New York: Acad. Press 1981, 295—307.

[14] PASCALI, D.: Approximation solvability of a semilinear wave equation. Libertas Math. **4** (1984), 73—79.

[15] PETRYSHYN, W. V.: Remarks on the approximation-solvability of nonlinear functional equations. Arch. Rat. Mech. Anal. **26** (1967), 43—49.

[16] PETRYSHYN, W. V.: Using degree theory for densely defined A-proper maps in the solvability of semilinear equations with unbounded and noninvertible linear part. Nonlin. Anal. — Theory, Methods & Appl. **4** (1980), 259—281.

[17] PRUSZKO, T.: A coincidence degree for L-compact convex-valued mappings and its application to the Picard problem for orientor fields. Bull. Acad. Polon. Sci. **27** (1979), 895—902.

[18] PRUSZKO, T.: Topological degree methods in multi-valued boundary value problems. Nonlin. Anal. — Theory, Methods & Appl. 5 (1981), 859—973.
[19] TARAFDAR, E., and S. K. TEO: On the existence of solutions of the equation $Lx \in Nx$ and a coincidence degree theory. J. Austral. Math. Soc. (Ser. A) 28 (1979), 139—176.
[20] WILLEM, M.: Topology and semilinear equations at resonance in Hilbert space. Nonlin. Anal. — Theory, Methods & Appl. 5 (1981), 517—524.

Manuskripteingang: 22. 09. 1987

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